EE2023 TUTORIAL 1 (SOLUTIONS)

Solution to Q.1

Write z in polar form:

$$z = x + jy = |z|\exp(j\angle z).$$

Since adding integer multiples of 2π to $\angle z$ does not affect the value of z, we may also express z as

$$z = |z|\exp\big(j(\angle z + 2k\pi)\big)$$

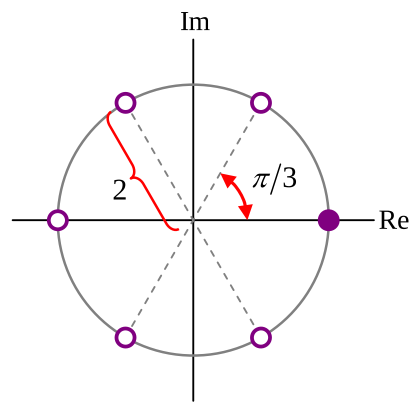
where k is an integer. This leads to

$$\sqrt[N]{z} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right); \quad k = 0,1,\cdots,N-1,$$

which yields the N distinct values of $\sqrt[N]{z}$.

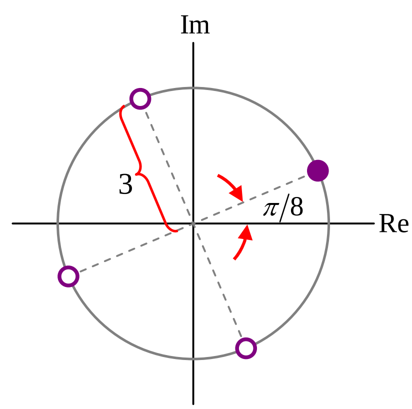
$\sqrt[6]{64}$:

$$\begin{cases} z = 64 \;\rightarrow\; \begin{cases}|z| = 64\\ \angle z = 0\end{cases} \\ \sqrt[6]{64} = |z|^{1/N}\exp\left(j\left(\dfrac{\angle z}{N} + \dfrac{2k\pi}{N}\right)\right)\Bigg|_{z=64,\ N=6} \\ \quad = 2\exp\left(j\left(\dfrac{k\pi}{3}\right)\right); \quad k = 0,1,\cdots,5 \\ \quad = \begin{cases}(\ 2),\ 2\exp\left(j\left(\dfrac{\pi}{3}\right)\right);\ 2\exp\left(j\left(\dfrac{2\pi}{3}\right)\right); \\ (-2),\ 2\exp\left(j\left(\dfrac{4\pi}{3}\right)\right);\ 2\exp\left(j\left(\dfrac{5\pi}{3}\right)\right)\end{cases}\end{cases}$$



$\sqrt[4]{j81}$:

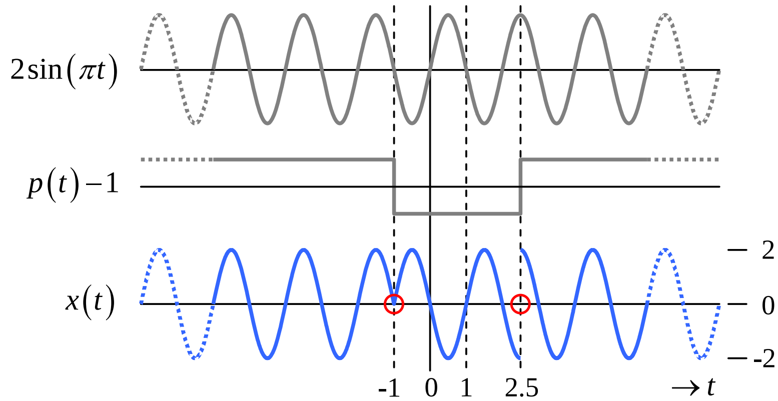
$$\begin{cases} z = j81 \;\rightarrow\; \begin{cases}|z| = 81\\ \angle z = \dfrac{\pi}{2}\end{cases} \\ \sqrt[4]{j81} = |z|^{1/N}\exp\left(j\left(\dfrac{\angle z}{N} + \dfrac{2k\pi}{N}\right)\right)\Bigg|_{z=81,\ N=4} \\ \quad = 3\exp\left(j\left(\dfrac{\pi}{8} + \dfrac{k\pi}{2}\right)\right); \quad k = 0,1,\cdots,3 \\ \quad = \begin{cases}3\exp\left(j\left(\dfrac{\pi}{8}\right)\right),\ 3\exp\left(j\left(\dfrac{5\pi}{8}\right)\right), \\ 3\exp\left(j\left(\dfrac{9\pi}{8}\right)\right),\ 3\exp\left(j\left(\dfrac{13\pi}{8}\right)\right)\end{cases}\end{cases}$$



Solution to Q.2

(a) $p(t) = 2 - 2\text{rect}\left(\frac{t-0.75}{3.5}\right)$

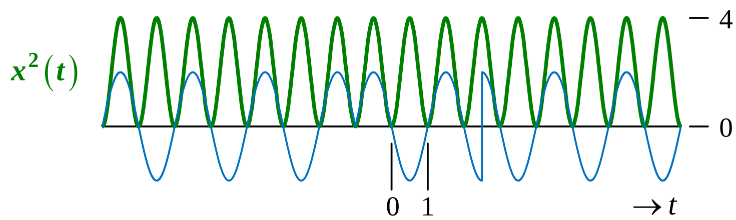
(b) By inspection, $x(t)$ is not periodic.



Notice the π rad $(\text{or } 180^\circ)$ phase jumps in $x(t)$ occurring at the zero crossings of $p(t)-1$.

(c)

$$\begin{aligned} x^2(t) &= 4\sin^2(\pi t)\underbrace{\left(p(t)-1\right)^2}_{1} \\ &= 4\sin^2(\pi t) \\ &= 2\left(1-\cos(2\pi t)\right) \end{aligned}$$



Note that $x^2(t)$ is periodic with a period of $T = 1$.

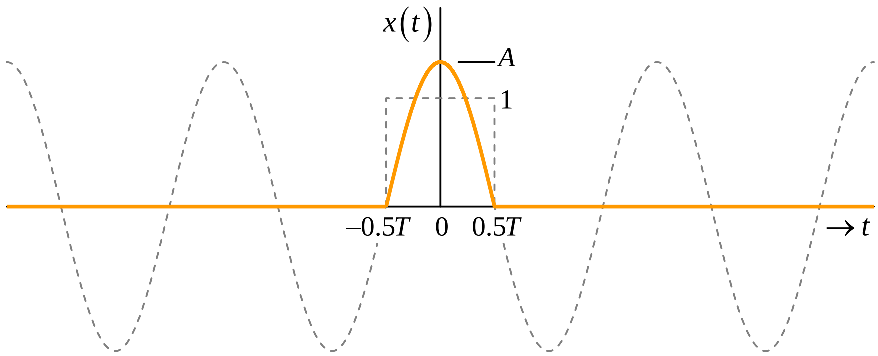
Total energy:
$$E = \int_{-\infty}^{\infty} x^2(t)\,dt = \sum_{n=-\infty}^{\infty} \underbrace{\int_{nT}^{(n+1)T} x^2(t)\,dt}_{\substack{\int \text{over one period} \\ \text{thus independent} \\ \text{of } n}} = \left(\underbrace{\int_0^T x^2(t)\,dt}_{\text{finite}}\right)\underbrace{\sum_{n=-\infty}^{\infty} 1}_{\infty} = \infty$$

Average Power:
$$P = \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt}_{\substack{x^2(t) \text{ is periodic. } \therefore \\ P \text{ can be obtained} \\ \text{by averaging over} \\ \text{one period.}}} = \int_{-0.5}^{0.5} 2\left(1-\cos(2\pi t)\right)dt = 2$$

Conclusion: $x(t)$ is an aperiodic power signal.

Solution to Q.3

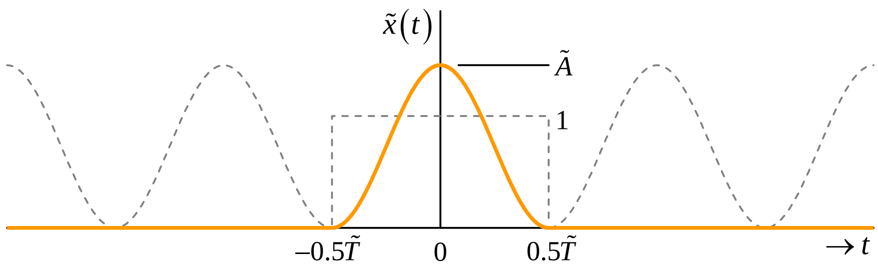
Half-cosine pulse: $x(t) = A\cos\left(\frac{\pi t}{T}\right)\text{rect}\left(\frac{t}{T}\right)$



$$x^2(t) = \frac{A^2}{2}\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]\text{rect}\left(\frac{t}{T}\right)$$

$$\textbf{Energy:}\ E = \frac{A^2}{2}\int_{-0.5T}^{0.5T} 1 + \underbrace{\cos\left(\frac{2\pi t}{T}\right)}_{\int \text{over one period} = 0} dt = \frac{1}{2}A^2 T$$

Raised-cosine pulse: $\tilde{x}(t) = \frac{\tilde{A}}{2}\left(1+\cos\left(\frac{2\pi t}{\tilde{T}}\right)\right)\text{rect}\left(\frac{t}{\tilde{T}}\right)$



$$\tilde{x}^2(t) = \frac{\tilde{A}^2}{4}\left[\frac{3}{2}+2\cos\left(\frac{2\pi t}{\tilde{T}}\right)+\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)\right]\text{rect}\left(\frac{t}{\tilde{T}}\right)$$

$$\textbf{Energy:}\ \tilde{E} = \frac{\tilde{A}^2}{4}\int_{-0.5\tilde{T}}^{0.5\tilde{T}} \frac{3}{2} + \underbrace{2\cos\left(\frac{2\pi t}{\tilde{T}}\right)}_{\int \text{over one period} = 0} + \underbrace{\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)}_{\int \text{over two periods} = 0} dt = \frac{3}{8}\tilde{A}^2\tilde{T}$$

Both $x(t)$ and $\tilde{x}(t)$ will have the same energy if $A^2T = \frac{3}{4}\tilde{A}^2\tilde{T}$.

Solution to Q.4

(a) Let m, n and k be positive integers. Based on the definition of a periodic signal, we have

$$\begin{aligned} x_1(t) &= x_1(t+T_1) = x_1(t+mT_1) \\ x_2(t) &= x_2(t+T_2) = x_2(t+nT_2) \end{aligned}.$$

Hence,

$$x_0(t) = x_1(t) + x_2(t) = x_1(t+mT_1) + x_2(t+nT_2).$$

If $mT_1 = nT_2 = kT_0$, then

$$x_0(t) = x_1(t+kT_0) + x_2(t+kT_0) = x_0(t+kT_0)$$

which shows that $x_0(t)$ is periodic with a period of T_0 and a fundamental frequency of $f_0 = \dfrac{1}{T_0}$.

Under this condition, $\left(\text{i.e. } mT_1 = nT_2 = kT_0\right)$,

$$\left.\begin{aligned} \frac{1}{T_1} &= m\frac{1}{kT_0} \\ \frac{1}{T_2} &= n\frac{1}{kT_0} \end{aligned}\right\} \text{ implying that } \begin{cases} \cdots\cdots \ \dfrac{1}{kT_0} \text{ are common factors of } \left\{\dfrac{1}{T_1}, \dfrac{1}{T_2}\right\} \ \cdots\cdots \\ \qquad\qquad\text{and} \\ \dfrac{1}{T_0} \text{ is the highest common factor (HCF) of } \left\{\dfrac{1}{T_1}, \dfrac{1}{T_2}\right\} \\ \qquad\qquad\text{or} \\ f_0 \text{ is the highest common factor (HCF) of } \{\, f_1,\ f_2 \,\} \end{cases}$$

Conclusion: For $x_0(t)$ to be periodic, $\{\, f_1,\ f_2 \,\}$ must have a HCF. In turn, this HCF is the fundamental frequency of $x_0(t)$

(b) i.

$$x(t) = \cos(3.2t) + \sin(1.6t) + \exp(j2.8t) \ \cdots \begin{cases} \cos(3.2t) & \text{has a frequency of 3.2 } rad/s \\ \sin(1.6t) & \text{has a frequency of 1.6 } rad/s \\ \exp(j2.8t) & \text{has a frequency of 2.8 } rad/s \end{cases}$$

Highest common factor (HCF) of $\{3.2,\ 1.6,\ 2.8\}$ exists and is equal to 0.4. Thus, $x(t)$ is periodic and has a fundamental frequency of $0.4\ rad/s$ $\left(\text{or } 0.2/\pi\ Hz\right)$ and a fundamental period of $5\pi\ s$.

REMARKS: *Although $x(t)$ is periodic with a fundamental frequency of $0.4\ rad/s$, it does not contain the fundamental frequency component itself.*

(b) ii.

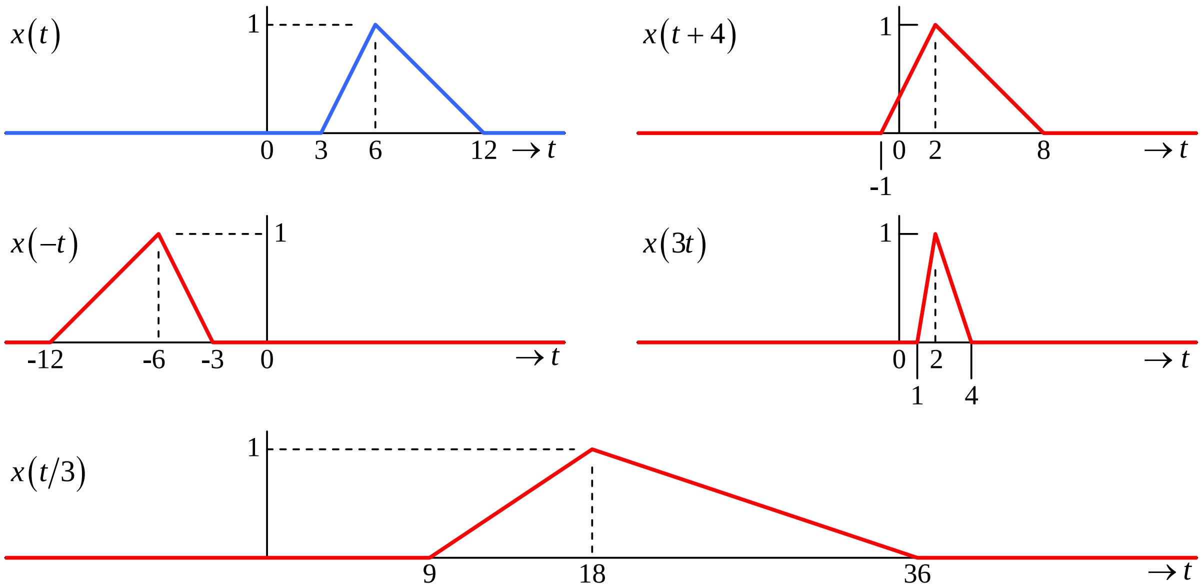
$$x(t) = \cos(4t) + \sin(\pi t) \ \cdots \begin{cases} \cos(4t) \text{ has a frequency of } 4\ rad/s \\ \sin(\pi t) \text{ has a frequency of } \pi\ rad/s \end{cases}$$

Highest common factor (HCF) of $\{4,\ \pi\}$ does not exist. Thus, $x(t)$ is not periodic.

REMARKS: *Summing sinusoids does not necessarily lead to a periodic signal unless the frequencies of the sinusoids are harmonics of a common fundamental frequency.*

Solution to Q.5

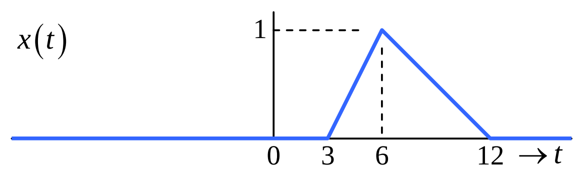
(a)



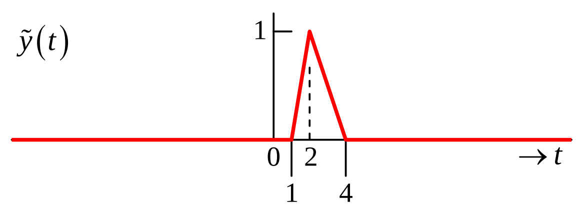
(b) We observe that $y(t)$ is a time-scaled, -reversed and –shifted version of $x(t)$.

For problems of this nature, we should start with time-scaling first since it involves linear warping of the time axis. If we were to start with time-shifting and/or time-reversal, we may have to redo them after time-scaling. However, this sequence of operation need not be followed if we are sketching the signal from the mathematical expression.

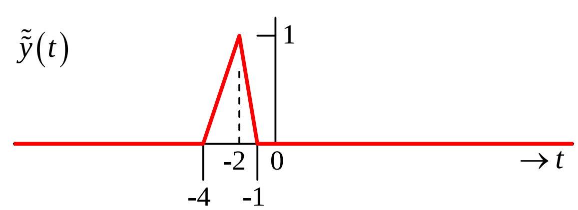
Comparing $x(t)$ and $y(t)$, we note that $y(t)$ involves time-scaling (or contraction) of $x(t)$ by a factor of 3.



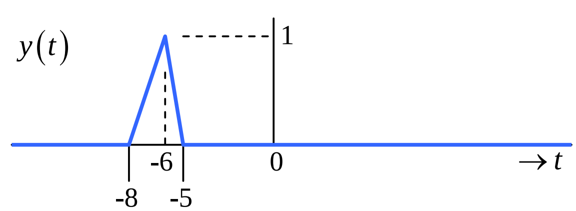
Time-scaling of $x(t)$: $\tilde{y}(t) = x(3t)$



Time-reversal of $\tilde{y}(t)$: $\tilde{\tilde{y}}(t) = \tilde{y}(-t) = x(-3t)$



Time shifting of $\tilde{\tilde{y}}(t)$: $\begin{cases} y(t) = \tilde{\tilde{y}}(t+4) \\ \quad\;\; = x(-3(t+4)) \end{cases}$



$\therefore\ \boldsymbol{y(t) = x(-3(t+4))}$

Solution to Q.6

Let

$$\delta(t) = \lim_{\Delta\to 0} x(t) \quad \text{where} \quad x(t) = \frac{1}{\Delta}\text{rect}\left(\frac{t}{\Delta}\right).$$

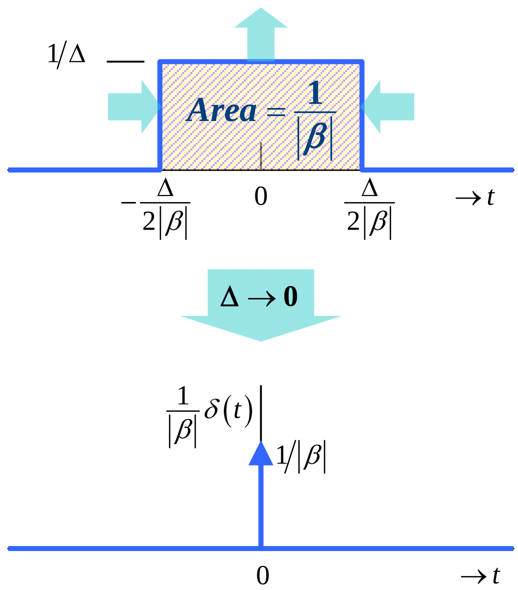
Here, we have implicitly assumed that $\Delta > 0$.

Due to symmetry, we have $\delta(\beta t) = \delta(|\beta| t)$.

$$\delta(\beta t) = \delta(|\beta| t)$$

$$= \lim_{\Delta\to 0} x(|\beta| t) = \lim_{\Delta\to 0} \frac{1}{\Delta}\text{rect}\left(\frac{t}{\Delta/|\beta|}\right)$$

But $\left(\begin{array}{l} \displaystyle\lim_{\Delta\to 0} \frac{1}{\Delta}\text{rect}\left(\frac{t}{\Delta/|\beta|}\right) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases} \\ \displaystyle\lim_{\Delta\to 0} \int_{-\infty}^{\infty} \frac{1}{\Delta}\text{rect}\left(\frac{t}{\Delta/|\beta|}\right) dt = \frac{1}{|\beta|} \end{array}\right).$



Hence, $\delta(\beta t) = \displaystyle\lim_{\Delta\to 0} \frac{1}{\Delta}\text{rect}\left(\frac{t}{\Delta/|\beta|}\right) = \frac{1}{|\beta|}\delta(t)$.

Domain-scaling of $\delta(\cdot)$ is often encountered in transforming a spectrum containing $\delta(\cdot)$ between cyclic-frequency (f) and radian frequency (ω) domain:

$$\left[\delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f)\right] \quad \text{or} \quad \left[\delta(f) = \delta\left(\frac{\omega}{2\pi}\right) = 2\pi\delta(\omega)\right].$$