1. PROBLEM SETTING

- There are N edge-devices/nodes and one Federating Server (FS).
- Let $\hat{\theta}_{i,t}$ denote the neural network of i^{th} node at time t and $\mathcal{L}_{i,t}(\theta_{j,t}, z_{i,t})$ denote the loss of node i at time t using the neural network of node j at time t, where $z_{i,t}$ denotes the data (or a batch of data points) seen by node i at time t.
- Let $\omega_{ij,t}$ denote the weight given by node i to node j based upon the loss incurred by node i upon using the neural network of node j.

2. ALGORITHM

for $t \in \{1, \ldots, T\}$ **do**

- Update $\omega_{ij,t+1} \leftarrow \omega_{ij,t} e^{-\eta \mathcal{L}_{i,t}(\hat{\theta}_{j,t}, z_{i,t})}$, $\forall\ j \in \{1, \ldots, N\}$, where $\eta > 0$ is a hyper-parameter and $\omega_{ij,1} = 1\ \forall\ i, j \in \{1, \ldots, N\}$
- Compute $\theta_{i,t} = arg\min_\theta \mathcal{L}_{i,t}(\theta, z_{i,t})$ and broadcast to all nodes
- Update $\hat{\theta}_{i,t+1} \leftarrow \dfrac{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1} \theta_{j,t}}{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}}$ and broadcast to all nodes, where $\mathcal{S}_{t+1} \subseteq \{1, \ldots, N\}$ is randomly chosen and is of cardinality K

3. GUARANTEES

Let $\omega_{ij,t+1} = \omega_{ij,t} e^{-\eta \mathcal{L}_{i,t}(\hat{\theta}_{j,t}, z_{i,t})} = e^{-\eta L_{j,t}}$. Define $\Phi_t = \log \sum_{j \in \mathcal{S}_t} \omega_{ij,t}$. Assume that $\mathcal{S}_1 = \{1, \ldots, N\}$ and hence $\Phi_1 = \log N$. Let $\boldsymbol{p}_{i,t}$ denote the distribution over $j = \{1, \ldots, N\}$ such that $p_{ij,t} = \dfrac{\omega_{ij,t}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t}}$

Consider $\mathbb{E}[\Phi_{t+1} - \Phi_t]$

$$= \mathbb{E}\Big[\log\Big(\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}\Big) - \log\Big(\sum_{j \in \mathcal{S}_t} \omega_{ij,t}\Big)\Big]$$

$$= \mathbb{E}\left[\log\left(\frac{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t}}\right)\right]$$

$$= \mathbb{E}\left[\log\left(\frac{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t}} \frac{\sum_{j \in \mathcal{S}_t} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t+1}}\right)\right]$$

$$= \mathbb{E}\left[\log\left(\frac{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t+1}}\right)\left(\frac{\sum_{j \in \mathcal{S}_t} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t}}\right)\right]$$

$$= \mathbb{E}\left[\log\left(\frac{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t+1}}\right)\left(\frac{\sum_{j \in \mathcal{S}_t} \omega_{ij,t} e^{-\eta \mathcal{L}_{i,t}(\hat{\theta}_{j,t}, z_{i,t})}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t}}\right)\right]$$

$$= \mathbb{E}\left[\log\left(\frac{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t+1}}\right)\left(\sum_{j \in \mathcal{S}_t} \left(\frac{\omega_{ij,t}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t}}\right) e^{-\eta \mathcal{L}_{i,t}(\hat{\theta}_{j,t}, z_{i,t})}\right)\right]$$

$$= \mathbb{E}\left[\log\left(\frac{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t+1}}\right)\left(\sum_{j \in \mathcal{S}_t} p_{ij,t} e^{-\eta \mathcal{L}_{i,t}(\hat{\theta}_{j,t}, z_{i,t})}\right)\right]$$

$$= \mathbb{E}\left[\log\left(\frac{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t+1}}\right)\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\right], \text{ where } X = -\mathcal{L}_{i,t}(\hat{\theta}_{j,t}, z_{i,t})$$

$$= \mathbb{E}\left[\log\left(\frac{\sum_{j \in \mathcal{S}_{t+1}} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t} \omega_{ij,t+1}}\right) + \log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\right], \text{ where } X = -\mathcal{L}_{i,t}(\hat{\theta}_{j,t}, z_{i,t})$$

$$= \mathbb{E}\left[\log\left(\frac{\sum_{j \in \mathcal{S}_{t+1} \setminus \mathcal{S}_t} \omega_{ij,t+1} + \sum_{j \in \mathcal{S}_{t+1} \cap \mathcal{S}_t} \omega_{ij,t+1}}{\sum_{j \in \mathcal{S}_t \setminus \mathcal{S}_{t+1}} \omega_{ij,t+1} + \sum_{j \in \mathcal{S}_t \cap \mathcal{S}_{t+1}} \omega_{ij,t+1}}\right) + \log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\log \frac{\left(\frac{\sum_{j\in\mathcal{S}_{t+1}\setminus\mathcal{S}_t}\omega_{ij,t+1}}{\sum_{j\in\mathcal{S}_{t+1}\cap\mathcal{S}_t}\omega_{ij,t+1}}\right)+1}{\left(\frac{\sum_{j\in\mathcal{S}_{t}\setminus\mathcal{S}_{t+1}}\omega_{ij,t+1}}{\sum_{j\in\mathcal{S}_{t}\cap\mathcal{S}_{t+1}}\omega_{ij,t+1}}\right)+1} + \log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\right] \\
&= \mathbb{E}\left[\log \frac{\left(\frac{\sum_{j\in\mathcal{S}_{t+1}\setminus\mathcal{S}_t}e^{-\eta L_{j,t}}}{\sum_{j\in\mathcal{S}_{t+1}\cap\mathcal{S}_t}e^{-\eta L_{j,t}}}\right)+1}{\left(\frac{\sum_{j\in\mathcal{S}_{t}\setminus\mathcal{S}_{t+1}}e^{-\eta L_{j,t}}}{\sum_{j\in\mathcal{S}_{t}\cap\mathcal{S}_{t+1}}e^{-\eta L_{j,t}}}\right)+1} + \log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\right] \\
&\leq \mathbb{E}\left[\log \frac{\left(\frac{|\mathcal{S}_{t+1}\setminus\mathcal{S}_t|e^{-\eta L_{min,t}}}{|\mathcal{S}_{t+1}\cap\mathcal{S}_t|e^{-\eta L_{max,t}}}\right)+1}{\left(\frac{|\mathcal{S}_{t}\setminus\mathcal{S}_{t+1}|e^{-\eta L_{max,t}}}{|\mathcal{S}_{t}\cap\mathcal{S}_{t+1}|e^{-\eta L_{min,t}}}\right)+1} + \log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\right] \\
&= \mathbb{E}\left[\log \frac{\left(\frac{|\mathcal{S}_{t+1}\setminus\mathcal{S}_t|}{|\mathcal{S}_{t+1}\cap\mathcal{S}_t|}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\frac{|\mathcal{S}_{t}\setminus\mathcal{S}_{t+1}|}{|\mathcal{S}_{t}\cap\mathcal{S}_{t+1}|}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\right] \\
&= \mathbb{E}\left[\log \frac{\left(\frac{|\mathcal{S}_{t+1}\setminus\mathcal{S}_t|}{|\mathcal{S}_{t+1}\cap\mathcal{S}_t|}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\frac{|\mathcal{S}_{t}\setminus\mathcal{S}_{t+1}|}{|\mathcal{S}_{t}\cap\mathcal{S}_{t+1}|}e^{\eta(L_{min,t}-L_{max,t})}\right)+1}\right] + \mathbb{E}\left[\log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\right] \\
&\leq \log \frac{\left(\mathbb{E}\Big[\frac{|\mathcal{S}_{t+1}\setminus\mathcal{S}_t|}{|\mathcal{S}_{t+1}\cap\mathcal{S}_t|}\Big]e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\mathbb{E}\Big[\frac{|\mathcal{S}_{t}\setminus\mathcal{S}_{t+1}|}{|\mathcal{S}_{t}\cap\mathcal{S}_{t+1}|}\Big]e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \mathbb{E}\Big[\log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\Big], \text{ by Jensen's inequality} \\
&= \log \frac{\left(\sum_{m=1}^{K-1}\Big(\frac{K-m}{m}\Big)\mathbb{P}\Big[|\mathcal{S}_t\cap\mathcal{S}_{t+1}|=m\Big]e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1}\Big(\frac{K-m}{m}\Big)\mathbb{P}\Big[|\mathcal{S}_t\cap\mathcal{S}_{t+1}|=m\Big]e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \mathbb{E}\Big[\log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\Big], \text{ since } |\mathcal{S}_{t+1}|=|\mathcal{S}_t|=K \\
&= \log \frac{\left(\sum_{m=1}^{K-1}\Big(\frac{K-m}{m}\Big)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_K}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1}\Big(\frac{K-m}{m}\Big)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_K}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \mathbb{E}\Big[\log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta X}]\right)\Big] \\
&= \log \frac{\left(\sum_{m=1}^{K-1}\Big(\frac{K-m}{m}\Big)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_K}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1}\Big(\frac{K-m}{m}\Big)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_K}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \mathbb{E}\Big[\log\left(\mathbb{E}_{\boldsymbol{p}_{i,t}}[e^{\eta(X-\mathbb{E}_{\boldsymbol{p}_{i,t}}[X])+\eta\mathbb{E}_{\boldsymbol{p}_{i,t}}[X]}]\right)\Big] \\
&\leq \log \frac{\left(\sum_{m=1}^{K-1}\Big(\frac{K-m}{m}\Big)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_K}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1}\Big(\frac{K-m}{m}\Big)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_K}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \frac{\eta^2}{8} + \eta\mathbb{E}\Big[\mathbb{E}_{\boldsymbol{p}_{i,t}}[X]\Big], \text{ by Hoeffding's Lemma}
\end{aligned}
$$

$$= \log \frac{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \frac{\eta^2}{8} - \eta\mathbb{E}\Big[\mathbb{E}_{\boldsymbol{p}_{i,t}}[\mathcal{L}_{i,t}(\hat{\theta}_{j,t}, z_{i,t})]\Big]$$

$$\leq \log \frac{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \frac{\eta^2}{8} - \eta\mathbb{E}\Big[\mathcal{L}_{i,t}(\mathbb{E}_{\boldsymbol{p}_{i,t}}[\hat{\theta}_{j,t}], z_{i,t})\Big]$$, by convexity of loss function in the first argument

So we have the upper bound as,

$$\mathbb{E}[\Phi_{T+1}-\Phi_T] \leq \sum_{t=1}^{T} \log \frac{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \frac{\eta^2 T}{8} - \eta\mathbb{E}[\textstyle\sum_{t=1}^{T}\mathcal{L}_{i,t}(\mathbb{E}_{\boldsymbol{p}_{i,t}}[\hat{\theta}_{j,t}], z_{i,t})],$$

i.e.,

$$\mathbb{E}[\Phi_{T+1}]-\mathbb{E}[\Phi_T] \leq \sum_{t=1}^{T} \log \frac{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \frac{\eta^2 T}{8} - \eta\mathbb{E}[\textstyle\sum_{t=1}^{T}\mathcal{L}_{i,t}(\mathbb{E}_{\boldsymbol{p}_{i,t}}[\hat{\theta}_{j,t}], z_{i,t})]$$

Again consider $\mathbb{E}[\Phi_{T+1}] - \mathbb{E}[\Phi_1]$
$= \mathbb{E}[\log(\sum_{j\in\mathcal{S}_{T+1}} \omega_{ij,T+1})] - \log N$
$= \mathbb{E}[\log(\sum_{j\in\mathcal{S}_{T+1}} e^{-\eta L_{j,T}})] - \log N$
$\geq \mathbb{E}[\log(\max_{j\in\mathcal{S}_{T+1}} e^{-\eta L_{j,T}})] - \log N$
$= -\eta\mathbb{E}[\min_{j\in\mathcal{S}_{T+1}}(L_{j,T})] - \log N$
$= -\eta\mathbb{E}[\min_{j\in\mathcal{S}_{T+1}}(L_{j,T}) - \min_{j\in\{1,\dots,N\}}(L_{j,T})] - \eta\mathbb{E}[\min_{j\in\{1,\dots,N\}}(L_{j,T})] - \log N$
$= -\eta\Big[\mathbb{P}[arg\min_{j\in\{1,\dots,N\}}(L_{j,T}) \in \mathcal{S}_{T+1}][\min_{j\in\{1,\dots,N\}}(L_{j,T}) - \min_{j\in\{1,\dots,N\}}(L_{j,T})] + \mathbb{P}(arg\min_{j\in\{1,\dots,N\}}(L_{j,T}) \notin \mathcal{S}_{T+1}][\min_{j\in\mathcal{S}}(L_{j,T}) - \min_{j\in\{1,\dots,N\}}(L_{j,T})]\Big] - \eta\mathbb{E}[\min_{j\in\{1,\dots,N\}}(L_{j,T})] - \log N$
$= -\eta\left(1 - \frac{{}^{N-1}C_{K-1}}{{}^{N}C_{K}}\right)\Big[\min_{j\in\mathcal{S}_{T+1}}(L_{j,T}) - \min_{j\in\{1,\dots,N\}}(L_{j,T})\Big] - \eta\mathbb{E}[\min_{j\in\{1,\dots,N\}}(L_{j,T})] - \log N$, where $K = |\mathcal{S}_{T+1}|$
$= -\eta\left(1 - \frac{K}{N}\right)\Big[\min_{j\in\mathcal{S}_{T+1}}(L_{j,T}) - \min_{j\in\{1,\dots,N\}}(L_{j,T})\Big] - \eta\mathbb{E}[\min_{j\in\{1,\dots,N\}}(L_{j,T})] - \log N$, where $K = |\mathcal{S}_{T+1}|$

So we have the lower bound as, $\mathbb{E}[\Phi_{T+1}] - \mathbb{E}[\Phi_T] \geq -\eta\left(1 - \frac{K}{N}\right)\Big[\min_{j\in\mathcal{S}_{T+1}}(L_{j,T}) - \min_{j\in\{1,\dots,N\}}(L_{j,T})\Big] - \eta\mathbb{E}[\min_{j\in\{1,\dots,N\}}(L_{j,T})] - \log N$

Combining both the bounds yields,

$$-\eta\left(1 - \frac{K}{N}\right)\Big[\min_{j\in\mathcal{S}_{T+1}}(L_{j,T}) - \min_{j\in\{1,\dots,N\}}(L_{j,T})\Big] - \eta\mathbb{E}[\min_{j\in\{1,\dots,N\}}(L_{j,T})] - \log N \leq$$
$$\sum_{t=1}^{T} \log \frac{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1}\left(\frac{K-m}{m}\right)\frac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \frac{\eta^2 T}{8} - \eta\mathbb{E}[\textstyle\sum_{t=1}^{T}\mathcal{L}_{i,t}(\mathbb{E}_{\boldsymbol{p}_{i,t}}[\hat{\theta}_{j,t}], z_{i,t})]$$

On re-arranging terms, we get
$\mathbb{E}[\sum_{t=1}^{T}\mathcal{L}_{i,t}(\mathbb{E}_{\boldsymbol{p}_{i,t}}[\hat{\theta}_{j,t}], z_{i,t})] - \mathbb{E}[\min_{j\in\{1,\dots,N\}}(L_{j,T})] \leq$

$\frac{1}{\eta}\sum_{t=1}^{T}\log\frac{\left(\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \frac{\eta T}{8} + \frac{\log N}{\eta} + \left(1 - \frac{K}{N}\right)\Big[\min_{j\in\mathcal{S}_{T+1}}(L_{j,T}) -$
$\min_{j\in\{1,\ldots,N\}}(L_{j,T})\Big]$
i.e.,
$\mathbb{E}[\sum_{t=1}^{T}\mathcal{L}_{i,t}(\mathbb{E}_{\boldsymbol{p}_{i,t}}[\hat{\theta}_{j,t}], z_{i,t})] - \mathbb{E}[\min_{j\in\{1,\ldots,N\}}(L_{j,T})] \leq$
$\frac{1}{\eta}\sum_{t=1}^{T}\log\frac{\left(\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{\eta(L_{min,t}-L_{max,t})}\right)+1} + \frac{\eta T}{8} + \frac{\log N}{\eta} + \left(1 - \frac{K}{N}\right)L_T,$
where $\min_{j\in\mathcal{S}_{T+1}}(L_{j,T}) - \min_{j\in\{1,\ldots,N\}}(L_{j,T}) \leq L_T$
Let us further upper bound the following term
$\sum_{t=1}^{T}\log\frac{\left(\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}e^{\eta(L_{min,t}-L_{max,t})}\right)+1}$
Observe that $\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}} = \prod\limits_{i=1}^{m}\dfrac{K-i+1}{N-i+1}$ and as $0 \leq K \leq N$, we have $\prod\limits_{i=1}^{m}\dfrac{K-i+1}{N-i+1} \leq 1\;\forall\, m \in \{1,\ldots K-1\}$
This means $\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}} \leq 1\;\forall\, m \in \{1,\ldots K-1\}$
So,
$\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}$
$\leq \sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)$
$= \sum\limits_{m=1}^{K-1}\left(\dfrac{K}{m} - 1\right)$
$= K\sum\limits_{m=1}^{K-1}\left(\dfrac{1}{m}\right) - (K-1)$
$\leq K\sum\limits_{m=1}^{K-1}\left(1\right) - (K-1)$
$= K(K-1) - (K-1)$
$= (K-1)^2$
We thus have, $\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}} \leq (K-1)^2$
Also, observe that $\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}} = \prod\limits_{i=1}^{m}\dfrac{K-i+1}{N-i+1}$ and as $0 \leq K \leq N$, we have $\prod\limits_{i=1}^{m}\dfrac{K-i+1}{N-i+1} > 0\;\forall\, m \in \{1,\ldots K-1\}$
This means $\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}} > 0\;\forall\, m \in \{1,\ldots K-1\}$
So,
$\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}}$
$> \sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right).0$
$= 0$
We thus have, $\sum\limits_{m=1}^{K-1}\left(\dfrac{K-m}{m}\right)\dfrac{{}^{N-m}C_{K-m}}{{}^{N}C_{K}} > 0$

Using these results in $\sum_{t=1}^{T} \log \frac{\left(\sum_{m=1}^{K-1} \left(\frac{K-m}{m}\right) \frac{{}^{N-m}C_{K-m}}{{}^{N}C_K} e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1} \left(\frac{K-m}{m}\right) \frac{{}^{N-m}C_{K-m}}{{}^{N}C_K} e^{\eta(L_{min,t}-L_{max,t})}\right)+1}$, we get

$\sum_{t=1}^{T} \log \frac{\left(\sum_{m=1}^{K-1} \left(\frac{K-m}{m}\right) \frac{{}^{N-m}C_{K-m}}{{}^{N}C_K} e^{-\eta(L_{min,t}-L_{max,t})}\right)+1}{\left(\sum_{m=1}^{K-1} \left(\frac{K-m}{m}\right) \frac{{}^{N-m}C_{K-m}}{{}^{N}C_K} e^{\eta(L_{min,t}-L_{max,t})}\right)+1} < \sum_{t=1}^{T} \log \left((K-1)^2 e^{-\eta(L_{min,t}-L_{max,t})}+1\right)$

So we have,

$\mathbb{E}[\sum_{t=1}^{T} \mathcal{L}_{i,t}(\mathbb{E}_{\boldsymbol{p}_{i,t}}[\hat{\theta}_{j,t}], z_{i,t})] - \mathbb{E}[\min_{j\in\{1,\dots,N\}}(L_{j,T})] <$

$\frac{1}{\eta}\sum_{t=1}^{T} \log \left((K-1)^2 e^{-\eta(L_{min,t}-L_{max,t})}+1\right) + \frac{\eta T}{8} + \frac{\log N}{\eta} + \left(1-\frac{K}{N}\right) L_T,$

where $\min_{j\in\mathcal{S}_{T+1}}(L_{j,T}) - \min_{j\in\{1,\dots,N\}}(L_{j,T}) \leq L_T$

Further, if we allow $(K-1)^2 e^{-\eta(L_{min,t}-L_{max,t})}+1 \approx (K-1)^2 e^{-\eta(L_{min,t}-L_{max,t})} = e^{-\eta(L_{min,t}-L_{max,t})+2\log(K-1)}$, then

$\frac{1}{\eta}\sum_{t=1}^{T} \log \left((K-1)^2 e^{-\eta(L_{min,t}-L_{max,t})}+1\right) \approx -\sum_{t=1}^{T} \eta(L_{min,t}-L_{max,t}) + \frac{2T}{\eta}\log(K-1)$

Finally, we have

$\mathbb{E}[\sum_{t=1}^{T} \mathcal{L}_{i,t}(\mathbb{E}_{\boldsymbol{p}_{i,t}}[\hat{\theta}_{j,t}], z_{i,t})] - \mathbb{E}[\min_{j\in\{1,\dots,N\}}(L_{j,T})] \lessapprox$

$-\sum_{t=1}^{T} \eta(L_{min,t}-L_{max,t}) + \frac{2T}{\eta}\log(K-1) + \frac{\eta T}{8} + \frac{\log N}{\eta} + \left(1-\frac{K}{N}\right) L_T,$

where $\min_{j\in\mathcal{S}_{T+1}}(L_{j,T}) - \min_{j\in\{1,\dots,N\}}(L_{j,T}) \leq L_T$