Submission for HW 3

CS: 427 Mathematics for Data Science

Autumn 2020-21

K. Sai Anuroop, 170030035

Sunday, October 18, 2020.

Problem 1

Prove that a function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its hessian is positive semi-definite.

Answer Assuming that f is twice differentiable and domain of f is convex.

Sketch of the proof:

(1) We first prove it for the case of $f: \mathbb{R} \to \mathbb{R}$ and then

(2) We use the first-order condition for convexity, i.e., f is convex if and only if dom f is convex and

$$f(y) \geq f(x) + \nabla f(x)^T (y-x) \text{ holds } \forall\, x, y \in \text{dom} f.$$

to prove the statement in problem.

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(1) Let us assume $n=1$.

$\Rightarrow$) Suppose $f: \mathbb{R} \to \mathbb{R}$ is convex. Let $x, y \in$ dom f where $y > x$. By the first order condition

$f(y) \geq f(x) + f'(x)(y-x)$ and $f(x) \geq f(y) + f'(y)(x-y)$

This means $f(x) + f'(x)(y-x) \leq f(y) \leq f(x) + f'(y)(y-x)$

$$\Rightarrow f'(x)(y-x) \leq f(y) - f(x) \leq f'(y)(y-x)$$

So we get $f'(y)(y-x) \geq f'(x)(y-x)$

$\Rightarrow \quad f'(y)(y-x) - f'(x)(y-x) \geq 0$

$\Rightarrow \quad \dfrac{f'(y) - f'(x)}{y-x} \geq 0$ (Note that $y>x$).

Taking $y \to x$,

$$\lim_{y \to x} \frac{f'(y)-f'(x)}{y-x} = f''(x)$$

So, $\displaystyle\lim_{y \to x} \frac{f'(y)-f'(x)}{y-x} = f''(x) \geq 0$.

Therefore $f''(x) \geq 0$.

($\Leftarrow$) Suppose $f''(z) \geq 0 \ \forall z \in \operatorname{dom} f$.

Consider any two points $x, y \in \operatorname{dom} f$ where $x<y$.

Now consider $\displaystyle\int_x^y f''(z)(y-z)\,dz$

Note that $f''(z) \geq 0 \ \forall z \in \operatorname{dom} f$.

and $y-z \geq 0 \ \forall \ z \in [x, y]$.

So, $0 \leq \displaystyle\int_x^y f''(z)(y-z)\,dz$

We now apply $\boxed{\int u\,dv = uv - \int v\,du}$ rule to solve the above integral.

$$0 \leq \int_x^y f''(z)(y-z)\,dz$$

$$= \int_x^y (y-z)\, d(f'(z))$$

$$= (y-z)f'(z)\Big|_x^y - \int_x^y f'(z)(-dz)$$

$$= -(y-x)f'(x) + \int_x^y f'(z)\,dz$$

$$= -(y-x)f'(x) + f(z)\Big|_x^y$$

$$= -(y-x)f'(x) + f(y) - f(x)$$

$$\Rightarrow \quad f(y) - f(x) - (y-x)f'(x) \geq 0$$

i.e.; $\quad f(y) \geq f(x) + f'(x)(y-x).$

Therefore f is convex.

$\square$

(end of ① proof).

② Consider that a function is convex if and only if it is convex on all lines. In other words, the function $g(t) = f(x_0 + tv)$ is convex in t for all $x_0 \in \operatorname{dom} f$ and all v.

Therefore, for any $n \geq 1$, f is convex if and only if $g''(t) = v^T \nabla^2 f(x_0 + tv)\, v \geq 0$

$\forall\, x_0 \in \mathbf{dom} f$, $v \in \mathbb{R}^n$ and t such that $x_0 + tv \in \mathbf{dom} f$

Here, it is necessary and sufficient that

$$\nabla^2 f(x) \succeq 0 \quad \forall\, x \in \mathbf{dom} f.$$

Therefore, $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if if $\nabla^2 f(x) \succeq 0$ (Hessian of f is positive semi-definite).

$\square$

(end of ② proof).

Problem 2 When is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

Answer An epigraph of a function f, denoted by epi f is defined as follows: $(f : \mathbb{R}^n \to \mathbb{R})$

$$\mathbf{epi} f := \{(x, t) : x \in \mathbb{R}^n,\ t \in \mathbb{R},\ f(x) \leq t\}$$

Strictly speaking

$$\boxed{\mathbf{epi} f := \{(x, t) \mid x \in \mathbf{dom} f,\ f(x) \leq t\}}$$

A halfspace is a set of the form $\{x \mid a^T x \leq b\}$ where $x \in \mathbb{R}^n$, $a (\neq 0) \in \mathbb{R}^n$, $b \in \mathbb{R}$.

A function $f: \mathbb{R}^m \to \mathbb{R}^n$ is affine if it has the form $f(x) = Ax+b$ where $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$.

Let us take $n=1$, then

$f(x) = Ax+b$ where $A \in \mathbb{R}^{1\times m}$ and $b \in \mathbb{R}$

is an affine function from $\mathbb{R}^m \to \mathbb{R}$.

We can now figure out that in the definition of an epigraph of a function, if we constrain f to be affine, then $\text{epi} f$ = halfspace defined by

$\{a \mid f(x) \leq t, x \in \text{dom} f\}$.

$\therefore$ The epigraph of a function is a halfspace if f is affine.

A convex cone is defined as follows:

A set C is convex if for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$ we have

$\theta_1 x_1 + \theta_2 x_2 \in C$.

We know that a function is convex if and only if its epigraph is convex. Let f be positive homogeneous, i.e; $f(\alpha x) = \alpha f(x)$ $\forall$ $\alpha \geq 0$

for all $a > 0$ we have:

$(x, t) \in \text{epi} f \Leftrightarrow f(x) \leq t$

$\Leftrightarrow af(x) = f(ax) \leq at$

$\Leftrightarrow (ax, at) \in \text{epi} f$.

Thus, epigraph is a cone.

(PTO)

Suppose epigraph is a cone.

$\Rightarrow$ $\forall\ a > 0$, if $(x, t) \in \text{epi} f$ then $(ax, at) \in \text{epi} f$

Clearly, $(x, f(x)) \in \text{epi} f$.

So, $(ax, af(x)) \in \text{epi} f$.

$\Rightarrow$ $f(ax) \leq af(x)$ — (i)

Similarly, $(ax, f(ax)) \in \text{epi} f$. So, if the epigraph is a cone, $(x, f(ax)/a) \in \text{epi} f$

i.e., $f(x) \leq f(ax)/a$. — (ii)

By combining (i) and (ii) we get

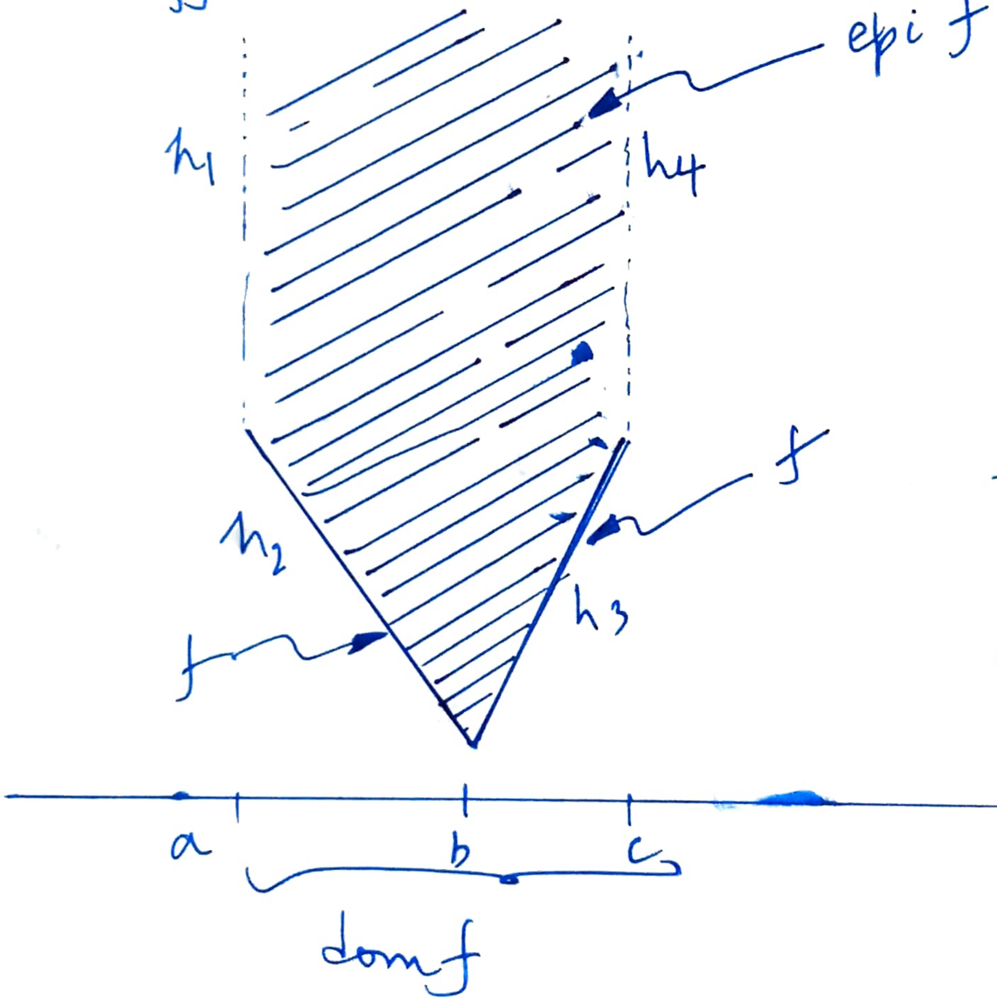
$$af(x) \leq f(ax) \leq af(x).$$

$\Rightarrow$ $f(ax) = af(x)$. So this says f is positive homogenous.

$\therefore$ Epigraph of a function is a convex cone if f is ~~positively~~ homogenous.

A polyhedron is the intersection of a finite number of halfspaces and hyperplanes.

we just proved that the epigraph of f is a halfspace if f is affine. Thus for the epigraph to be a polyhedron, we require that f be 'piecewise affine'.



Note in this figure that f is piecewise affine. The shaded region is epi f which is a polyhedron too. The halfspaces which form epigraph are: h_1, h_2, h_3 & h_4 in the figure.

Problem 3 A function $f: \mathbb{R}^n \to \mathbb{R}^n$ is monotone if $\forall x, y \in$ dom

$$(f(x) - f(y))^T (x-y) \geq 0.$$

Show that gradient ∇f of f is monotone.
Is every monotone mapping a gradient of some convex function?

Answer The question is slightly incorrect. It should have been as follows:

[A function $f: \mathbb{R}^n \to \mathbb{R}^n$ is monotone if $\forall x, y \in$ dom

$$(f(x) - f(y))^T (x-y) \geq 0.$$

Suppose $g: \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function. Show that ∇g is monotone. Is every monotone mapping the gradient of some convex function?]

Answer

Since g is convex, we have

$$g(x) \geq g(y) + \nabla g(y)^T (x-y) \text{ and}$$

$$g(y) \geq g(x) + \nabla g(x)^T (y-x), \text{ for any } x, y \in \text{dom} f$$

(by first order convexity).

Combining the above two inequalities we get

$$(\nabla g(x) - \nabla g(y))^T (x-y) \geq 0$$ which says

∇g is monotone.

For the other part of the question, consider

$$\phi(x) = \begin{bmatrix} x_1 \\ x_1/2 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

claim

$\phi(x)$ is monotone.

$$\phi(x) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\phi(y) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\phi(x) - \phi(y) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - y_1 \\ \frac{x_1 - y_1}{2} + x_2 - y_2 \end{bmatrix}$$

$$(\phi(x) - \phi(y))^T (x - y) = \begin{bmatrix} x_1 - y_1 & \frac{x_1 - y_1}{2} + x_2 - y_2 \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}$$

$$= (x_1 - y_1)^2 + (x_2 - y_2)^2 + \frac{(x_2 - y_2)(x_1 - y_1)}{2}$$

$$\geq \frac{(x_1 - y_1)^2}{4} + \frac{(x_2 - y_2)^2}{4} + \frac{(x_2 - y_2)(x_1 - y_1)}{2}$$

$$= \left[\frac{(x_1 - y_1) + (x_2 - y_2)}{2}\right]^2$$

≥ 0. $\square$

(P.T.O)

† We know that for a given field $F=(F_1, \cdots, f_n)$ which is smooth (C^1) to be gradient, it is necessary that $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$, $1 \le i < j \le n$.

Let us check if ϕ satisfies this condition.

$$\frac{\partial \phi_1}{\partial x_2} = \frac{\partial x_1}{\partial x_2} = 0$$

$$\frac{\partial \phi_2}{\partial x_1} = \frac{\partial (x_{1/2} + x_2)}{\partial x_1} = \frac{1}{2}$$

$$\Rightarrow \frac{\partial \phi_1}{\partial x_2} \neq \frac{\partial \phi_2}{\partial x_1}.$$

$\therefore$ ϕ cannot be a gradient.

So, we have shown a counter example where a monotone function is not the gradient of a convex function.

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† This comes from the fact that curl of a gradient is 0

Problem 4 Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is convex, $g: \mathbb{R}^n \to \mathbb{R}$ is concave, $\operatorname{dom} f = \operatorname{dom} g = \mathbb{R}^n$, and for all x, $g(x) \le f(x)$. Show that there exists an affine function h such that for all x, $g(x) \le h(x) \le f(x)$. In other words, if a concave function g is an under-estimator of a convex function f, then we can fit an affine function between f and g.

Answer Recall that $\operatorname{epi} f := \{(x,t) : x \in \operatorname{dom} f, t \ge f(x)\}$

$$\operatorname{hypo} f := \{(x,t) : x \in \operatorname{dom} f, t \le f(x)\}$$

Notice that $\operatorname{int}(\operatorname{epi} f) \ne \emptyset$ (interior of epi f is not empty)

as $\operatorname{dom} f = \mathbb{R}^n$.

Also, $\operatorname{int}(\operatorname{epi} f) \cap \operatorname{hypo} g = \Phi$ as

$f(x) < t$ for $(x,t) \in \operatorname{int} \operatorname{epi} f$ and

$t \le g(x)$ for $(x,t) \in \operatorname{hypo} g$.

Thus, these two sets ($\operatorname{int}(\operatorname{epi} f)$ and $\operatorname{hypo} g$) can be separated by a hyperplane. This means there exist $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ ~~($a \ne 0$, $b \ne 0$) su~~ (where both a and b are not zero at the same time), and $c \in \mathbb{R}$ such that

$$a^T x + bt \ge c \ge a^T y + bv \quad \text{where}$$

$t \ge f(x)$ and $v \le g(y)$.

Note that if $b=0$, $a^T x \geq a^T y \Rightarrow a=0$.
$\forall x, y$

So, $b \neq 0$. Let $x=y$.

Then,
$$bt \geq c \geq bv$$
$$\Rightarrow bt \geq bv \quad -(i)$$

But we have $t > f(x)$ and $v \leq g(x)$

and $g(x) \leq f(x)$

So, $v \leq g(x) \leq f(x) < t$

$$\Rightarrow t > v \quad -(ii)$$

Plugging (ii) in (i) we get,

$$b(t-v) \geq 0$$
$$\Rightarrow \underline{\underline{b > 0}}$$

Let us now separate int(epi f) and hypo g by a hyperplane.

Consider a point ~~(x,t) int epi~~
$(x,t) \in \text{int}(\text{epi } f)$ and
$(y,v) = (x, g(x)) \in \text{hypo } g$.

Then $a^T x + bt \geq c \geq a^T x + b g(x)$.

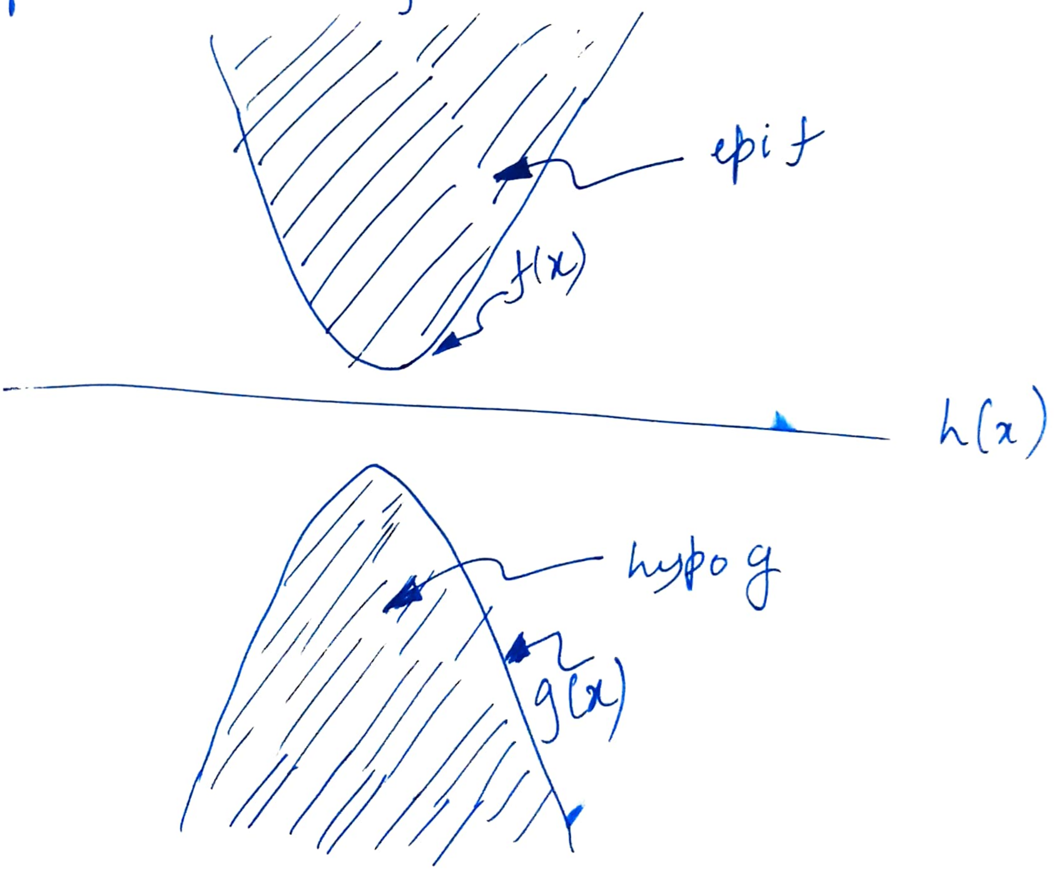
Dividing by b. (note that we proved $b > 0$)

$$\frac{a^T x}{b} + t \geq \frac{c}{b} \geq \frac{a^T x}{b} + g(x)$$

$$\Rightarrow \boxed{t \geq \frac{-a^T x + c}{b} \geq g(x) \quad \forall\ t \geq f(x)}.$$

Thus, we have $h(x) = \dfrac{c - a^T x}{b}$, an affine function which lies between f and g.

A representation of the theorem is as follows:



Submission for HW 3 (Programming Assignment)

CS 427: Mathematics for Data Science, Autumn 2020-21

K. Sai Anuroop, 170030035

October 18, 2020

1 Question 1

Plot a 3D graph and a contour map of $f(x, y) = x^2 - y^2 \ \forall \ x, y \in [-5, 5]$

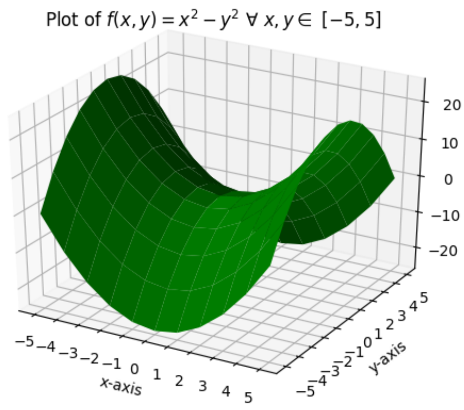


Figure 1: 3D graph

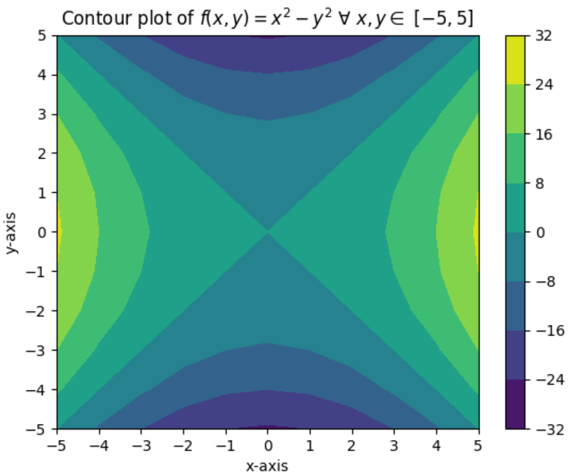


Figure 2: Contour map

2 Question 2

Randomly generate a set of 24 points that belong to the set $\{(x, y) : x, y \in [-5, 5]\}$. Create a scatter plot and outline the convex hull of the set you just created.

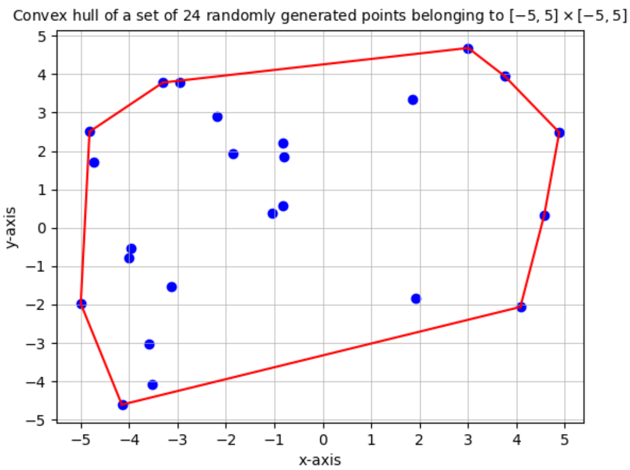


Figure 3: Scatter plot and convex hull

3 Question 3

Check if the function $f(x) = x^T Ax$ for $A \in R^{2\times 2}$ where all components of x are integers in $[-10, 10]$, is convex. Find 11 counter examples if it is not.

I take a random vector x where $x_{ij} \in [-10, 10]$ for $i, j = \{1, 2\}$. I then rotate and scale it using the matrix $A := \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Here $f(x) = x^T Ax = (x^2 + y^2)\cos\theta + 2xy\sin\theta$. There is a result which states that $f(x) = x^T Ax$ is convex if and only if A is positive semidefinite. In the manner I defined the matrix A, it is symmetric. However, it can be analysed that for some values of θ, f takes negative values. So I provide 11 such values of A for which f is not convex. For better visualisation, I plotted the rotated and scaled vector Ax for some values of A. I also plotted the corresponding surfaces of f.
[1]Here are the counter-examples in terms of the matrix A:
$\begin{pmatrix} 0.27 & 0.96 \\ 0.96 & 0.27 \end{pmatrix} \begin{pmatrix} 0.17 & 0.99 \\ 0.99 & 0.17 \end{pmatrix} \begin{pmatrix} 0.07 & 1.0 \\ 1.0 & 0.07 \end{pmatrix} \begin{pmatrix} -0.03 & 1.0 \\ 1.0 & -0.03 \end{pmatrix} \begin{pmatrix} -0.13 & 0.99 \\ 0.99 & -0.13 \end{pmatrix} \begin{pmatrix} -0.23 & 0.97 \\ 0.97 & -0.23 \end{pmatrix} \begin{pmatrix} -0.32 & 0.95 \\ 0.95 & -0.32 \end{pmatrix}$
$\begin{pmatrix} -0.42 & 0.91 \\ 0.91 & -0.42 \end{pmatrix} \begin{pmatrix} -0.5 & 0.86 \\ 0.86 & -0.5 \end{pmatrix} \begin{pmatrix} -0.59 & 0.81 \\ -0.59 & 0.81 \end{pmatrix} \begin{pmatrix} -0.67 & 0.75 \\ 0.75 & -0.67 \end{pmatrix}$
Clearly as seen in the below plots, f loses its convexity as we vary the value of θ from 0 to π.

[1]Please refer to `plots.py` in my GitHub repository at https://github.com/ksanu1998/MDS_HW_Solutions to view the code used to generate counter-examples and plots in this assignment.

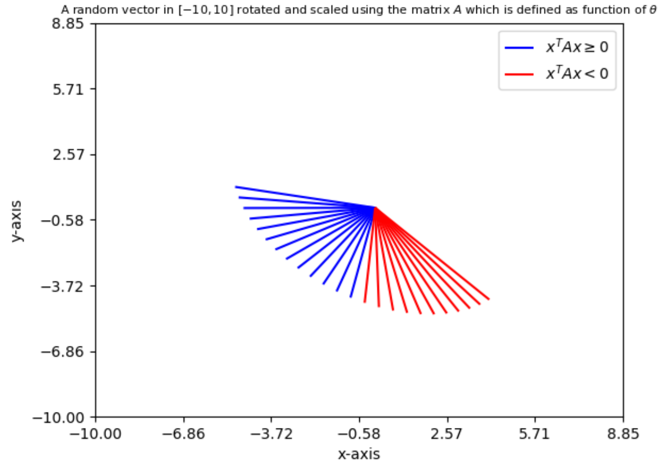


Figure 4: Rotated and scaled vectors Ax for different matrices A

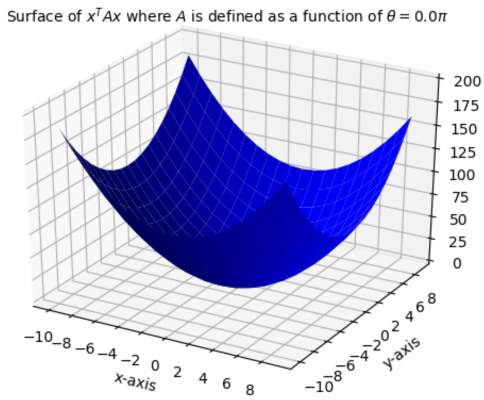


Figure 5: Surface of f for $\theta = 0$

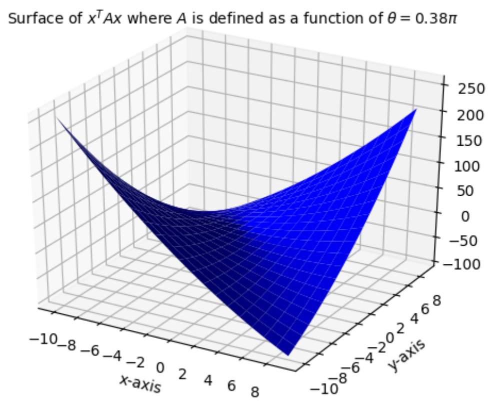


Figure 6: Surface of f for $\theta = 0.38\pi$

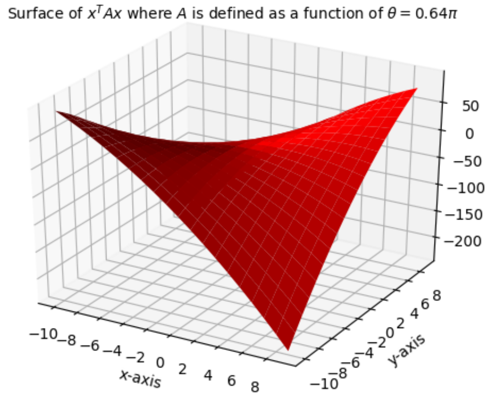


Figure 7: Surface of f for $\theta = 0.64\pi$

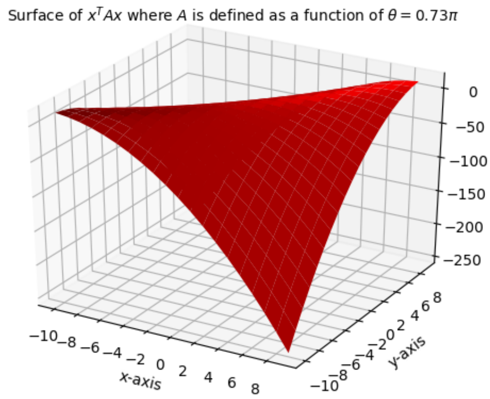


Figure 8: Surface of f for $\theta = 0.73\pi$

Scan this QR code to access the GitHub repository of my homweork solutions at https://github.com/ksanu1998/MDS_HW_Solutions