LINEAR DISCRIMINANT ANALYSIS (LDA)

Goal : To project a feature space (N dimensional data) onto a smaller subspace k ($k \leq n-1$) while maintaining the class discriminatory info.

Example :-

Consider 2-D dataset with class information.

x_1	x_2	Y
4	1	c_1
2	4	c_1
2	3	c_1
3	6	c_1
4	4	c_1
9	10	c_2
6	8	c_2
9	5	c_2
8	7	c_2
10	8	c_2

Step 1 : Compute within class scattered matrix (S_W)

$$S_W = S_1 + S_2$$

S_1 is the covariance matrix for class c_1,
S_2 is the covariance matrix for class c_2

So, let's compute the covariance matrices of each class

$$S_1 = \sum_{x \in c_1} (x - \mu_1)(x - \mu_1)^T$$

μ_1 is the mean of class C1, which is computed by

$$\mu_1 = \left\{ \frac{4+2+2+3+4}{5}, \frac{1+4+3+6+4}{5} \right\}$$

$$= [3.00, 3.60]$$

similarly $\mu_2 = \left\{ \frac{9+6+9+8+10}{5}, \frac{10+8+5+7+8}{5} \right\}$

$$= [8.4, 7.60]$$

Now Again, $S_1 = \sum_{x \in C_1} (x - \mu_1)(x - \mu_1)^T$

$$(x - \mu_1) = \begin{bmatrix} 4-3 & 2-3 & 2-3 & 3-3 & 4-3 \\ 1-3.6 & 4-3.6 & 3-3.6 & 6-3.6 & 4-3.6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 & 0 & 1 \\ -2.6 & 0.4 & -0.6 & 2.4 & 0.4 \end{bmatrix}$$

Now, For each x, we are going to calculate $(x - \mu_1)(x - \mu_1)^T$. So, we will have 5 such matrices

we have to calculate one by one for each,

$$\begin{bmatrix} 1 \\ -2.6 \end{bmatrix} \begin{bmatrix} 1 & -2.6 \end{bmatrix} = \begin{bmatrix} 1 & -2.6 \\ -2.6 & 6.76 \end{bmatrix} \quad \text{--- (1)}$$

First matrix

similarly, for rest we get

$$\begin{bmatrix} -1 \\ 0.4 \end{bmatrix} \begin{bmatrix} -1 & 0.4 \end{bmatrix} = \begin{bmatrix} 1 & -0.4 \\ -0.4 & 0.16 \end{bmatrix} \quad — (2)$$

$$\begin{bmatrix} -1 \\ -0.6 \end{bmatrix} \begin{bmatrix} -1 & -0.6 \end{bmatrix} = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.36 \end{bmatrix} \quad — (3)$$

$$\begin{bmatrix} 0 \\ 2.4 \end{bmatrix} \begin{bmatrix} 0 & 2.4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5.76 \end{bmatrix} \quad — (4)$$

$$\begin{bmatrix} 1 \\ 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0.4 \end{bmatrix} = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 0.16 \end{bmatrix} \quad — (5)$$

Adding (1), (2), (3), (4), and (5) and taking average we get covariance matrix S_1

$$S_1 = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 2.6 \end{bmatrix}$$

similarly, for the class 2, the covariance matrix is given by

$$S_2 = \begin{bmatrix} 1.84 & -0.04 \\ -0.04 & 2.64 \end{bmatrix}$$

Now, $S_W = S_1 + S_2$

$$= \begin{bmatrix} 2.64 & -0.44 \\ -0.44 & 5.28 \end{bmatrix}$$

Step 2 :- Compute between class scattered matrix S_B

$$S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$$

$$= \begin{pmatrix} -5.4 \\ -4 \end{pmatrix} \begin{pmatrix} -5.4 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 29.16 & 21.6 \\ 21.6 & 16.00 \end{pmatrix}$$

Step 3 :- Find the best LDA projection vector.

Similar to PCA, we find this using eigen vectors having largest eigen value :

$$\underbrace{S_W^{-1}}_{\text{within class scattered matrix}} . \underbrace{S_B}_{\text{between class scattered matrix}} V = \lambda \overset{\text{projection vector.}}{V}$$

$$|S_W^{-1} . S_B - \lambda I| = 0$$

$$\begin{vmatrix} 11.89-\lambda & 8.81 \\ 5.08 & 3.76-\lambda \end{vmatrix} = 0$$

Solving λ, we get $\lambda = 15.65$

Substitute λ value and

$$\begin{bmatrix} 11.89 & 8.81 \\ 5.08 & 3.76 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 15.65 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

we get, $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0.91 \\ 0.39 \end{bmatrix}$

$\boxed{\text{or}}$ we get directly by solving

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = S_W^{-1} (\mu_1 - \mu_2)$$

$$= \begin{bmatrix} 0.1921 & -0.032 \\ -0.031 & 0.38 \end{bmatrix} \begin{bmatrix} -5.4 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} -0.91 & -0.39 \end{bmatrix}^T$$

S_W^{-1} is found by using the formula,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Step 4 :- Dimension Reduction

$y = w^T x \leftarrow$ Input data sample

$\uparrow$

projection vector

$$= \begin{bmatrix} 4 & 1 \\ 2 & 4 \\ 2 & 3 \\ 3 & 6 \\ 4 & 4 \\ 9 & 10 \\ 6 & 8 \\ 9 & 5 \\ 8 & 7 \\ 10 & 8 \end{bmatrix} \cdot \begin{bmatrix} 0.91 \\ 0.39 \end{bmatrix} = \begin{bmatrix} 4.03 \\ 3.38 \\ 2.99 \\ 5.07 \\ 5.2 \\ 12.09 \\ 8.58 \\ 10.14 \\ 10.01 \\ 12.22 \end{bmatrix}$$