Discretization using finite volume method

Governing equation for the two D heat conduction with volumetric heat source is

$$k\nabla^2 T + q^{'''} = 0 \tag{1}$$

In a two dimensional case this equation transforms to,

$$\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + q^{'''} = 0 \tag{2}$$

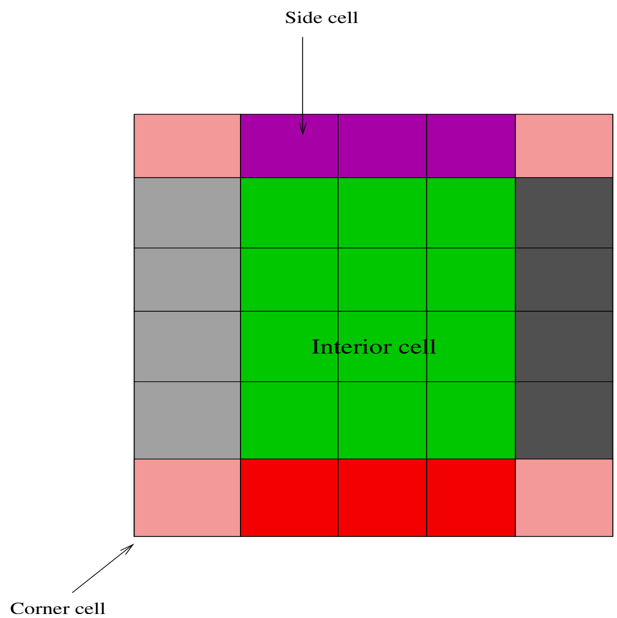

Figure 1: Different types of cells in a rectangular block

In a rectangular block with convective surface boundary condition there are nine different cells depending upon neighboring cells.

1. Four corner cells with two convective boundaries
2. Four side cells with one convective boundaries and other three neighboring cells
3. Interior cells surrounded by cell from all sides and no boundary cell

Discretizing the equations for the interior cells first. Integrating the equations over the control volume O with neighboring cells North, South, East and West. Cell walls of the respective cells are denoted by the lower case letters.

$$\int \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})dxdy + \int \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})dxdy + \int q^{'''}dv = 0 \tag{3}$$

Using the Gausses divergence theorm we get,

$$k\frac{\partial T}{\partial x}|_e \Delta y - k\frac{\partial T}{\partial x}|_w \Delta y + k\frac{\partial T}{\partial y}|_n \Delta x - k\frac{\partial T}{\partial y}|_s \Delta x + q^{'''}\Delta x \Delta y = 0 \tag{4}$$

Expanding the gradients at the cell walls in terms of the values at the cell centers,

$$k\Delta y\frac{(T_E - T_O)}{\Delta x} - k\Delta y\frac{(T_O - T_W)}{\Delta x} + k\Delta x\frac{(T_N - T_O)}{\Delta y} - k\Delta x\frac{(T_O - T_S)}{\Delta y} + q^{'''}\Delta x\Delta y = 0 \tag{5}$$

Dividing the equation by the $\Delta x\Delta y$ we get,

$$k\frac{(T_E - T_O)}{\Delta x^2} - k\frac{(T_O - T_W)}{\Delta x^2} + k\frac{(T_N - T_O)}{\Delta y^2} - k\frac{(T_O - T_S)}{\Delta y^2} + q^{'''} = 0 \tag{6}$$

In terms of the index k and conductivity, cnd, the above expression can be written as

$$cnd\frac{(T_{k+1} - T_k)}{\Delta x^2} - cnd\frac{(T_k - T_{k-1})}{\Delta x^2} + cnd\frac{(T_{k+nx} - T_k)}{\Delta y^2} - cnd\frac{(T_k - T_{k-nx})}{\Delta y^2} + q_k^{'''} = 0 \tag{7}$$

Rearranging the terms we get,

$$\frac{cnd}{\Delta x^2}T_{k+1} + \frac{cnd}{\Delta x^2}T_{k-1} - (\frac{2cnd}{\Delta x^2} + \frac{2cnd}{\Delta y^2})T_k + \frac{cnd}{\Delta y^2}T_{k+nx} + \frac{cnd}{\Delta y^2}T_{k-nx} + q_k^{'''} = 0 \tag{8}$$

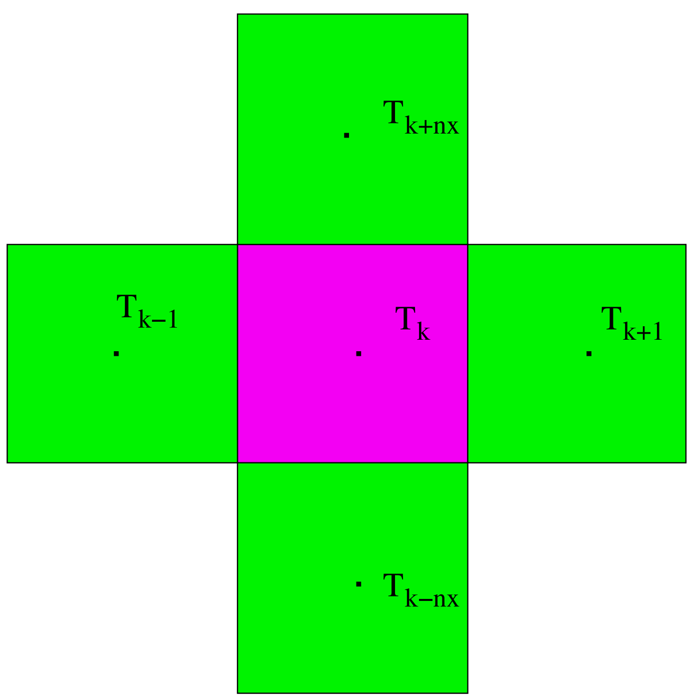

Figure 2: Interior cells in a rectangular block

The boundary cell

For the cell at the convective boundary at the RHS or LHS wall, we can write equation matching the heat flux,

$$-cnd\frac{\partial T}{\partial x}|_{surf} = h_x(T_{surf} - T_f) \tag{9}$$

Discretizing over the length $\Delta x/2$ we obtain,

$$-cnd\frac{T_{surf} - T_k}{\Delta x/2} = h_x T_{surf} - h_x T_f \tag{10}$$

Rearranging the terms

$$T_{surf} = [\frac{2cnd}{\Delta x}T_k + h_x T_f]/(\frac{2cnd}{\Delta x} + h_x) \tag{11}$$

Similarly the other cell which is at the top or bottom wall we can write discretized equation as,

$$T_{surf} = [\frac{2cnd}{\Delta y}T_k + h_y T_f]/(\frac{2cnd}{\Delta y} + h_y) \tag{12}$$

$$T_{surf} = \frac{2cnd}{\Delta y(\frac{2cnd}{\Delta y} + hy)}T_k + \frac{h_y}{(\frac{2cnd}{\Delta y} + hy)}T_f \tag{13}$$

$$T_{surf} = \frac{2cnd}{(2cnd + hy\Delta y)}T_k + \frac{h_y \Delta y}{(2cnd + hy\Delta y)}T_f \tag{14}$$

The corner cell

Consider the top left corner cell for discretization.

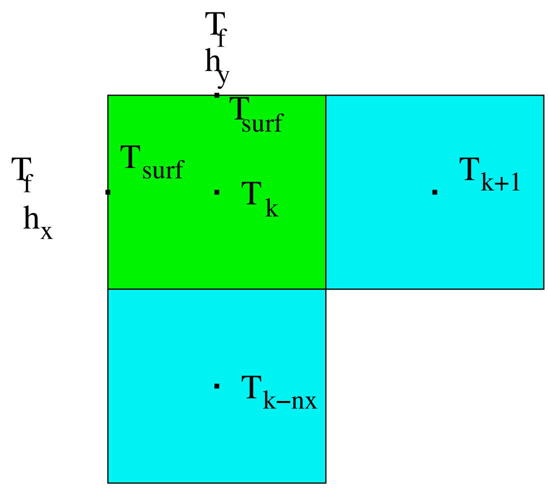

Figure 3: Cell at the top left corner of the the rectangular block

Discretized governing equation for the cell is,

$$k\Delta y\frac{(T_E-T_O)}{\Delta x}-k\Delta y\frac{(T_O-T_{surf})}{\Delta x/2}+k\Delta x\frac{(T_{surf}-T_O)}{\Delta y/2}-k\Delta x\frac{(T_O-T_S)}{\Delta y}+q^{'''}\Delta x\Delta y=0 \tag{15}$$

Dividing the equation by the $\Delta x\Delta y$ and rearranging the terms in terms of index k we get,

$$\frac{cnd}{\Delta x^2}T_{k+1}-(\frac{3cnd}{\Delta x^2}+\frac{3cnd}{\Delta y^2})T_k-(\frac{2cnd}{\Delta x^2}+\frac{2cnd}{\Delta y^2})T_{surf}+\frac{cnd}{\Delta y^2}T_{k-nx}+q_k^{'''}=0 \tag{16}$$

Putting expression for T_{surf}

$$\frac{cnd}{\Delta x^2}T_{k+1}-(\frac{3cnd}{\Delta x^2}+\frac{3cnd}{\Delta y^2})T_k+\frac{2cnd}{\Delta x^2}[\frac{2cnd}{\Delta x}\frac{T_k}{(\frac{2cnd}{\Delta x}+h_x)}+h_x\frac{T_f}{(\frac{2cnd}{\Delta x}+h_x)}]$$
$$+\frac{2cnd}{\Delta y^2}[\frac{2cnd}{\Delta y}\frac{T_k}{(\frac{2cnd}{\Delta y}+h_y)}+h_y\frac{T_f}{(\frac{2cnd}{\Delta y}+h_y)}]+\frac{cnd}{\Delta y^2}T_{k-nx}+q_k^{'''}=0$$

Similarly the discretized equation for the top right corner is given by,

$$\frac{cnd}{\Delta x^2}T_{k-1}-(\frac{3cnd}{\Delta x^2}+\frac{3cnd}{\Delta y^2})T_k+(\frac{2cnd}{\Delta x^2}+\frac{2cnd}{\Delta y^2})T_{surf}+\frac{cnd}{\Delta y^2}T_{k-nx}+q_k^{'''}=0 \tag{17}$$

For the bottom left and right corner we can write as,

$$\frac{cnd}{\Delta x^2}T_{k+1} - (\frac{3cnd}{\Delta x^2} + \frac{3cnd}{\Delta y^2})T_k + (\frac{2cnd}{\Delta x^2} + \frac{2cnd}{\Delta y^2})T_{surf} + \frac{cnd}{\Delta y^2}T_{k+nx} + q_k^{'''} = 0 \quad (18)$$

$$\frac{cnd}{\Delta x^2}T_{k-1} - (\frac{3cnd}{\Delta x^2} + \frac{3cnd}{\Delta y^2})T_k + (\frac{2cnd}{\Delta x^2} + \frac{2cnd}{\Delta y^2})T_{surf} + \frac{cnd}{\Delta y^2}T_{k+nx} + q_k^{'''} = 0 \quad (19)$$

In each of the equation of T_{surf} can be substituted to obtain an expression similar to top left corner.

Cells adjacent the wall

There are four types of cell groups each with RHS, LHS, top, bottom of the wall. The discretized equation for the cells adjacent to top wall is,

$$k\Delta y\frac{(T_E - T_O)}{\Delta x} - k\Delta y\frac{(T_O - T_W)}{\Delta x} + k\Delta x\frac{(T_{surf} - T_O)}{\Delta y/2} - k\Delta x\frac{(T_O - T_S)}{\Delta y} + q^{'''}\Delta x\Delta y = 0 \tag{20}$$

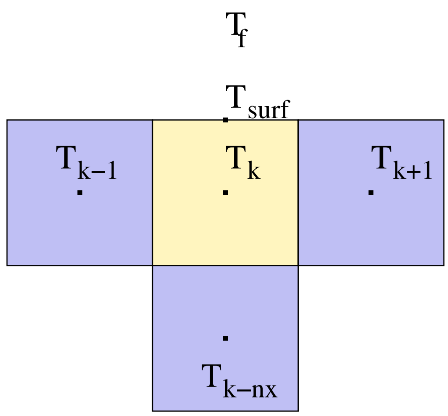

Figure 4: Cell at the top side of the the rectangular block

Dividing the equation by the $\Delta x \Delta y$, rearranging, and rewriting the equations in the k index form we get,

$$\frac{cnd}{\Delta x^2}T_{k+1} - (\frac{2cnd}{\Delta x^2} + \frac{3cnd}{\Delta y^2})T_k + \frac{cnd}{\Delta x^2}T_{k-1} + \frac{2cnd}{\Delta y^2}T_{surf} + \frac{cnd}{\Delta y^2}T_{k-nx} + q_k^{'''} = 0 \tag{21}$$

For the cells adjacent to bottom wall we can write,

$$\frac{cnd}{\Delta x^2}T_{k+1} - (\frac{2cnd}{\Delta x^2} + \frac{3cnd}{\Delta y^2})T_k + \frac{cnd}{\Delta x^2}T_{k-1} + \frac{2cnd}{\Delta y^2}T_{surf} + \frac{cnd}{\Delta y^2}T_{k+nx} + q_k^{'''} = 0 \tag{22}$$

For the cells adjacent to LHS walls discretized equation will be,

$$\frac{cnd}{\Delta x^2}T_{k+1} - (\frac{3cnd}{\Delta x^2} + \frac{2cnd}{\Delta y^2})T_k + \frac{cnd}{\Delta y^2}T_{k+nx} + \frac{2cnd}{\Delta x^2}T_{surf} + \frac{cnd}{\Delta y^2}T_{k-nx} + q_k^{'''} = 0 \tag{23}$$

For the cells adjacent to RHS walls discretized equation will be,

$$\frac{cnd}{\Delta x^2}T_{k-1} - (\frac{3cnd}{\Delta x^2} + \frac{2cnd}{\Delta y^2})T_k + \frac{cnd}{\Delta y^2}T_{k+nx} + \frac{2cnd}{\Delta x^2}T_{surf} + \frac{cnd}{\Delta y^2}T_{k-nx} + q_k^{'''} = 0 \tag{24}$$

The analytical solution to the problem

In one dimensional case governing equation transforms to,

$$\frac{\partial}{\partial x}(cnd\frac{\partial T}{\partial x}) + q^{'''} = 0 \tag{25}$$

$$\frac{\partial}{\partial x}(cnd\frac{\partial T}{\partial x}) = -q^{'''} \tag{26}$$

Intgrating both the sides with respect to x twice we get,

$$T = -\frac{q^{'''}}{cnd}\frac{x^2}{2} + c_1 x + c_2 \tag{27}$$

Where c_1 and c_2 are constants.

Boundary conditions are, at $x = a, T = Tsurf_1$ and $x = -a, T = Tsurf_2$

Here a is the distance of the surface from center of the plate.

Putting the boundary conditions we get,

$$T_{surf1} = -\frac{q^{'''}}{cnd}\frac{a^2}{2} + c_1 a + c_2 \tag{28}$$

$$T_{surf2} = -\frac{q^{'''}}{cnd}\frac{a^2}{2} - c_1 a + c_2 \tag{29}$$

Solving for c_1 and c_2 gives,

$$T = -\frac{q^{'''}}{2cnd}(a^2 - x^2) + \frac{(T_{surf1} - T_{surf2})}{2a}x + \frac{(T_{surf1} + T_{surf2})}{2} \tag{30}$$

As the $h_x = h_y, T_{surf1} = T_{surf2}$. With this the above euations becomes,

$$T = \frac{q^{'''}}{2cnd}(a^2 - x^2) + T_{surf} \tag{31}$$

At the convective boundary

$$-cnd\frac{dT}{dx}|_{surf} = h_x(T_{surf} - T_f) \tag{32}$$

Differentiating eq. (31) and putting in the above equation we obtain,

$$-cnd\frac{q^{'''}}{2cnd}(-2a) = h_x(T_{surf} - T_f) \tag{33}$$

Rearranging terms,

$$T_{surf} = T_f + \frac{q^{'''}a}{h_x} \tag{34}$$

The analytical solution thus becomes,

$$T = \frac{q^{'''}}{2cnd}(a^2 - x^2) + T_f + \frac{q^{'''}a}{h_x} \tag{35}$$