Part A

Inner ring cell

This cell is near the center of the rod. The two dimensional heat conduction governing equation for the cell is,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)+q^{'''}=0 \tag{1}$$

Integrating the equations over the control volume O with neighboring cells North, East and West. Cell walls of the respective cells are denoted by the lower case letters.

$$\int\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)rdrd\theta+\int\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)rdrd\theta+\int q^{'''}rdrd\theta=0 \tag{2}$$

Using the Gauss divergence theorm we get,

$$k\frac{\partial T}{\partial r}|_n A_n+k\frac{\partial T}{\partial \theta}|_e A_e-k\frac{\partial T}{\partial \theta}|_w A_w+q^{'''}\Delta r^2\Delta\theta/2=0 \tag{3}$$

Expanding the gradients at the cell walls in terms of the values at the cell centers,

$$k\frac{T_N-T_O}{\Delta r}\Delta r\Delta\theta+k\frac{T_E-T_O}{\Delta\theta}\Delta r-k\frac{T_O-T_W}{\Delta\theta}\Delta r+q^{'''}\Delta r^2\Delta\theta/2=0 \tag{4}$$

Rearranging and dividing both sides by $\Delta r^2\Delta\theta/2$ we get,

$$2k\frac{T_N-T_O}{\Delta r^2}+2k\frac{T_E-T_O}{\Delta\theta^2\Delta r}-2k\frac{T_O-T_W}{\Delta\theta^2\Delta r}+q^{'''}=0 \tag{5}$$

$$\frac{2k}{\Delta r^2}T_N+\frac{2k}{\Delta\theta^2\Delta r}T_E-\left(\frac{4k}{\Delta\theta^2\Delta r}+\frac{2k}{\Delta r^2}\right)T_O+\frac{2k}{\Delta\theta^2\Delta r}T_W+q^{'''}=0 \tag{6}$$

Interior cell

Governing equation for the interior cell remains the same as Eq. (1). But divergence is calculated over all the four sides North, South, East and West of the cell.

$$k\frac{\partial T}{\partial r}|_n A_n-k\frac{\partial T}{\partial r}|_s A_s+k\frac{\partial T}{\partial \theta}|_e A_e-k\frac{\partial T}{\partial \theta}|_w A_w+q^{'''}r_O\Delta r\Delta\theta=0 \tag{7}$$

Expanding the gradients at the cell walls in terms of the values at the cell centers,

$$k\frac{T_N-T_O}{\Delta r}(r_O+\Delta r/2)\Delta\theta-k\frac{T_O-T_S}{\Delta r}(r_O-\Delta r/2)\Delta\theta+k\frac{T_E-T_O}{\Delta\theta}\Delta r-k\frac{T_O-T_W}{\Delta\theta}\Delta r+q^{'''}r_O\Delta r\Delta\theta=0 \tag{8}$$

Rearranging and dividing both sides by $r_O\Delta r\Delta\theta$ we get,

$$\frac{k}{r_O\Delta r^2}(r_O+\Delta r/2)T_N+\frac{k}{r_O\Delta r^2}(r_O-\Delta r/2)T_S-(\frac{k(r_O+\Delta r/2)}{r_O\Delta r^2}+\frac{k(r_O-\Delta r/2)}{r_O\Delta r^2}+\frac{2k}{r_O\Delta\theta^2})T_O+\frac{k}{\Delta\theta^2 r_O}T_E+\frac{k}{\Delta\theta^2 r_O}T_W+q^{'''}=0$$

Discretization for the boundary cell

For the cell at the convective boundary at the outer wall, we can write equation matching the heat flux,

$$-k\frac{\partial T}{\partial r}|_{surf} = h(\theta)(T_{surf} - T_f) \tag{9}$$

Writing the Taylor series expansion around T_{surf} in terms of two inner nodes one at the distance $\Delta r/2$ and other at the $3\Delta r/2$, we get,

$$T_O = T_{surf} - \frac{\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + \frac{(\frac{\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{10}$$

$$T_{ss} = T_{surf} - \frac{3\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + \frac{(\frac{3\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{11}$$

Multiply eq. (10) by A and eq. (11) by B, and solving for A and B by setting the coefficient of $\frac{\partial T}{\partial r}$ equal to 1 and $\frac{\partial T^2}{\partial r^2}$ equal to zero

$$AT_O = AT_{surf} - A\frac{\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + A\frac{(\frac{\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{12}$$

$$BT_{ss} = BT_{surf} - B\frac{3\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + B\frac{(\frac{3\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{13}$$

Thus we need to solve for,

$$-A\frac{\Delta r}{2} - B\frac{3\Delta r}{2} = 1 \tag{14}$$

$$A\frac{(\frac{\Delta r}{2})^2}{2!} + B\frac{(\frac{3\Delta r}{2})^2}{2!} = 0 \tag{15}$$

Solving for A and B in Mathcad (see file Boundary cell.xmcd), we obtain, $A = -\frac{3}{\Delta r}$ and $B = \frac{1}{3\Delta r}$

Substituting these we get

$$-\frac{3}{\Delta r}T_O = -\frac{3}{\Delta r}T_{surf} + \frac{3}{\Delta r}\frac{\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} - \frac{3}{\Delta r}\frac{(\frac{\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{16}$$

$$\frac{1}{3\Delta r}T_{ss} = \frac{1}{3\Delta r}T_{surf} - \frac{1}{3\Delta r}\frac{3\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + \frac{1}{3\Delta r}\frac{(\frac{3\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{17}$$

Solving for $\frac{\partial T}{\partial r}|_{surf}$ gives,

$$\frac{\partial T}{\partial r}|_{surf} = -\frac{3}{\Delta r}T_O + \frac{1}{3\Delta r}T_{ss} + \frac{8}{3\Delta r}T_{surf} \tag{18}$$

Putting this in the governing equation of flux matching, Eq.(9) we get,

$$-k\{-\frac{3}{\Delta r}T_O + \frac{1}{3\Delta r}T_{ss} + \frac{8}{3\Delta r}T_{surf}\} = h(\theta)(T_{surf} - T_f) \tag{19}$$

$$\left(h(\theta) + \frac{8k}{3\Delta r}\right) T_{surf} = h(\theta)T_f + \frac{3k}{\Delta r}T_O - \frac{k}{3\Delta r}T_{ss} \tag{20}$$

$$T_{surf} = \frac{1}{\left(h(\theta) + \frac{8k}{3\Delta r}\right)} \left[h(\theta)T_f + \frac{3k}{\Delta r}T_O - \frac{k}{3\Delta r}T_{ss}\right] \tag{21}$$

Error term calcualtion

The error term will be,

$$\epsilon = -A\frac{(\frac{\Delta r}{2})^3}{3!}\frac{\partial^3 T}{\partial r^3}|_{surf} + A\frac{(\frac{\Delta r}{2})^4}{4!}\frac{\partial^4 T}{\partial r^4}|_{surf} - B\frac{(\frac{3\Delta r}{2})^3}{3!}\frac{\partial^3 T}{\partial r^3}|_{surf} + B\frac{(\frac{3\Delta r}{2})^4}{4!}\frac{\partial T^4}{\partial r^4}|_{surf} \tag{22}$$

Putting A and B we get,

$$\epsilon = \frac{3}{8}\frac{\Delta r^2}{3!}\frac{\partial^3 T}{\partial r^3}|_{surf} - \frac{3}{16}\frac{\Delta r^3}{4!}\frac{\partial^4 T}{\partial r^4}|_{surf} - \frac{9}{8}\frac{\Delta r^2}{3!}\frac{\partial^3 T}{\partial r^3}|_{surf} + \frac{27}{16}\frac{\Delta r^3}{4!}\frac{\partial^4 T}{\partial r^4}|_{surf} \tag{23}$$

$$\epsilon = \frac{-1}{8}\Delta r^2\frac{\partial^3 T}{\partial r^3}|_{surf} + \frac{1}{16}\Delta r^3\frac{\partial^4 T}{\partial r^4}|_{surf} \tag{24}$$

Energy Balance

Writing down the energy balance for the cell we obtain,

$$k\frac{T_{surf} - T_O}{\Delta r}(r_O + \Delta r/2)\Delta\theta - k\frac{T_O - T_S}{\Delta r}(r_O - \Delta r/2)\Delta\theta + k\frac{T_E - T_O}{\Delta\theta}\Delta r - k\frac{T_O - T_W}{\Delta\theta}\Delta r + q^{'''} r_O\Delta r\Delta\theta = 0 \tag{25}$$

Rearranging and dividing both sides by $r_O\Delta r\Delta\theta$ we get,

$$\frac{k}{r_O\Delta r^2}(r_O + \Delta r/2)T_{surf} + \frac{k}{r_O\Delta r^2}(r_O - \Delta r/2)T_S - (\frac{k(r_O + \Delta r/2)}{r_O\Delta r^2} + \frac{k(r_O - \Delta r/2)}{r_O\Delta r^2} + \frac{2k}{r_O\Delta\theta^2})T_O + \frac{k}{\Delta\theta^2 r_O}T_E + \frac{k}{\Delta\theta^2 r_O}T_W + q^{'''} = 0$$

The expression for the T_{surf}, Eq.(21) can be put into this to obtain final discretized equation. This derivation of T_{surf} introduced one unknown neighbouring cell temperature in the derivation.

Part B

For the third order accuracy we have to consider the first five terms of the Taylor series negelecting the terms $o(5)$. We consider the eight points at the grid crossing in the neighbourhood of the point given (T_O). Let us denote them by T_E, T_W, T_N, T_S, $T_{EE}, T_{WW}, T_{NN}, T_{SS}$

$$T_E = T_O + \frac{r_o \Delta\theta}{r_o}\frac{\partial T}{\partial \theta} + \frac{(r_o \Delta\theta)^2}{r_o^2 2!}\frac{\partial^2 T}{\partial \theta^2} + \frac{(r_o \Delta\theta)^3}{r_o^3 3!}\frac{\partial^3 T}{\partial \theta^3} + \frac{(r_o \Delta\theta)^4}{r_o^4 4!}\frac{\partial^4 T}{\partial \theta^4} \quad (26)$$

$$T_W = T_O - \frac{r_o \Delta\theta}{r_o}\frac{\partial T}{\partial \theta} + \frac{(r_o \Delta\theta)^2}{r_o^2 2!}\frac{\partial^2 T}{\partial \theta^2} - \frac{(r_o \Delta\theta)^3}{r_o^3 3!}\frac{\partial^3 T}{\partial \theta^3} + \frac{(r_o \Delta\theta)^4}{r_o^4 4!}\frac{\partial^4 T}{\partial \theta^4} \quad (27)$$

$$T_{EE} = T_O + \frac{2r_o \Delta\theta}{r_o}\frac{\partial T}{\partial \theta} + \frac{(2r_o \Delta\theta)^2}{r_o^2 2!}\frac{\partial^2 T}{\partial \theta^2} + \frac{(2r_o \Delta\theta)^3}{r_o^3 3!}\frac{\partial^3 T}{\partial \theta^3} + \frac{(2r_o \Delta\theta)^4}{r_o^4 4!}\frac{\partial^4 T}{\partial \theta^4} \quad (28)$$

$$T_{WW} = T_O - \frac{2r_o \Delta\theta}{r_o}\frac{\partial T}{\partial \theta} + \frac{(2r_o \Delta\theta)^2}{r_o^2 2!}\frac{\partial^2 T}{\partial \theta^2} - \frac{(2r_o \Delta\theta)^3}{r_o^3 3!}\frac{\partial^3 T}{\partial \theta^3} + \frac{(2r_o \Delta\theta)^4}{r_o^4 4!}\frac{\partial^4 T}{\partial \theta^4} \quad (29)$$

$$T_N = T_O + \Delta r\frac{\partial T}{\partial r} + \frac{\Delta r^2}{2!}\frac{\partial^2 T}{\partial r^2} + \frac{\Delta r^3}{3!}\frac{\partial^3 T}{\partial r^3} + \frac{\Delta r^4}{4!}\frac{\partial^4 T}{\partial r^4} \quad (30)$$

$$T_S = T_O - \Delta r\frac{\partial T}{\partial r} + \frac{\Delta r^2}{2!}\frac{\partial^2 T}{\partial r^2} - \frac{\Delta r^3}{3!}\frac{\partial^3 T}{\partial r^3} + \frac{\Delta r^4}{4!}\frac{\partial^4 T}{\partial r^4} \quad (31)$$

$$T_{NN} = T_O + 2\Delta r\frac{\partial T}{\partial r} + \frac{(2\Delta r)^2}{2!}\frac{\partial^2 T}{\partial r^2} + \frac{(2\Delta r)^3}{3!}\frac{\partial^3 T}{\partial r^3} + \frac{(2\Delta r)^4}{4!}\frac{\partial^4 T}{\partial r^4} \quad (32)$$

$$T_{SS} = T_O - 2\Delta r\frac{\partial T}{\partial r} + \frac{(2\Delta r)^2}{2!}\frac{\partial^2 T}{\partial r^2} - \frac{(2\Delta r)^3}{3!}\frac{\partial^3 T}{\partial r^3} + \frac{(2\Delta r)^4}{4!}\frac{\partial^4 T}{\partial r^4} \quad (33)$$

Multiplying the Eq. (26) by A, Eq. (27) by B, Eq.(28) by C, Eq. (29) by D. Adding and setting the coefficient of,

$\frac{\partial T}{\partial \theta}$ equal to 0,
$\frac{\partial^2 T}{\partial \theta^2}$ equal to $\frac{1}{r_o^2}$,
$\frac{\partial^3 T}{\partial \theta^3}$ & $\frac{\partial^4 T}{\partial \theta^4}$ equal to 0.

Solving for A, B, C, D in *Mathcad* (see the Mathcad file MeshCrossingPoint.xmcd) we get,

$$A = \frac{4}{3r_o^2 \Delta\theta^2} \quad (34)$$

$$B = \frac{4}{3r_o^2\Delta\theta^2} \tag{35}$$

$$C = \frac{-1}{12r_o^2\Delta\theta^2} \tag{36}$$

$$D = \frac{-1}{12r_o^2\Delta\theta^2} \tag{37}$$

Similarly multiplying the Eq. (30) by E, Eq. (31) by F, Eq.(32) by G, Eq. (33) by H. Adding and setting the coefficient of,

$\frac{\partial T}{\partial r}$ equal to $\frac{1}{r_o}$,
$\frac{\partial^2 T}{\partial r_o^2}$ equal to 1,
$\frac{\partial^3 T}{\partial r_o^3}$ & $\frac{\partial^4 T}{\partial r_o^4}$ equal to 0.

Solving for E, F, G, H in *Mathcad* we get,

$$E = \frac{2}{3}\frac{2r_o + \Delta r}{\Delta r^2 r_o} \tag{38}$$

$$F = \frac{2}{3}\frac{2r_o - \Delta r}{\Delta r^2 r_o} \tag{39}$$

$$G = \frac{-1}{12}\frac{r_o + \Delta r}{\Delta r^2 r_o} \tag{40}$$

$$H = \frac{-1}{12}\frac{r_o - \Delta r}{\Delta r^2 r_o} \tag{41}$$

Therefore the discretized form of the governing equation becomes,

$$\frac{4}{3r_o^2\Delta\theta^2}T_E + \frac{4}{3r_o^2\Delta\theta^2}T_W - T_{EE}\frac{1}{12r_o^2\Delta\theta^2} - T_{WW}\frac{1}{12r_o^2\Delta\theta^2} + T_N\frac{2}{3}\frac{2r_o + \Delta r}{\Delta r^2 r_o} + T_S\frac{2}{3}\frac{2r_o - \Delta r}{\Delta r^2 r_o} + T_{NN}\frac{-1}{12}\frac{r_o + \Delta r}{\Delta r^2 r_o} + T_{SS}\frac{-1}{12}\frac{r_o - \Delta r}{\Delta r^2 r_o}$$

Error term calcualtion

Since the coefficients are equal, the fifth order error terms will cancel each other in the θ equations,

$$\epsilon = A\frac{(r_o\Delta\theta)^6}{r_o^6 6!}\frac{\partial^6 T}{\partial\theta^6} + B\frac{(r_o\Delta\theta)^6}{r_o^6 6!}\frac{\partial^6 T}{\partial\theta^6} + C\frac{(2r_o\Delta\theta)^6}{r_o^6 6!}\frac{\partial^6 T}{\partial\theta^6} + D\frac{(2r_o\Delta\theta)^6}{r_o^6 6!}\frac{\partial^6 T}{\partial\theta^6} + E\frac{\Delta r^5}{5!}\frac{\partial^5 T}{\partial r^5} - F\frac{\Delta r^5}{5!}\frac{\partial^5 T}{\partial r^5} + G\frac{(2\Delta r)^5}{5!}\frac{\partial^5 T}{\partial r^5} - G\frac{(2\Delta r)^5}{5!}\frac{\partial^5 T}{\partial r^5} + E\frac{\Delta r^6}{6!}\frac{\partial^6 T}{\partial r^6} + F\frac{\Delta r^6}{6!}\frac{\partial^6 T}{\partial r^6} + G\frac{(2\Delta r)^6}{6!}\frac{\partial^6 T}{\partial r^6} + G\frac{(2\Delta r)^6}{6!}\frac{\partial^6 T}{\partial r^6}$$

After putting A, B, C, D, E, F, G, H and simplifying we get,

$$\epsilon = -\frac{1}{90}\frac{\Delta\theta^4}{r_o^2}\frac{\partial^6 T}{\partial\theta^6} - \frac{1}{30r_o}\frac{\Delta r^4}{r_o}\frac{\partial^5 T}{\partial r^5} - \frac{1}{90}\Delta r^4\frac{\partial^6 T}{\partial r^6} \tag{42}$$