Discretization using finite volume method for a tansient conduction problem

Governing equation for the transient, two D heat conduction with volumetric heat source is

$$\rho C_p \frac{\partial T}{\partial t} = k\nabla^2 T + q^{'''} \tag{1}$$

In a two dimensional case this equation transforms to,

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + q^{'''} \tag{2}$$

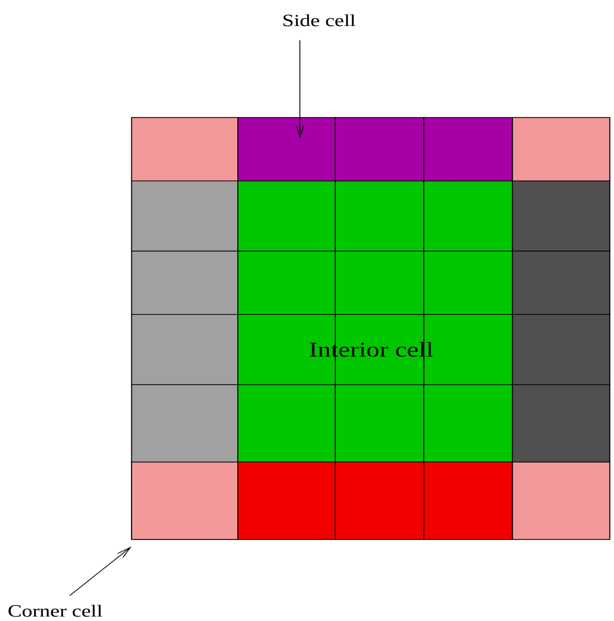

Figure 1: Different types of cells in a rectangular block

In a rectangular block with convective surface boundary condition there are nine different cells depending upon neighboring cells.

1. Four corner cells with two convective boundaries
2. Four side cells with one convective boundaries and other three neighboring cells
3. Interior cells surrounded by cell from all sides and no boundary cell

Discretizing the equations for the interior cells first. Integrating the equations over the control volume O with neighboring cells North, South, East and West in space and time we get. Cell walls of the respective cells are denoted by the lower case letters.

$$\int_t^{t+\Delta t}\int_V \rho C_p \frac{\partial T}{\partial t} = \int_t^{t+\Delta t}\int_V \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})dxdy + \int_t^{t+\Delta t}\int_V \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})dxdy + \int_t^{t+\Delta t}\int_V q^{'''}dv$$

(3)

Using the Gausses divergence theorm we get,

$$\rho C_p(T^{t+\Delta t} - T^t)\Delta x \Delta y = \left[k\frac{\partial T}{\partial x}|_e \Delta y - k\frac{\partial T}{\partial x}|_w \Delta y\right]^t \Delta t + \left[k\frac{\partial T}{\partial y}|_n \Delta x - k\frac{\partial T}{\partial y}|_s \Delta x\right]^t \Delta t + q^{'''}\Delta x \Delta y \Delta t \tag{4}$$

Expanding the gradients at the cell walls in terms of the values at the cell centers, and rearranging terms we get

$$\rho C_p(T^{t+\Delta t} - T^t)\Delta x \Delta y = \left[k\Delta y\frac{(T_E^t - T_O^t)}{\Delta x} - k\Delta y\frac{(T_O^t - T_W^t)}{\Delta x}\right]\Delta t + \left[k\Delta x\frac{(T_N^t - T_O^t)}{\Delta y} - k\Delta x\frac{(T_O^t - T_S^t)}{\Delta y}\right]\Delta t + q^{'''}\Delta x \Delta y \Delta t$$

Dividing the equation by the $\Delta x \Delta y \rho C p$ we get, and denoting thermal diffusivity as $\alpha = \frac{k}{\rho C_p}$,

$$T^{t+\Delta t} - T^t = \alpha\Delta t\frac{(T_E^t - T_O^t)}{\Delta x^2} - \alpha\Delta t\frac{(T_O^t - T_W^t)}{\Delta x^2} + \alpha\Delta t\frac{(T_N^t - T_O^t)}{\Delta y^2} - \alpha\Delta t\frac{(T_O^t - T_S^t)}{\Delta y^2} + \frac{\Delta t}{\rho C_p}q^{'''} \tag{5}$$

In terms of the index k and conductivity, cnd, the above expression can be written as

$$T^{n+1} - T^n = \alpha\Delta t\frac{(T_{k+1}^t - T_k^t)}{\Delta x^2} - \alpha\Delta t\frac{(T_k^t - T_{k-1}^t)}{\Delta x^2} + \alpha\Delta t\frac{(T_{k+nx}^t - T_k^t)}{\Delta y^2} - \alpha\Delta t\frac{(T_k^t - T_{k-nx}^t)}{\Delta y^2} + \frac{\Delta t}{\rho C_p}q^{'''} \tag{6}$$

Rearranging the terms we get,

$$T^{n+1} = \alpha\Delta t\frac{(T_{k+1}^n + T_{k-1}^n)}{\Delta x^2} + \alpha\Delta t\frac{(T_{k+nx}^n + T_{k-nx}^n)}{\Delta y^2} + \left[1 - 2\alpha\Delta t\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right]\right]T_k^n \tag{7}$$

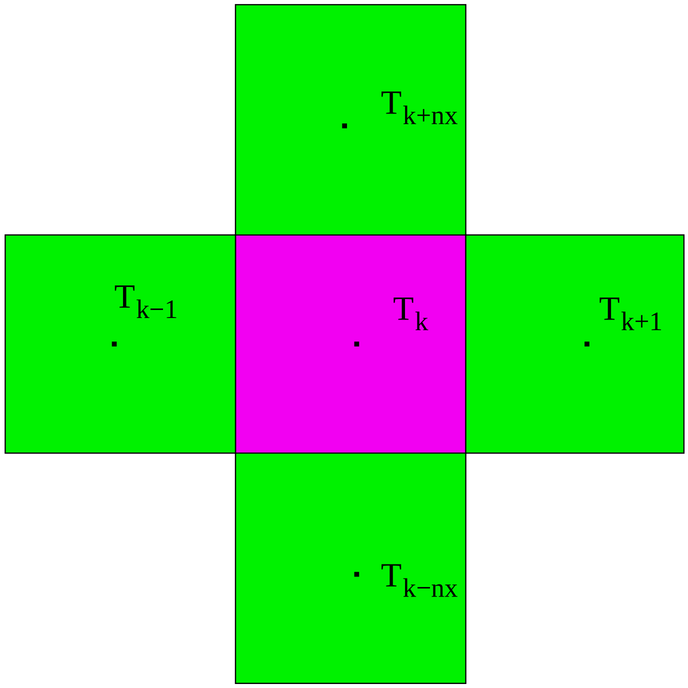

Figure 2: Interior cells in a rectangular block

The boundary cell

RHS cell

For the cell at the convective boundary at the RHS wall, we can write equation matching the heat flux,

$$-cnd\frac{\partial T}{\partial x}|_{surf} = h_x(T_{surf} - T_f) \tag{8}$$

Using the Taylor Series we can write,

$$T_O = T_{surf} - \frac{\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + \frac{(\frac{\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{9}$$

$$T_W = T_{surf} - \frac{3\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + \frac{(\frac{3\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{10}$$

Multiply eq. (9) by A and eq. (10) by B, and solving for A and B by setting the coefficient of $\frac{\partial T}{\partial x}$ equal to 1 and $\frac{\partial T^2}{\partial x^2}$ equal to zero

$$AT_O = AT_{surf} - A\frac{\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + A\frac{(\frac{\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{11}$$

$$BT_{ss} = BT_{surf} - B\frac{3\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + B\frac{(\frac{3\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{12}$$

Thus we need to solve for,

$$-A\frac{\Delta x}{2} - B\frac{3\Delta x}{2} = 1 \tag{13}$$

$$A\frac{(\frac{\Delta x}{2})^2}{2!} + B\frac{(\frac{3\Delta x}{2})^2}{2!} = 0 \tag{14}$$

Solving for A and B, we obtain, $A = -\frac{3}{\Delta x}$ and $B = \frac{1}{3\Delta x}$
Substituting these we get

$$-\frac{3}{\Delta x}T_O = -\frac{3}{\Delta x}T_{surf} + \frac{3}{\Delta x}\frac{\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} - \frac{3}{\Delta x}\frac{(\frac{\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{15}$$

$$\frac{1}{3\Delta x}T_W = \frac{1}{3\Delta x}T_{surf} - \frac{1}{3\Delta x}\frac{3\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + \frac{1}{3\Delta x}\frac{(\frac{3\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{16}$$

Solving for $\frac{\partial T}{\partial x}|_{surf}$ gives,

$$\frac{\partial T}{\partial x}|_{surf} = -\frac{3}{\Delta x}T_O + \frac{1}{3\Delta x}T_W + \frac{8}{3\Delta x}T_{surf} \tag{17}$$

simplfying we get,

$$\frac{\partial T}{\partial x}|_{surf} = \frac{-9T_O + T_W + 8T_{surf}}{3\Delta x} \tag{18}$$

Putting this in the governing equation of flux matching, Eq.(8) we get,

$$-cnd\frac{(-9T_O + T_W + 8T_{surf})}{3\Delta x} = h_x(T_{surf} - T_f) \tag{19}$$

Rearranging the terms

$$T_{surf} = \frac{(9cndT_k - cndT_W + 3\Delta xhT_f)}{8cnd + 3\Delta xh_x} \tag{20}$$

and eliminating the T_{surf} from the heat flux equation to obtain gradient at the surface we get,

$$-cnd\frac{\partial T}{\partial x}|_{surf} = h_x(T_{surf} - T_f) = h_x(\frac{(9cndT_k - cndT_W + 3\Delta xhT_f)}{8cnd + 3\Delta xh_x} - T_f) \tag{21}$$

$$= h_x cnd\frac{(9T_k - T_W - 8T_f)}{8cnd + 3\Delta xh_x} \tag{22}$$

Similarly the other cell which is at the top wall we can write discretized equation as,

$$T_{surf} = \frac{(9cndT_k - cndT_S + 3\Delta yhT_f)}{8cnd + 3\Delta yh_y} \tag{23}$$

and the heat flux in the y direction

$$-cnd\frac{\partial T}{\partial y}|_{surf} = h_y(T_{surf} - T_f) = h_y(\frac{(9cndT_k - cndT_S + 3\Delta y h T_f)}{8cnd + 3\Delta y h_y} - T_f) \quad (24)$$

$$= h_y cnd\frac{(9T_k - T_S - 8T_f)}{8cnd + 3\Delta y h_y} \quad (25)$$

The corner cell

Consider the top left corner cell for discretization.

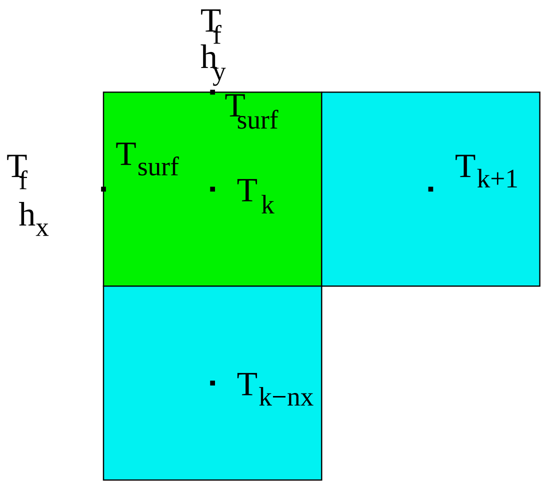

Figure 3: Cell at the top left corner of the the rectangular block

Discretized governing equation for the cell is,

$$\rho C_p\frac{(T^{n+1} - T^n)}{\Delta t}\Delta x\Delta y = cnd\Delta y\frac{(T_E^n - T_O^n)}{\Delta x} - h_y cnd\frac{(9T_O^n - T_S^n - 8T_f)}{8k + 3\Delta y h_y}\Delta x - h_x cnd\frac{(9T_O^n - T_E^n - 8T_f)}{8cnd + 3\Delta x h_x} - cnd\Delta x\frac{(T_O^n - T_S^n)}{\Delta y} + q^{'''}\Delta x\Delta y$$

Dividing the equation by the $\Delta x\Delta y\rho C_p$, substituting the value of T_S and rearranging the terms in terms of index k we get,

$$\frac{(T^{n+1} - T^n)}{\Delta t} = \left(\frac{\alpha}{\Delta x^2} + \frac{h_x\alpha}{8cnd + 3\Delta x h_x}\right)T_{k+1}^n - (\frac{\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_x\alpha}{\Delta x(8cnd + 3\Delta x h_x)} + \frac{9h_y\alpha}{\Delta y(8cnd + 3\Delta y h_y)})T_k^n + (\frac{8h_x\alpha}{\Delta x(8cnd + 3\Delta x h_x)} + \frac{8h_y\alpha}{\Delta y(8cnd + 3\Delta y h_y)})T_f + \left(\frac{\alpha}{\Delta y^2} + \frac{h_y\alpha}{\Delta y(8cnd + 3\Delta y h_y)}\right)T_{k-nx}^n + \frac{q_k^{'''}}{\rho Cp} \quad (26)$$

Similarly the discretized equation for the top right corner is given by,

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \left(\frac{\alpha}{\Delta x^2} + \frac{h_x\alpha}{8cnd+3\Delta x h_x}\right)T^n_{k-1} - (\frac{\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + (\frac{8h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)} + \frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T_f + \left(\frac{\alpha}{\Delta y^2} + \frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}\right)T^n_{k-nx} + \frac{q_k^{'''}}{\rho Cp} \quad (27)$$

For the bottom left and right corner we can write as,

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \left(\frac{\alpha}{\Delta x^2} + \frac{h_x\alpha}{8cnd+3\Delta x h_x}\right)T^n_{k+1} - (\frac{\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + (\frac{8h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)} + \frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T_f + \left(\frac{\alpha}{\Delta y^2} + \frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}\right)T^n_{k+nx} + \frac{q_k^{'''}}{\rho Cp} \quad (28)$$

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \left(\frac{\alpha}{\Delta x^2} + \frac{h_x\alpha}{8cnd+3\Delta x h_x}\right)T^n_{k-1} - (\frac{\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + (\frac{8h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)} + \frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T_f + \left(\frac{\alpha}{\Delta y^2} + \frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}\right)T^n_{k+nx} + \frac{q_k^{'''}}{\rho Cp} \quad (29)$$

Cells adjacent the wall

There are four types of cell groups each with RHS, LHS, top, bottom of the wall. The discretized equation for the cells adjacent to top wall is,

$$\rho C_p \frac{(T^{n+1}-T^{n})}{\Delta t}\Delta x \Delta y = cnd\Delta y\frac{(T_E - T_O)}{\Delta x} - cnd\Delta y \frac{(T_O - T_W)}{\Delta x} - h_y cnd \frac{(9T_O - T_S - 8T_f)}{8cnd+3\Delta y h_y}\Delta x - cnd\Delta x\frac{(T_O - T_S)}{\Delta y} + q^{'''}\Delta x\Delta y$$

Dividing the equation by the $\Delta x \Delta y \rho C_p$, rearranging, and rewriting the equations in the k index form we get,

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \frac{\alpha}{\Delta x^2}T^n_{k+1} - (\frac{2\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + \frac{\alpha}{\Delta x^2}T^n_{k-1} + \quad (30)$$

$$(\frac{\alpha}{\Delta y^2} + \frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_{k-nx} + \frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}T_f + \frac{q_k^{'''}}{\rho Cp} \quad (31)$$

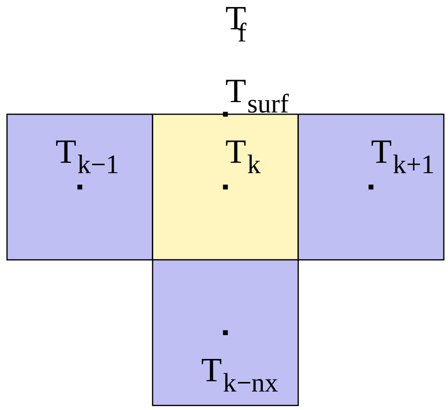

Figure 4: Cell at the top side of the the rectangular block

For the cells adjacent to bottom wall we can write,

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \frac{\alpha}{\Delta x^2}T^n_{k+1} - (\frac{2\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + \frac{\alpha}{\Delta x^2}T^n_{k-1} + \tag{32}$$

$$(\frac{\alpha}{\Delta y^2} + \frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_{k+nx} + \frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}T_f + \frac{q_k^{'''}}{\rho Cp} \tag{33}$$

For the cells adjacent to LHS walls discretized equation will be,

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \frac{\alpha}{\Delta y^2}T^n_{k+nx} - (\frac{2\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + \frac{\alpha}{\Delta y^2}T^n_{k-nx} + \tag{34}$$

$$(\frac{\alpha}{\Delta x^2} + \frac{h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)})T^n_{k+1} + \frac{8h_x k}{\Delta x(8cnd+3\Delta x h_x)}T_f + \frac{q_k^{'''}}{\rho Cp} \tag{35}$$

For the cells adjacent to RHS walls discretized equation will be,

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \frac{\alpha}{\Delta y^2}T^n_{k+nx} - (\frac{2\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + \frac{\alpha}{\Delta y^2}T^n_{k-nx} + \tag{36}$$

$$(\frac{\alpha}{\Delta x^2} + \frac{h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)})T^n_{k-1} + \frac{8h_x k}{\Delta x(8cnd+3\Delta x h_x)}T_f + \frac{q_k^{'''}}{\rho Cp} \tag{37}$$

Results and discussion

The temperature at all the points in the conductor is set to 300 K at time=0.

1. To test the code the heat transfer coeffiecient in the both the direction is set to zero, and solution is tested for 5 s transient simulation with the analytical solution. sFig. 5 shows the comparison of the analytical solution with numerical solution. Numerical solution is very close to the analytical solution. Analytical solution is obtained by intefrating the governing equation with no heat transfer, with given initial condition.

$$T_{analytical} = \frac{q}{\rho C_p}t + T_f \tag{38}$$

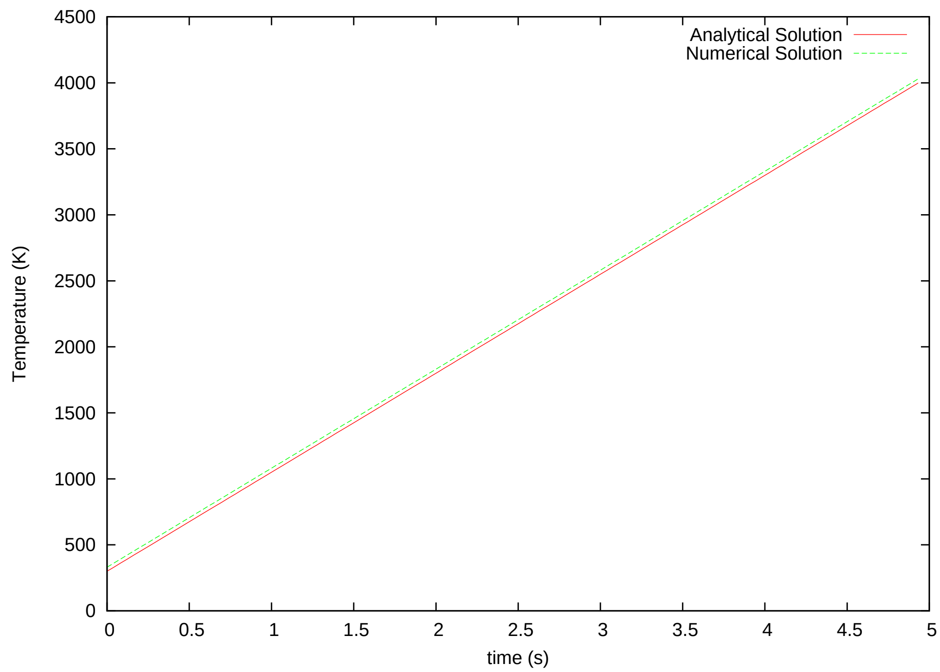

Figure 5: Comparison of the numerical solution with analytical solution

2. The program is run for 3 by 3 mesh, for five second. Time history plots of maximum value of abs $(T_{ss} - T)$ are plotted in Fig. 6 for four different time steps. When the time step is at the stability limit or one tenth of the stability limit, the solution converges smoothly to the steady state solution. But when the time step is double the stability limit the solution diverges rapidly. The same is observed in the plot of the difference of $T_{ss} - T$ at the central volume in FIg. 7. At the small number of volumes increase of 20 % above the stability limit doesnot affect the stability of the solution

3. Fig. 8 shows the plot of maximum ABS value of $(T_{ss} - T)$ for a 9 by 9 mesh with different time steps. The solution converges smoothly with the time step value equal to the stability limit. But a little increase of 5 % leads to the divergence. The same is reflected in the plot of the difference of $T_{ss} - T$ at the central volume in Fig. 9.

4. Time required for the simulation with a 9 by 9 mesh is 0.722819 s. Simulation time increases with large mesh of 99 by 99 volumes to 1.192819 s. Thus there is 65 % increase in the simulation time. The time required could have been reduced by the use higher time step. But at the higher mesh size a small increase in the time step above the stability limit will lead to a diverged solution. Thus in case of large cell volumes it would be suitable to use implicit scheme with no limit on the stability and use of larger time step will be possible.

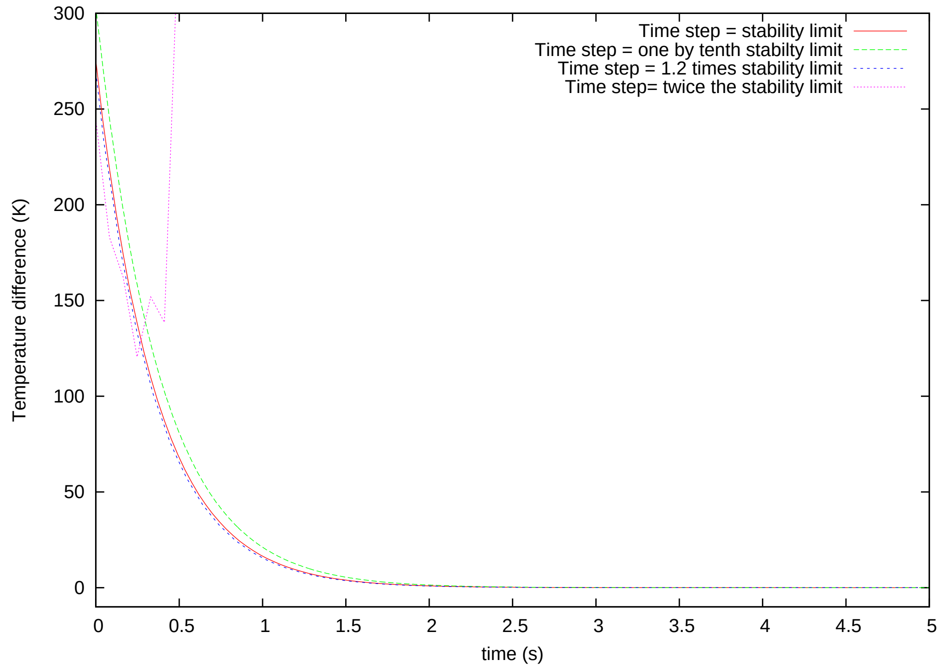

Figure 6: Maximum ABS value of $(T_{ss} - T)$ for a 3 by 3 mesh with different time steps

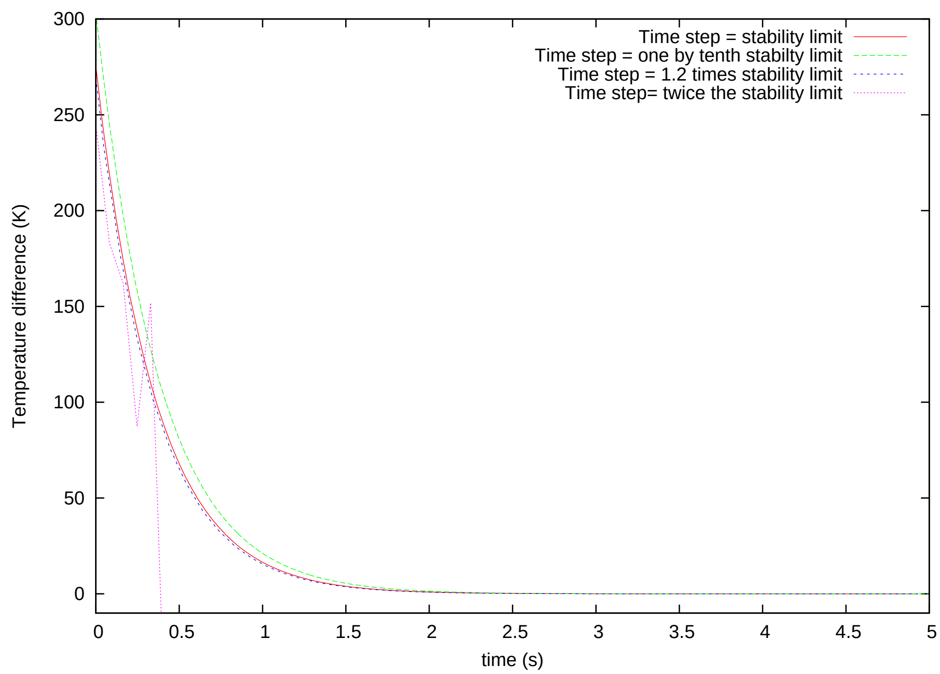

Figure 7: Difference of $T_{ss} - T$ at the central volume for a 3 by 3 mesh

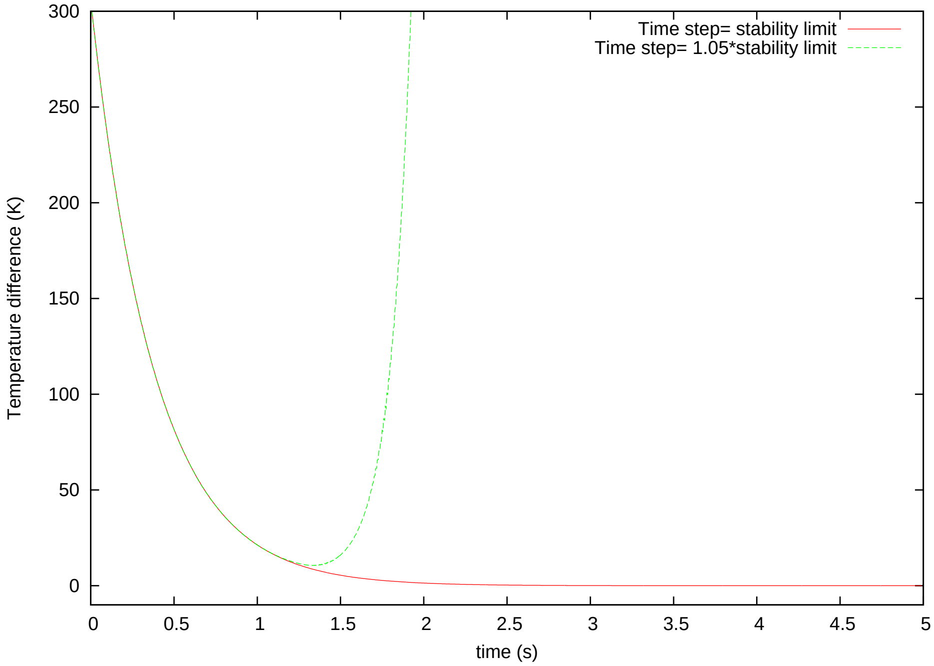

Figure 8: Maximum ABS value of $(T_{ss} - T)$ for a 9 by 9 mesh with different time steps

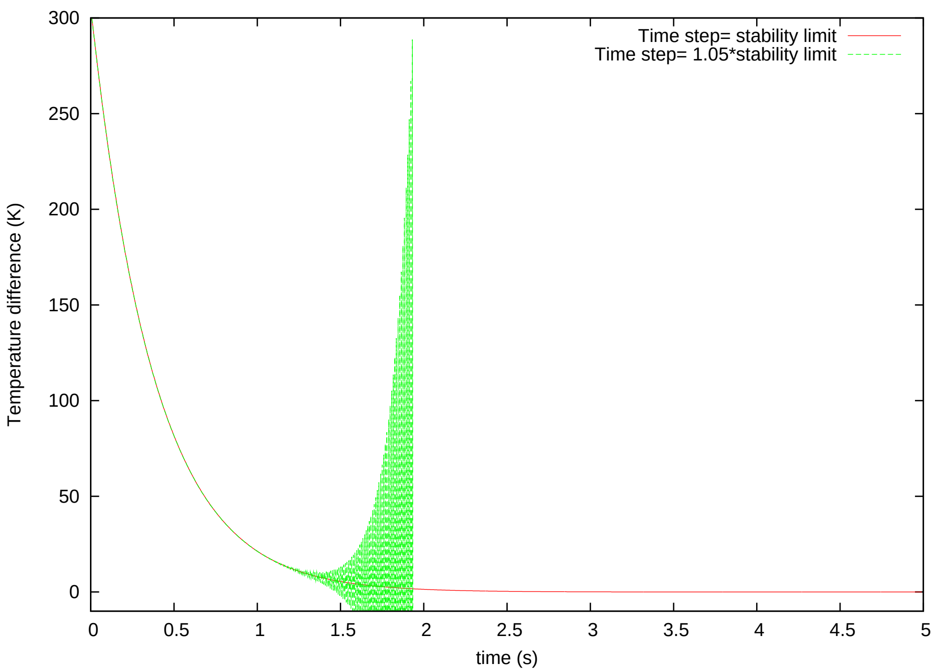

Figure 9: Difference of $T_{ss} - T$ at the central volume for 9 by 9 mesh