Discretization using finite volume method for a tansient conduction problem

Governing equation for the transient, two D heat conduction with volumetric heat source is

$$\rho C_p \frac{\partial T}{\partial t} = k\nabla^2 T + q''' \tag{1}$$

In a two dimensional case this equation transforms to,

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + q''' \tag{2}$$

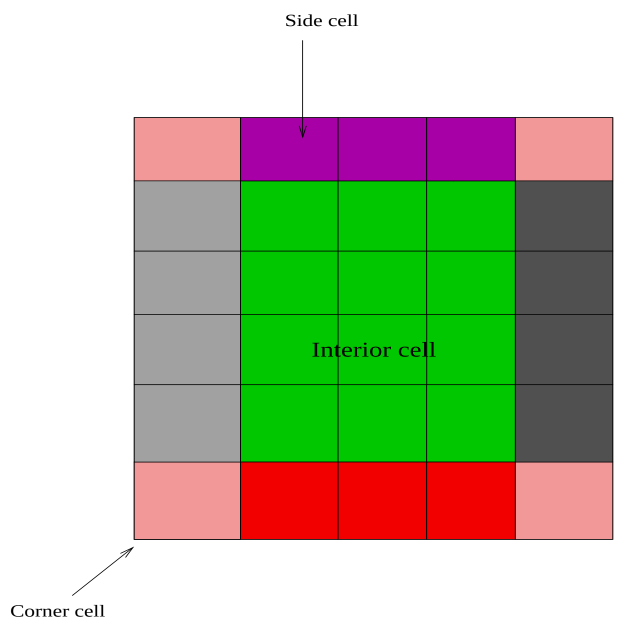

Figure 1: Different types of cells in a rectangular block

In a rectangular block with convective surface boundary condition there are nine different cells depending upon neighboring cells.

1. Four corner cells with two convective boundaries
2. Four side cells with one convective boundaries and other three neighboring cells
3. Interior cells surrounded by cell from all sides and no boundary cell

Discretizing the equations for the interior cells first. Integrating the equations over the control volume O with neighboring cells North, South, East and West in space and time we get. Cell walls of the respective cells are denoted by the lower case letters.

$$\int_t^{t+\Delta t}\int_V \rho C_p \frac{\partial T}{\partial t} = \int_t^{t+\Delta t}\int_V \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})dxdy + \int_t^{t+\Delta t}\int_V \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})dxdy + \int_t^{t+\Delta t}\int_V q''' dv$$

(3)

Using the Gausses divergence theorm we get,

$$\rho C_p(T^{t+\Delta t}-T^t)\Delta x\Delta y = \left[k\frac{\partial T}{\partial x}|_e\Delta y - k\frac{\partial T}{\partial x}|_w\Delta y\right]^t \Delta t+\left[k\frac{\partial T}{\partial y}|_n\Delta x - k\frac{\partial T}{\partial y}|_s\Delta x\right]^t \Delta t+q^{'''}\Delta x\Delta y\Delta t \tag{4}$$

Expanding the gradients at the cell walls in terms of the values at the cell centers, and rearranging terms we get

$$\rho C_p(T^{t+\Delta t} - T^t)\Delta x\Delta y = \left[k\Delta y\frac{(T^t_E - T^t_O)}{\Delta x} - k\Delta y\frac{(T^t_O - T^t_W)}{\Delta x}\right]\Delta t + \left[k\Delta x\frac{(T^t_N - T^t_O)}{\Delta y} - k\Delta x\frac{(T^t_O - T^t_S)}{\Delta y}\right]\Delta t + q^{'''}\Delta x\Delta y\Delta t$$

Dividing the equation by the $\Delta x\Delta y\rho Cp$ we get, and denoting thermal diffusivity as $\alpha = \frac{k}{\rho C_p}$,

$$T^{t+\Delta t}-T^t = \alpha\Delta t\frac{(T^t_E - T^t_O)}{\Delta x^2}-\alpha\Delta t\frac{(T^t_O - T^t_W)}{\Delta x^2}+\alpha\Delta t\frac{(T^t_N - T^t_O)}{\Delta y^2}-\alpha\Delta t\frac{(T^t_O - T^t_S)}{\Delta y^2}+\frac{\Delta t}{\rho C_p}q^{'''} \tag{5}$$

In terms of the index k and conductivity, cnd, the above expression can be written as

$$T^{n+1}-T^n = \alpha\Delta t\frac{(T^n_{k+1} - T^n_k)}{\Delta x^2}-\alpha\Delta t\frac{(T^n_k - T^n_{k-1})}{\Delta x^2}+\alpha\Delta t\frac{(T^n_{k+nx} - T^n_k)}{\Delta y^2}-\alpha\Delta t\frac{(T^n_k - T^n_{k-nx})}{\Delta y^2}+\frac{\Delta t}{\rho C_p}q^{'''} \tag{6}$$

Rearranging the terms we get,

$$T^{n+1} = \alpha\Delta t\frac{(T^n_{k+1} + T^n_{k-1})}{\Delta x^2}+\alpha\Delta t\frac{(T^n_{k+nx} + T^n_{k-nx})}{\Delta y^2}+\left[1 - 2\alpha\Delta t\left[\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right]\right]T^n_k \tag{7}$$

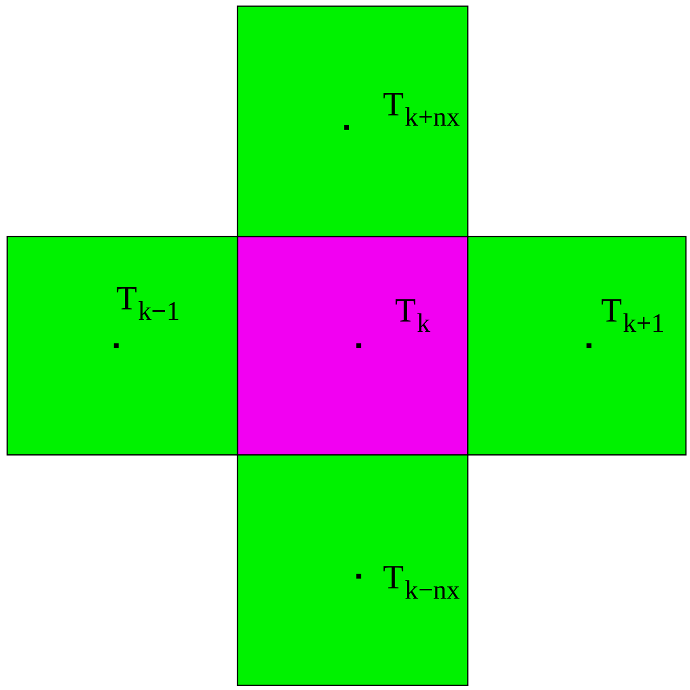

Figure 2: Interior cells in a rectangular block

The boundary cell

RHS cell

For the cell at the convective boundary at the RHS wall, we can write equation matching the heat flux,

$$-cnd\frac{\partial T}{\partial x}|_{surf} = h_x(T_{surf} - T_f) \tag{8}$$

Using the Taylor Series we can write,

$$T_O = T_{surf} - \frac{\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + \frac{(\frac{\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{9}$$

$$T_W = T_{surf} - \frac{3\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + \frac{(\frac{3\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{10}$$

Multiply eq. (9) by A and eq. (10) by B, and solving for A and B by setting the coefficient of $\frac{\partial T}{\partial x}$ equal to 1 and $\frac{\partial T^2}{\partial x^2}$ equal to zero

$$AT_O = AT_{surf} - A\frac{\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + A\frac{(\frac{\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{11}$$

$$BT_{ss} = BT_{surf} - B\frac{3\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + B\frac{(\frac{3\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{12}$$

Thus we need to solve for,

$$-A\frac{\Delta x}{2} - B\frac{3\Delta x}{2} = 1 \tag{13}$$

$$A\frac{(\frac{\Delta x}{2})^2}{2!} + B\frac{(\frac{3\Delta x}{2})^2}{2!} = 0 \tag{14}$$

Solving for A and B, we obtain, $A = -\frac{3}{\Delta x}$ and $B = \frac{1}{3\Delta x}$
Substituting these we get

$$-\frac{3}{\Delta x}T_O = -\frac{3}{\Delta x}T_{surf} + \frac{3}{\Delta x}\frac{\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} - \frac{3}{\Delta x}\frac{(\frac{\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{15}$$

$$\frac{1}{3\Delta x}T_W = \frac{1}{3\Delta x}T_{surf} - \frac{1}{3\Delta x}\frac{3\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + \frac{1}{3\Delta x}\frac{(\frac{3\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{16}$$

Solving for $\frac{\partial T}{\partial x}|_{surf}$ gives,

$$\frac{\partial T}{\partial x}|_{surf} = -\frac{3}{\Delta x}T_O + \frac{1}{3\Delta x}T_W + \frac{8}{3\Delta x}T_{surf} \tag{17}$$

simplfying we get,

$$\frac{\partial T}{\partial x}|_{surf} = \frac{-9T_O + T_W + 8T_{surf}}{3\Delta x} \tag{18}$$

Putting this in the governing equation of flux matching, Eq.(8) we get,

$$-cnd\frac{(-9T_O + T_W + 8T_{surf})}{3\Delta x} = h_x(T_{surf} - T_f) \tag{19}$$

Rearranging the terms

$$T_{surf} = \frac{(9cndT_k - cndT_W + 3\Delta xhT_f)}{8cnd + 3\Delta xh_x} \tag{20}$$

and eliminating the T_{surf} from the heat flux equation to obtain gradient at the surface we get,

$$-cnd\frac{\partial T}{\partial x}|_{surf} = h_x(T_{surf} - T_f) = h_x(\frac{(9cndT_k - cndT_W + 3\Delta xhT_f)}{8cnd + 3\Delta xh_x} - T_f) \tag{21}$$

$$= h_x cnd\frac{(9T_k - T_W - 8T_f)}{8cnd + 3\Delta xh_x} \tag{22}$$

Similarly the other cell which is at the top wall we can write discretized equation as,

$$T_{surf} = \frac{(9cndT_k - cndT_S + 3\Delta yhT_f)}{8cnd + 3\Delta yh_y} \tag{23}$$

and the heat flux in the y direction

$$-cnd\frac{\partial T}{\partial y}|_{surf} = h_y(T_{surf} - T_f) = h_y(\frac{(9cndT_k - cndT_S + 3\Delta yhT_f)}{8cnd + 3\Delta yh_y} - T_f) \quad (24)$$

$$= h_y cnd\frac{(9T_k - T_S - 8T_f)}{8cnd + 3\Delta yh_y} \quad (25)$$

The corner cell
Consider the top left corner cell for discretization.

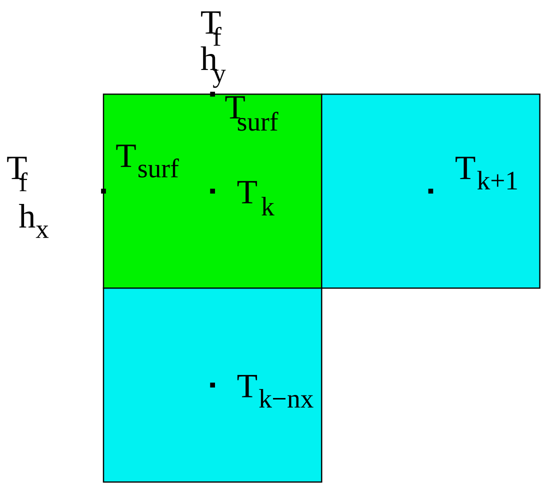

Figure 3: Cell at the top left corner of the the rectangular block

Discretized governing equation for the cell is,

$$\rho C_p\frac{(T^{n+1} - T^n)}{\Delta t}\Delta x\Delta y = cnd\Delta y\frac{(T_E^n - T_O^n)}{\Delta x} - h_y cnd\frac{(9T_O^n - T_S^n - 8T_f)}{8k + 3\Delta yh_y}\Delta x - h_x cnd\frac{(9T_O^n - T_E^n - 8T_f)}{8cnd + 3\Delta xh_x} - cnd\Delta x\frac{(T_O^n - T_S^n)}{\Delta y} + q^{'''}\Delta x\Delta y$$

Dividing the equation by the $\Delta x\Delta y\rho C_p$, substituting the value of T_S and rearranging the terms in terms of index k we get,

$$\frac{(T^{n+1} - T^n)}{\Delta t} = \left(\frac{\alpha}{\Delta x^2} + \frac{h_x\alpha}{8cnd + 3\Delta xh_x}\right)T_{k+1}^n - (\frac{\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_x\alpha}{\Delta x(8cnd + 3\Delta xh_x)} + \frac{9h_y\alpha}{\Delta y(8cnd + 3\Delta yh_y)})T_k^n + (\frac{8h_x\alpha}{\Delta x(8cnd + 3\Delta xh_x)} + \frac{8h_y\alpha}{\Delta y(8cnd + 3\Delta yh_y)})T_f + \left(\frac{\alpha}{\Delta y^2} + \frac{h_y\alpha}{\Delta y(8cnd + 3\Delta yh_y)}\right)T_{k-nx}^n + \frac{q_k^{'''}}{\rho Cp} \quad (26)$$

Similarly the discretized equation for the top right corner is given by,

$$\frac{(T^{n+1}-T^n)}{\Delta t} = \left(\frac{\alpha}{\Delta x^2}+\frac{h_x\alpha}{8cnd+3\Delta x h_x}\right)T^n_{k-1} - (\frac{\alpha}{\Delta x^2}+\frac{\alpha}{\Delta y^2}+ \tag{27}$$

$$\frac{9h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)}+\frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + (\frac{8h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)}+\frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T_f$$

$$+\left(\frac{\alpha}{\Delta y^2}+\frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}\right)T^n_{k-nx}+\frac{q_k^{'''}}{\rho Cp}$$

For the bottom left and right corner we can write as,

$$\frac{(T^{n+1}-T^n)}{\Delta t} = \left(\frac{\alpha}{\Delta x^2}+\frac{h_x\alpha}{8cnd+3\Delta x h_x}\right)T^n_{k+1} - (\frac{\alpha}{\Delta x^2}+\frac{\alpha}{\Delta y^2}+ \tag{28}$$

$$\frac{9h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)}+\frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + (\frac{8h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)}+\frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T_f$$

$$+\left(\frac{\alpha}{\Delta y^2}+\frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}\right)T^n_{k+nx}+\frac{q_k^{'''}}{\rho Cp}$$

$$\frac{(T^{n+1}-T^n)}{\Delta t} = \left(\frac{\alpha}{\Delta x^2}+\frac{h_x\alpha}{8cnd+3\Delta x h_x}\right)T^n_{k-1} - (\frac{\alpha}{\Delta x^2}+\frac{\alpha}{\Delta y^2}+ \tag{29}$$

$$\frac{9h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)}+\frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + (\frac{8h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)}+\frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T_f$$

$$+\left(\frac{\alpha}{\Delta y^2}+\frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}\right)T^n_{k+nx}+\frac{q_k^{'''}}{\rho Cp}$$

Cells adjacent the wall

There are four types of cell groups each with RHS, LHS, top, bottom of the wall. The discretized equation for the cells adjacent to top wall is,

$$\rho C_p\frac{(T^{n+1}-T^n)}{\Delta t}\Delta x\Delta y = cnd\Delta y\frac{(T_E-T_O)}{\Delta x} - cnd\Delta y\frac{(T_O-T_W)}{\Delta x} - h_y cnd\frac{(9T_O-T_S-8T_f)}{8cnd+3\Delta y h_y}\Delta x - cnd\Delta x\frac{(T_O-T_S)}{\Delta y}+q^{'''}\Delta x\Delta y$$

Dividing the equation by the $\Delta x\Delta y\rho C_p$, rearranging, and rewriting the equations in the k index form we get,

$$\frac{(T^{n+1}-T^n)}{\Delta t} = \frac{\alpha}{\Delta x^2}T^n_{k+1} - (\frac{2\alpha}{\Delta x^2}+\frac{\alpha}{\Delta y^2}+\frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + \frac{\alpha}{\Delta x^2}T^n_{k-1} + \tag{30}$$

$$(\frac{\alpha}{\Delta y^2}+\frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_{k-nx} + \frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}T_f + \frac{q_k^{'''}}{\rho Cp} \tag{31}$$

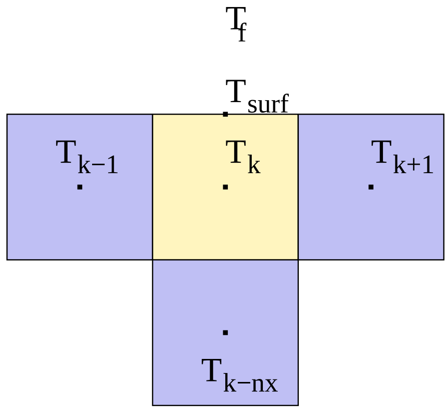

Figure 4: Cell at the top side of the the rectangular block

For the cells adjacent to bottom wall we can write,

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \frac{\alpha}{\Delta x^2}T^n_{k+1} - (\frac{2\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + \frac{\alpha}{\Delta x^2}T^n_{k-1} + \quad (32)$$

$$(\frac{\alpha}{\Delta y^2} + \frac{h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_{k+nx} + \frac{8h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)}T_f + \frac{q_k^{'''}}{\rho Cp} \quad (33)$$

For the cells adjacent to LHS walls discretized equation will be,

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \frac{\alpha}{\Delta y^2}T^n_{k+nx} - (\frac{2\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + \frac{\alpha}{\Delta y^2}T^n_{k-nx} + \quad (34)$$

$$(\frac{\alpha}{\Delta x^2} + \frac{h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)})T^n_{k+1} + \frac{8h_x k}{\Delta x(8cnd+3\Delta x h_x)}T_f + \frac{q_k^{'''}}{\rho Cp} \quad (35)$$

For the cells adjacent to RHS walls discretized equation will be,

$$\frac{(T^{n+1}-T^{n})}{\Delta t} = \frac{\alpha}{\Delta y^2}T^n_{k+nx} - (\frac{2\alpha}{\Delta x^2} + \frac{\alpha}{\Delta y^2} + \frac{9h_y\alpha}{\Delta y(8cnd+3\Delta y h_y)})T^n_k + \frac{\alpha}{\Delta y^2}T^n_{k-nx} + \quad (36)$$

$$(\frac{\alpha}{\Delta x^2} + \frac{h_x\alpha}{\Delta x(8cnd+3\Delta x h_x)})T^n_{k-1} + \frac{8h_x k}{\Delta x(8cnd+3\Delta x h_x)}T_f + \frac{q_k^{'''}}{\rho Cp} \quad (37)$$

Part B

Derivation of the stability limit for the explicit method using von Neumann (Fourier) approach

Let us express the T as,

$$T = \sum_{o}^{\infty} a_n e^{kx} e^{ky} \quad (38)$$

The two dimensional governing equation of heat flow is

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) \quad (39)$$

Application of the finite volume apporoach as discussed above yields the following expression,

$$T^{n+1}-T^{n}=\alpha\Delta t\frac{(T^{n}_{k+1}-T^{n}_{k})}{\Delta x^{2}}-\alpha\Delta t\frac{(T^{n}_{k}-T^{n}_{k-1})}{\Delta x^{2}}+\alpha\Delta t\frac{(T^{n}_{k+nx}-T^{n}_{k})}{\Delta y^{2}}-\alpha\Delta t\frac{(T^{n}_{k}-T^{n}_{k-nx})}{\Delta y^{2}} \tag{40}$$

Which can be written as,

$$\frac{T^{n+1}-T^{n}}{\alpha\Delta t}=\frac{(T^{n}_{k+1}-2T^{n}_{k}+T^{n}_{k-1})}{\Delta x^{2}}+\frac{(T^{n}_{k+nx}-2T^{n}_{k}+T^{n}_{k-nx})}{\Delta y^{2}} \tag{41}$$

Let us purturb the temperature with small error δT, with this purberation the above equation becomes,

$$\frac{\delta T^{n+1}-\delta T^{n}}{\alpha\Delta t}=\frac{(\delta T^{n}_{k+1}-2\delta T^{n}_{k}+\delta T^{n}_{k-1})}{\Delta x^{2}}+\frac{(\delta T^{n}_{k+nx}-2\delta T^{n}_{k}+\delta T^{n}_{k-nx})}{\Delta y^{2}} \tag{42}$$

δT will be,

$$\delta T^{n}=\delta T e^{ik\delta x(i-1)}e^{ik\delta y(j-1)} \tag{43}$$

$$\frac{\delta T^{n+1}e^{ik\delta x(i-1)}e^{ik\delta y(j-1)}-\delta T^{n}e^{ik\delta x(i-1)}e^{ik\delta y(j-1)}}{\alpha\Delta t}=\frac{\delta T e^{ik\delta y(j-1)}}{\Delta x^{2}}(e^{ik\delta x}-2e^{ik\delta x(i-1)}+e^{ik\delta x(i-2)})+\frac{\delta T e^{ik\delta x(i-1)}}{\Delta y^{2}}(e^{ikj\delta y}-2e^{ik\delta x(j-1)}+e^{ik\delta x(j-2)}) \tag{44}$$

Rearranging the above equation we get,

$$\delta T^{n+1}=\delta T^{n}\left(1-2\alpha\Delta t[\frac{1}{\Delta x^{2}}+\frac{1}{\Delta x^{2}}]+2\alpha\Delta t[\frac{cosk\delta x}{\delta x^{2}}+\frac{cosk\delta x}{\delta x^{2}}]\right) \tag{45}$$

let

$$G=\left(1-2\alpha\Delta t[\frac{1}{\Delta x^{2}}+\frac{1}{\Delta x^{2}}]+2\alpha\Delta t[\frac{cosk\delta x}{\delta x^{2}}+\frac{cosk\delta x}{\delta x^{2}}]\right) \tag{46}$$

For stability $|G|\leq 1$
which is same as writting $-1\leq G\leq 1$
Hence,

$$-1\leq\left(1-2\alpha\Delta t[\frac{1}{\Delta x^{2}}+\frac{1}{\Delta x^{2}}]+2\alpha\Delta t[\frac{cosk\delta x}{\delta x^{2}}+\frac{cosk\delta x}{\delta x^{2}}]\right)\leq 1 \tag{47}$$

For all $k\Delta x$ and $k\Delta y$, G is positive, Applying lower limit,

$$-1\leq\left(1-2\alpha\Delta t[\frac{1}{\Delta x^{2}}+\frac{1}{\Delta x^{2}}]+2\alpha\Delta t[\frac{cosk\delta x}{\delta x^{2}}+\frac{cosk\delta x}{\delta x^{2}}]\right) \tag{48}$$

$$-2 \leq \left(-2\alpha\Delta t[\frac{1}{\Delta x^2} + \frac{1}{\Delta x^2}] + 2\alpha\Delta t[\frac{cosk\delta x}{\delta x^2} + \frac{cosk\delta x}{\delta x^2}] \right) \quad (49)$$

$$\left(\alpha\Delta t[\frac{1}{\Delta x^2} + \frac{1}{\Delta x^2}] - \alpha\Delta t[\frac{cosk\delta x}{\delta x^2} + \frac{cosk\delta x}{\delta x^2}] \right) \leq 1 \quad (50)$$

The above expression is maximum at the $cosk\delta x = cosk\delta y = -1$ which gives the condition for stability as,

$$\left(2\alpha\Delta t[\frac{1}{\delta x^2} + \frac{1}{\delta x^2}] \right) \leq 1 \quad (51)$$

or

$$\Delta t \leq \frac{1}{2\alpha[\frac{1}{\delta x^2} + \frac{1}{\delta x^2}]} \quad (52)$$

Results and discussion

1. Figure 5 shows below the time history plot of centeral volume teperature of all the three methods for a 3 by 3 grid for 5 s transient and with a time step of 0.0001 seconds. With this very small time steps transient history plots of three methods coincide.

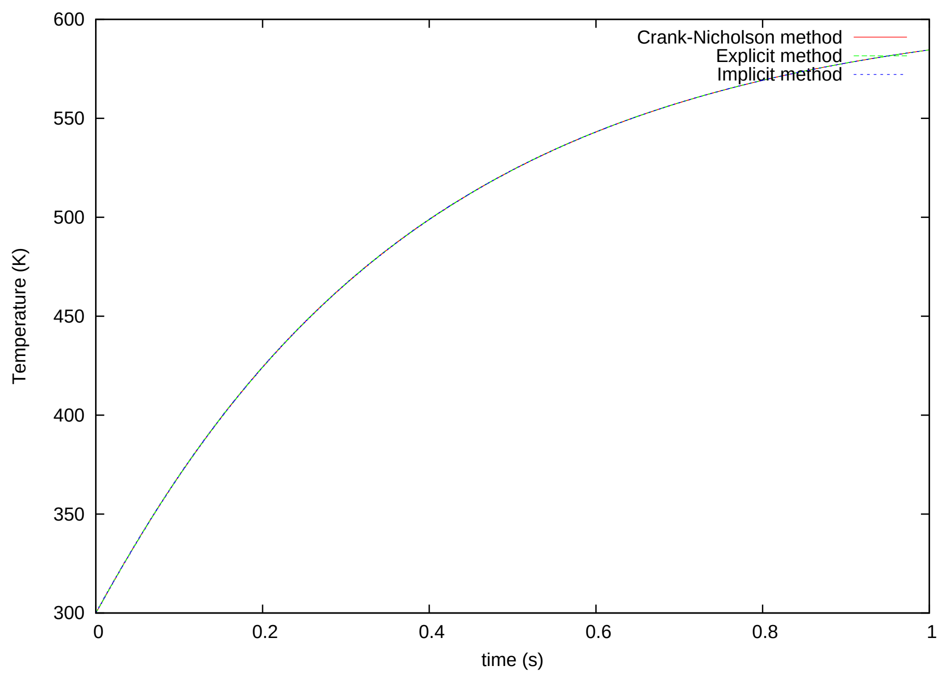

Figure 5: Time history plot of centeral volume teperature for all the three methods

2. Figure 6 shows the time history plot of the central volume temperature with different time steps of 0.05, 0.1, 0.2 and 0.4s along with 0.0001s. With the largest time step of 0.4 there are small weigles in the plot but method finally converges to a stable solution.

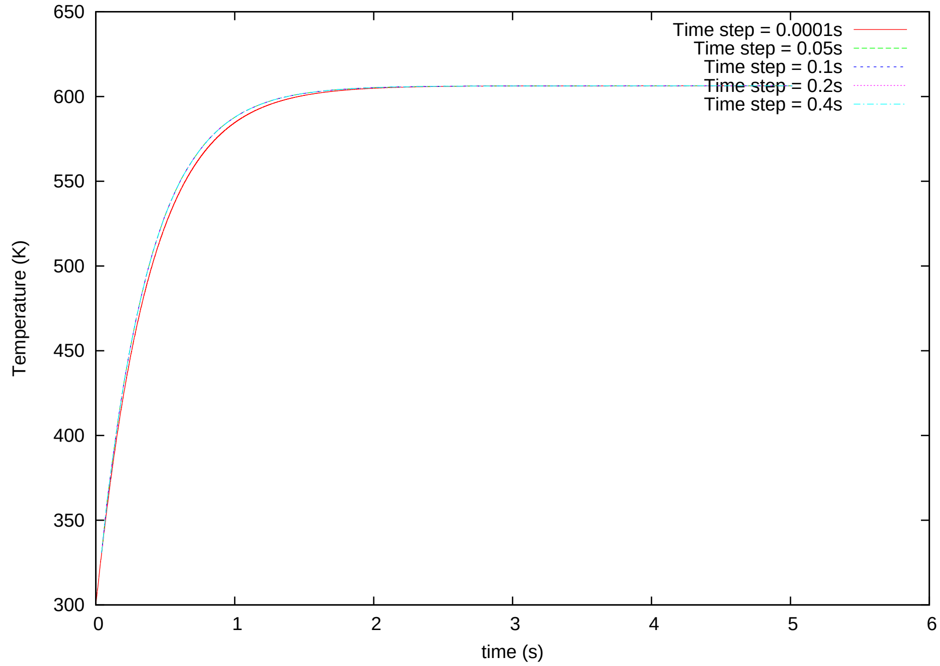

Figure 6: Fully implicit method with different time steps

3. Figure 7 shows the time history plot for Crank-Nicholson method with different time steps. Even with largest time step of 0.4s method is stable and converges to a stable solution.

4. Richardson extrapolation analysis of time discretization error to was applied to all points in the mesh. A subroutine "TemporalRichardson" was written for this purpose. Transient simulation was carried out over a 9 by 9 mesh for 0.8s with three different time steps of 0.0001. 0.002 and 0.004 s. Temperatures at the end of time 0.8s at the three different time steps are used for estimation of order of accuracy and error associated with lowest time step in the solution with explicit method. Following is the summary for the Richardson analysis.

- Time Step 1.00E-03
- Average error 1.06E-01
- Average order 1.00E+00
- Maximum error 1.06E-01
- Maximum order 1.00E+00
- Minimum error 5.49E-02
- Minimum order 1.00E+00

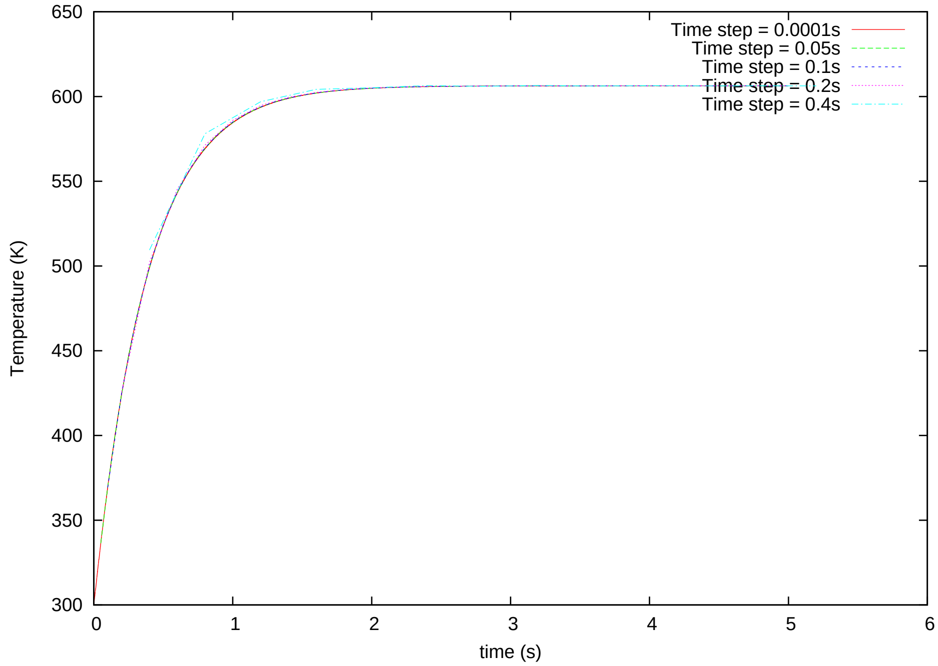

Figure 7: Crank-Nicholson method with different time steps

The order of the explicit method is one as expected.

5. Similar analysis is carried out for the two time sets 0.01, 002, 0.04s and 0.1, 0.2, and 0.4s. Results are summarised below. The order of the method is 2 for the higher set of time steps, and with small time steps it is 1.31. If the solution with the small time steps made to run for the longer time it gives order of 2 for Implicit method as expected.

Time steps- 0.01, 0.02, 0.04 s

- Time Step 1.00E-02
- Average error 6.17E-05
- Average order 1.31E+00
- Maximum error 6.17E-05
- Maximum order 1.31E+00
- Minimum error 2.07E-05
- Minimum order 1.31E+00

Time steps- 0.1, 0.2, 0.4 s

- Time Step 1.00E-01
- Average error 1.52E-03
- Average order 1.97E+00

- Maximum error 1.52E-03
- Maximum order 1.99E+00
- Minimum error 5.00E-04
- Minimum order 1.99E+00

6. Results of Richardson analysis for the Crank-Nicholson is summarised below for two set of time steps. Compared to the earlier results, maximum error is higher for this method. Order of the method is 2 as expected.

Time steps- 0.01, 0.02, 0.04 s

- Time Step 1.00E-02
- Average error 3.34E-07
- Average order 1.99E+00
- Maximum error 3.34E-07
- Maximum order 1.99E+00
- Minimum error 1.12E-07
- Minimum order 1.99E+00

Time steps- 0.1, 0.2, 0.4 s

- Time Step 1.00E-01
- Average error 9.99E+02
- Average order 1.84E+00
- Maximum error 9.99E+02
- Maximum order 4.30E+00
- Minimum error 1.98E-04
- Minimum order 4.30E+00