## Modeling of flow in a circular tube with heated walls

Water flows in a circular tube with wall heated externally by the supply of constant heat flux. It is intended to predict the transient behavior of the fluid in the tube for 30s.

**Data**

- Tube id= 0.004m
- Thickness of the tube wall is 0.001 m
- Tube metal density = 8000 kg/m$^3$
- Specific heat of tube metal, C$_p$ =500 J/kg
- Conductivity= 80 W/m$^2$K
- Perfectly insulated outer wall of the tube

  **Initial Condition**

- Area averaged inlet velocity = 0.1 m/s
- Fully developed, laminar flow
- Inlet water temperature= 300K, initial condition
- Outlet pressure= 2 MPa
- Inlet temperature changes as below for transient time from 0 through 2 seconds

$$T_{in} = 300 + 75t^2 - 25t^3 \tag{1}$$

for transient time greater than 2 seconds

$$T_{in} = 400 \tag{2}$$

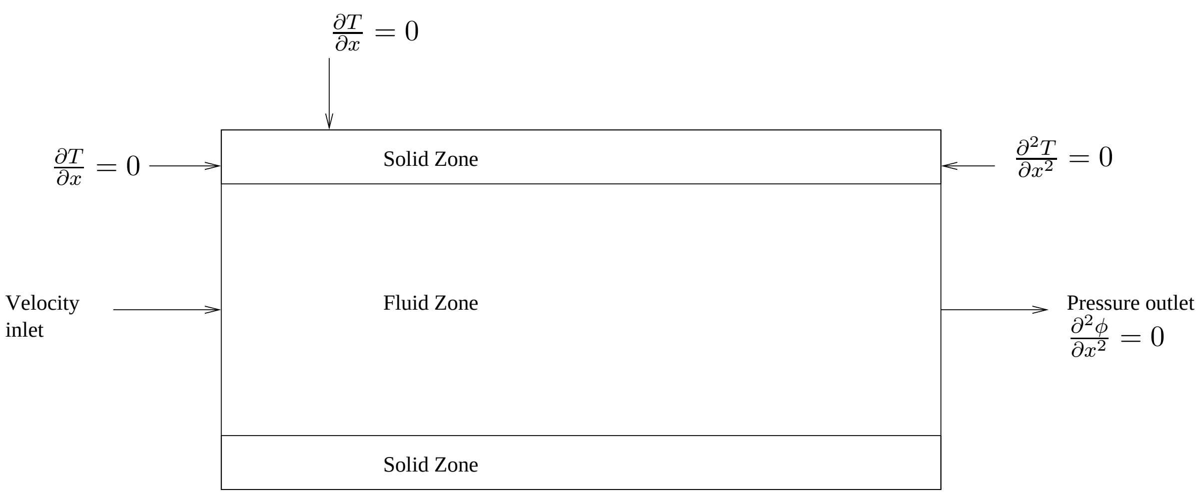


Figure 1: Boundary conditions for the domain

**Assumptions in the solution procedure**

- One dimensional flow
- Laminar flow
- Zero curvature boundary condition for the flow outlet
- Temperature dependent liquid density
- Incompressible fluid flow
- Zero heat flux at the other side walls of the solid zone facing inlet and outlet

**Procedure**

1. The heat source to the energy equation appears in the form $h(T_{wall} - T_{fluid})$ in energy equation of the fluid where the heat transfer coefficient is based upon laminar flow.

2. 2D transient conduction model for the wall, which will yield the wall temperature and temperature distribution in the tube wall.

3. Second order accurate difference formulation in space for interior cells

4. Either a fully implicit time level method for transient calculations

5. The fluid/metal boundary second order flux matching

6. Steady state through the execution of a sufficiently long transient

7. Sparse (direct or iterative) solution package to solve the linearized equations associated with the Newton iteration

8. Staggered grid with thermodynamic properties stored at the cell centers and velocities stored at the cell faces

9. Use of Implicit Quick method for spatial discretization

10. Independent variables are P, T, V

11. Solve for $\rho$ at each step using Newton Iteration method

**Description of the model equations**

**Solid Domain**

In the solid domain, two dimensional heat conduction equation is solved. Transient heat conduction equation with source in radial co-ordinate system is as below.

$$\rho C_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + q^{'''} \tag{3}$$

At the inner wall of the tube there is heat transfer to the fluid by convection. In the equation below, $h$ is heat transfer coefficient for laminar flow under constant heat flux condition.

at $r = r_i$,

$$-k\frac{\partial T}{\partial r}|_{surf} = h(T_w - T_f) \tag{4}$$

Heat flux is zero at the outer wall of the tube boundary conditions, i.e. at $r = r_o$,

$$-k\frac{\partial T}{\partial r}|_{outer} = 0 \tag{5}$$

Also heat flux is zero at the left wall and right wall,

$$-k\frac{\partial T}{\partial r}|_{l} = 0 \tag{6}$$

$$-k\frac{\partial T}{\partial r}|_{r} = 0 \tag{7}$$

**Difference equation for the interior cell of the solid zone** For the interior cell there are four neighboring cells, north, south, east and west. Integrating the governing equation for the heat flow over the control volume in time and space we obtain.

$$\int_o^t \int_V \rho C_p \frac{\partial T}{\partial t} r dr d\theta dx dt = \int_o^t \int_V \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) r dr d\theta dx dt + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) r dr d\theta dx dt + \int_o^t \int_V q^{'''} r dr dx dt \tag{8}$$

Using the Gauss divergence theorem and using implicit method for time integration we get,

$$\rho C_p \frac{T_O^{t+1} - T_O^t}{\Delta t} r_o \Delta r \Delta\theta \Delta x = k\frac{\partial T^{t+1}}{\partial r}|_n A_n - k\frac{\partial T^{t+1}}{\partial r}|_s A_s + k\frac{\partial T^{t+1}}{\partial x}|_e A_e - k\frac{\partial T^{t+1}}{\partial x}|_w A_w + q^{'''} r_O \Delta r \Delta\theta \Delta x \tag{9}$$

Expanding the gradients at the cell walls in terms of the values at the cell centers,

$$\rho C_p \frac{T_O^{t+1} - T_O^t}{\Delta t} r_O \Delta r \Delta\theta \Delta x = k\frac{T_N^{t+1} - T_O^{t+1}}{\Delta r}(r_O + \Delta r/2)\Delta\theta\Delta x - k\frac{T_O^{t+1} - T_S^{t+1}}{\Delta r}(r_O - \Delta r/2)\Delta\theta\Delta x + k\frac{T_E^{t+1} - T_O^{t+1}}{\Delta x} r_O \Delta r_O \Delta\theta - k\frac{T_O^{t+1} - T_W^{t+1}}{\Delta\theta} r_O \Delta r \Delta\theta + q^{'''} r_O \Delta r \Delta\theta \Delta x \tag{10}$$

Dividing by $\rho C_p r_O \Delta r \Delta\theta \Delta x$ we obtain,

$$\frac{T_O^{t+1} - T_O^t}{\Delta t} = \alpha\frac{T_N^{t+1} - T_O^{t+1}}{r_O \Delta r^2}(r_O + \Delta r/2) - \alpha\frac{T_O^{t+1} - T_S^{t+1}}{r_O \Delta r^2}(r_O - \Delta r/2) + \alpha\frac{T_E^{t+1} - T_O^{t+1}}{\Delta x^2} - \alpha\frac{T_O^{t+1} - T_W^{t+1}}{\Delta x^2} + \frac{q^{'''}}{\rho C_p} \tag{11}$$

Where $\alpha = \frac{k}{\rho C_p}$ is thermal diffusivity. Rearranging equations we obtain implicit form of the difference equation,

$$T_O^{t+1}\left(1+\frac{\alpha\Delta t(r_O+\Delta r/2)}{r_O\Delta r^2}+\frac{\alpha\Delta t(r_O-\Delta r/2)}{r_O\Delta r^2}+\frac{2\alpha\Delta t}{\Delta x^2}\right)-\frac{\alpha\Delta t(r_O+\Delta r/2)}{r_O\Delta r^2}T_N^{t+1} \\ -\frac{\alpha\Delta t(r_O-\Delta r/2)}{r_O\Delta r^2}T_S^{t+1}-\frac{\alpha\Delta t}{\Delta x^2}T_E^{t+1}-\frac{\alpha\Delta t}{\Delta x^2}T_W^{t+1}=\frac{q^{'''}}{\rho C_p}\Delta t+T_O^t \quad (12)$$

**Difference equation for the outer boundary cell of the solid zone**

For the outer boundary cell, there is no surface heat flux due to adiabatic wall. The discretized equation after application of Gauss's divergence theorem,

$$\rho C_p\frac{T_O^{t+1}-T_O^t}{\Delta t}r_O\Delta r\Delta\theta\Delta x=k\frac{\partial T^{t+1}}{\partial r}|_n A_n-k\frac{\partial T^{t+1}}{\partial r}|_s A_s+k\frac{\partial T^{t+1}}{\partial x}|_e A_e-k\frac{\partial T^{t+1}}{\partial x}|_w A_w \\ +q^{'''}r_O\Delta r\Delta\theta\Delta x \quad (13)$$

Since $k(\frac{\partial T}{\partial r})|_n^{t+1}=0$

Expanding the gradients at the cell walls in terms of the values at the cell centers,

$$\rho C_p\frac{T_O^{t+1}-T_O^t}{\Delta t}r_O\Delta r\Delta\theta\Delta x=k\frac{T_E^{t+1}-T_O^{t+1}}{\Delta x}r_O\Delta r\Delta\theta-k\frac{T_O^{t+1}-T_W^{t+1}}{\Delta x}r_O\Delta r\Delta\theta \\ -k\frac{T_O^{t+1}-T_S^{t+1}}{\Delta r}(r_O-\Delta r/2)\Delta\theta\Delta x+q^{'''}r_O\Delta r\Delta\theta\Delta x \quad (14)$$

Dividing by $\rho C_p r_o\Delta r\Delta\theta\Delta x$ we obtain,

$$\frac{T_O^{t+1}-T_O^t}{\Delta t}=\alpha\frac{T_E^{t+1}-T_O^{t+1}}{\Delta x^2}-k\frac{T_O^{t+1}-T_W^{t+1}}{\Delta x^2}-\alpha\frac{T_O^{t+1}-T_S^{t+1}}{r_O\Delta r^2}(r_O-\Delta r/2)+\frac{q^{'''}}{\rho C_p} \quad (15)$$

Where $\alpha = \frac{k}{\rho C_p}$ is thermal diffusivity. Rearranging equations we obtain implicit form of the difference equation,

$$T_O^{t+1}\left(1+\frac{\alpha\Delta t(r_O+\Delta r/2)}{r_O\Delta r^2}+\frac{2\alpha\Delta t}{\Delta x^2}\right)-\frac{\alpha\Delta t(r_O-\Delta r/2)}{r_O\Delta r^2}T_S^{t+1} \\ -\frac{\alpha\Delta t}{\Delta x^2}T_E^{t+1}-\frac{\alpha\Delta t}{\Delta x^2}T_W^{t+1}=\frac{q^{'''}}{\rho C_p}\Delta t+T_O^t \quad (16)$$

**Difference equation for the inner wall cell of the solid zone**

**Calculation of surface temperature**

For the cell at the convective boundary at the inner wall, we can write equation matching the heat flux as,

$$k\frac{\partial T}{\partial r}|_{surf} = h(T_{surf} - T_f) \tag{17}$$

Writing the Taylor series expansion around $T_{surf}$ in terms of two inner nodes one at the distance $\Delta r/2$ and other at the $3\Delta r/2$, we get,

$$T_O = T_{surf} + \frac{\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + \frac{(\frac{\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{18}$$

$$T_N = T_{surf} + \frac{3\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + \frac{(\frac{3\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{19}$$

Multiply eq. (18) by A and eq. (19) by B, and solving for A and B by setting the coefficient of $\frac{\partial T}{\partial r}$ equal to 1 and $\frac{\partial T^2}{\partial r^2}$ equal to zero

$$AT_O = AT_{surf} + A\frac{\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + A\frac{(\frac{\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{20}$$

$$BT_N = BT_{surf} + B\frac{3\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + B\frac{(\frac{3\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{21}$$

Thus we need to solve for,

$$A\frac{\Delta r}{2} + B\frac{3\Delta r}{2} = 1 \tag{22}$$

$$A\frac{(\frac{\Delta r}{2})^2}{2!} + B\frac{(\frac{3\Delta r}{2})^2}{2!} = 0 \tag{23}$$

Solving for A and B, we obtain, $A = \frac{3}{\Delta r}$ and $B = -\frac{1}{3\Delta r}$

Substituting these we get

$$\frac{3}{\Delta r}T_O = \frac{3}{\Delta r}T_{surf} + \frac{3}{\Delta r}\frac{\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} + \frac{3}{\Delta r}\frac{(\frac{\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{24}$$

$$-\frac{1}{3\Delta r}T_N = -\frac{1}{3\Delta r}T_{surf} - \frac{1}{3\Delta r}\frac{3\Delta r}{2}\frac{\partial T}{\partial r}|_{surf} - \frac{1}{3\Delta r}\frac{(\frac{3\Delta r}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ...$$

$$ \tag{25} $$

Solving for $\frac{\partial T}{\partial r}|_{surf}$ gives,

$$\frac{\partial T}{\partial r}|_{surf} = \frac{3}{\Delta r}T_O - \frac{1}{3\Delta r}T_N - \frac{8}{3\Delta r}T_{surf} \tag{26}$$

Putting this in the governing equation of flux matching, Eq.(17) we get,

$$k\{\frac{3}{\Delta r}T_O - \frac{1}{3\Delta r}T_N - \frac{8}{3\Delta r}T_{surf}\} = h(T_{surf} - T_f) \tag{27}$$

$$\left(h + \frac{8k}{3\Delta r}\right)T_{surf} = hT_f + \frac{3k}{\Delta r}T_O - \frac{k}{3\Delta r}T_N \tag{28}$$

$$T_{surf} = \frac{1}{(h + \frac{8k}{3\Delta r})}\left[hT_f + \frac{3k}{\Delta r}T_O - \frac{k}{3\Delta r}T_N\right] \tag{29}$$

Putting this in the gradient term we obtain,

$$\frac{\partial T}{\partial r}|_{surf} = \frac{3}{\Delta r}T_O - \frac{1}{3\Delta r}T_N - \frac{8}{3\Delta r h + 8k}\left[hT_f + \frac{3k}{\Delta r}T_O - \frac{k}{3\Delta r}T_N\right] \tag{30}$$

Simplifying this we obtain expression as below,

$$\frac{\partial T}{\partial r}|_{surf} = \frac{9h}{(3\Delta r h + 8k)}T_O - \frac{h}{3\Delta r h + 8k}T_N - \frac{8h}{3\Delta r h + 8k}T_f \tag{31}$$

**Error term calculation**

The error term will be,

$$\epsilon = -A\frac{(\frac{\Delta r}{2})^3}{3!}\frac{\partial^3 T}{\partial r^3}|_{surf} + A\frac{(\frac{\Delta r}{2})^4}{4!}\frac{\partial^4 T}{\partial r^4}|_{surf} - B\frac{(\frac{3\Delta r}{2})^3}{3!}\frac{\partial^3 T}{\partial r^3}|_{surf} + B\frac{(\frac{3\Delta r}{2})^4}{4!}\frac{\partial T^4}{\partial r^4}|_{surf} \tag{32}$$

Putting A and B we get,

$$\epsilon = -\frac{3}{8}\frac{\Delta r^2}{3!}\frac{\partial^3 T}{\partial r^3}|_{surf} + \frac{3}{16}\frac{\Delta r^3}{4!}\frac{\partial^4 T}{\partial r^4}|_{surf} + \frac{9}{8}\frac{\Delta r^2}{3!}\frac{\partial^3 T}{\partial r^3}|_{surf} - \frac{27}{16}\frac{\Delta r^3}{4!}\frac{\partial^4 T}{\partial r^4}|_{surf} \tag{33}$$

$$\epsilon = \frac{1}{8}\Delta r^2\frac{\partial^3 T}{\partial r^3}|_{surf} - \frac{1}{16}\Delta r^3\frac{\partial^4 T}{\partial r^4}|_{surf} \tag{34}$$

**Difference equation**

The governing equation of the internal cell is same as interior cell, except the south face is replaced by a convective boundary, After integration over the control volume the difference equation takes the following form.

$$\rho C_p \frac{T_O^{t+1} - T_O^t}{\Delta t} r_O \Delta r \Delta\theta \Delta x = k\frac{T_N^{t+1} - T_O^{t+1}}{\Delta r}(r_O + \Delta r/2)\Delta\theta\Delta x - k\frac{\partial T}{\partial r}|_{surf}(r_O - \Delta r/2)\Delta\theta\Delta x \quad (35)$$
$$+k\frac{T_E^{t+1} - T_O^{t+1}}{\Delta x} r_O\Delta r\Delta\theta - k\frac{T_O^{t+1} - T_W^{t+1}}{\Delta x} r_O\Delta r\Delta\theta + q^{'''} r_O \Delta r\Delta\theta\Delta x$$

Dividing by $\rho C_p r_O \Delta r \Delta\theta \Delta x$ we obtain,

$$\frac{T_O^{t+1} - T_O^t}{\Delta t} = \alpha\frac{T_N^{t+1} - T_O^t}{r_O\Delta r^2}(r_O + \Delta r/2) - \frac{\alpha}{r_O\Delta r}(r_O - \Delta r/2)\frac{\partial T}{\partial r}|_{surf} \quad (36)$$
$$+\alpha\frac{T_E^{t+1} - T_O^{t+1}}{\Delta x^2} - \alpha\frac{T_O^{t+1} - T_W^t}{\Delta x^2} + \frac{q^{'''}}{\rho C_p}$$

Substituting value of $k\frac{\partial T}{\partial r}|_{surf}$ from Eq. 26in the above equation we get,

$$\frac{T_O^{t+1} - T_O^t}{\Delta t} = \alpha(r_O + \Delta r/2)\frac{T_N^{t+1} - T_O^t}{r_O\Delta r^2} - \frac{\alpha(r_O - \Delta r/2)}{r_O\Delta r}(\frac{9h}{(3\Delta rh + 8k)}T_O^{t+1} - \frac{h}{3\Delta rh + 8k}T_N^{t+1} \quad (37)$$
$$-\frac{8h}{3\Delta rh + 8k}T_f) + \alpha\frac{T_E^{t+1} - T_O^{t+1}}{\Delta x^2} - \alpha\frac{T_O^{t+1} - T_W^{t+1}}{\Delta x^2} + \frac{q^{'''}}{\rho C_p}$$

Where $\alpha = \frac{k}{\rho C_p}$ is thermal diffusivity. Rearranging equations we obtain implicit form of the difference equation,

$$T_O^{t+1}\left(1 + \frac{\alpha\Delta t(r_O + \Delta r/2)}{r_O\Delta r^2} + \frac{9h\alpha\Delta t(r_O - \Delta r/2)}{r_O\Delta r(3\Delta rh + 8k)} + \frac{2\alpha\Delta t}{\Delta x^2}\right) - (\frac{\alpha\Delta t(r_O + \Delta r/2)}{r_O\Delta r^2} + \quad (38)$$
$$\frac{h\alpha\Delta t(r_O - \Delta r/2)}{r_O\Delta r(3\Delta rh + 8k)})T_N^{t+1} - \frac{\alpha\Delta t}{x^2}T_E^{t+1} - \frac{\alpha\Delta t}{\Delta x^2}T_W^t = \frac{q^{'''}}{\rho C_p}\Delta t + T_O^t + \frac{8h\alpha\Delta t(r_O - \Delta r/2)}{r_O\Delta r(3\Delta rh + 8k)}T_f$$

**Fluid domain**

Governing equations for the transient, one dimensional, laminar fluid flow are as below,

**Continuity Equation**

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \tag{39}$$

**Momentum Equation**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial x^2} \tag{40}$$

**Energy Equation**

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2} + h(\theta)(T_w - T_f) \tag{41}$$

Nusselt number for fully developed laminar flow with constant wall heat flux is 4.36. Hence heat transfer coefficient is given by following equation.

$$h = Nu_D\frac{k}{D} = 4.36 \times \frac{0.62}{0.004} = 675.8\ W/m^2K \tag{42}$$

**Difference equations for the fluid zone**

Implicit method is used for the transient calculations because of its property of unconditional stability. For the spatial differencing Quick method is used for higher order of accuracy.

**Continuity Equation**

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{(\rho V)_{j+1/2} - (\rho V)_{j-1/2}}{\Delta x_j} = 0 \tag{43}$$

**Momentum Equation**

$$\frac{V_{j+1/2}^{n+1} - V_{j+1/2}^{n}}{\Delta t} + V_{j+1/2}^{n+1}\frac{V_r^{n+1} - V_l^{n+1}}{\Delta x_j} = \frac{-1}{\rho_{j+1/2}^{n+1}}\frac{P_{j+1}^{n+1} - P_j^{n+1}}{\Delta x_j} + K_{j+1/2}^{n+1}V_{j+1/2}^{n+1}|V_{j+1/2}^{n+1}| \tag{44}$$

In the above Eq. 44 $K$, is wall friction coefficient, which is function of velocity and fluid properties given by Fanning friction factor. For laminar flow it is given by,

$$K_{j+1/2}^{n+1} = \frac{f}{2D} \tag{45}$$

In the above Eq.(45), $f$ is Darcy's friction factor. For laminar flow it is $\frac{64}{Re}$.

**Calculation of $K$**

Since there is little variation in the velocity as the fluid passes through the tube we can calculate the $K$ based on the inlet velocity and temperature condition.

$$Re = \frac{\rho V D}{\mu} = \frac{993.69 \times 0.1 \times 0.004}{0.00086} = 462.18 \tag{46}$$

$$f = \frac{64}{Re} = 0.1384 \tag{47}$$

Hence,

$$K = \frac{f}{2D} = \frac{0.1384}{2 \times 0.004} = 17.3 \tag{48}$$

This K is kept same throughout the length of the tube.

**Energy Equation**

$$\frac{\rho_j^{n+1} e_j^{n+1} - \rho_j^n e_j^n}{\Delta t} + \frac{(\rho e V)_{j+1/2}^{n+1} - (\rho e V)_{j-1/2}^{n+1}}{\Delta x_j} + P_j^{n+1} \frac{V_{j+1/2}^{n+1} - V_{j-1/2}^{n+1}}{\Delta x_j} = h_{vol}(T_{surf} - T_j^{n+1}) \tag{49}$$

In the above Eq. 49, $h_{vol}$ is heat transfer coefficient multiplied by the heat transfer area per volume of fluid.

Using Quick method we can write an equation for any property of the fluid stored at the cell center. In above discretized governing equations for the fluid flow, $\rho_{j+1/2}^{n+1}$ and $\rho_{j-1/2}^{n+1}$ are given as,

$$\rho_{j+1/2}^{n+1} = \frac{3}{8}\rho_{j+1}^{n+1} + \frac{3}{4}\rho_j^{n+1} - \frac{1}{8}\rho_{j-1}^{n+1} \tag{50}$$

$$\rho_{j-1/2}^{n+1} = \frac{3}{8}\rho_j^{n+1} + \frac{3}{4}\rho_{j-1}^{n+1} - \frac{1}{8}\rho_{j-2}^{n+1} \tag{51}$$

Similarly $(\rho e)_{j-1/2}^{n+1}$ and $(\rho e)_{j+1/2}^{n+1}$ are obtained as,

$$(\rho e)_{j+1/2}^{n+1} = \frac{3}{8}(\rho e)_{j+1}^{n+1} + \frac{3}{4}(\rho e)_j^{n+1} - \frac{1}{8}(\rho e)_{j-1}^{n+1} \tag{52}$$

$$(\rho e)_{j-1/2}^{n+1} = \frac{3}{8}(\rho e)_j^{n+1} + \frac{3}{4}(\rho e)_{j-1}^{n+1} - \frac{1}{8}(\rho e)_{j-2}^{n+1} \tag{53}$$

$V_r$ and $V_l$ are evaluated as below,

$$V_r^{n+1} = \frac{3}{8}V_{j+3/2}^{n+1} + \frac{3}{4}V_{j+1/2}^{n+1} - \frac{1}{8}V_{j-1/2}^{n+1} \tag{54}$$

$$V_l^{n+1} = \frac{3}{8}V_{j+1/2}^{n+1} + \frac{3}{4}V_{j-1/2}^{n+1} - \frac{1}{8}V_{j-3/2}^{n+1} \tag{55}$$

For the inlet boundary cell and cell immediately next to the boundary these equations remain same except ghost cells with cell properties equal to the inlet boundary conditions are introduced for calculation of j-1 and j-2 cell properties. For calculation of the velocity at the ghost cell at the outlet zero curvature boundary condition is used. By which,

$$\frac{\partial^2 V}{\partial x^2} = 0 \tag{56}$$

or,

$$T_E = 2T_O - T_W \tag{57}$$

Where $T_E$ will be ghost cells.

**Stability analysis**

Stability analysis is performed for the implicit difference equation of the interior cell in the solid zone. Girchgorin's theorem is applied to the equation. By this theorem, eigenvalues of the matrix are calculated as shown below. For an implicit method to be stable the eigenvalues has to be greater than or equal to one.

$$\begin{aligned} \mid \lambda - [1 + \frac{2\alpha\Delta t}{\Delta r^2} + \frac{2\alpha\Delta t}{\Delta x^2}] \mid \leq \mid -\frac{\alpha\Delta t}{r_O\Delta r^2}(r_O + \frac{\Delta r}{2}) \mid + \mid -\frac{\alpha\Delta t}{r_O\Delta r^2}(r_O - \frac{\Delta r}{2}) \mid \\ + \mid -\frac{\alpha\Delta t}{\Delta x^2} \mid + \mid -\frac{\alpha\Delta t}{\Delta x^2} \mid \end{aligned} \tag{58}$$

$$\mid \lambda - [1 + \frac{2\alpha\Delta t}{\Delta r^2} + \frac{2\alpha\Delta t}{\Delta x^2}] \mid \leq \frac{\alpha\Delta t}{r_O\Delta r^2}(r_O + \frac{\Delta r}{2}) + \frac{\alpha\Delta t}{r_O\Delta r^2}(r_O - \frac{\Delta r}{2}) + \frac{\alpha\Delta t}{\Delta x^2} + \frac{\alpha\Delta t}{\Delta x^2} \tag{59}$$

First eigenvalue will be,

$$\lambda_1 - [1 + \frac{2\alpha\Delta t}{\Delta r^2} + \frac{2\alpha\Delta t}{\Delta x^2}] = \frac{\alpha\Delta t}{r_O\Delta r^2}(r_O + \frac{\Delta r}{2}) + \frac{\alpha\Delta t}{r_O\Delta r^2}(r_O - \frac{\Delta r}{2}) + \frac{\alpha\Delta t}{\Delta x^2} + \frac{\alpha\Delta t}{\Delta x^2} \tag{60}$$

$$\lambda_1 = 1 + \frac{4\alpha\Delta t}{\Delta r^2} + \frac{4\alpha\Delta t}{r_O\Delta x^2}] \tag{61}$$

Second eigenvalue will be,

$$\lambda_2 - [1 + \frac{2\alpha\Delta t}{\Delta r^2} + \frac{2\alpha\Delta t}{\Delta x^2}] = -\frac{\alpha\Delta t}{r_O\Delta r^2}(r_O + \frac{\Delta r}{2}) - \frac{\alpha\Delta t}{r_O\Delta r^2}(r_O - \frac{\Delta r}{2}) - \frac{\alpha\Delta t}{\Delta x^2} - \frac{\alpha\Delta t}{\Delta x^2} \tag{62}$$

$$\lambda_2 - [1 + \frac{2\alpha\Delta t}{\Delta r^2} + \frac{2\alpha\Delta t}{\Delta x^2}] = -\frac{2\alpha\Delta t}{\Delta r^2} - \frac{2\alpha\Delta t}{\Delta x^2} \tag{63}$$

$$\lambda_2 = 1 \tag{64}$$

Hence the stability condition is satisfied by the both eigenvalues, hence a unconditionally stable difference scheme.

### Error analysis of the energy equation for the solid domain

Error analysis for the surface node is already shown to be order 2 in space in the difference equation formulation. In this section error for the interior node is performed by using the Taylor series method.

Expanding the $T_O, T_N, T_S$ and $T_E$ terms around the point O, as following,

$$T_N = T_O + \Delta r\frac{\partial T}{\partial r} + \frac{\Delta r^2}{2!}\frac{\partial^2 T}{\partial r^2}|_O + \frac{\Delta r^3}{3!}\frac{\partial^3 T}{\partial r^3}|_O + \frac{\Delta r^4}{4!}\frac{\partial^4 T}{\partial r^4}|_O + ... \quad (65)$$

$$T_S = T_O - \Delta r\frac{\partial T}{\partial r} + \frac{\Delta r^2}{2!}\frac{\partial^2 T}{\partial r^2}|_O - \frac{\Delta r^3}{3!}\frac{\partial^3 T}{\partial r^3}|_O + \frac{\Delta r^4}{4!}\frac{\partial^4 T}{\partial r^4}|_O + ... \quad (66)$$

$$T_E = T_O + \frac{\Delta x}{1!}\frac{\partial T}{\partial x} + \frac{\Delta x^2}{x^2 2!}\frac{\partial^2 T}{\partial x^2}|_O + \frac{(\Delta x)^3}{x^3 3!}\frac{\partial^3 T}{\partial \Delta x^3}|_O + \frac{(\Delta)^4}{x^4 4!}\frac{\partial^4 T}{\partial x^4}|_O + ... \quad (67)$$

$$T_W = T_O - \frac{\Delta x}{1!}\frac{\partial T}{\partial x} + \frac{\Delta x^2}{x^2 2!}\frac{\partial^2 T}{\partial x^2}|_O - \frac{(\Delta x)^3}{x^3 3!}\frac{\partial^3 T}{\partial \Delta x^3}|_O + \frac{(\Delta)^4}{x^4 4!}\frac{\partial^4 T}{\partial x^4}|_O + ... \quad (68)$$

With this,

$$\alpha\frac{T_N^{t+1} - T_O^{t+1}}{r_O \Delta r^2}(r_O + \Delta r/2) - \alpha\frac{T_O^{t+1} - T_S^{t+1}}{r_O \Delta r^2}(r_O - \Delta r/2) = \frac{\alpha}{\Delta r^2}(T_N - 2T_O + T_S) + \frac{\alpha}{2 r_O \Delta r}(T_N - T_S) \quad (69)$$

Putting the Taylor series expansions in the above equation we obtain,

$$\alpha\frac{T_N^{t+1} - T_O^{t+1}}{r_O \Delta r^2}(r_O + \Delta r/2) - \alpha\frac{T_O^{t+1} - T_S^{t+1}}{r_O \Delta r^2}(r_O - \Delta r/2) = \alpha\left(\frac{\partial^2 T}{\partial r^2}|_O + 2\frac{\Delta r^2}{4!}\frac{\partial^4 T}{\partial r^4}|_O\right) + \quad (70)$$

$$\frac{\alpha}{r_O}\left(\Delta r\frac{\partial T}{\partial r}|_O + \frac{\Delta r^2}{3!}\frac{\partial^3 T}{\partial r^3}|_O\right) \quad (71)$$

Similarly

$$\alpha\frac{T_E^{t+1} - T_O^{t+1}}{\Delta x^2} - \alpha\frac{T_O^{t+1} - T_W^{t+1}}{\Delta x^2} = \frac{\alpha}{\Delta x^2}(T_E - 2T_O + T_W) \quad (72)$$

$$\alpha\frac{T_E^{t+1} - T_O^{t+1}}{\Delta x^2} - \alpha\frac{T_O^{t+1} - T_W^{t+1}}{\Delta x^2} = \alpha\left(\frac{\partial^2 T}{\partial x^2}|_O + 2\frac{\Delta x^2}{4!}\frac{\partial^4 T}{\partial x^2}|_O\right) \quad (73)$$

For the transient term, the $T_O^{t+1}$ is expanded around $O$ in time,

$$T_O^{n+1} = T_O^n + \Delta t\frac{\partial T}{\partial t} + \frac{\Delta t^2}{2!}\frac{\partial^2 T}{\partial t^2} + ... \quad (74)$$

Rearranging we get

$$\frac{T_O^{n+1} - T_O^n}{\Delta t} = \frac{\partial T}{\partial t} + \frac{\Delta t}{2!}\frac{\partial^2 T}{\partial t^2} + ... \tag{75}$$

Hence the difference form of the energy equation is of first order in time and second order in space.

**Error analysis of the continuity equation**

Expanding the density in time around cell $j$ we obtain,

$$\rho_j^{n+1} = \rho_j^n + \Delta t\frac{\partial \rho}{\partial t} + \frac{\Delta t^2}{2!}\frac{\partial^2 \rho}{\partial t^2} + ... \tag{76}$$

Rearranging we get

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} = \frac{\partial \rho}{\partial t} + \frac{\Delta t}{2!}\frac{\partial^2 \rho}{\partial t^2} + ... \tag{77}$$

Hence the difference equation of the continuity equation is of order one in time.

For evaluation of the order of error in time for the difference form of the continuity equation, expanding terms by Taylor series expansion.

$$(\rho V)_{j+1/2}^{n+1} = (\rho V)_j^{n+1} + \frac{\Delta x_j}{2}\frac{\partial (\rho V)_j^{n+1}}{\partial x_j} + \frac{(\frac{\Delta x_j}{2})^2}{2!}\frac{\partial^2 (\rho V)_j^{n+1}}{\partial x_j^2} + \frac{(\frac{\Delta x_j}{2})^3}{3!}\frac{\partial^3 (\rho V)_j^{n+1}}{\partial x_j^3} + ... \tag{78}$$

$$(\rho V)_{j-1/2}^{n+1} = (\rho V)_j^{n+1} - \frac{\Delta x_j}{2}\frac{\partial (\rho V)_j^{n+1}}{\partial x_j} + \frac{(\frac{\Delta x_j}{2})^2}{2!}\frac{\partial^2 (\rho V)_j^{n+1}}{\partial x_j^2} - \frac{(\frac{\Delta x_j}{2})^3}{3!}\frac{\partial^3 (\rho V)_j^{n+1}}{\partial x_j^3} + ... \tag{79}$$

Eq.(78)-Eq.(79) gives,

$$\frac{(\rho V)_{j+1/2}^{n+1} - (\rho V)_{j-1/2}^{n+1}}{\Delta x_j} = \frac{\partial (\rho V)_j^{n+1}}{\partial x_j} + 2\frac{(\frac{\Delta x_j}{2})^2}{3!}\frac{\partial^3 (\rho V)_j^{n+1}}{\partial x_j^3} + ... \tag{80}$$

Hence the order of accuracy in space is 2 for the continuity equation.

**Analytical solution for fluid problem**

For an internal flow in a tube we can write analytical solution for the axial variation of bulk fluid and metal temperature as,

$$Tb(x) = T_i + \frac{4q^{``}x}{Du_{av}\rho C_v} \tag{81}$$

and, metal temperature given by,

$$Tw(x) = Tb(x) + \frac{11}{48}\frac{q^{''}D}{k} \tag{82}$$

Fluid properties are evaluated at the mean bulk fluid temperature, which is $T_i + T_o$/2, Assuming Outlet temperature of 330 K, we get bulk temperature as, 430 K. At this temperature $\rho$ is 845.8 kg/m$^3$ In Eq.(84), $T_i$ is inlet fluid temperature, $q^{``}$ is applied heat flux, $x$ is axial location from inlet of the tube, $u_{av}$ is average velocity. Here $q^{''}$ is,

$$q^{''} = \frac{Q}{\pi DL} = \frac{200}{\pi \times 0.004 \times 2} = 7957.7W/m^2 \tag{83}$$

and when inlet temperature is 300 K, outlet fluid temperature is,

$$Tb(x) = 300 + \frac{4 \times 7957.7x}{0.004 \times 0.1 \times 892.356 \times 3720} = 300 + 23.97x \tag{84}$$

$$Tb(2) = 300 + 23.97 \times 2 = 347.94\ K \tag{85}$$

and, metal temperature given by,

$$Tw(x) = Tb(x) + \frac{11}{48}\frac{q^{''}D}{k} = 300 + 23.97x + \frac{11}{48}\frac{7957.7 \times 0.004}{0.62} = 300 + 23.97x + 11.76 \tag{86}$$

$$Tw(2) = 300 + 23.97 \times 2 + 11.76 = 359.7\ K \tag{87}$$

**Analytical solution for conduction problem**

In the steady state, there is heat transfer in radial direction only. The governing equation for the one dimensional heat conduction in r direction with heat source is,

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + q^{'''} = 0 \tag{88}$$

Integrating w. r. t. $r$ we get,

$$\frac{\partial T}{\partial r} = -\frac{q^{'''}r}{2k} + \frac{c_1}{r} \tag{89}$$

Integrating the above expression once again w.r.t. $r$ we obtain

$$T = -\frac{q^{'''}}{2k}(\frac{r^2}{2}) + c_1 logr + c_2 \tag{90}$$

Applying the boundary condion at $r = r_o$ we have zero heat flux and at $r = r_i$ we have by heat balance,

$$q^{'''}\pi(r_o^2 - r_i^2)L = h(2\pi r_i L)(T_{surf} - T_f) \tag{91}$$

Applying the boundary conditions and heat balance condition we can solve for $c_1$ and $c_2$ and obtain the final form of expression for temperature distribution in radial direction,

$$T = \frac{q^{'''}}{4k}(r_i^2 - r^2) + \frac{q^{'''}}{2k}r_o^2 ln(\frac{r}{r_i}) + T_f + \frac{q^{'''}}{2h}\frac{(r_o^2 - r_i^2)}{r_i} \tag{92}$$

**Scale Analysis**

Time scale required by the flow for the acheiving the steady state is calculated as,

$$\tau_f = \frac{l}{v} = \frac{2m}{0.1m/s} = 20s \tag{93}$$

Time scale required for the conduction problem to achieve the complete penetration of heat in the radial, direction is calculated by using scale analysis of the governing equation. The governing equation for the heat conduction is,

$$\rho C_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + q''' \tag{94}$$

First term of this equation scales as,

$$\rho C_p \frac{\Delta T}{\Delta t} \tag{95}$$

The second term of the Eq. (94) after expanding and taking the largest order term becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = k\frac{\Delta T}{\Delta r^2} + \frac{k}{r}\frac{\Delta T}{\Delta r} \sim k\frac{\Delta T}{\Delta r^2} \tag{96}$$

Third term which is negligible compared to previous term, as $1/\Delta r \gg\ 1/\Delta x^2$ is,

$$k\frac{\Delta T}{\Delta r^2} = \frac{\Delta T}{\Delta x^2} \tag{97}$$

Which reduces to,

$$\rho C_p \frac{\Delta T}{\Delta t} = k\frac{\Delta T}{\Delta r^2} \tag{98}$$

$$\tau_m = \frac{r_o^2 - r_i^2}{\alpha} = 0.25s \tag{99}$$

Hence for achieving steady state solution the fluid time scale is the important, and time for steady state has be to be greater than this time. Figure 2 shows the axial variation of the centerline temperature at different time intervals. With increasing time solution finally reaches a steady state.

### Steady state results

Transient variation of temperature is shown in Fig. 2 Axial variation of the tube and fluid temperature at the steady state, is shown in the Fig. 3 below. Both the outlet fluid and wall temperature shows a good agreement with the analytical solution. There is a little energy imbalance 0.48% in the fluid solution, which remains always due to the coupled fluid solution with metal solution at every iteration. Variation of the tube wall temperatures at three different location is plotted in the Fig. 4. Which confirms the scale analysis results that very little time is required for radial heat transfer penetration.

Pressure drop along the length of the tube is 316.83 Pa.

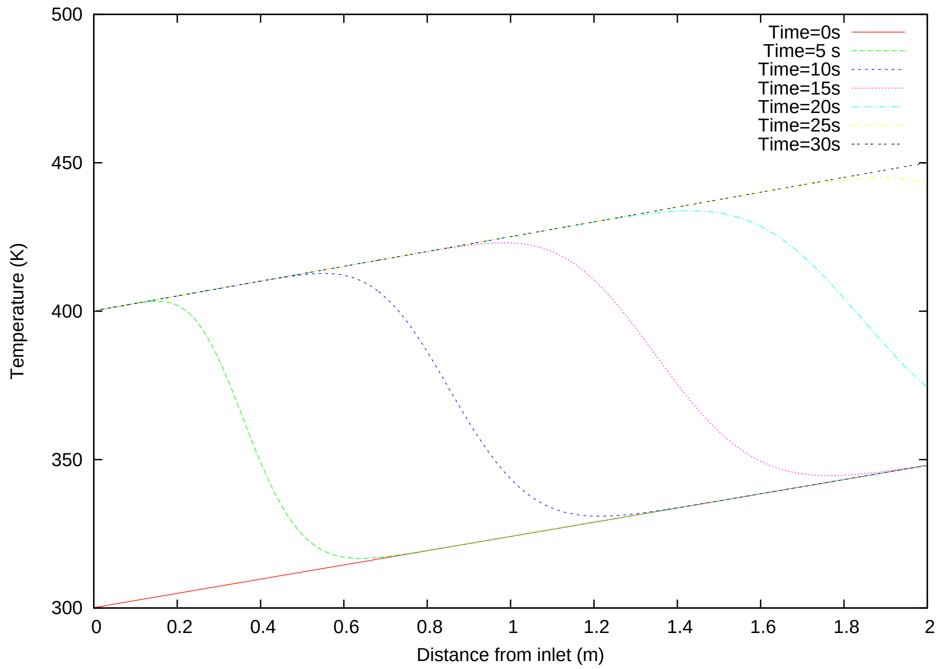


Figure 2: Axial variation of the centerline temperature at different time intervals

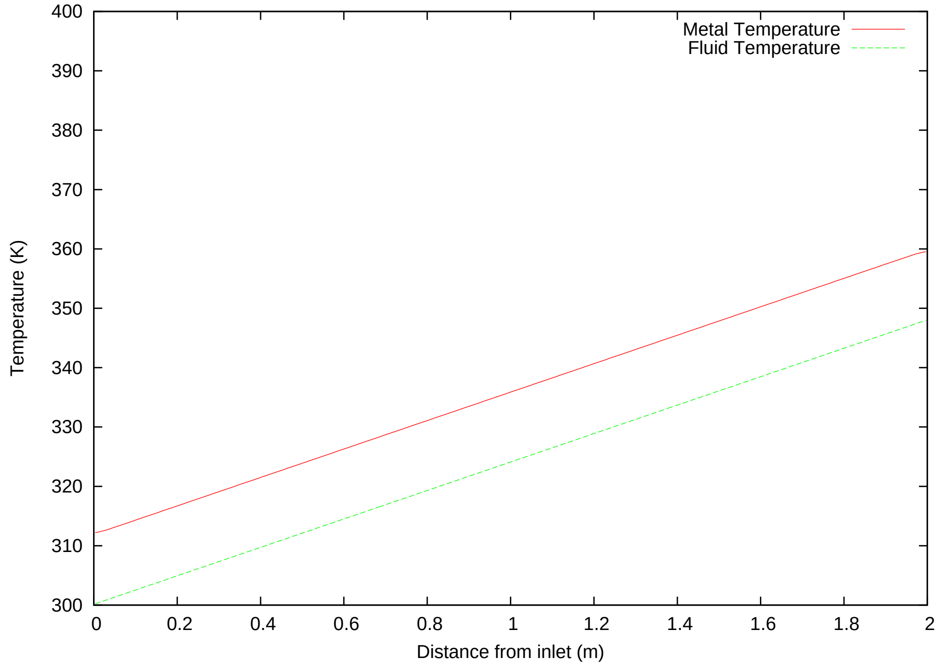


Figure 3: Axial variation of tube wall and fluid temperature at the steady state, run for 300 cells and 100 s

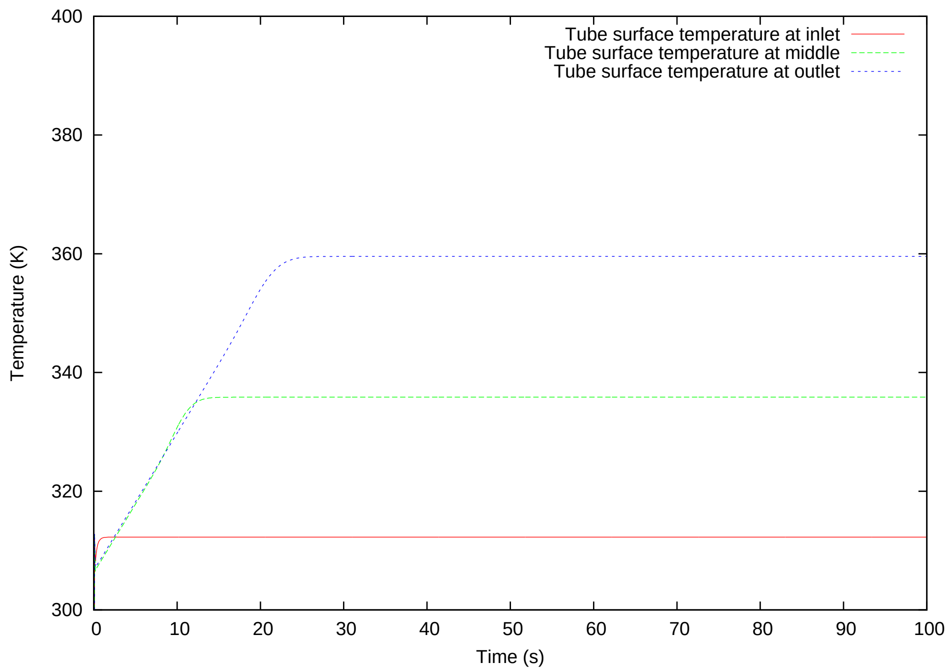


Figure 4: Variation of tube wall temperatures at outer wall with time

### Comparison of the solution with analytical solution

Figure 5 below shows the comparison of the metal and fluid temperature along the axis of the tube at the steady state. There is good agreement between the analytical and numerical solution obtained from the program. As shown in Fig. 6, there is a very small difference of 0.25 K, between the temperature in the radial direction obtained by numerical solution and the analytical solution for temperature.

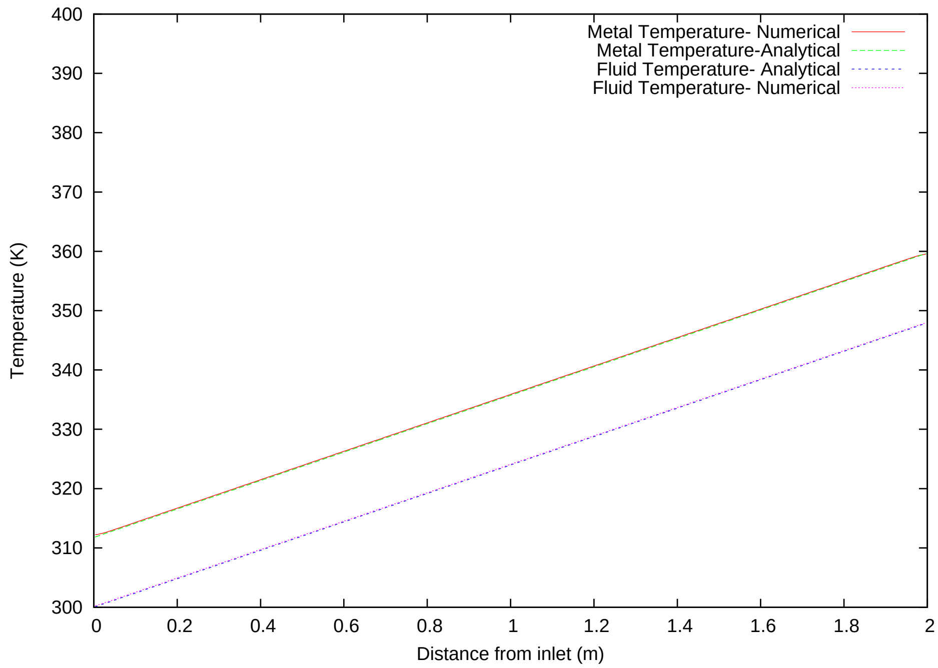


Figure 5: Comparison of the numerical solution with the analytical solution at the end of 100s and 300 cells and 30 cells in radial direction

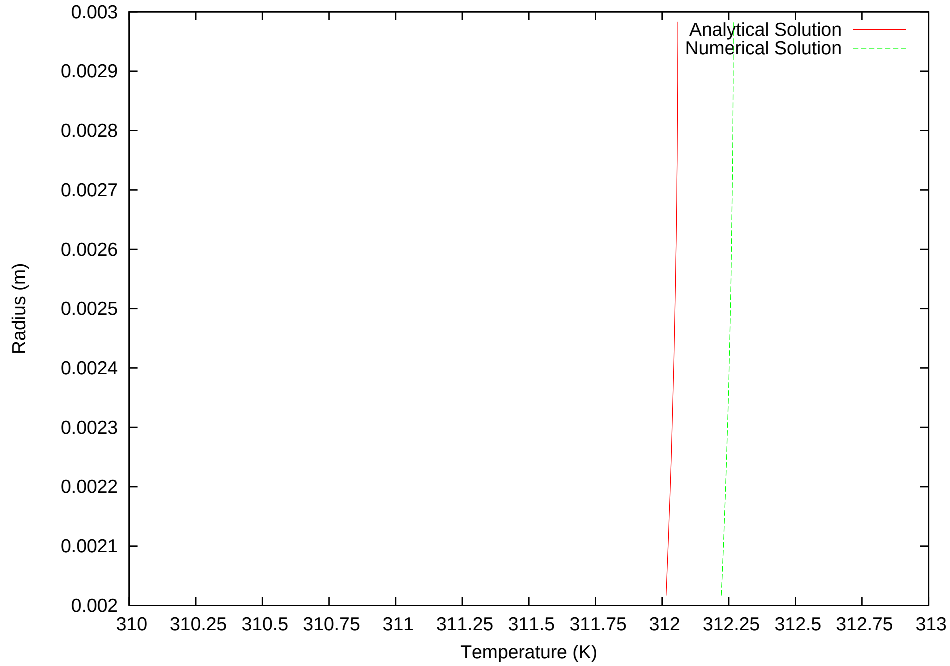


Figure 6: Comparison of the numerical solution with the analytical solution in radial the end of 30s and 300 cells in x direction and 30 cells in radial direction

**Test case**

Test case solution is generated to validate the solution. In the test case, fluid and metal are allowed to achieve steady solution first, after which the heat source in the tube metal is switched off. Temperature of the fluid at the outlet is monitored with time. With zero heat source in the wall the this temperature will be same as inlet fluid temperature.

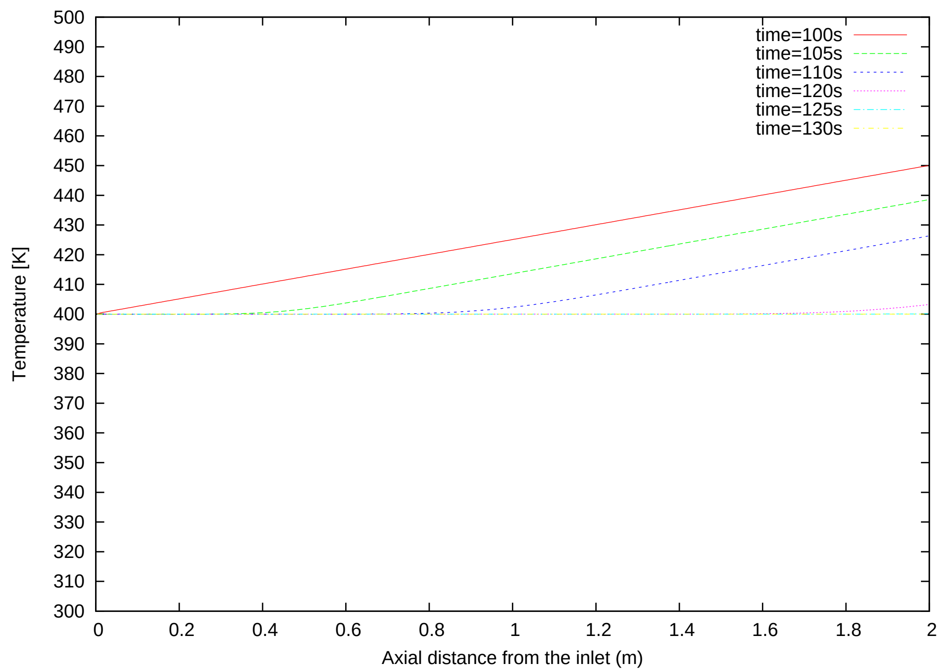


Figure 7: Variation of fluid temperature with time when heat source is set to zero after 100s with 300 cells

**Richardson Analysis**

Richardson analysis is performed to estimate the order of accuracy in space and time for the fluid solution. Please refer to the excell sheets for the details. Spatial analysis is carried out at three different mesh sizes of 20, 60 and 180 cells. Analysis is performed at the same cell center of each of the mesh. For cells with low error, the mean order of accuracy is close 1.98 near outlet cells which is close to 2, as predicted by Taylor's error analysis.

Using the same method mean order of error in time is estimated for the three different time steps. Time steps used are 0.01, 0.03. and 0.09. Number of cells used in the analysis are 300. Average order of accuracy is 0.98, which is close to 1, as predicted by Taylor series analysis. GCI for the analysis was close to 0.2 at the outlet cells.