Discretization using finite volume method

Governing equation for the two D heat conduction with volumetric heat source is

$$k\nabla^2 T + q^{'''} = 0 \tag{1}$$

In a two dimensional case this equation transforms to,

$$\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + q^{'''} = 0 \tag{2}$$

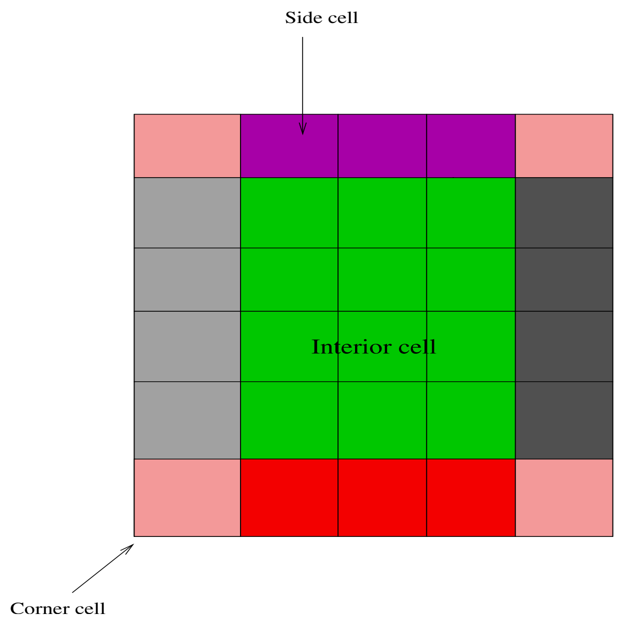

Figure 1: Different types of cells in a rectangular block

In a rectangular block with convective surface boundary condition there are nine different cells depending upon neighboring cells.

1. Four corner cells with two convective boundaries
2. Four side cells with one convective boundaries and other three neighboring cells
3. Interior cells surrounded by cell from all sides and no boundary cell

Discretizing the equations for the interior cells first. Integrating the equations over the control volume O with neighboring cells North, South, East and West. Cell walls of the respective cells are denoted by the lower case letters.

$$\int \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})dxdy + \int \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})dxdy + \int q^{'''}dv = 0 \tag{3}$$

Using the Gausses divergence theorem we get,

$$k\frac{\partial T}{\partial x}|_e \Delta y - k\frac{\partial T}{\partial x}|_w \Delta y + k\frac{\partial T}{\partial y}|_n \Delta x - k\frac{\partial T}{\partial y}|_s \Delta x + q^{'''}\Delta x \Delta y = 0 \tag{4}$$

Expanding the gradients at the cell walls in terms of the values at the cell centers,

$$k\Delta y\frac{(T_E - T_O)}{\Delta x} - k\Delta y\frac{(T_O - T_W)}{\Delta x} + k\Delta x\frac{(T_N - T_O)}{\Delta y} - k\Delta x\frac{(T_O - T_S)}{\Delta y} + q^{'''}\Delta x\Delta y = 0 \tag{5}$$

Dividing the equation by the $\Delta x\Delta y$ we get,

$$k\frac{(T_E - T_O)}{\Delta x^2} - k\frac{(T_O - T_W)}{\Delta x^2} + k\frac{(T_N - T_O)}{\Delta y^2} - k\frac{(T_O - T_S)}{\Delta y^2} + q^{'''} = 0 \tag{6}$$

In terms of the index k and conductivity, cnd, the above expression can be written as

$$cnd\frac{(T_{k+1} - T_k)}{\Delta x^2} - cnd\frac{(T_k - T_{k-1})}{\Delta x^2} + cnd\frac{(T_{k+nx} - T_k)}{\Delta y^2} - cnd\frac{(T_k - T_{k-nx})}{\Delta y^2} + q_k^{'''} = 0 \tag{7}$$

Rearranging the terms we get,

$$\frac{cnd}{\Delta x^2}T_{k+1} + \frac{cnd}{\Delta x^2}T_{k-1} - \left(\frac{2cnd}{\Delta x^2} + \frac{2cnd}{\Delta y^2}\right)T_k + \frac{cnd}{\Delta y^2}T_{k+nx} + \frac{cnd}{\Delta y^2}T_{k-nx} + q_k^{'''} = 0 \tag{8}$$

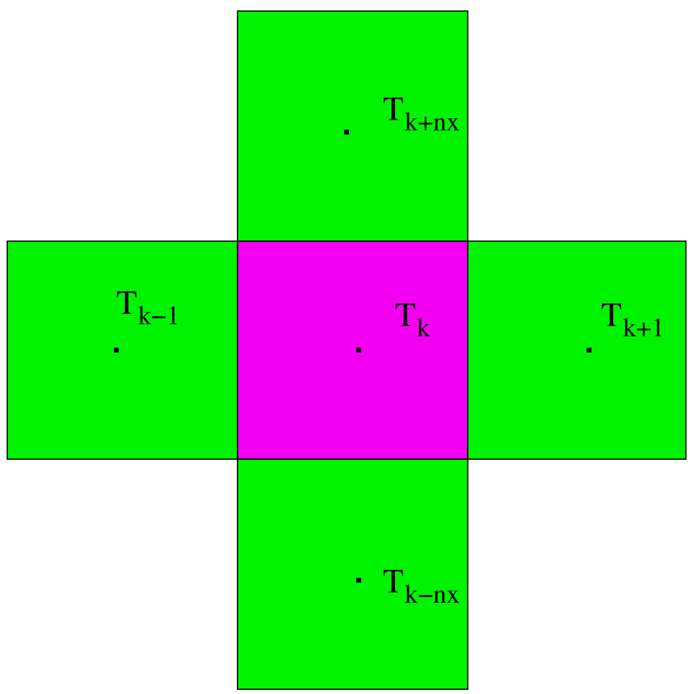

Figure 2: Interior cells in a rectangular block

The boundary cell
RHS cell

For the cell at the convective boundary at the RHS wall, we can write equation matching the heat flux,

$$-cnd\frac{\partial T}{\partial x}|_{surf} = h_x(T_{surf} - T_f) \tag{9}$$

Using the Taylor Series we can write,

$$T_O = T_{surf} - \frac{\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + \frac{(\frac{\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial r^2}|_{surf} + ... \tag{10}$$

$$T_W = T_{surf} - \frac{3\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + \frac{(\frac{3\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{11}$$

Multiply eq. (10) by A and eq. (11) by B, and solving for A and B by setting the coefficient of $\frac{\partial T}{\partial x}$ equal to 1 and $\frac{\partial T^2}{\partial x^2}$ equal to zero

$$AT_O = AT_{surf} - A\frac{\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + A\frac{(\frac{\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{12}$$

$$BT_{ss} = BT_{surf} - B\frac{3\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + B\frac{(\frac{3\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{13}$$

Thus we need to solve for,

$$-A\frac{\Delta x}{2} - B\frac{3\Delta x}{2} = 1 \tag{14}$$

$$A\frac{(\frac{\Delta x}{2})^2}{2!} + B\frac{(\frac{3\Delta x}{2})^2}{2!} = 0 \tag{15}$$

Solving for A and B, we obtain, $A = -\frac{3}{\Delta x}$ and $B = \frac{1}{3\Delta x}$
Substituting these we get

$$-\frac{3}{\Delta x}T_O = -\frac{3}{\Delta x}T_{surf} + \frac{3}{\Delta x}\frac{\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} - \frac{3}{\Delta x}\frac{(\frac{\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{16}$$

$$\frac{1}{3\Delta x}T_W = \frac{1}{3\Delta x}T_{surf} - \frac{1}{3\Delta x}\frac{3\Delta x}{2}\frac{\partial T}{\partial x}|_{surf} + \frac{1}{3\Delta x}\frac{(\frac{3\Delta x}{2})^2}{2!}\frac{\partial^2 T}{\partial x^2}|_{surf} + ... \tag{17}$$

Solving for $\frac{\partial T}{\partial x}|_{surf}$ gives,

$$\frac{\partial T}{\partial x}|_{surf} = -\frac{3}{\Delta x}T_O + \frac{1}{3\Delta x}T_W + \frac{8}{3\Delta x}T_{surf} \tag{18}$$

simplifying we get,

$$\frac{\partial T}{\partial x}|_{surf} = \frac{-9T_O + T_W + 8T_{surf}}{3\Delta x} \tag{19}$$

Putting this in the governing equation of flux matching, Eq.(9) we get,

$$-cnd\frac{(-9T_O + T_W + 8T_{surf})}{3\Delta x} = h_x(T_{surf} - T_f) \tag{20}$$

Rearranging the terms

$$T_{surf} = \frac{(9cndT_k - cndT_W + 3\Delta xhT_f)}{8cnd + 3\Delta xh_x} \tag{21}$$

and eliminating the T_{surf} from the heat flux equation to obtain gradient at the surface we get,

$$-cnd\frac{\partial T}{\partial x}|_{surf} = h_x(T_{surf} - T_f) = h_x(\frac{(9cndT_k - cndT_W + 3\Delta xhT_f)}{8cnd + 3\Delta xh_x} - T_f) \tag{22}$$

$$= h_x cnd\frac{(9T_k - T_W - 8T_f)}{8cnd + 3\Delta xh_x} \tag{23}$$

Similarly the other cell which is at the top wall we can write discretized equation as,

$$T_{surf} = \frac{(9cndT_k - cndT_S + 3\Delta yhT_f)}{8cnd + 3\Delta yh_y} \tag{24}$$

and the heat flux in the y direction

$$cnd\frac{\partial T}{\partial y}|_{surf} = h_y(T_{surf} - T_f) = h_y(\frac{(9cndT_k - cndT_S + 3\Delta yhT_f)}{8cnd + 3\Delta yh_y} - T_f) \tag{25}$$

$$= h_y cnd\frac{(9T_k - T_S - 8T_f)}{8cnd + 3\Delta yh_y} \tag{26}$$

The corner cell

Consider the top left corner cell for discretization.

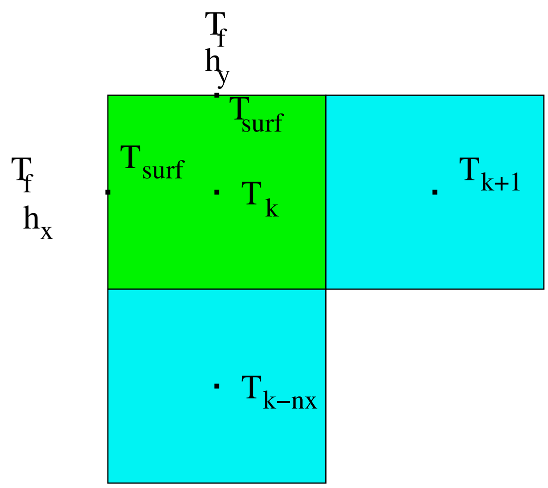

Figure 3: Cell at the top left corner of the the rectangular block

Discretized governing equation for the cell is,

$$k\Delta y\frac{(T_E - T_O)}{\Delta x} - h_y k\frac{(9T_O - T_S - 8T_f)}{8k + 3\Delta y h_y}\Delta x - h_x k\frac{(9T_O - T_E - 8T_f)}{8k + 3\Delta x h_x} - k\Delta x\frac{(T_O - T_S)}{\Delta y} + q^{'''}\Delta x\Delta y = 0 \tag{27}$$

Dividing the equation by the $\Delta x\Delta y$ and rearranging the terms in terms of index k we get,

$$(\frac{cnd}{\Delta x^2} + \frac{h_x k}{8k + 3\Delta x h_x})T_{k+1} - (\frac{cnd}{\Delta x^2} + \frac{cnd}{\Delta y^2} + \frac{9h_x k}{\Delta x(8k + 3\Delta x h_x)} + \frac{9h_y k}{\Delta y(8k + 3\Delta y h_y)})T_k \tag{28}$$

$$+(\frac{cnd}{\Delta y^2} + \frac{h_y k}{\Delta y(8k + 3\Delta y h_y)})T_{k-nx} + (\frac{8h_x k}{\Delta x(8k + 3\Delta x h_x)} + \frac{8h_y k}{\Delta y(8k + 3\Delta y h_y)})T_f + q_k^{'''} = 0 \tag{29}$$

Similarly the discretized equation for the top right corner is given by,

$$(\frac{cnd}{\Delta x^2} + \frac{h_x k}{8k + 3\Delta x h_x})T_{k-1} - (\frac{cnd}{\Delta x^2} + \frac{cnd}{\Delta y^2} + \frac{9h_x k}{\Delta x(8k + 3\Delta x h_x)} + \frac{9h_y k}{\Delta y(8k + 3\Delta y h_y)})T_k \tag{30}$$

$$+(\frac{cnd}{\Delta y^2} + \frac{h_y k}{\Delta y(8k + 3\Delta y h_y)})T_{k-nx} + (\frac{8h_x k}{\Delta x(8k + 3\Delta x h_x)} + \frac{8h_y k}{\Delta y(8k + 3\Delta y h_y)})T_f + q_k^{'''} = 0 \tag{31}$$

For the bottom left and right corner we can write as,

$$(\frac{cnd}{\Delta x^2} + \frac{h_x k}{8k + 3\Delta x h_x})T_{k+1} - (\frac{cnd}{\Delta x^2} + \frac{cnd}{\Delta y^2} + \frac{9h_x k}{\Delta x(8k + 3\Delta x h_x)} + \frac{9h_y k}{\Delta y(8k + 3\Delta y h_y)})T_k \tag{32}$$

$$+(\frac{cnd}{\Delta y^2} + \frac{h_y k}{\Delta y(8k + 3\Delta y h_y)})T_{k+nx} + (\frac{8h_x k}{\Delta x(8k + 3\Delta x h_x)} + \frac{8h_y k}{\Delta y(8k + 3\Delta y h_y)})T_f + q_k^{'''} = 0 \tag{33}$$

$$(\frac{cnd}{\Delta x^2} + \frac{h_x k}{8k + 3\Delta x h_x})T_{k-1} - (\frac{cnd}{\Delta x^2} + \frac{cnd}{\Delta y^2} + \frac{9h_x k}{\Delta x(8k + 3\Delta x h_x)} + \frac{9h_y k}{\Delta y(8k + 3\Delta y h_y)})T_k \quad (34)$$

$$+(\frac{cnd}{\Delta y^2} + \frac{h_y k}{\Delta y(8k + 3\Delta y h_y)})T_{k+nx} + (\frac{8h_x k}{\Delta x(8k + 3\Delta x h_x)} + \frac{8h_y k}{\Delta y(8k + 3\Delta y h_y)})T_f + q_k^{'''} = 0 \quad (35)$$

Cells adjacent the wall

There are four types of cell groups each with RHS, LHS, top, bottom of the wall. The discretized equation for the cells adjacent to top wall is,

$$k\Delta y\frac{(T_E-T_O)}{\Delta x}-k\Delta y\frac{(T_O-T_W)}{\Delta x}-h_yk\frac{(9T_k-T_S-8T_f)}{8k+3\Delta yh_y}\Delta x-k\Delta x\frac{(T_O-T_S)}{\Delta y}+q^{'''}\Delta x\Delta y=0 \tag{36}$$

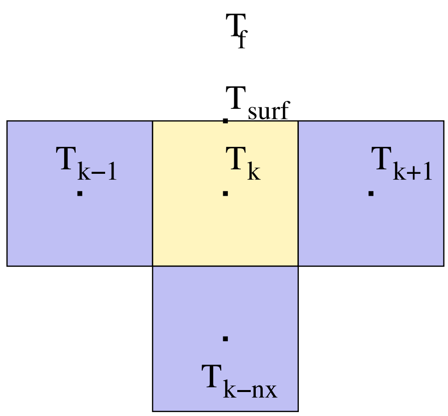

Figure 4: Cell at the top side of the the rectangular block

Dividing the equation by the $\Delta x\Delta y$, rearranging, and rewriting the equations in the k index form we get,

$$\frac{cnd}{\Delta x^2}T_{k+1}-(\frac{2cnd}{\Delta x^2}+\frac{cnd}{\Delta y^2}+\frac{9h_yk}{\Delta y(8k+3\Delta yh_y)})T_k+\frac{cnd}{\Delta x^2}T_{k-1}+ \tag{37}$$

$$(\frac{cnd}{\Delta y^2}+\frac{h_yk}{\Delta y(8k+3\Delta yh_y)})T_{k-nx}+\frac{8h_yk}{\Delta y(8k+3\Delta yh_y)}T_f+q_k^{'''}=0 \tag{38}$$

For the cells adjacent to bottom wall we can write,

$$\frac{cnd}{\Delta x^2}T_{k+1}-(\frac{2cnd}{\Delta x^2}+\frac{cnd}{\Delta y^2}+\frac{9h_yk}{\Delta y(8k+3\Delta yh_y)})T_k+\frac{cnd}{\Delta x^2}T_{k-1}+ \tag{39}$$

$$(\frac{cnd}{\Delta y^2}+\frac{h_yk}{\Delta y(8k+3\Delta yh_y)})T_{k+nx}+\frac{8h_yk}{\Delta y(8k+3\Delta yh_y)}T_f+q_k^{'''}=0 \tag{40}$$

For the cells adjacent to LHS walls discretized equation will be,

$$\frac{cnd}{\Delta y^2}T_{k+nx}-(\frac{2cnd}{\Delta y^2}+\frac{cnd}{\Delta x^2}+\frac{9h_xk}{\Delta x(8k+3\Delta xh_x)})T_k+\frac{cnd}{\Delta y^2}T_{k-nx}+ \tag{41}$$

$$(\frac{cnd}{\Delta x^2}+\frac{h_xk}{\Delta x(8k+3\Delta xh_x)})T_{k+1}+\frac{8h_xk}{\Delta x(8k+3\Delta xh_x)}T_f+q_k^{'''}=0 \tag{42}$$

For the cells adjacent to RHS walls discretized equation will be,

$$\frac{cnd}{\Delta y^2}T_{k+nx} - (\frac{2cnd}{\Delta y^2} + \frac{cnd}{\Delta x^2} + \frac{9h_x k}{\Delta x(8k + 3\Delta x h_x)})T_k + \frac{cnd}{\Delta y^2}T_{k-nx} + \tag{43}$$

$$(\frac{cnd}{\Delta x^2} + \frac{h_x k}{\Delta x(8k + 3\Delta x h_x)})T_{k-1} + \frac{8h_x k}{\Delta x(8k + 3\Delta x h_x)}T_f + q_k^{'''} = 0 \tag{44}$$

The analytical solution to the problem

In one dimensional case governing equation transforms to,

$$\frac{\partial}{\partial x}(cnd\frac{\partial T}{\partial x}) + q^{'''} = 0 \tag{45}$$

$$\frac{\partial}{\partial x}(cnd\frac{\partial T}{\partial x}) = -q^{'''} \tag{46}$$

Integrating both the sides with respect to x twice we get,

$$T = -\frac{q^{'''}}{cnd}\frac{x^2}{2} + c_1 x + c_2 \tag{47}$$

Where c_1 and c_2 are constants.

Boundary conditions are, at $x = a, T = Tsurf_1$ and $x = -a, T = Tsurf_2$

Here a is the distance of the surface from center of the plate.

Putting the boundary conditions we get,

$$T_{surf1} = -\frac{q^{'''}}{cnd}\frac{a^2}{2} + c_1 a + c_2 \tag{48}$$

$$T_{surf2} = -\frac{q^{'''}}{cnd}\frac{a^2}{2} - c_1 a + c_2 \tag{49}$$

Solving for c_1 and c_2 gives,

$$T = -\frac{q^{'''}}{2cnd}(a^2 - x^2) + \frac{(T_{surf1} - T_{surf2})}{2a}x + \frac{(T_{surf1} + T_{surf2})}{2} \tag{50}$$

As the $h_x = h_y, T_{surf1} = T_{surf2}$. With this the above equations becomes,

$$T = \frac{q^{'''}}{2cnd}(a^2 - x^2) + T_{surf} \tag{51}$$

At the convective boundary

$$-cnd\frac{dT}{dx}|_{surf} = h_x(T_{surf} - T_f) \tag{52}$$

Differentiating eq. (51) and putting in the above equation we obtain,

$$-cnd\frac{q^{'''}}{2cnd}(-2a) = h_x(T_{surf} - T_f) \tag{53}$$

Rearranging terms,

$$T_{surf} = T_f + \frac{q^{'''}a}{h_x} \tag{54}$$

The analytical solution thus becomes,

$$T = \frac{q^{'''}}{2cnd}(a^2 - x^2) + T_f + \frac{q^{'''}a}{h_x} \tag{55}$$

Richardson extrapolation

For Richardson extrapolation method derivations in the class are used for the calculation of the coefficients r, a, p. Exact solution is calculated for each of the location, center, mid of the side, and mid of bottom face. The difference between the exact solution and numerical solution gives the predicted error.

Observations

With second order scheme the the solution is closer to the analytical solution, which indicates the higher accuracy with the use of higher order scheme. This is also reflected in Richardson analysis. Use of finer mesh and second order scheme, reduces the error even further. From the predicted error for 9/9 and 27/27 we can conclude that with mesh refinement error reduces. Predicted error is very small at the final refined mesh of 27/27.