Advection Equation

Advection equation is,

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0 \tag{1}$$

with boundary conditions, for $0 \leq t \leq 1$

$$\rho(0,t) = 3t^2 - 2t^3$$

for $t > 1$

$$\rho(0,t) = 1$$

Initial condition is

$$\rho(x > 0, 0) = 0 \tag{2}$$

With velocity of 1 m/s. The length of the pipe is 5 m. Total time for the simulation is 2.5s.

Analytical Solution

Analytical solution obtained with method of characteristics with $v = 1$ is,

$$\rho(x,t) = f(x_0) = f(x - vt) = f(x - t) \tag{3}$$

with initial condition, for $(x - t) > 0$

$$\rho(x,t) = f(x - t) \tag{4}$$

With boundary condition, for $(x - t) \leq 0$,

$$\rho(x,t) = g(x - t) \tag{5}$$

So,for $x > t$

$$\rho(x,t) = 0 \tag{6}$$

for $0 \leq (t - x) \leq 1$

$$\rho(x,t) = 3(t - x)^2 - 2(t - x)^3 \tag{7}$$

for $(t - x) > 1$

$$\rho(x,t) = 1 \tag{8}$$

Hence for $x > 2.5$, $\rho = 0$
for $1.5 \leq 2.5$, $\rho = 3(2.5 - x)^2 - 2(2.5 - x)^3$
for $x \leq 1.5$, $\rho = 1$

Donor Cell Method

Following is the difference scheme for explicit donor cell method. Discretized equation for the explicit method is,

$$\frac{\rho_i^{n+1} - \rho_i^n}{\delta t} + v\frac{\rho_i^n - \rho_{i-1}^n}{\delta x} = 0 \tag{9}$$

which can be written as,

$$\rho_i^{n+1} = \rho_i^n(1-c) + c\rho_{i-1}^n \tag{10}$$

In the above equation c is Courant number, $\frac{v\delta t}{\delta x}$.
Implicit formulation of the above method is as follows,

$$\frac{\rho_i^{n+1} - \rho_i^n}{\delta t} + v\frac{\rho_i^{n+1} - \rho_{i-1}^{n+1}}{\delta x} = 0 \tag{11}$$

which can be written as,

$$\rho_i^{n+1}(1+c) - c\rho_{i-1}^{n+1} = c\rho_i^n \tag{12}$$

Explicit and Implicit Quick Method
Formulation of the Explicit Quick method as described by Leonard, is as follows,

$$\frac{\rho_i^{n+1} - \rho_i^n}{\delta t} + v\frac{\rho_r^n - \rho_l^n}{\delta x} = 0 \tag{13}$$

Where ρ_r and ρ_l are given as,

$$\rho_r^n = \frac{3}{8}\rho_{i+1}^n + \frac{3}{4}\rho_i^n - \frac{1}{8}\rho_{i-1}^n \tag{14}$$

$$\rho_l^n = \frac{3}{8}\rho_i^n + \frac{3}{4}\rho_{i-1}^n - \frac{1}{8}\rho_{i-2}^n \tag{15}$$

Substituting and rearranging the terms we get explicit Quick formulation as,

$$\rho_i^{n+1} = \rho_i^n - c(\frac{3}{8}\rho_{i+1}^n - \frac{7}{8}\rho_{i-1}^n + \frac{1}{8}\rho_{i-2}^n + \frac{3}{8}\rho_i^n) \tag{16}$$

Implicit Quick method is as below,

$$\frac{\rho_i^{n+1} - \rho_i^n}{\delta t} + v\frac{\rho_r^{n+1} - \rho_l^{n+1}}{\delta x} = 0 \tag{17}$$

Where ρ_r and ρ_l are given as,

$$\rho_r^{n+1} = \frac{3}{8}\rho_{i+1}^{n+1} + \frac{3}{4}\rho_i^{n+1} - \frac{1}{8}\rho_{i-1}^{n+1} \tag{18}$$

$$\rho_l^{n+1} = \frac{3}{8}\rho_i^{n+1} + \frac{3}{4}\rho_{i-1}^{n+1} - \frac{1}{8}\rho_{i-2}^{n+1} \tag{19}$$

Substituting and rearranging the terms we get Implicit Quick formulation as,

$$(1+\frac{3c}{8})\rho_i^{n+1} + \frac{3c}{8}\rho_{i+1}^n + \frac{c}{8}\rho_{i-2}^{n+1} - \frac{7c}{8}\rho_{i-1}^n = \rho_i^n \tag{20}$$

Quickest Method
Quickest method is follows,

$$grad_l = \frac{\rho_i^n - \rho_{i-1}^n}{dx} \tag{21}$$

$$grad_r = \frac{\rho_{i+1}^n - \rho_i^n}{dx} \tag{22}$$

$$curv_l = \frac{\rho_{i-2}^n + \rho_i^n - 2\rho_{i-1}^n}{dx^2} \tag{23}$$

$$curv_r = \frac{\rho_{i-1}^n + \rho_{i+1}^n - 2\rho_i^n}{dx^2} \tag{24}$$

$$\rho_i^{n+1} = \rho_i^n - c((0.5(\rho_i^n + \rho_{i+1}^n) - 0.5c \times dx \times grad_r - 0.167dx^2(1-c^2)curv_r) - (0.5(\rho_{i-1}^n + \rho_i^n) - 0.5c \times dx \times grad_l - 0.167dx^2(1-c^2)curv_l))$$

Leith's Method
Leith's method is follows,

$$\rho i^{n+1} = \rho_i^n - 0.5c(\rho_{i+1}^n - \rho_{i-1}^n) + 0.5c^2(\rho_{i-1}^n + \rho_{i+1}^n - 2\rho_i^n) \tag{25}$$

Results and Discussion
Part A

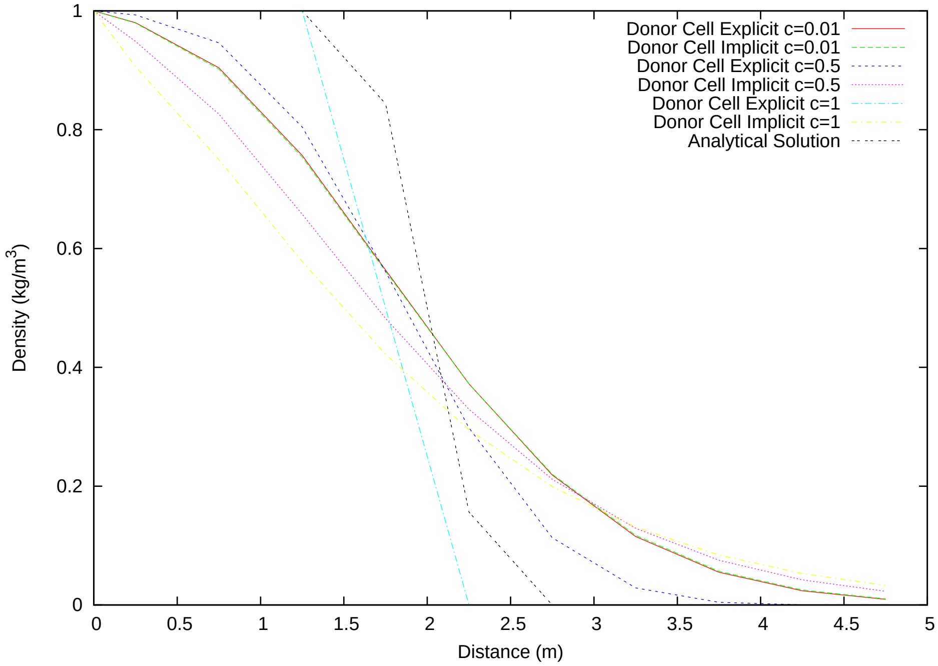

Figure 1: Results of explicit and implicit donor cell method for c = 0.01, 0.5, 1 and 10 cells

Figure 1 shows the comparison of the donor cell implicit and explicit method for three different Courant numbers, 0.01, 0.5 and 1, with analytical solution. At smaller Courant number equal to 0.01, both the methods shows similar results. At Courant number equal to 0.5, Implicit donor cell method shows higher numerical diffusion compared to the explicit donor cell method. The same behavior is observed with Courant number of one.

Part B

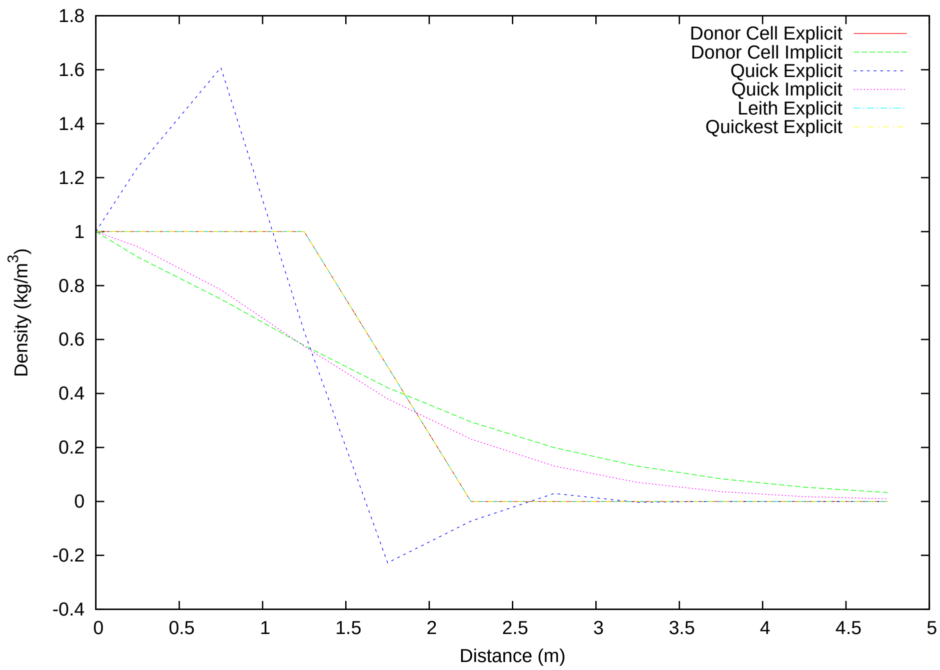

Figure 2: Results of different methods for c = 1 and 10 cells

Figure 2 shows the results of the various schemes with Courant number=1 and ten cells. Results of Leith, Donor Cell Explcit and Quickest Explicit coincide. Again the donor cell and Quick Implict shows the highest numerical diffusion. Quick explicit shows weigles. Figure 3 shows the results of the various schemes with Courant number=0.5 and ten cells. Among all the methods, Quick explicit method shows weigles. This is absent in the Quickest method, with lowest numerical diffusion. Both implicit methods show higher numerical diffusion compared to explicit methods. Results of the Leith method closely follow results of Quikest.

Figure 4 shows the results of the different method with Courant number of 0.5 and 100 cells. Compared to the case with the 10 cells and Courant number of 0.5, this case uses a time step one tenth of the earlier case. Donor cell implicit still shows the higher numerical diffusion. Quick Explicit is unstable at this condition and results are not plotted here. Leith method shows weigles. Quikest method shows least numerical diffusion. Quick Implicit and Donor Cell Explicit follows solution closely. Although in both the method density doesnot approaches to zero at time t = 2.5s due to numerical diffusion.

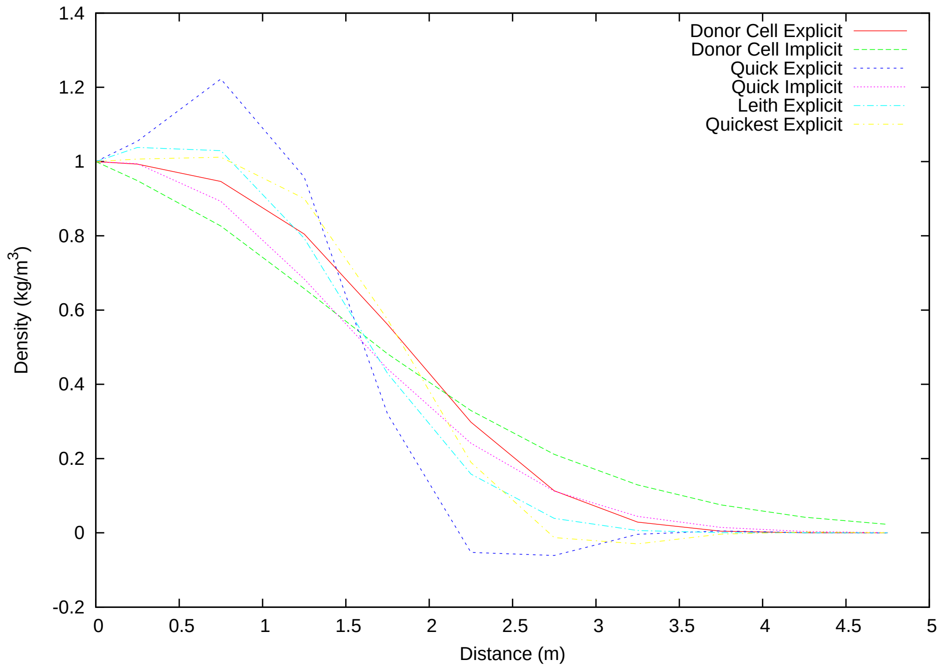

Figure 3: Results of different methods for c = 0.5 and 10 cells

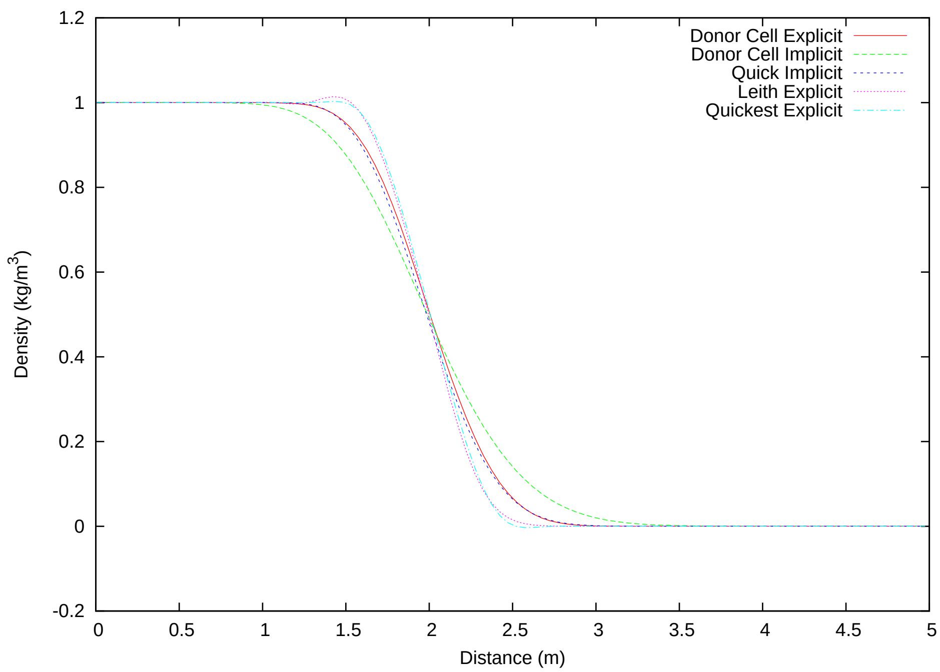

Figure 4: Results of different methods for c = 0.5 and 100 cells

Figure 5 and 6 shows the results of the Donor cell explicit and Implicit methods at two different mesh sizes. Implicit method shows very high diffusion at the low mesh size. Lower diffusion is present in the case of explicit method

Figure 7 shows that Leith's method has little diffusion with finer mesh but weigles are present. Results of Quickest method at two different mesh size are shown in Fig. 8. Quickest method has little numerical numerical diffusion. Quick Implicit method as shown in Fig. 9, shows very high numerical diffusion at small number of cells. Even with higher number of cells numerical diffusion is present.

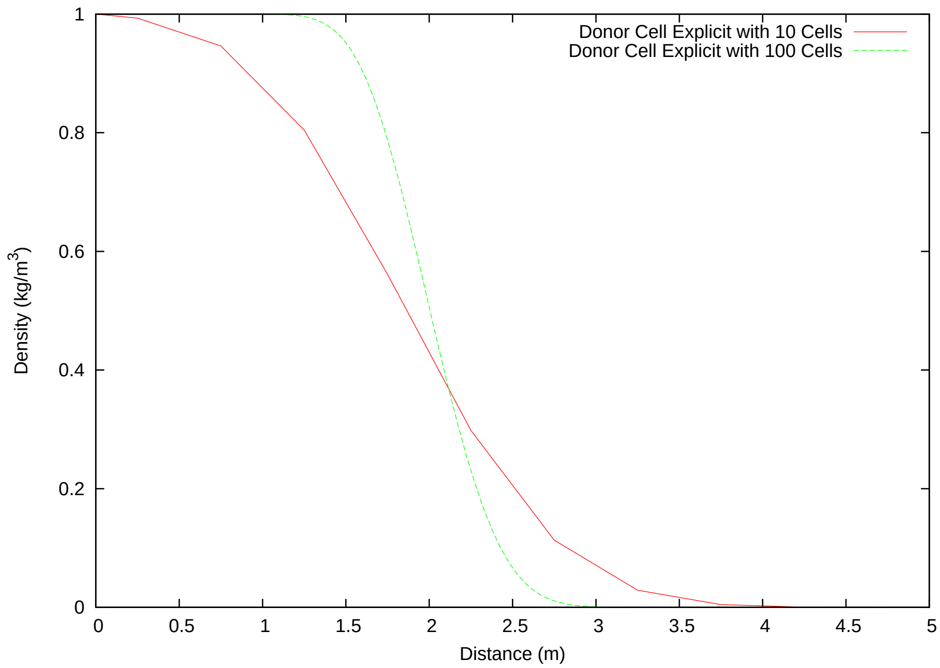

Figure 5: Results of Donor Explicit method for 10 and 100 cells with c = 0.5

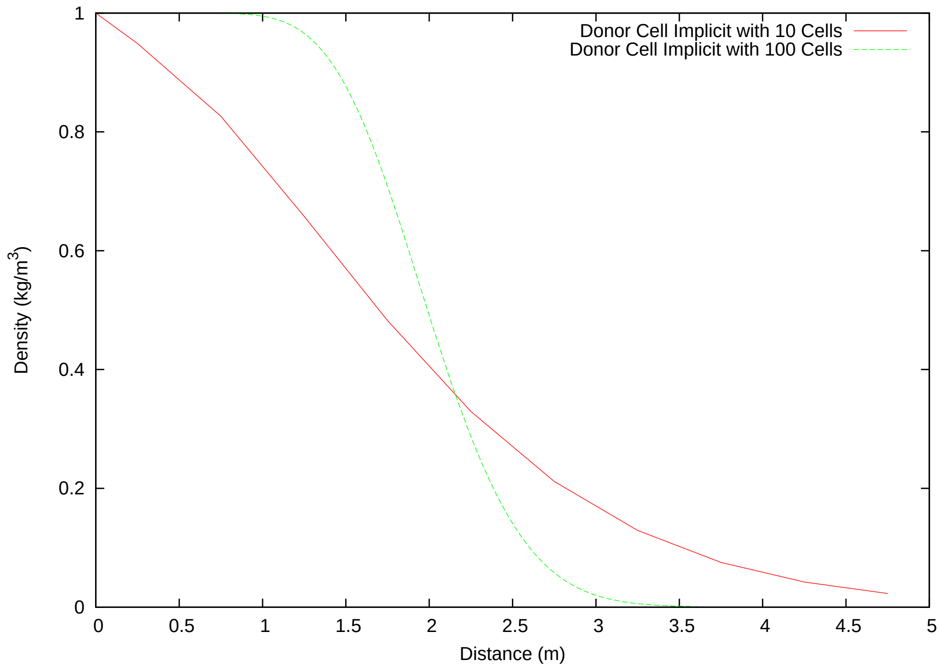

Figure 6: Results of Donor Implicit method for 10 and 100 cells with c = 0.5

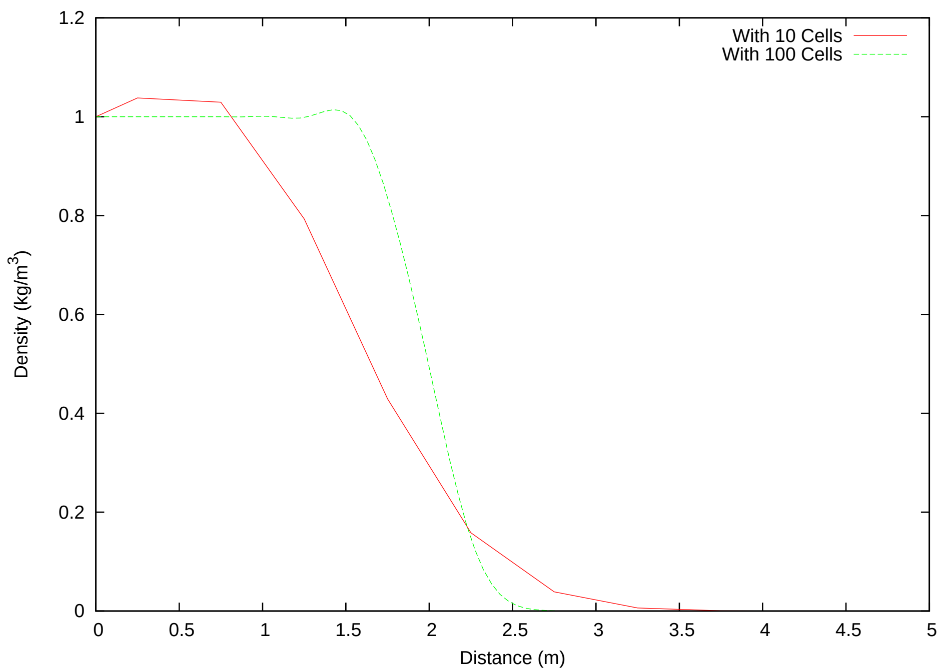

Figure 7: Results of Leith Explicit method for 10 and 100 cells with c = 0.5

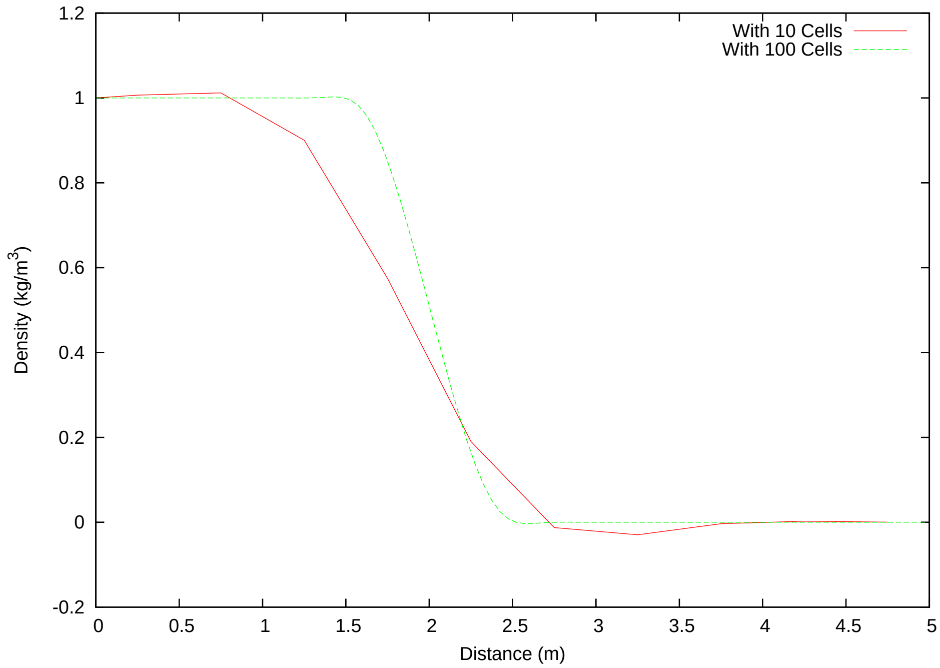

Figure 8: Results of Quickest Explicit method for 10 and 100 cells with c = 0.5

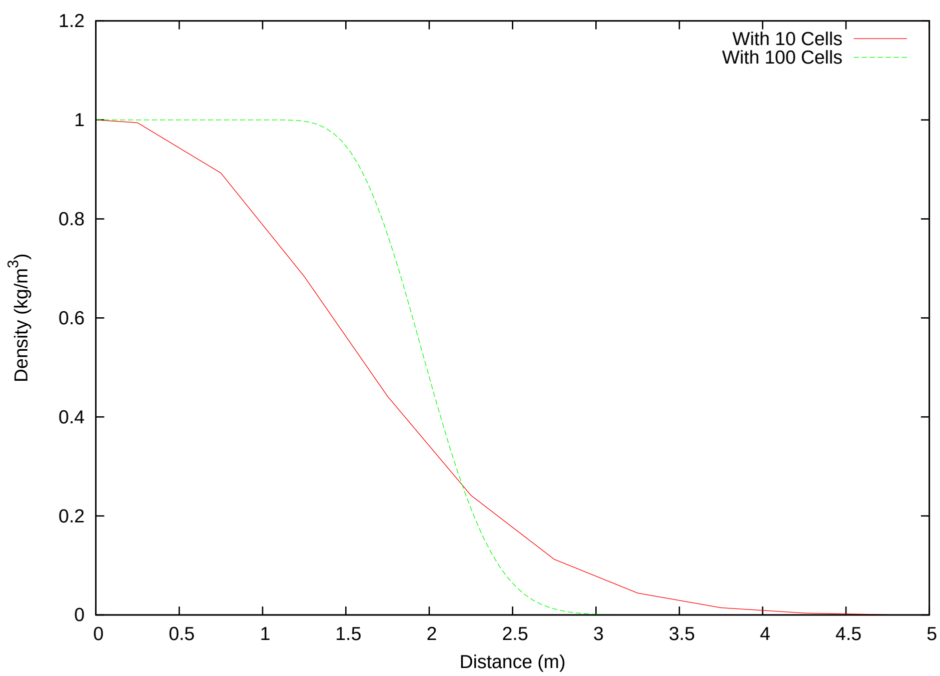

Figure 9: Results of Quick Implicit method for 10 and 100 cells with c = 0.5

Part C

Richardson analysis is carried out on implicit Quick and explicit Quickest to check the effective order of accuracy in time and space of the two methods. For calculation of spatial order of accuracy, three different mesh sizes of 20, 60, 80 are used. For temporal order of accuracy calculation, time steps of 0.001, 0.003, 0.009 are used. Spatial order of accuracy for the Quickest method fluctuates from 1.26 to 5.9, as shown in Fig. 10. Figure 11 shows that temporal order of accuracy for the Quickest method fluctuates from 0.6 to 3.57. For Quick Implicit method, order of accuracy in space fluctuates from 1.17 to 4.69, as shown in Fig. 12. For temporal, it flctuates from 0.5 to 1.65, which is shown in Fig 13.

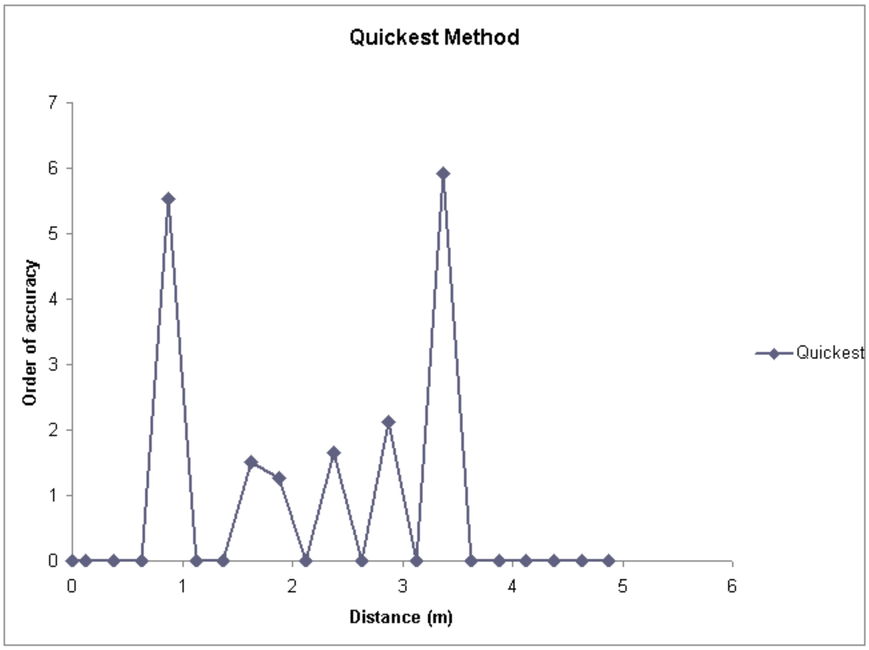

Figure 10: Order of accuracy in space for Quickest method

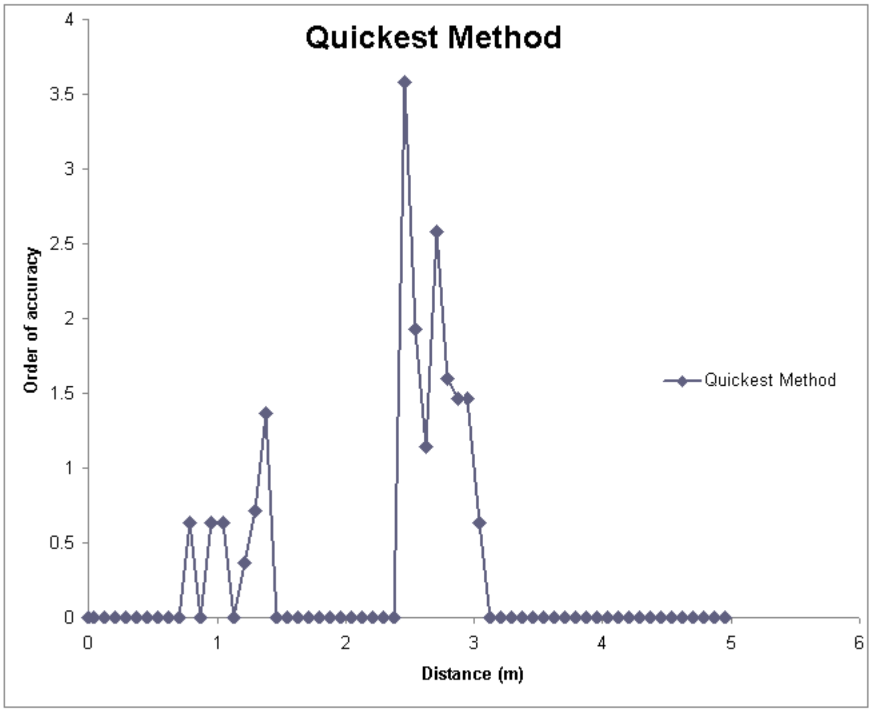

Figure 11: Order of accuracy in time for Quickest method

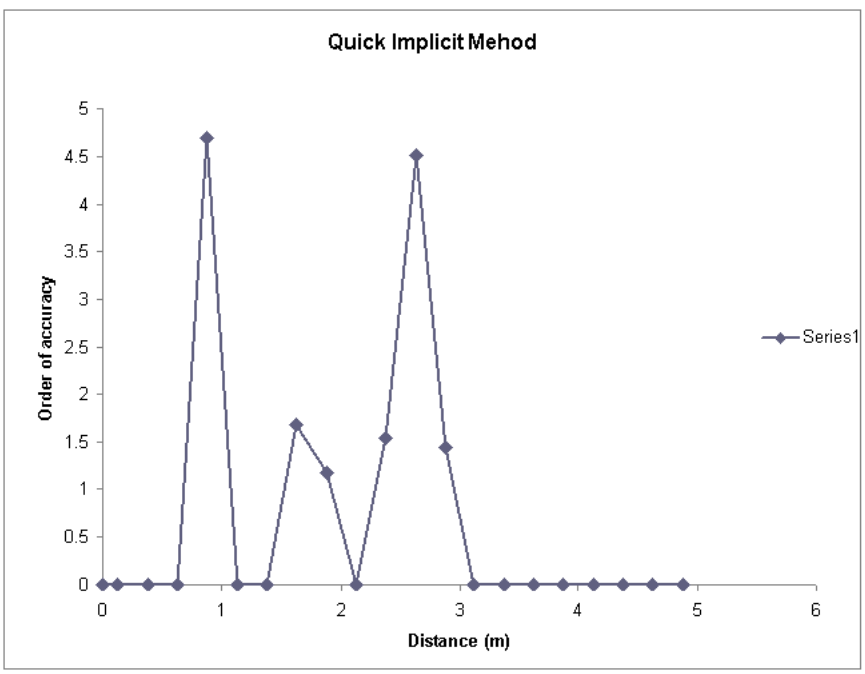

Figure 12: Order of accuracy in space for Quick Implicit method

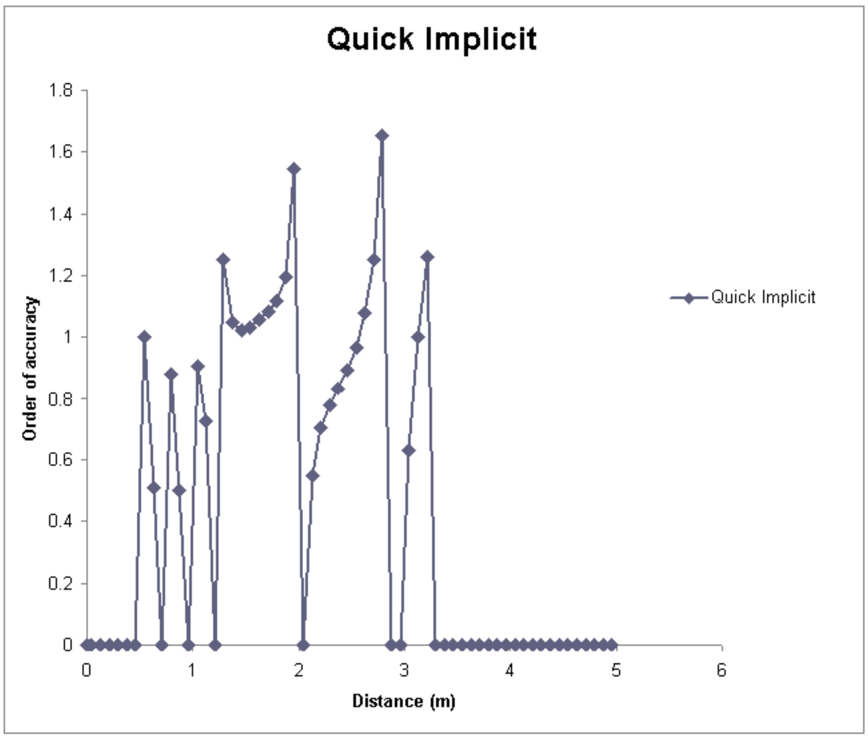

Figure 13: Order of accuracy in time for Quick Implicit method