Mosher-Tangora
Fuchs-Fomenko
Whitehead.
McCleary
for Steenrod ops.

$P^i : H^*(-;\mathbb{Z}/p) \to H^{*+2i(p-1)}$
$\beta : H^* \to H^{*+1}$ Bockstein.
$P^0 = id$, $x \in H^{2n} \Rightarrow P^n(x) = x^p$,
$x \in H^n$, $2k > n \Rightarrow P^k(x) = 0$.
(*) $P^k(x \cup y) = \sum_{j=0}^{k} P^j(x) \cup P^{k-j}(y)$

From geoff@math.ucla.edu Thu Dec 11 19:32:35 1997

Here are a few exercises from my course this fall. Unfortunately no one did any of these, but they did do some problems from Hatcher's book - online at math.cornell.edu/~hatcher ! More fun than video games !

1. Compute the action of the Steenrod powers P^i on the cohomology of CP^n. Consider the fiber bundle CP^{2k+1} -> HP^k with fiber a 2-sphere. What is the induced map on cohomology ? Find the action of the Steenrod powers P^i for p = 3 on the Z/3 cohomology of HP^k. Show that there is no fiber bundle M^{20} -> CaP^2 with fiber a 4 sphere such that the cohomology ring of M^{20} is that of HP^5.
ANY 20-MFD.

2.a) Use the Steenrod powers, for p = 3, to show that there is no self map f of HP^k, k > 1, such that H*f = -1 on H^4; in fact H*f acts on H^4 by multiplication by n, n = 0 or 1 mod 3.
[In fact, in addition n = 0 or 1 mod 8, if k = 2, for the _standard_ HP^2. For this some information in unstable homotopy theory is needed: -1 circ nu = - nu + [i_4, i_4]; k circ nu = k nu + C(k, 2)[i_4, i_4]; and [i_4, i_4] = 2 nu +x where x generates Sigma pi_6S^3. A "nonstandard" HP^2 would mean a space obtained by attaching an 8-cell to a 4-sphere by a map of

Hopf invariant 1 in a different homotopy class than the Hopf map. There are 12 HP^2 s altogether, and they can all be realized as manifolds, not necessarily smoothable.]
(In contrast, there is an involution on CP^n which induces -1 on H^2. If n=2k+1 is odd, the involution can be chosen to be free and preserve each of the fibers CP^1 of the fiber bundle CP^n -> HP^k. If n is even, the involution can be chosen as complex conjugation.)

b) Show that if f: HP^3 -> HP^3 satisfies H*f = m on H^4, then m = 0, 1, or 4 mod 5, and m = 0 or 1 mod 3.
****
Recall the space S^1 x CP^{inf}/(S^1 x {x_0}).
Use Sq^2 to show that it is not homotopy equivalent to S^3 x CP^{inf}.
Remark: The two spaces have isomorphic homotopy groups. For both spaces, the third Postnikov approximation is K(Z, 2) x K(Z, 3). The third Postnikov invariant distinguishes them.
NO MAP inducing $\cong$.

"Problem 4" from the previous email asst is particularly recommended; in fact it is a group of problems. Here are some more problems, mainly on the homotopy exact sequence of a fiber bundle:

1. There are fiber bundles SO(n) -> SO(n+1) -> S^n, since SO(n+1) acts transitively on S^n with point stabilizer SO(n). Suppose the fiber bundle has a section. Then show S^n is parallelizable (almost a tautology). Also show that if S^n is parallelizable, then the fiber bundle has a section (Gram-Schmidt). Now show that if the fiber bundle has a section, there is a bidegree (1, 1) map from S^n x S^n to S^n.
ADJOINT of $S^n \to SO(n+1)$

2. Given a map f: X x Y -> Z, the Hopf construction Hf is a map from CX x Y union X x CY -> SZ, defined on CX x Y by coning the map and on X x CY by coning the map, sending X x CY to the lower cone. Show that if f : S^n x S^n -> S^n has bidegree (a, b) then Hf has Hopf invariant ab. CLEAR $n=1$ (HOPF)

$Hf : CS^n \times S^n \cup S^n \times CS^n \to \Sigma S^n = S^{n+1}$
$Hf : D^{n+1} \times S^n \cup S^n \times D^{n+1} \to S^{n+1}$
The JOIN $= S^{2n+1}$

3. Show that if S^n is parallelizable then S^n is an H-space and therefore n+1 is a power of 2. (Use problem 2.) STND.

$\Rightarrow \pi_4 Spin 5 = \pi_4 S^3$.
Acts on $\mathbb{R}^8$, S^7, stab $S^3 = Sp(1)$
4. Assume known that Spin 5 (= Sp(2)) fibers over S^7 with fiber S^3. Using prob. 3, show that pi_4 of Spin 5 is Z/2. Deduce that pi_4 Sp(k) = Z/2 for all k > 1. Also deduce that pi_4 SO(6) = 0 using the homotopy exact sequence of a fiber bundle. Using Spin 6 = SU4, deduce that pi_4 SUn = 0 for all n > 3, so pi_4 SU infinity = 0

$Y = D^{2n} \cup_{Hf} S^{n-1}$
$H^* Y = \begin{matrix} 2n \\ n \\ 0 \end{matrix}$ $x^2 = \underset{\text{Hopf inv.}}{\sim} y$
$x^2 = Sq^n x$.
Use Adem relations

SU(n), SU(n+1) cmp.
✓ ok.

SO(5), SO(6) bundle –
use SO(5) virtually Spin5
AND S^5 not parallel $\Rightarrow \pi_5 S^5 \twoheadrightarrow \pi_4 SO(5)$

(a case of Bott's calculation of the homotopy groups of SU infinity). Also deduce that pi_4 SO infinity = 0.

5. Now consider the fiber bundle SU2 -> SU3 -> S^5. 8D [Note that SU3 is a subgroup of SO(6).] Show that pi_4 SU3 = 0. Deduce that SU3 is not the product bundle SU2 x S^5. ↖ FROM $SU3 \to SU4 \to S^7$ bundle!! "Deduce" comes from $SU2 \to SU3$ bundle.

TANG. BUNDLE is cx. vector bundle.

6. Show that an almost complex structure on S^6 exists provided there is a section of the bundle SO(7)/U3 -> SO(7)/SO(6) = S^6. Use obstruction theory to show that there is a section. Lies in $H^6(S^6; \pi_5(SO(7)/U_3))$. vanishes

A.C.S. means a lift

$$\begin{tikzcd} SO(7)/U_3 \arrow[r] \arrow[d] & BU_3 \arrow[d] \\ S^6 \arrow[r] \arrow[ur, dotted] & BSO_6 \end{tikzcd}$$

$s: S^6 \to SO(7)/U3$. But $SO(7)/U_3$ is base of a bundle w/ U_3 fiber $\Rightarrow$ map to BU_3. Check square commutes.

7. (This is really several problems.)
Consider the monoid Maps(S^2, S^2) of all continuous maps from S^2 to itself, not necessarily taking the basepoint to the basepoint.
usual ARG a) The components of Maps(S^2, S^2) correspond to integers, using degree.
b) Let Maps_n be the degree n component. Show that evaluation at the basepoint * is a fibration Maps_n S^2 -> S^2 and the fiber over * is PtdMaps_n(S^2, S^2), the basepoint preserving maps.
c) Show that wedging-on a degree 1 or -1 map determines maps
R: PtdMaps_n(S^2, S^2) -> PtdMaps_{n+1}(S^2, S^2) and
L: PtdMaps_{n+1}(S^2, S^2) -> PtdMaps_n(S^2, S^2).
which are homotopy inverses of each other.
d) Find pi_1 PtdMaps_0(S^2, S^2). $= \pi_3 S^2 = \mathbb{Z}$!
e) ~~Use the Hopf invariant to~~ show that pi_1 Maps_0(S^2, S^2) = Z. EASY. Section!!
f) Find the fundamental group of the homotopy fiber of the inclusion SO(3) -> Maps_1(S^2, S^2). Deduce that the inclusion is not a homotopy equivalence. $\pi_2 SO_3 = 0$ $\pi_1 SO3 = \mathbb{Z}_2$ $\pi_2 Maps_1 =$? $\pi_1 Maps_1 = \mathbb{Z}_2$
*g) Show that pi_1 Maps_n(S^2, S^2) is Z/(2n). (Use the fibration over S^2.) $\nwarrow \chi(S^2)$

Homotopy Fiber - pretend it's a bundle and write down LESH.

8. a) Show that there is a bundle X over S^4 whose fiber over a point is the space of almost complex structures. Then show that up to homotopy equivalence X = X_1:= SO(5)/U(2), and the projection to S^4 comes from the map SO(5)/U(2) -> SO(5)/SO(4) with fiber SO(4)/U(2) = S^2.
b) Show that SO(5)/U(2) = CP^3. A hint: use the double cover Spin(5) of SO(5); it is a subgroup of Spin(6)= SU(4).
c) Skip b) if you aren't familiar enough with the linear algebra involved. Conclude that S^4 has no almost complex structure.

p. 104, or IVB.12. "For odd p we choose omega_2 to be the Z/p reduction of a generator of H^2(L^inf, Z), so omega_2 is determined up to sign"

This doesn't seem quite right, because H^2(L^inf, Z) is Z/p and has p-1 generators, which are permuted by the homotopy self equivalences of L^inf. It seems to me that one should say something like this. We suppose that an epimorphism Z -> Z/p has been fixed. That determines a map CP^inf -> K(Z/p, 2), and the homotopy fiber of that map has a canonical idenitification of its pi_1 with Z/p. Thus we have a fixed identification of the homotopy fiber with L^inf. Omega_2 is the reduction of the pull back to the homotopy fiber of the universal element i_2 of H^2(CP^inf), and then choose omega_1 so that the bockstein, etc.

ex. 16, IV.124, Dec. 1995 version. This can also be done using the long exact sequence in homotopy. It might be instructive to propose it as an ex. for chapter IV and then again for chapter IVA, so the reader is encouraged to work out both arguments.

9. We now know that eta^2 is nonzero in pi^s_2, and from the exercises, pi^s_2 must be isomorphic to Z/2. Alternatively this can be proven using differential topology. But you still have an unrequited yearning to work out an elementary proof - one that doesn't use the Steenrod squares or framed cobordism or a spectral sequence argument.

a) Consider the homotopy exact sequence of the pair (J_2S^2, S^2). Use homotopy excision to show that there is an exact sequence

? -> pi_5 S^4 -> pi_4 S^2 -> pi_4 J_2S^2 ->0

b) Show that the map from pi_5 S^4 to pi_4 S^2 takes Sigma^2 eta to
2 eta \circ Sigma eta.

c) It suffices to show that 2 eta \circ Sigma eta = 0. Be careful;
composition is not in general linear in the first factor.
Show that if 2 is the degree 2 map from S^3 to S^3, then
2 \circ Sigma eta = 4 Sigma eta = 0. Then show that the map from pi_5 S^4
to pi_4 S^2 is zero. Then show that pi_4 J(S^2) = pi_4 (S^2) = Z/2.

Here's an exercise to illustrate proposition 4.46, IV.88.
I would have given it at the time, if I'd thought of it.
a) isn't very hard, and is the main part of the exercise.
I recommend reading the rest of this in any case.

a) Recall that J(S^n) is homotopy equivalent to Omega(S^{n+1}).
Use the H-space structure to define a map
S^1 x J(S^2) -> J(S^1) which induces an isomorphism on homology
groups. Then show that the map is a homotopy equivalence.
b) Give an alternative proof using the loop space sequence from IV.72.
That is, show that there is a map inducing an isomorphism on homotopy
without using homology. Similarly, show that S^3 x Omega(S^7) is
homotopy equivalent to Omega(S^4) and that S^7 x Omega(S^15) is
homotopy equivalent to Omega(S^8). You could also use corollary 4.21 (instead
of prop 4.46) and not use the loop space sequence, for the latter two
examples.

If X and Y are homotopy equivalent H spaces and f: X -> Y is
a homotopy equivalence, then it makes sense to ask whether f
transports the H space structure on X to that on Y. The group
of homotopy self equivalence classes of X (or Y) might act
nontrivially on the set of H space structures up to homotopy on X (or Y), so
f (or its homotopy class) has to be specified. If so then X and Y
are called equivalent H spaces.

NO. c) Try showing that Omega(S^2) is equivalent as an H space to the
product H space of S^1 and Omega(S^3). Omega(S^4) is not equivalent
to a product H space in which both factors are noncontractible, and
neither is Omega(S^8). I haven't explained the idea of the classifying
space of a "group like H space" so those facts are a bit out of reach for now.

d) We know that Omega(S^6) has the integer cohomology of S^5 x Omega(S^11)
There is no homotopy equivalence. You can show that using the J construction
and ex. 25 of chapter III and the fact that S^5 is not an H space.
Similarly Omega(S^n) factors as a product only for n= 1,2,4,8.

In a fit of feeblemindedness I wrote:

>c) Try showing that Omega(S^2) is equivalent as an H space to the
>product H space of S^1 and Omega(S^3).

It isn't. I doubt any of you have lost any sleep over this though.
Geoffrey Mess

1. $H^*(\mathbb{C}P^n; \mathbb{Z}_p) \cong \mathbb{Z}_p[x_2]/x_2^{n+1}$

$x_2 \in H^2(\mathbb{C}P^n; \mathbb{Z}_p)$.

$\Rightarrow P^1(x_2) = x_2^p$ and $P^j(x_2) = 0$.

$\Rightarrow P^1(x_2^2) = x_2 x_2^p + x_2^p x_2 = 2x_2^{p+1}$

$P^1(x_2^3) = 4x^{p+2}$

$P^1(x_2^k) = 2^{k-1} x^{p+k-1}$

$$P^2(x_2^2) = x_2 \cdot 0 + P^1(x_2) \cdot P^1(x_2) + 0 \cdot x_2$$
$$= x_2^{2p}$$

$$P^2(x_2^3) = x_2 \cdot P^2(x_2^2) + x_2^p P^1(x_2^2) + 0 \cdot x_2^2$$
$$= x_2^{2p+1} + 2x_2^{2p+1} = 3x_2^{2p+1}$$

$$P^2(x_2^k) = x_2 \cdot P^2(x_2^{k-1}) + x_2^p \cdot P^1(x_2^{k-1})$$
$$= x_2 \cdot P^2(x_2^{k-1}) + x_2^p \cdot 2^{k-2} x_2^{p+k-2}$$
$$= x_2 P^2(x_2^{k-1}) + 2^{k-2} x_2^{2p+k-2}.$$
$$= (2^{k-1} - 1) x^{2p+k-2}.$$

etc...

$$\mathbb{C}P^{2k+1} \longrightarrow \mathbb{H}P^k.$$

$$[z_0, \dots z_{2k+2}] \longrightarrow [z_0 + jz_1, \dots, z_{2k+1} + jz_{2k+2}]$$

CLEARLY $\mathbb{C}P^1$ fibers...

H^* $\quad 0, 2, 4, \dots \qquad\qquad 0, 4, 8 \dots$

Induced map on cohomology is like including $(p,0)$ row in E_2 term into E_∞.

$\Rightarrow$ MAP is 0 in $4k+2$ dims and $\cong$ in $4k$ dims.

$$f^*: H^*(\mathbb{H}P^k; \mathbb{Z}_3) \to H^*(\mathbb{C}P^{2k+1}; \mathbb{Z}_3).$$

$x \in H^4(\mathbb{H}P^k; \mathbb{Z}_3)$ generator; $\quad f^*(x) = x_{\mathbb{C}}^2 \in H^4(\mathbb{C}P^{2k+1} \dots)$

$\qquad P^1(x_{\mathbb{C}}^2) = 2x_{\mathbb{C}}^4$

$P^2(x) = x^3 \qquad P^k(x) = 0 \qquad k > 2. \qquad \Rightarrow P^1(x) = 2x^2$

Adem $\Rightarrow$ $P^3 = P^1P^2 = 0 \quad \rightarrow \quad P^1(x^3) = 0$

$P^2(x^2) = x^4 + 4x^4 + x^4 = 6x^4.$

SAME remarks as above hold if $M^{20} \to \mathbb{C}aP^2$ (isom in cohomology in dims $0, 8, 16$).

Get a contradiction by computing P^i's for $\mathbb{C}aP^2$.

$\qquad H^q \to H^{q+4i}$

$P^1(y) = 0$ (dimensions)

$P^2(y) = 6y^2. \quad$ But $\quad P^2 = P^1P^1$ (Adem)

$\qquad = 0 \pmod 3 \qquad\qquad\qquad = 0.$

<u>2a.</u> If $f^*x = ax$, then

$$f^*(P^1(x)) \underset{nat.}{=} P^1(ax) = 2ax^2$$

$$\Vert$$

$$f^*(2ax^2) = 2a^2x^2 \Rightarrow a^2 \equiv a \pmod{3}$$

$$\Rightarrow a = 0 \text{ or } 1 \pmod{3}.$$