Singularities defined by the Frobenius map

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Updates will be posted here:

https://github.com/kschwede/FrobeniusSingularitiesBook

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Dedication

This book is dedicated to Mel Hochster on the occasion of his eightieth birthday.

Mel is truly the founder of the subject of F-singularities: an idea ultimately attributable to him underlies nearly every sentence of this manuscript.

Mel's work was foundational in establishing the importance of the Cohen-Macaulay property and using prime characteristic to establish it. He introduced F-purity as a way to understand singularities and his lectures [**Hoc75b**] have driven research in commutative algebra for nearly 50 years. Mel's profound ideas crystalized with the theory of tight closure and F-regularity (in collaboration with Huneke) in the late 1980's. Since then, tight closure and F-singularities have continued to evolve through the work of an ever-broadening circle of mathematicians, finding deep applications in prime characteristic birational geometry and now even in mixed characteristic. There can be no doubt that Mel Hochster's genius is the driving force of all this progress in commutative algebra, algebraic geometry and arithmetic geometry. His contributions will endure as long as mathematics continues to fascinate humans, or whatever intelligence comes next.

Most importantly, Mel Hochster is kind and gifted mentor, who graduated forty-nine PhD students and championed countless other young mathematicians, including the authors of this book. He generously shares his insight and encourages all to embrace their own mathematical journey. Not only mathematics, but also the world, are much better for Mel's dedicated service to us all.

Notation

- The canonical module ω_R of a ring or canonical sheaf ω_X of a scheme.
- The dualizing complex $\omega_R^{\bullet}$ of a ring, or scheme $\omega_X^{\bullet}$.
- $D_+(f) = Y \setminus \mathbb{V}(f)$, is the open set of $Y = \operatorname{Proj} S$ assuming $f \in S$ is homogeneous.
- for $\phi \in \operatorname{Hom}_R(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ and $\psi \in \operatorname{Hom}_R(F^d_* \mathcal{O}_X, \mathcal{O}_X)$,

  $$\phi \star \psi := \phi \circ F^e_* \psi \in \operatorname{Hom}_R(F^{e+d}_* \mathcal{O}_X, \mathcal{O}_X)$$

  in other words the map corresponding to multiplication in the Cartier algebra. See Chapter 1 Subsection 4.2.
- Weil divisor associated to $\phi \in \operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$, D_ϕ. See Chapter 5 Definition 1.1.
- The anti-canonical $\mathbb{Q}$-divisor associated to $\phi \in \operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$, Δ_ϕ. See Chapter 5 Definition 2.1.
- The finitistic tight closure of an ideal $I^{*\,\mathrm{fg}}$ or of a submodule $N_M^{*\,\mathrm{fg}}$. Chapter 7 Definition 1.12.
- F-pure threshold $\operatorname{fpt}(f^t), \operatorname{fpt}(\mathfrak{a}^t)$. See Chapter 4 Definition 3.1 Definition 4.28.
- Gorenstein in codimension n, G_n.
- Hilbert-Kunz multiplicity of I along a module M, $e_{\mathrm{HK}}(I; M)$. Chapter 9 Definition 2.15.
- $\mathbf{R}\operatorname{Hom}$ into a dualizing complex, $\mathbf{R}\operatorname{Hom}_R(-, \omega_R^{\bullet})$ or $\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(-, \omega_X^{\bullet})$, $\mathbf{D}(-)$.
- Hom into a canonical module, $\operatorname{Hom}_R(-, \omega_R)$ or $\mathscr{H}om_{\mathcal{O}_X}(-, \omega_X)$, $(-)^{\vee_\omega}$.
- ith cohomology of a complex $C^{\bullet}$, $\mathcal{H}^i(C^{\bullet})$.
- $\mathcal{K}(X)$ or $\mathcal{K}(R)$, the function field of X or the total ring of fractions of the ring R.
- length of an R-module M, $\ell_R(M)$.
- Minimal number of generators of M as an R-module, $\mu_R(M)$.
- Matlis dual $\operatorname{Hom}_R(-, E) = (-)^{\vee}$.
- Regular in codimension n, R_n.
- Right derived functor of F, $\mathbf{R}F$.
- Ring dual: $\operatorname{Hom}_R(-, R) = (-)^*$, $\mathscr{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X) = (-)^*$. See Appendix B (4.1.1) and (4.1.2).
- reflexification/S_2-ification, $(-)^{\mathrm{S}_2}$.

- Serre's condition S_n.
- Sheaf of units of a sheaf of rings $\mathcal{O}_X$, $\mathcal{O}_X^{\times}$, or group of units of a ring R, $R^{\times}$.
- The tight closure of an ideal I^* or submodule N_M^*. See Chapter 7 Definition 1.1.
- Vanishing locus of an ideal I, $\mathbb{V}(I)$.
- Vector space dual $\mathrm{Hom}_k(-,k) = (-)^{\vee}$.

Preface

The goal of this book is to introduce the *Frobenius morphism* and its uses in commutative algebra and algebraic geometry.

Let X be a Noetherian scheme of prime characteristic p. The **Frobenius** morphism is the scheme map

$$F : X \to X \qquad \mathcal{O}_X \to F_*\mathcal{O}_X$$

defined to be the *identity map* on the underlying topological space X and the p^{th} power map on sections. For example, when $X = \operatorname{Spec} R$ is affine, the Frobenius map is essentially just the ring homomorphism $R \to R$ sending $r \mapsto r^p$. An important special case to keep in mind is the case where X is a variety[1] over an algebraically closed field k of characteristic $p > 0$.

A major theme of this book is the use of Frobenius in studying *singularities* of schemes. This story begins with a famous 1964 theorem of Ernst Kunz: a Noetherian scheme X is regular if and only if the Frobenius morphism is *flat* [**Kun69a**]. For varieties, this says that X is smooth if and only if $F_*\mathcal{O}_X$ is a locally free sheaf (or vector bundle).

By relaxing the flatness condition for the Frobenius in various ways, we get a host of other "mild singularity" classes in prime characteristic. Many theorems that hold for smooth varieties over fields of characteristic p can be extended to these classes of "F-singularities." We will prove that these classes of singularities are analogous to many of the classes of "mild singularities" in the minimal model program. In fact, in many cases, Frobenius can be used to detect the singularity class of complex varieties as well, by reduction to characteristic p. For example, Kawamata log terminal singularities are equivalent, after suitably "reducing to prime characteristic," to strongly F-regular singularities. Likewise, there is a conjectural association between log canonical and F-pure (or locally Frobenius split) singularities in prime characteristic.

[1] The Frobenius map is not a map of varieties, however, because it is not k-linear but rather raises scalars to the p-th power. Some authors prefer to work with a variant called the **relative Frobenius** which is k-linear. In this book, we will work in the category of schemes, and so always mean the absolute Frobenius map when referring to Frobenius.

Vast bodies of techniques in birational algebraic geometry that had been developed for complex varieties using differential forms and L^2-analysis have "characteristic p" analogs defined using the Frobenius map. The multiplier ideals, made famous by complex geometers (*cf.* **[GR70, Koh79, EV83, Kol86, Nad90, Lip94, Siu09, DEL00]**), for example, share many of the properties of *test ideals*, which first arose in Hochster and Huneke's theory of tight closure, and can be used to prove the same kinds of theorems. Likewise, a numerical invariant of singularities called the log canonical threshold, which is defined using convergence of certain integrals, has a "characteristic p analog" called the F-pure threshold which beautifully captures some of the more subtle kinds of singularities that can arise in prime characteristic. As one concrete example, consider the simple cusp defined by $y^2 = x^3$. This singularity has F-pure threshold equal to

$$\begin{array}{cc} 1/2 & \text{in char } 2 \\ 2/3 & \text{in char } 3 \\ \frac{5}{6} - \frac{1}{6p} & \text{if char } p = 5 \bmod 6 \\ \frac{5}{6} & \text{if char } p = 1 \bmod 6. \end{array}$$

We see that cusp is most singular in characteristic 2, because the F-pure threshold is smallest then. It is slightly less singular in characteristic 3, and even less singular for larger p. For infinitely many p (namely, those p congruent to 5 mod 6), it it is not really anymore singular than it is over complex numbers, in the sense that the *log canonical threshold* of the cusp over $\mathbb{C}$ is also 5/6. The fact that the cusp is "more singular" in some characteristics than it is over $\mathbb{C}$ is deeply connected to arithmetic issues such as supersingularity for elliptic curves.

Characteristic p techniques can be used to recover or prove characteristic zero theorems, by standard (or sometimes by especially clever) reduction to prime characteristic techniques. But even better, tools developed in F-singularity theory can serve as replacements to the analytic techniques, allowing us to extend results known for complex varieties to varieties over an arbitrary field. For example, the Ein-Lazarsfeld-Smith theorem on the uniform behavior of symbolic powers of ideal sheaves on smooth varieties, first proved with the help of *multiplier ideals* for complex varieties, can be proven in prime characteristic with the test ideal. In a different direction, recent progress in the minimal model program for varieties of prime characteristic p relies in key steps on restricting attention to the class of *strongly F-regular pairs* (X, D), rather than the traditional log terminal pairs used in the minimal model program over $\mathbb{C}$ **[HX15, Bir16, BW17, Wal18, DW19, HW22, HW23, Cas21]**.

The local story of the Frobenius map has a storied history in commutative algebra, going back to the work of Kunz as mentioned above **[Kun69a]**, to work of Peskine and Szpiro **[PS73]**, and especially to the work Hochster and

Robert's proof that the ring of invariants of a linearly reductive group acting linearly on a polynomial ring (over a field of any characteristic) is Cohen-Macaulay [**HR74**]. Hochster later used Frobenius to prove the existence of (big) Cohen-Macaulay modules [**Hoc73b, Hoc75b, Hoc75a**], and later with Huneke, that the absolute integral closure R^+ of any excellent local domain R of characteristic $p > 0$ is a Cohen-Macaulay R-algebra [**HH92**]. The subject eventually matured into Hochster and Huneke's theory of tight closure, which has inspired much of the theory we present here, even when the roots are not always visible. Although it arose independently, tight closure is reminiscent of Faltings' theory of "almost ring theory" in arithmetic geometry [**Fal02, GR03**], which is used in Scholze's theory of perfectoid algebras. Ideas pioneered in tight closure theory have contributed to recent progress on the minimal model program in mixed characteristic; see [**TY23, BMP$^+$23, HW23**].

When employed globally, the Frobenius map can be used to prove vanishing theorems for cohomology of line bundles and other results about the global geometry of projective varieties. Here, the emphasis is not so much on *singularities*—many such applications are interesting even in the realm of *smooth* projective varieties—but rather on *positivity* of the geometry. For example, Frobenius splitting implies a type of positivity for the anti-canonical sheaf, allowing us to bypass tricky situations in prime characteristic such as the failure of Kodaira Vanishing, or the lack of a resolution of singularities in characteristic p. Combined with its usefulness in delicately quantifying singularities in prime characteristic, this explains why the Frobenius techniques of this book have generated intense interest in the current quest for the minimal model program in prime characteristic.

At the time of this writing, one frontiers for the subject lies in extending and developing the theory to *mixed characteristic*—meaning for local rings that do not contain any field, or for schemes over $\mathbb{Z}_p$. Much of the progress in this direction comes from the connection tight closure and big Cohen-Macaulay algebras developed in [**HH92**] and [**Smi94**], in light of the new theory of perfectoid algebras and its applications to big Cohen-Macaulay Algebras due to André, Bhatt, Gabber, and Scholze, as well as others [**And20, BS22**]. We briefly discuss the relevant tight closure theory in characteristic p in Chapter 7, pointing out the relevant parts of the theory that are now being generalized to mixed characteristic by Hacon, Ma, Schwede, Tucker, and others.

0.1. Conventions. The letter p always denotes a (positive) prime integer.

The word "ring" in these notes always means a commutative ring with unity (unless explicitly stated otherwise). Typically, our rings and schemes are Noetherian as well, but we will include this as an explicit hypothesis. A ring is called *local* if it has a unique maximal ideal; we do not assume a local ring is Noetherian. The notation $(R, \mathfrak{m})$ or $(R, \mathfrak{m}, k)$ will denote a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

For a field k, the notation $\overline{k}$ denotes an algebraic closure of k. By *variety*, we mean a reduced irreducible separated scheme of finite type over an algebraically closed field. Often, the fact that the ground field k is algebraically closed is not important, and may be replaced by the assumption that k is perfect, or even that $[k : k^p]$ is finite.

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CHAPTER 1

Introduction to the local theory of Frobenius splitting

1. Frobenius on rings and schemes

Fix a positive prime integer p. A commutative ring R (with unity) has characteristic p if the subring generated by 1_R is a field of cardinality p.

A simple but powerful feature of rings of characteristic p is that the p-th power map

$$R \xrightarrow{F} R \quad \text{sending} \quad r \mapsto r^p$$

is a *ring* homomorphism. This is the *Frobenius map* of R and the subject of this book.

1.1. Frobenius on Spec. Like any ring homomorphism, the Frobenius map induces a morphism of the corresponding scheme $\operatorname{Spec} R$, which we also denote by F. It is easy to check that this induced map on Spectra

$$\operatorname{Spec} R \xrightarrow{F} \operatorname{Spec} R \qquad \text{sending} \qquad P \mapsto F^{-1}(P) = \{r \in R \,|\, r^p \in P\}$$

is the identity map. Thus, the Frobenius map on the affine scheme $X = \operatorname{Spec} R$ is the identity map on the underlying topological space, while the corresponding map of structure sheaves

$$\mathcal{O}_X \to F_*\mathcal{O}_X$$

is the p-th power map locally on sections. While using the same notation F for both the map of rings and its corresponding (dual) map of schemes is an abuse of notation, the intended meaning is usually clear from the context. In any case, this abuse is ubiquitous throughout the literature.

1.2. Frobenius as a map of modules. Whenever we have a ring homomorphism $R \xrightarrow{f} S$, we can view S as an R-module via restriction of scalars: by definition, $r \in R$ acts on $s \in S$ by $r \cdot s = f(r)s$. This is true in particular for the Frobenius map.

Because both the source and target of Frobenius are the same ring R, this can be confusing. We therefore usually use the notation F_*R to denote the target ring. This is consistent with the notation for the induced map of sheaves $\mathcal{O}_X \to F_*\mathcal{O}_X$ for the corresponding affine scheme morphism. We consider F_*R as an R-module via restriction of scalars for the Frobenius map.

As an Abelian group, the R-module F_*R is precisely R, but its R-module structure is defined by

$$r \in R \text{ acts on } x \in F_*R \text{ by } r \cdot x = r^p x \in F_*R. \tag{1.0.1}$$

With this R-module structure on F_*R, the Frobenius map

$$R \to F_*R \qquad r \mapsto r^p \tag{1.0.2}$$

is not only a ring map, but also a map of R-modules. In particular, the Frobenius map (1.0.2) defines an R-*algebra* structure on F_*R.

Example 1.1. Suppose $R = \mathbb{F}_p[x]$, and consider the R-module F_*R. Using the notation (1.0.1), then, $2 \cdot x = 2x$ since $2^p \equiv 2 \pmod p$. However, $x \cdot x = x^p x = x^{p+1}$. This example emphasizes the need for care in distinguishing elements of R and F_*R, even though these are the same sets. We will soon address the notation further; see Notation 1.4.

1.3. Frobenius on schemes. The following fact is so fundamental that we highlight it as a proposition despite its straightforward proof:

Proposition 1.2. *Let $R \xrightarrow{\phi} S$ be any ring homomorphism between commutative rings of prime characteristic p. Then the diagram*

$$\begin{tikzcd} R \arrow[r, "F"] \arrow[d, "\phi"] & R \arrow[d, "\phi"] \\ S \arrow[r, "F"] & S \end{tikzcd}$$

commutes, where rightward arrows are the Frobenius maps on their respective rings.

As a special case, we see that the Frobenius map commutes with the localization map $R \to W^{-1}R$ at any multiplicative system W. This allows us to easily generalize the Frobenius morphism to arbitrary schemes.

Given a scheme X of prime characteristic, we can cover X by affine charts, each of which is endowed with its own Frobenius morphism. Because the Frobenius map commutes with localization, these Frobenius maps on charts restrict to the Frobenius map on every open subset of each chart. So they uniquely patch together to define a Frobenius map on X, which we denote $F : X \to X$. The map of the underlying topological space is the *identity map* by Paragraph 1.1, whereas the map of sheaves of rings $\mathcal{O}_X \to F_*\mathcal{O}_X$ is the p-th power map over *every* open set $U \subseteq X$. That is, the ring map

$$\mathcal{O}_X(U) \to F_*\mathcal{O}_X(U) = \mathcal{O}_X(F^{-1}(U)) = \mathcal{O}_X(U)$$

sends a section $f \in \mathcal{O}_X(U)$ to $f^p \in \mathcal{O}_X(U)$.

Remark 1.3. The Frobenius morphism is clearly an affine morphism: the preimage of an affine open set U is again U, so affine. Even more, the Frobenius is an integral morphism: the ring maps $\mathcal{O}_X(U) \to F_*\mathcal{O}_X(U)$ are integral because every element of the target ring satisfies a polynomial of the form $X^p - f$, where $f \in \mathcal{O}_X(U)$. However, the Frobenius map need not be finite in general, although it is in most cases of interest; See Definition 1.17.

1.4. The Frobenius pushforward functor. Fix a ring R of prime characteristic p. For any R-module M, we denote by

$$F_*M$$

the R-module that is M as an abelian group, but whose R-module structure is twisted by the Frobenius map on R. That is, the R-module structure on F_*M is defined by

$$r \in R \quad \text{acts on} \quad m \in F_*M \quad \text{by} \quad r \cdot m = r^p m.$$

The case $M = R$ recovers the notation F_*R above. The notation F_*M is consistent with the standard notation for the pushforward of the coherent sheaf defined by M on the affine scheme $\operatorname{Spec} R$ under the Frobenius map on $\operatorname{Spec} R$.

Clearly, the operation F_* defines a functor

$$\begin{aligned} \{ \ R\text{-modules} \ \} &\longrightarrow \{ \ R\text{-modules} \ \} \\ M &\longmapsto F_*M \end{aligned}$$

from R-modules to R-modules. It is exact because it does nothing to the underlying abelian group structure.

Notation 1.4. It is convenient to use the notation F_*m for elements of F_*M. Remembering that, as sets, $F_*M = M$, the element F_*m denotes the element $m \in M$ viewed as an element of F_*M, so that the R-module action on F_*M can be written

$$rF_*m = F_*(r^p m)$$

Observe how the notation works: for $x, y \in M$, we have

$$F_*x + F_*y = F_*(x + y) \quad \text{in} \quad F_*M$$

and if $x, y \in R$ (or some other R-algebra), furthermore,

$$F_*xF_*y = F_*xy \quad \text{in} \quad F_*R.$$

Note that despite our notation, the functor F_* does not actually act on *elements* of M: the symbol F_*m is just notation for the element m viewed in the module F_*M.

Example 1.5. Rewriting Example 1.1 using this new notation, we have

$$2F_*x = F_*2x \qquad \text{and} \qquad xF_*x = F_*x^{p+1}.$$

This notation emphasizes the different roles of the elements in the ring R and in the R-module F_*R (even though as sets, R and F_*R are identical). For this reason, it is our preferred notation for most of the book.

Of course, the Frobenius map similarly defines a natural push-forward functor on the category of quasi-coherent sheaves on a scheme X of characteristic p:

$$\left\{ \begin{array}{c} \text{quasi-coherent} \\ \text{sheaves on } X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{quasi-coherent} \\ \text{sheaves on } X \end{array} \right\}$$

$$\mathcal{M} \longmapsto F_*\mathcal{M}$$

Again, this functor is the identity on the underlying sheaves of abelian groups, so it is exact.

Remark 1.6. If $R \to S$ is a homomorphism of rings of characteristic $p > 0$, then any S-module M may also be considered an R-module by restriction of scalars. Fortunately, there is no ambiguity in the notation F_*M in this case: we *could* write $F_{R*}M$ (respectively $F_{S*}M$) for the Frobenius pushforward functor on R-modules (respectively S-modules) applied to M. But these agree: viewing M as an R-module, the R-module $F_{R*}M$ is precisely the S-module $F_{S*}M$ viewed as an R-module via restriction of scalars. For this reason, we usually write simply F_*M in this situation. Of course, as an abelian group F_*M is always simply M—only the R-module (or S-module) structure is changed by twisting by Frobenius.

To emphasize this point, we reiterate using the notation for schemes. If $X \xrightarrow{g} Y$ is a morphism of schemes of prime characteristic, then Proposition 1.2 ensures that there is a commutative diagram of schemes

(1.6.1)
$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "g"] & X \arrow[d, "g"] \\ Y \arrow[r, "F_Y"'] & Y. \end{tikzcd}$$

So for any sheaf $\mathcal{M}$ on X, we have that

(1.6.2)
$$F_{Y*}g_*\mathcal{M} = g_*F_{X*}\mathcal{M}.$$

Again, there is little risk of confusion if we simply write F for both F_X and F_Y: Frobenius commutes with any morphism $X \xrightarrow{g} Y$ of schemes, and $F_*g_* = g_*F_*$ as functors of quasicoherent sheaves on X.

Example 1.7. For a local ring R with maximal ideal $\mathfrak{m}$ and residue field k, the ring F_*R has maximal ideal $F_*\mathfrak{m}$ and residue field F_*k, which is a degree $[k : k^p]$ extension of k. Here k^p is the field of pth powers of elements of k. Note that the notation F_*k can have two different meanings, since k is a ring of characteristic p in its own right as well as an R-module. Fortunately, there is little danger of confusion, since there is a canonical identification $F_{k*}k = F_{R*}k$ (Remark 1.6).

1.5. Alternative notations. An alternative notation makes F_* more concrete in the case R where is a domain (or reduced). In this case, we can identify F_*R with the subring $R^{1/p}$ of an algebraic closure of the fraction field (or total quotient ring) of R consisting of the p-th roots of elements of R. Since every element has a unique p-th root, there is a canonical isomorphism of R-algebras

$$F_*R \to R^{1/p} \quad \text{sending} \quad F_*r \mapsto r^{1/p}.$$

Here, the R-module structure on $R^{1/p}$ is the obvious one: $r \in R$ acts on $x^{1/p}$ in $R^{1/p}$ by $r \cdot x^{1/p} = rx^{1/p}$ (which is the same as $(r^p x)^{1/p}$). Under this identification, the Frobenius map

$$R \to F_*R \quad \text{sending} \quad r \mapsto F_*r^p$$

becomes simply the inclusion

$$R \hookrightarrow R^{1/p} \quad r \mapsto r = (r^p)^{1/p}.$$

Here we see concretely that the ring $R^{1/p}$ is an R-algebra in the obvious way, since it is an extension ring of R. This R-algebra is canonically isomorphic to the R-algebra F_*R. Elements of the R-algebra $R^{1/p}$ will be denoted by $x^{1/p}$ (instead of F_*x) when using this notation.

Alternately, we can let R^p denote the subring of pth powers of elements of R. Then the Frobenius map $R \xrightarrow{F} R$ has image R^p, and if R is reduced we have that $R^p \cong R$ as rings. In this case, the inclusion $R^p \hookrightarrow R$ is also identified with the Frobenius map. We won't use this observation much, though occasionally it adds insight.

For certain R-modules M, the notation $M^{1/p}$ makes sense literally as p-th roots and can be used in place of F_*M. For example, if I is an ideal in a domain R, then we can write $I^{1/p}$ to denote the p-th roots of elements of I. Note that $I^{1/p}$ is an ideal of $R^{1/p}$, and under the identification of F_*R with $R^{1/p}$, the R-submodule F_*I of F_*R is identified with $I^{1/p}$. Similarly, if M is an R-submodule of the fraction field K of R, then $M^{1/p}$ makes sense as a subset of the field $K^{1/p}$.

Even more generally, some authors use the notation $R^{1/p}$ to denote the target copy of R under the Frobenius map even when R is not a domain (or reduced). In this case, the elements of $R^{1/p}$ are the symbols $r^{1/p}$ with the obvious ring structure, so that the Frobenius map becomes

$$R \to R^{1/p} \quad r \mapsto (r^p)^{1/p}.$$

We caution the reader, however, that this notation can be misleading when R is not reduced since $(r^p)^{1/p}$ may be zero instead of r if $r^p = 0$. In this book, we will use this notation when we feel it is illuminating, but never in the non-reduced case.

Similarly, some authors use the notation $M^{1/p}$ in place of F_*M for arbitrary modules M. There is little chance of confusion when R is a domain and M is a submodule of its fraction field, for example, but in general, this notation can get confusing so we will generally avoid it.

Example 1.8. Let R be the polynomial ring $\mathbb{F}_p[x]$ over the field $\mathbb{F}_p$ of p elements. Then it is easy to see that $R^{1/p}$ is a free R-module with basis $1, x^{1/p}, \dots, x^{(p-1)/p}$. Equivalently F_*R is a free R-module with basis $F_*1, F_*x, \dots, F_*x^{p-1}$.

1.6. Iterates of Frobenius. The Frobenius homomorphism on a ring of characteristic p can be iterated:

$$R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} \cdots \xrightarrow{F} R$$
$$r \mapsto r^p \mapsto (r^p)^p = r^{p^2} \mapsto \cdots \mapsto r^{p^e}.$$

Iterating e times, the resulting map F^e raises elements to the p^e-th power, and is of course also a ring homomorphism. Just as in the case of Frobenius, we denote the target ring by F^e_*R, and endow it with the unique R-module structure that makes the ring map

$$R \longrightarrow F^e_*R \qquad r \mapsto F^e_* r^{p^e} \tag{1.8.1}$$

also an R-module homomorphism. Again, we use the notation F^e_*x to denote the element $x \in R$ viewed in the target ring F^e_*R. In particular, note that

$$rF^e_*x = F^e_*(r^{p^e}x),$$

for all elements $r \in R$ and $F^e_*x \in F^e_*R$, exactly as in Notation 1.4.

The iterates of Frobenius are defined similarly on any scheme X of characteristic p. The map $F^e : X \to X$ is the identity map on the underlying topological space and the p^e-th power map locally on sections.

Likewise, for each $e \in \mathbb{N}$, we have the Frobenius pushforward functor F^e_* on R-modules (or quasi-coherent sheaves on the scheme X). Since this functor does nothing to the underlying abelian groups (or sheaves of abelian groups), it is exact. Note also that $F_* \circ F^e_* = F^{e+1}_*$ as functors for all $e \in \mathbb{N}$.

Notation 1.9. For any ideal in any ring R of characteristic p, the notation $I^{[p^e]}$ denotes the expansion of the ideal I under the (iterated) Frobenius map $F^e : R \to R$ sending $r \mapsto r^{p^e}$. Explicitly, $I^{[p^e]}$ is the ideal generated by the p^e-powers of all elements (or equivalently, any set of generators) of I. Practicing the notation, note that

$$IR^{1/p^e} = (I^{[p^e]}R)^{1/p^e} \quad \text{and} \quad IF^e_*R = F^e_* I^{[p^e]}.$$

We prove the following to help practice this notation. This observation will become important to us later in Chapter 9.

Proposition 1.10. *Let $(R, \mathfrak{m}, k)$ be an arbitrary local ring of characteristic p. If F_*R is a finitely generated module, then it is minimally generated by*

$$[k : k^p] \dim_k(R/\mathfrak{m}^{[p]})$$

elements. Here k^p is the subfield of k consisting of all pth powers.

Proof. By Nakayama's Lemma, the minimal number of generators for F_*R is the dimension of the R-module $R/\mathfrak{m} \otimes_R F_*R$. This is isomorphic to $F_*R/\mathfrak{m}F_*R \cong F_*(R/\mathfrak{m}^{[p]})$. Considered as a vector space over its residue field F_*k, the ring $F_*(R/\mathfrak{m}^{[p]})$ has dimension equal to $\dim_k(R/\mathfrak{m}^{[p]})$. So to find its dimension over k we must multiply by the degree of the field extension $k \to F_*k$. Remembering that $k \mapsto F_*k$ is the p-th power map, we see that $[F_*k : k] = [k : k^p]$, and the formula follows. $\square$

An easy, but crucial, point is that the Frobenius pushforward functor F^e_* commutes with localization in every possible sense.

Lemma 1.11. *Let R be a ring of prime characteristic p and W a multiplicative set in R. Suppose that M is any R-module. Then for every $e \in \mathbb{N}$, there is a natural $W^{-1}R$-module isomorphism*

$$W^{-1}F^e_*M \longrightarrow F^e_*(W^{-1}M) \qquad \frac{r}{w} \otimes F^e_*m \mapsto F^e_*\left(\frac{r^{p^e}m}{w^{p^e}}\right), \tag{1.11.1}$$

where the second F^e_* can be viewed as the Frobenius pushforward functor for either $W^{-1}R$-modules or for R-modules.

Proof. The basic point is that inverting all the elements of W is the same as inverting only their p-th-powers. The map (1.11.1) is clearly linear over $W^{-1}R$. It is surjective because

$$F^e_*\left(\frac{m}{w}\right) = F^e_*\left(\frac{mw^{p^e-1}}{w^{p^e}}\right) = \frac{1}{w}\, F^e_*(mw^{p^e-1}),$$

which is the image of $\frac{1}{w} \otimes F^e_*(mw^{p^e-1})$ under (1.11.1). It is easily seen to be injective as well: an arbitrary element,

$$\sum_{i=1}^{t} \frac{r_i}{w_i} \otimes F^e_* m_i \in W^{-1} F^e_* M = W^{-1} R \otimes_R F^e_* M$$

can be written in the form $\frac{1}{w} \otimes F^e_* m$ for suitable $w \in W$ and $m \in M$ (by finding a common denominator for the $\frac{r_i}{w_i}$ and pulling the numerators across the tensor symbol). Now if $\frac{1}{w} \otimes F^e_* m$ is in the kernel, then $F^e_*(\frac{m}{w^{p^e}}) = 0$, which in turn means that $\frac{m}{w^{p^e}} \in W^{-1}M$ is zero, since F^e_* does not affect the underlying abelian group. This means that there exists $h \in W$ such that $hm = 0$, so

$$\frac{1}{w} F^e_* m = \frac{h}{hw} F^e_* m = \frac{1}{hw} F^e_*(h^{p^e} m) = \frac{1}{hw} F^e_* \ 0 = 0$$

in $W^{-1} F^e_* M$ as well. □

Example 1.12 (Polynomial and power series rings)**.** Let k be any perfect field of characteristic $p > 0$, and consider the ring R of either polynomials $k[x_1, \ldots, x_n]$ or power series $k[\![x_1, \ldots, x_n]\!]$ in n variables over k. It is not hard to check that in both cases, $F^e_* R$ is a free R-module on the basis $\{F^e_* x_1^{a_1} \cdots x_n^{a_n} \mid 0 \le a_i \le p^e - 1\}$. In particular, the module $F^e_* R$ is free over R of rank p^{ne}.

The situation is more complicated for non-polynomial rings.

Example 1.13. Working over the field $\mathbb{F}_2$ of two elements, consider the cuspidal ring $R = \mathbb{F}_2[s,t]/(s^3 - t^2) = \mathbb{F}_2[x^2, x^3] \subseteq \mathbb{F}_2[x]$. Let us examine the structure of $R^{1/2}$ as an R-module.

Observe that R has an $\mathbb{F}_2$-vector space basis consisting of all monomials x^n where n is any positive integer *except* 1. Thus $R^{1/2}$ has an $\mathbb{F}_2$-vector space basis consisting of all monomials $x^{n/2}$ where n is any positive integer except 1, as well. Note that R is $\mathbb{N}$-graded in a natural way, and that this grading is compatible with a natural $\frac{1}{2}\mathbb{N}$-grading on $R^{1/2}$.

It is not hard to see that $\{1, x, x^{3/2}, x^{5/2}\}$ is a generating set for $R^{1/2}$ over R. Because each basis element has a different degree, this is a minimal generating set for the graded R-module $R^{1/2}$.

However these polynomials are *not a free basis* over R. The ring R and the ring $\mathbb{F}_2[x]$ have the same fraction field $\mathbb{F}_2(x)$, so if $R^{1/2}$ is free over R of some rank d, then localizing we see that also $(\mathbb{F}_2(x))^{1/2}$ is free over $\mathbb{F}_2(x)$ of the same rank d. This is also the free rank of $F_*\mathbb{F}_2[x]$ over $\mathbb{F}_2[x]$, which is two, from Example 1.8. But we have seen that $R^{1/2}$ over R requires at least four generators. So in this case, F_*R is *not free* over R: it is minimally generated by four elements in a neighborhood of the maximal ideal (s,t) despite having generic rank two.

1.7. Frobenius and completion. Let R be a ring and $\mathfrak{n} \subseteq R$ be a finitely generated ideal and let $\widehat{R} = \widehat{R}^{\mathfrak{n}}$ denote the $\mathfrak{n}$-adic completion of R (most typically $(R, \mathfrak{n})$ will be a Noetherian local ring with maximal ideal $\mathfrak{n}$ and $\widehat{R}$ will be the $\mathfrak{n}$-adic completion of R). Consider the following two compositions

$$R \to F_*R \to \widehat{(F_*R)}$$

$$R \to \widehat{R} \to F_*\widehat{R}.$$

In the first, we follow Frobenius $R \to F_*R$ by the natural completion map[1] for the R-module F_*R at J. In the second, we first complete at J, then follow with the Frobenius map for $\widehat{R}$. Fortunately, these two compositions are essentially the same:

Lemma 1.14. *Let R be a commutative ring of prime characteristic, and let $\widehat{R}^{\mathfrak{n}}$ denote its completion at an arbitrary finitely generated ideal $\mathfrak{n}$. Then we have a canonical identification of the maps*

$$\widehat{R}^{\mathfrak{n}} \to \widehat{(F_*R)}^{\mathfrak{n}} \qquad \text{and} \qquad \widehat{R}^{\mathfrak{n}} \to F_*\widehat{R}^{\mathfrak{n}}$$

where the first map is the completion of the Frobenius map for R and the second map is the Frobenius map on $\widehat{R}^{\mathfrak{n}}$. Likewise, the same statement holds for any iterate F^e of Frobenius.

Proof. By definition, the completion $\widehat{(F_*R)}^{\mathfrak{n}}$ at $\mathfrak{n}$ is $\varprojlim F_*R/\mathfrak{n}^n F_*R$ and the map

$$\widehat{R}^{\mathfrak{n}} \to \widehat{(F_*R)}^{\mathfrak{n}}$$

is obtained by taking the inverse limit of the natural maps $R/\mathfrak{n}^n \to (F_*R)/\mathfrak{n}^n(F_*R)$.

There are natural isomorphisms

$$F_*R/\mathfrak{n}^n F_*R \cong F_*R/F_*(\mathfrak{n}^n)^{[p]} \cong F_*(R/(\mathfrak{n}^n)^{[p]})$$

for all n. Now the point is that the ideals $\{(\mathfrak{n}^n)^{[p]}\}_n$ and $\{\mathfrak{n}^n\}_n$ are cofinal with each other as we range over all n. Indeed, since $(\mathfrak{n}^n)^{[p]} \subseteq \mathfrak{n}^n$ for all n, we we only need to check that for each n, there is some N such that $\mathfrak{n}^N \subseteq (\mathfrak{n}^n)^{[p]}$. Indeed, it is easy to check that $N = pdn$ works, where d is the number of generators for $\mathfrak{n}$.

[1] See [**AM69**, §10] for basic material on completion.

Because $\{(\mathfrak{n}^n)^{[p]}\}_n$ and $\{\mathfrak{n}^n\}_n$ are cofinal, they define the same inverse limits. That is, $\varprojlim F_*R/\mathfrak{n}^n F_*R \cong F_*\widehat{R}^{\mathfrak{n}}$. Thus the completion of the Frobenius map $R \to F_*R$ produces a natural map

$$\widehat{R}^{\mathfrak{n}} \to \varprojlim F_*(R/(\mathfrak{n}^n)^{[p]}) \cong F_*\widehat{R}^{\mathfrak{n}}$$

which is the p-th power map on $\widehat{R}^{\mathfrak{n}}$. $\square$

Corollary 1.15. *If R is Noetherian and the Frobenius map is* finite*, then for any ideal $\mathfrak{n}$, there is a natural isomorphism*

$$\widehat{R}^{\mathfrak{n}} \otimes_R F_*R \cong F_*\widehat{R}^{\mathfrak{n}}.$$

Proof. This follows immediately from Lemma 1.14 because for Noetherian rings, the completion functor on finitely generated modules is equivalent to the functor $- \otimes_R \widehat{R}^{\mathfrak{n}}$ ([**AM69**, Prop 10.13] or [**Sta19**, Tag 00MA]). $\square$

Caution 1.16. If F_*R is not finitely generated, then it need not be the case that $\widehat{F_*R} \cong \widehat{R}^{\mathfrak{n}} \otimes_R F_*R$; see Exercise 1.6.

1.8. Finiteness of Frobenius. While there are many potential pathologies of arbitrary Noetherian rings (see, for example, [**Nag62**]), there is a simple condition in prime characteristic which eliminates nearly all of these—namely that Frobenius map is *finite*:

Definition 1.17. A ring of prime characteristic is **F-finite** if Frobenius is a finite map—that is, if F_*R is a finitely generated R-module. Likewise, a scheme of prime characteristic is **F-finite** if Frobenius is a finite morphism of schemes.

Most of the rings we encounter in algebraic geometry are F-finite:

Example 1.18. A field k of characteristic p is F-finite if and only if the extension $k^p \subseteq k$ has finite degree. In particular, perfect—including algebraically closed— fields are F-finite.

Example 1.19. Any polynomial ring $k[x_1, \ldots, x_n]$ or power series ring $k[\![x_1, \ldots, x_n]\!]$ over an F-finite field is F-finite. Indeed, a generating set for F_*R in this case can be taken to be $F_*\lambda x_1^{a_1} \cdots x_n^{a_n}$ where λ ranges through some (finite) basis for k over k^p and $0 \leq a_i \leq p-1$ for each i. In fact, this computation shows that shows that F_*R is a free R-module of rank $[F_*k : k]p^n$. Hence, when k is perfect, F_*R has rank p^n. See also Example 1.8.

Example 1.20. The cuspidal ring $\mathbb{F}_2[x^2, x^3]$ is F-finite, as F_*R is generated over R by four elements by Example 1.13.

Indeed, all prime characteristic varieties, their local rings at any point, and the completions of those local rings are also F-finite:

Proposition 1.21. *(a) Any homomorphic image of an F-finite ring is F-finite. In particular, any finitely generated algebra over an F-finite field is F-finite.*
(b) Any Noetherian complete local ring of characteristic $p > 0$ with F-finite residue field is F-finite.
(c) A module finite extension of an F-finite ring is F-finite.
(d) Any localization of an F-finite ring is F-finite. In particular, rings essentially of finite type over an F-finite field are F-finite.
(e) The completion of a Noetherian F-finite ring at any ideal is F-finite. In particular, if $(R, \mathfrak{m})$ is an F-finite Noetherian local ring, then its $\mathfrak{m}$-adic completion $\widehat{R}$ is F-finite.

Proof of Proposition 1.21. Statements (a) holds since the images of generators of F_*R over R will generate $F_*\frac{R}{I}$ over $\frac{R}{I}$ for any ideal I; the second sentence follows immediately from Example 1.19. Likewise, (b) follows from (a) by using the Cohen Structure theorem to view the ring as a quotient of a formal power series over a field isomorphic to its residue field. Statement (c) follows from general properties about composition of finite maps, while (d) is immediate because Frobenius commutes with localization. For (e), note that a ring R is F-finite if and only if there is a surjective R-module map from a finitely generated free R-module

$$R^{\oplus d} \twoheadrightarrow F_*R.$$

Applying the completion functor $\varprojlim R/I^t \otimes_R -$ to this surjection, we see that $\widehat{F_*R}$ is a finitely generated $\widehat{R}$-module; see [**Mat89**, Theorem 8.1(ii)] or [**Sta19**, Tag 0315]. So by Lemma 1.14 also $F_*\widehat{R}$ is a finitely generated $\widehat{R}$-module—that is, the completion $\widehat{R}$ along I is F-finite. $\square$

On the other hand, it is not hard to find non-F-finite rings:

Example 1.22. A field of infinite transcendence degree over $\mathbb{F}_p$ is an example of a non-F-finite ring. For example, the field $K = \mathbb{F}_p(x_1, x_2, \dots)$, where we adjoin infinitely many variables, is not F-finite.

Some of the nice properties of F-finite rings (and schemes) are summarized below:

Theorem 1.23. *Let R be an F-finite Noetherian ring of prime characteristic.*

(a) The regular locus of R is open.
(b) The ring R is excellent.
(c) If R is reduced, then its normalization[2] is also F-finite.

[2] *i.e.*, integral closure in its total quotient ring

(d) *The ring R is a homomorphic image of a regular ring of finite Krull dimension, in particular R itself has finite Krull dimension.*
(e) *The ring R admits a dualizing complex and hence if it is locally equidimensional, it has a canonical module.*
(f) *If R is equidimensional, then it admits a canonical canonical module ω_R with the property that $F^!\omega_R \cong \omega_R$.*

PROOF. We will deduce (a) as a corollary of a theorem of Kunz in Corollary 2.3. For (b), see [**Kun76**]. Statement (c) follows from (b) using the commutative diagram involving the normalization map $R \xrightarrow{\nu} S$:

$$\begin{tikzcd} R \arrow[r, "F"] \arrow[d, "\nu"'] & F_*R \arrow[d, "F_*\nu"] \\ S \arrow[r, "F"'] & F_*S. \end{tikzcd}$$

Indeed, because R is excellent, the map ν, and hence $F_*\nu$, is finite; since the top arrow is finite, it follows that so is the bottom arrow. Statement (e) is due to Gabber; see [**Gab04**]; Gabber's construction immediately implies (f). □

Iterates of Frobenius are always finite in F-finite rings:

Proposition 1.24. *For a ring R (or a scheme X) of prime characteristic, the Frobenius map is finite if and only if the iterated Frobenius F^e is finite for some (equivalently, every) $e \in \mathbb{N}$.*

One more elementary property about F-finite rings that we use frequently is that the modules $\mathrm{Hom}_R(F^e_*R, R)$ commute with flat base:

Lemma 1.25. *Suppose R is an F-finite Noetherian ring. Then for any flat R-algebra S, there is a natural isomorphism*

$$\mathrm{Hom}_R(F^e_*R, R) \otimes_R S \cong \mathrm{Hom}_S(S \otimes_R F^e_*R, S).$$

In particular,

(a) $\mathrm{Hom}_R(F^e_*R, R) \otimes_R W^{-1}R \cong \mathrm{Hom}_{W^{-1}R}(F^e_*(W^{-1}R), W^{-1}R)$ *for any multiplicative system $W \subseteq R$; and*
(b) $\mathrm{Hom}_R(F^e_*R, R) \otimes_R \widehat{R} \cong \mathrm{Hom}_{\widehat{R}}(F^e_*\widehat{R}, \widehat{R})$ *where $\widehat{R}$ denotes the completion of R along any ideal.*

That is, the formation of the module $\mathrm{Hom}_R(F^e_*R, R)$ *commutes with localization and completion for F-finite Noetherian rings.*

PROOF. Our hypothesis on R ensures that for all $e \in \mathbb{N}$, the R-module $F^e_* R$ is *finitely presented.* Thus the statement follows from general results about flat base change in commutative rings [**Sta19**, Tag 087R]. For (b), we also invoke Corollary 1.15. □

1.9. Exercises.

Exercise 1.1. Prove Proposition 1.2. Use it to verify that there is a unique Frobenius map defined on an arbitrary scheme of characteristic p.

Exercise 1.2. Let k be any field of characteristic p. Find a minimal generating set for $F^e_* k[x_1, \dots, x_n]$ over $k[x_1, \dots, x_n]$, and prove that it is a *free basis.* If $[k : k^p] = d < \infty$, prove that the rank of both $F^e_* k[x_1, \dots, x_n]$ over $k[x_1, \dots, x_n]$ and $F^e_* k[\![x_1, \dots, x_n]\!]$ over $k[\![x_1, \dots, x_n]\!]$ is $d^e p^{ne}$.

Exercise 1.3. Let $R = \mathbb{F}_p[x^2, x^3]$, where $p > 2$. Find a minimal set of generators for the R-module $R^{1/p}$. Conclude that $F_* R$ is not free over R in any characteristic.

Exercise 1.4. Let $(R, \mathfrak{m}, k)$ be an F-finite Noetherian local ring of characteristic p. Prove that the minimal number of generators of the R-module $F^e_* R$ is

$$[k : k^p]^e \dim_k(R/\mathfrak{m}^{[p^e]}).$$

Exercise 1.5. To practice the (notoriously confusing!) notation associated with the Frobenius pushforward functor F^e_*, verify the following statements, where $F^e : R \to F^e_* R$ denotes the e-iterated Frobenius map:

(a) $F^2 = F_* F \circ F$ as R-module maps $R \longrightarrow F^2_* R$;
(b) $F^e = F_* F^{e-1} \circ F = F^{e-1}_* F \circ F^{e-1}$ for all $e > 0$ as maps $R \longrightarrow F^e_* R$;
(c) For any $c \in R$, if $\mu_c : R \xrightarrow{\text{mult by } c} R$ denotes the "multiplication by c" map, then $F^e_* \mu_c$ is the multiplication by $F^e_* c$ map in $\operatorname{Hom}_R(F^e_* R, F^e_* R)$.

Exercise 1.6. Let $k = \mathbb{F}_p(t_1, t_2, \dots)$ be a field generated over $\mathbb{F}_p$ by infinitely many indeterminates t_i, and let $R = k[x]$ by the polynomial ring over k.

(a) Show that $F_* R$ is not a finitely generated R-module.
(b) Show that the canonical map (of rings) $\widehat{R} \otimes F_* R \longrightarrow \widehat{F_* R}$ is not an isomorphism, where the notation $\widehat{M}$ denotes completion of the R module M at the maximal ideal (x).

Exercise 1.7. Suppose A is a ring of characteristic $p > 0$ and R is an A-algebra.

(a) In the case $A = \mathbb{F}_p$, show that the Frobenius map $F : R \longrightarrow R$ is a map of A-algebras.

(b) In the case that $A \neq \mathbb{F}_p$ is a field, construct an A-algebra R such that the Frobenius map $F : R \longrightarrow R$ is not an A-algebra map.

Exercise 1.8 (Relative Frobenius)**.** Suppose that A is a ring of characteristic $p > 0$ and R is an A-algebra. The previous exercise demonstrated that $F : R \longrightarrow R$ is not generally a map of A-algebras. Define a map of F_*A-algebras

$$F_{R/A} : R \otimes_A (F_*A) \longrightarrow F_*R$$

$$(r \otimes F_*a) \longmapsto F_*(r^p a).$$

This map is called the *relative Frobenius of R over A*. Show that it is a map of F_*A-algebras. Further show that for any A-algebra B, the base change map $F_{R/A} \otimes_{F_*A} F_*B$ is equal to $F_{R \otimes_A B/B}$. In other words, the relative Frobenius is compatible with base change.

Exercise 1.9. Consider $A = \mathbb{F}_p[t]$ and $R = A[x]/(x^p - t)$. Write down explicitly the relative Frobenius map $F_{R/A}$ and show that the source and target are not isomorphic as rings.

Exercise 1.10. Suppose A is a ring of characteristic $p > 0$ and R is an A-algebra. Describe explicitly the dotted vertical map that makes the following diagram commute:

$$\begin{tikzcd} R \arrow[r, "F"] \arrow[d, dotted] & F_*R \arrow[d, equal] \\ R \otimes_A (F_*A) \arrow[r, "F_{R/A}"'] & F_*R \end{tikzcd}$$

Suppose, furthermore, that A is perfect, in other words that the Frobenius map on A is an isomorphism. Prove in this case that the dotted arrow is an isomorphism of rings.

2. Frobenius and regularity

A major theme of this book is that the Frobenius map can detect singularities of a Noetherian ring R of characteristic p.

Already Frobenius detects the most basic restriction on singularities: it is easy to see that R is *reduced* if and only if its Frobenius map is *injective.* Put differently, R is reduced if and only if the R-module F_*R is *faithful.*

A much deeper fact, due to Ernst Kunz, is that Frobenius can detect whether or not a ring has any singularities at all:

Theorem 2.1. **[Kun69b, Kun69a]** *If R is a Noetherian ring of prime characteristic, then R is regular if and only if F_*R is a flat R-module.*

For a variety over a perfect field, or more generally any F-finite scheme of characteristic p, Kunz's theorem can be stated as follows:

Corollary 2.2. *A Noetherian F-finite scheme X is regular if and only if the coherent sheaf $F_*\mathcal{O}_X$ is locally free.*

PROOF. Corollary 2.2 follows immediately from the fact that a finitely generated flat module over a Noetherian ring is locally free [**Mat89**, Thm 7.10]. $\square$

Kunz's Theorem has the following important consequence:

Corollary 2.3. *The regular locus of a Noetherian F-finite scheme is open. That is, the locus of points $P \in X$ where $\mathcal{O}_{X,P}$ is regular is an open subset of X.*

PROOF. Because openness can be checked on an affine cover, the statement immediately reduces to the affine case. By Kunz's theorem, we know that R_P is regular if and only if $F_*R_P = (F_*R)_P$ is flat. Since F_*R is a finitely generated R-module, flatness and freeness of $(F_*R)_P$ over R_P are equivalent. So the locus of points in $\operatorname{Spec} R$ such that R_P is regular is the same as the locus of points in $\operatorname{Spec} R$ such that $(F_*R)_P$ is free over R_P. But the free locus for any coherent sheaf on any Noetherian scheme is open ([**Har77**, Ex II 5.7]). $\square$

Caution 2.4. There exist Noetherian schemes with non-open regular loci. The following example is due to Hochster [**Hoc73a**]. Fix any field k. Consider the subring R' of the polynomial ring over k in countably many variables $\{x_i \mid i \in \mathbb{N}\}$ generated by $\{x_i^2, x_i^3 \mid i \in \mathbb{N}\}$. The set

$$U = R' \setminus \bigcup_{i=1}^{\infty} (x_i^2, x_i^3)R'$$

is the complement of the union of (countably many) prime ideals, hence multiplicatively closed. The localization

$$R = U^{-1}R'$$

is a Noetherian domain whose maximal ideals $\mathfrak{m}_i$ are indexed by the x_i. Each localization $R_{\mathfrak{m}_i}$ is a one-dimensional domain of the form

$$L_i[x_i^2, x_i^3]_{(x_i^2, x_i^3)}$$

where L_i is the field of infinite transcendence degree over k generated by the variables x_j (where $j \neq i$). In particular, each $R_{\mathfrak{m}_i}$ is non-normal (and hence non-regular). Thus R is a one dimensional Noetherian domain whose regular locus consists only of the generic point $(0) \in \operatorname{Spec} R$. Since the proper closed sets of $\operatorname{Spec} R$ are finite, the regular locus of $\operatorname{Spec} R$ is not open.

We now turn to the proof of Kunz's Theorem, proving in fact the following superficially stronger result:

Theorem 2.5. *If R is a Noetherian ring of characteristic $p > 0$, then R is regular if and only if $F^e_* R$ is a flat R-module for some (equivalently, every) $e \in \mathbb{N}$.*

The rest of this section will be devoted to the proof of Theorem 2.5. We will present two proofs: the original, fairly elementary Kunz, and a recent more machinery-heavy proof of Bhatt and Scholze which provides new insight.

Proof of Theorem 2.5. Since both flatness and regularity can be checked locally, Theorem 2.5 immediately reduces to the local case. Furthermore, since a local Noetherian ring $(R, \mathfrak{m})$ is regular if and only if its completion $\widehat{R}$ is regular, the proof of Theorem 2.5 reduces to the complete local case by the following:

Lemma 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p. Then the Frobenius map for R is flat if and only if the Frobenius map for $\widehat{R}$ is flat. Likewise, the same holds for the composition F^e of Frobenius with itself e times, for any $e \in \mathbb{N}$.*

Proof. Assume first that $F^e_* R$ is flat over R. Completing at the maximal ideal of R, we have also that $\widehat{F^e_* R}$ is flat over $\widehat{R}$ [**Sta19**, Tag 06LD]. Thus $F^e_* \widehat{R}$ is flat over $\widehat{R}$ (Lemma 1.14).

Conversely, assume that $F^e_* \widehat{R}$ is flat over $\widehat{R}$. Consider the following diagram:

$$\begin{array}{ccc} \widehat{R} & \xrightarrow{F^e} & F^e_* \widehat{R} \\ \uparrow & & \uparrow \\ R & \xrightarrow[F^e]{} & F^e_* R \end{array}$$

The vertical arrows, both being completion, are faithfully flat (note that right vertical arrow is essentially just a renaming of the left arrow). Because we are assuming that the upper arrow is flat, and a composition of flat maps is flat, we know that the composition $R \to F^e_* R \to F^e_* \widehat{R}$ is flat. Thus, the bottom horizontal arrow $R \to F^e_* R$ is flat by [**Sta19**, Tag 039V]. $\square$

Continuing with the proof of Theorem 2.5, we now assume that $(R, \mathfrak{m})$ is a Noetherian complete local ring, whence the Cohen Structure Theorem[3]

[3]See [**Mat89**, Theorem 29.4(iii)] or [**Sta19**, Tag 032A].

allows us to write $R = k[\![x_1, \ldots, x_n]\!]/I$ where I is some ideal of the power series ring $k[\![x_1, \ldots, x_n]\!]$, and n is the embedding dimension of R.

First, assuming that R is regular, we may assume $R = k[\![x_1, \ldots, x_n]\!]$. To see that Frobenius is flat on R, we factor the Frobenius map as

$$k[\![x_1, \ldots, x_n]\!] \subseteq k[\![x_1^{1/p}, \ldots, x_n^{1/p}]\!] \subseteq k^{1/p} \otimes_k k[\![x_1^{1/p}, \ldots, x_n^{1/p}]\!] \subseteq k^{1/p}[\![x_1^{1/p}, \ldots, x_n^{1/p}]\!].$$

Now, the first extension is flat because the set

$$\{(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n})^{1/p} \mid 0 \leq a_i \leq p-1\}$$

is a free basis for for $k[\![x_1^{1/p}, \ldots, x_n^{1/p}]\!]$ over $k[\![x_1, \ldots, x_n]\!]$. And the second extension is flat because $k^{1/p}$ is a flat k-module and flatness is preserved by base change. The final extension is flat because it is completion (see [**Sta19**, Tag 0AGW] taking $M = k^{1/p} \otimes_k k[\![x_1^{1/p}, \ldots, x_n^{1/p}]\!]$ in their notation). Since a composition of flat maps is flat, we conclude that Frobenius (and all its iterates F^e) is flat for R.

For the converse, we assume that $F_*^e R$ is flat for some natural number e. We will give two different proofs that $R = k[\![x_1, \ldots, x_n]\!]/I$ is regular.

2.1. Kunz's proof that flatness of Frobenius implies regularity.

This proof is elementary but relies on explicitly understanding minimal generators of the maximal ideal. We make use of the following notion:

Definition 2.7. A set of elements $\{f_1, \ldots, f_s\}$ of a commutative ring is called **Lech independent** if whenever $a_1 f_1 + \ldots + a_s f_s = 0$ for some elements a_i in the ring, then each $a_i \in (f_1, \ldots, f_s)$.

The most obvious example of a Lech independent set is a set of minimal generators for the maximal ideal of a local Noetherian ring. Given a Lech independent set, the next results (whose proofs are left as exercises) let us construct others:

Lemma 2.8. *Let R be an arbitrary ring, and assume elements $f_1, \ldots, f_n \in R$ generate an ideal J with the property that J/J^2 is a free R/J-module of rank n. Then the images $\{\phi(f_1), \ldots, \phi(f_n)\}$ under any flat map $R \xrightarrow{\phi} S$ form a Lech independent set in S.*

In particular, if Frobenius is flat in a local ring R whose maximal ideal is minimally generated by $x_1, \ldots, x_n$, then $\{x_1^{p^e}, \ldots, x_n^{p^e}\}$ is a Lech independent set for all $e \in \mathbb{N}$..

Lemma 2.9. [**Lec64**, Lemma 3] *Let $\{f_1, \ldots, f_s\}$ be a Lech independent set in some ring, and assume that g_1 divides f_1. Then $\{g_1, f_2, \ldots, f_s\}$ is also Lech independent.*

In particular, *any* set of powers of the minimal generators for the maximal ideal of a local Noetherian ring in which Frobenius is flat form a Lech independent set. In other words, combining the two lemmas we have:

Lemma 2.10. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic p, and let $\{x_1, \ldots, x_n\}$ be a minimal set of generators for $\mathfrak{m}$. If some iterate of Frobenius on R is flat, then the set*

$$\{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\}$$

is Lech independent for any positive $\alpha_1, \ldots, \alpha_n$.

Next, we note that the lengths of modules obtained by killing Lech independent sets are controlled:

Lemma 2.11. [**Lec64**, Lemma 4] *Let $\{f_1, \ldots, f_s\}$ be a Lech independent set of an arbitrary ring, and suppose that $f_1 = g_1 h_1$. If $\ell_R(R/(f_1, \ldots, f_s))$ is finite, then*

$$\ell_R(R/(f_1, \ldots, f_s)) = \ell_R(R/(g_1, f_2, \ldots, f_s)) + \ell_R(R/(h_1, f_2, \ldots, f_s)),$$

where here $\ell_R(M)$ denotes the length[4] *of M as an R-module.*

Proof. This follows from the short exact sequence

$$0 \to R/(h_1, f_2, \ldots, f_s) \xrightarrow{\text{mult by } g_1} R/(f_1, \ldots, f_s) \to R/(g_1, f_2, \ldots, f_s) \longrightarrow 0,$$

where the non-zero mapping on the right is the natural quotient map, and the non-zero mapping on the left is multiplication by g_1. To see the latter is injective, let r represent an element in its kernel, so that $rg_1 \in (f_1, \ldots f_s)$. Rearranging, $g_1(r - a_1 h_1) \in (f_2, \ldots f_s)$, so that multiplying by h_1 and using the fact that $f_1, \ldots, f_s$ is Lech independent, we conclude that $r - a_1 h_1 \in (f_1, f_2, \ldots f_s) \subseteq (h_1, f_2, \ldots f_s)$. Thus $r \in (h_1, f_2, \ldots f_s)$, and the class of r is zero. $\square$

The next result follows immediately from Lemma 2.11 by induction on $\alpha_1 + \cdots + \alpha_n$:

Lemma 2.12. *Let $(R, \mathfrak{m})$ be a complete local ring of prime characteristic and assume that some power of the Frobenius map on R is flat. Let $x_1, \ldots, x_n$ be minimal generators for the maximal ideal $\mathfrak{m}$. Then*

$$\ell_R\left(\frac{R}{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})}\right) = \alpha_1 \cdot \alpha_2 \cdots \alpha_n$$

for all $\alpha_i \in \mathbb{N}$. In particular, $R/\mathfrak{m}^{[p^t]}$ has length p^{nt} for all $t \in \mathbb{N}$.

[4]For instance, if R is a local ring containing a field k isomorphic to its residue field (for instance, a complete local ring), then $\ell_R(M) = \dim_k(M)$. See [**Sta19**, Tag 00IU] for basics about length.

Now, to complete the proof of Kunz's theorem, suppose that some power of Frobenius is flat for the complete local ring $R \cong k[\![x_1, \ldots, x_n]\!]/I$. By Lemma 2.12, the quotient $R/(x_1^{p^t}, \ldots, x_n^{p^t})$ has dimension $(p^t)^n = p^{tn}$ over k for all t. But also

$$\dim_k k[\![x_1, \ldots, x_n]\!]/(x_1^{p^t}, \ldots, x_n^{p^t}) = p^{tn}$$

for all t, since the set $\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \mid 0 \leq a_i < p^t\}$ is a k-basis. So the quotient map

$$\frac{k[\![x_1, \ldots, x_n]\!]}{(x_1^{p^t}, \ldots, x_n^{p^t})} \twoheadrightarrow \frac{k[\![x_1, \ldots, x_n]\!]}{I + (x_1^{p^t}, \ldots, x_n^{p^t})}$$

must be an isomorphism for every t. This means that

$$I \subseteq \bigcap_t (x_1^{p^t}, \ldots, x_n^{p^t}) \subseteq \bigcap_t (x_1, \ldots, x_n)^{p^t}$$

in the power series ring $k[\![x_1, \ldots, x_n]\!]$, and so $I = 0$ by the Krull intersection theorem [**Sta19**, Tag 00IP]. We conclude that $R = k[\![x_1, \ldots, x_n]\!]$, so that R is regular, completing the proof of Kunz's theorem. □

Remark 2.13 (Hilbert-Kunz Multiplicity)**.** For a Noetherian local ring $(R, \mathfrak{m})$ of dimension d, the Hilbert-Kunz multiplicity is defined to be the limit:

$$e_{\mathrm{HK}}(R) = \lim_{e \longrightarrow \infty} \frac{\ell_R\big(R/\mathfrak{m}^{[p^e]}\big)}{p^{ed}}.$$

Notice that if F_*R is a flat R-module (that is, when R is regular), the argument above shows that $e_{\mathrm{HK}}(R) = 1$. We will see later that this limit exists in general, and indeed, is equal to 1 if and only if R is regular Chapter 9, or see [**WY00, HY02**].

2.2. Alternate proofs that flatness of Frobenius implies regularity.

There are several proofs of Kunz's Theorem; see, for example, [**Her74**] and [**MR10**, Theorem 4.4.2]. Of particular interest is a recent proof due to Bhatt and Scholze, which uses their result that the perfection of a complete local ring has finite global dimension [**BS17**]. We include it because it provides a different perspective on this material, though the machinery used here will not be needed later.

Definition 2.14. Let R be a ring of prime characteristic p. We say that R is **perfect** if the Frobenius map on R is an isomorphism. Likewise a scheme is perfect if the Frobenius map on its structure sheaf is an isomorphism.

Because Frobenius is injective if and only if R is reduced, we can also define a perfect ring as a reduced ring in which every element has a p-th root. This terminology agrees with the usual terminology for perfect fields. In particular, every perfect field— including every finite field and every algebraically closed field— is a perfect ring.

Example 2.15. The ring $\mathbb{F}_p[x, x^{1/p}, x^{1/p^2}, x^{1/p^3}, \dots]$ obtained by adjoining all p^e-th roots to $\mathbb{F}_p[x]$ is perfect.

Every ring of characteristic p admits a natural map to a perfect ring:

Definition 2.16. Let R be a ring of prime characteristic p. The **perfection** of R is the direct limit ring

$$R_{\text{perf}} = \varinjlim_F R = \varinjlim (R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} R \dots)$$

where each transition map F above is Frobenius.

Remark 2.17. The perfection R_{perf} of R is obviously a perfect ring, and there is obviously a natural ring homomorphism $R \to R_{\text{perf}}$. We can alternatively define a perfect ring as one such that the natural map $R \to R_{\text{perf}}$ is an isomorphism.

Remark 2.18. If R is reduced (for example, a domain), we can identify the perfection with the subring $\bigcup_{e \geq 0} R^{1/p^e}$ of the algebraic closure of the total quotient ring of R consisting of the p^e-th roots of every element of R. This ring is also denoted R^∞ or $R^{\frac{1}{p^\infty}}$ in the literature.

Example 2.19. The perfection of the ring $R = \mathbb{F}_p[x]$ is the perfect ring $R_{\text{perf}} = \mathbb{F}_p[x, x^{1/p}, x^{1/p^2}, x^{1/p^3}, \dots]$.

Remark 2.20. The perfection of R is rarely Noetherian, even when R is. Indeed, the only Noetherian perfect rings are finite products of perfect fields. On the other hand, it is easy to check that $\operatorname{Spec} R_{\text{perf}} \longrightarrow \operatorname{Spec} R$ is a homeomorphism of the underlying topological spaces. In particular, when R is a Noetherian local ring, its perfection has finite Krull dimension.

Remark 2.21. The perfection R_{perf} is always reduced even when R is not. Indeed, the Frobenius map on R_{perf} is always injective: if something is killed by Frobenius, it is also killed by some map defining the direct limit.

For any map of rings $R \to S$, there is a naturally induced compatible map of perfections $R_{\text{perf}} \to S_{\text{perf}}$. Similarly, if R is an arbitrary ring that admits a homomorphism to a perfect ring $R \to S_{\text{perf}}$, then this map must factor uniquely through the perfection of R:

$$R \to R_{\text{perf}} \to S_{\text{perf}}.$$

Remark 2.22. The perfection we have introduced is sometimes also called the *direct limit perfection* or *colimit perfection*. This is because there is another natural way to get a map from an arbitrary ring R of characteristic p to a perfect ring—namely, we get take the *inverse limit* of the Frobenius maps instead:

$$R^{\text{perf}} = \varprojlim_F R = \varprojlim (R \xleftarrow{F} R \xleftarrow{F} R \xleftarrow{F} \cdots).$$

The perfect ring R^{perf} is called the **inverse limit perfection**. For example, if $R = \mathbb{F}_p[x]$, then $R^{\mathrm{perf}} \cong \mathbb{F}_p$. Whereas the perfection adds p-th roots to get a perfect ring, the inverse limit perfection annihilates all elements that are not p-th powers to get a perfect ring. We will not need this construction here.

Bhatt and Scholze prove the following beautiful theorem:

Theorem 2.23. *Let R_{perf} be the perfection of a complete local Noetherian ring R of prime characteristic. Then R_{perf} has finite global dimension.*

A ring has **finite global dimension** if there is an upper bound on the projective dimension for all R-modules. Recall that for a Noetherian local ring, one of the many characterizations of regularity is precisely finite global dimension.[5]

Theorem 2.23 says that the perfection of a complete local ring, while not Noetherian, behaves homologically like a regular local ring.

To prove Theorem 2.23, we need the following:

Lemma 2.24. *Suppose*

$$R \xleftarrow{g} S \xrightarrow{h} R'$$

are surjections of Noetherian rings of characteristic $p > 0$ with induced surjections

$$R_{\mathrm{perf}} \xleftarrow{g} S_{\mathrm{perf}} \xrightarrow{h} R'_{\mathrm{perf}}$$

of perfect rings. Then $\mathrm{Tor}_i^{S_{\mathrm{perf}}}(R_{\mathrm{perf}}, R'_{\mathrm{perf}}) = 0$ for all $i \neq 0$; in other words, there is a isomorphism

$$R_{\mathrm{perf}} \otimes^{\mathbf{L}}_{S_{\mathrm{perf}}} R'_{\mathrm{perf}} \cong R_{\mathrm{perf}} \otimes_{S_{\mathrm{perf}}} R'_{\mathrm{perf}}$$

in the derived category.

Proof. The proof is carried out in a series of steps in the exercises; see Exercises 2.5, 2.6, 2.7, 2.8, 2.9. □

Proof of Theorem 2.23. Write $R = S/I$ for $S = k[\![x_1, \ldots, x_d]\!]$ and note that the induced map $S_{\mathrm{perf}} \longrightarrow R_{\mathrm{perf}}$ is also surjective.

Let M be an arbitrary R_{perf}-module. We will show that $\mathbf{R}\operatorname{Hom}_{R_{\mathrm{perf}}}(M, N)$ has cohomology in bounded degrees (independent of M and N). In other words, for all $i \gg 0$, $\mathrm{Ext}^i_{R_{\mathrm{perf}}}(M, N) = 0$.

[5]See [**Ser56, AB56**], or [**Sta19**, Tag 00OC].

We first show we may assume that $R = S$—that is, that it suffices to consider the power series ring. Indeed, notice that there are isomorphisms in the derived category:

$$M \cong M \otimes^{\mathbf{L}}_{R_{\mathrm{perf}}} R_{\mathrm{perf}} \cong M \otimes^{\mathbf{L}}_{R_{\mathrm{perf}}} (R_{\mathrm{perf}} \otimes^{\mathbf{L}}_{S_{\mathrm{perf}}} R_{\mathrm{perf}}) \cong M \otimes^{\mathbf{L}}_{S_{\mathrm{perf}}} R_{\mathrm{perf}}$$

by Lemma 2.24. Now, by the derived version of Hom-tensor adjointness, see [**Wei94**, Theorem 10.7.6] or [**Sta19**, Tag 0A65], there are natural isomorphisms

$$\begin{aligned} & \mathbf{R}\operatorname{Hom}_{R_{\mathrm{perf}}}(M, N) \\ \cong\; & \mathbf{R}\operatorname{Hom}_{R_{\mathrm{perf}}}(M \otimes^{\mathbf{L}}_{S_{\mathrm{perf}}} R_{\mathrm{perf}}, N) \\ \cong\; & \mathbf{R}\operatorname{Hom}_{S_{\mathrm{perf}}}(M, \mathbf{R}\operatorname{Hom}_{R_{\mathrm{perf}}}(R_{\mathrm{perf}}, N)) \\ \cong\; & \mathbf{R}\operatorname{Hom}_{S_{\mathrm{perf}}}(M, N). \end{aligned}$$

So it suffices to prove that the perfection S_{perf} of the power series ring S has finite global dimension. But note that when $S = k[\![x_1, \ldots, x_d]\!]$, the perfection

$$S_{\mathrm{perf}} = \varinjlim S^{1/p^e}$$

is a limit of the regular rings $k^{1/p^e}[\![x_1^{1/p^e}, \ldots, x_d^{1/p^e}]\!]$, each of which has global dimension d. So by a theorem of Israel Bernstein, the limit, S_{perf}, has global dimension at most $d+1$ [**Ber58**, Cor 1]. $\square$

Corollary 2.25. *Suppose R is a Noetherian ring of prime characteristic such that F_*R is a flat R-module. Then R is regular.*

Proof. The proof reduces to the complete local case by Lemma 2.6. In this case, it suffices to check that $(R, \mathfrak{m}, k)$ has finite global dimension by [**Ser56, AB56**], (or see [**Sta19**, Tag 00OC]). For this, it suffices to show that there is an $d \in \mathbb{N}$ such that for any finitely generated R-module M,

$$\operatorname{Tor}_n^R(M, k) = 0 \quad \text{for} \quad n > d.$$

We claim that we can take d to be the global dimension of R_{perf}, which is finite by Theorem 2.23 of Bhatt and Scholze. Indeed, assuming Frobenius is flat, we see that R_{perf} is a limit of flat R-modules, and so flat itself; moreover, R_{perf} is *faithfully* flat since $\mathfrak{m}R_{\mathrm{perf}} \neq R_{\mathrm{perf}}$ (note $\mathfrak{m}$ is a subset of the maximal ideal of R_{perf}). Thus it suffices to check that

$$R_{\mathrm{perf}} \otimes_R \operatorname{Tor}_n^R(M, k) = 0 \quad \text{for} \quad n > d.$$

By flatness of R_{perf}, this is the same as $\operatorname{Tor}_n^{R_{\mathrm{perf}}}(R_{\mathrm{perf}} \otimes_R M, R_{\mathrm{perf}} \otimes_R k) = 0$ for $n > d$, which is immediate by definition of global dimension d for R_{perf}. $\square$

2.3. Exercises.

Exercise 2.1. If $(R, \mathfrak{m}) \xrightarrow{\phi} (S, \mathfrak{n})$ is a local homomorphism of Noetherian local rings, then ϕ is flat if and only if the induced map on completions is $\widehat{R}^{\mathfrak{m}} \xrightarrow{\phi} \widehat{S}^{\mathfrak{n}}$ if flat. This generalizes the technique used to prove Proposition 2.6.

Exercise 2.2 (A relative form Kunz's theorem)**.** Let $f : X \to Y$ be a flat map of varieties over a field of prime characteristic. Then f is *smooth* if and only if the *relative Frobenius map* for X/Y (in the sense of Exercise 1.8) is flat,[6] in which case it is locally free of rank p^n where n is the relative dimension of X/Y.

Hint: The smoothness of f in this case is equivalent to regularity of the closed fibers. Use Exercise 1.8.

Exercise 2.3. Prove the two lemmas about Lech independence, Lemmas 2.8 and 2.9, used in Kunz's proof that flatness of Frobenius implies regularity.

Exercise 2.4. Suppose R is a perfect (but not necessarily Noetherian) ring of characteristic $p > 0$. Let $S = R[x_1, \dots, x_n]/I$ for some finitely generated ideal I. Show that F_*S is a finitely presented S-module.

Exercise 2.5. With notation as in Lemma 2.24, let $I = \ker g = (f_1, \dots, f_n)$ so that $R = S/I$. Show that

$$\ker g_{\mathrm{perf}} = \ker(S_{\mathrm{perf}} \longrightarrow R_{\mathrm{perf}}) = (f_1^{1/p^e}, \dots, f_n^{1/p^e})_{e \geq 0}.$$

Exercise 2.6. With notation as in the previous exercise, assume we can prove Lemma 2.24 in the case that I is generated by fewer than j elements. Set $I_j = (f_1^{1/p^e}, \dots, f_j^{1/p^e})_{e \geq 0} \subseteq S_{\mathrm{perf}}$ and fix $R_{\mathrm{perf},j} = S_{\mathrm{perf}}/I_j$. Show that

$$R_{\mathrm{perf},j} \otimes^{\mathbf{L}}_{S_{\mathrm{perf}}} R'_{\mathrm{perf}} \cong R_{\mathrm{perf},j} \otimes_{R_{\mathrm{perf},j-1}} (R_{\mathrm{perf},j-1} \otimes_{S_{\mathrm{perf}}} R'_{\mathrm{perf}})$$

Deduce that it suffices to prove the result in the case that $I = (f)$ is principal.

Exercise 2.7. With notation as in the previous exercise, consider the directed system

$$\{S_{\mathrm{perf}}, \cdot f^{\frac{p-1}{p^n}}\} = \left(S_{\mathrm{perf}} \xrightarrow{\cdot f^{\frac{p-1}{p^2}}} \cdots \xrightarrow{\cdot f^{\frac{p-1}{p^{n-1}}}} S_{\mathrm{perf}} \xrightarrow{\cdot f^{\frac{p-1}{p^n}}} S_{\mathrm{perf}} \xrightarrow{\cdot f^{\frac{p-1}{p^{n+1}}}} \cdots \right).$$

Verify that there is a map from this directed system to I_{perf} sending a (from the nth spot) to $f^{1/p^n}a$. Show that the induced map on the direct limit

$$\mu : \varinjlim \{S_{\mathrm{perf}}, \cdot f^{\frac{p-1}{p^n}}\} \longrightarrow I_{\mathrm{perf}}.$$

[6] More generally, a map of prime characteristic Noetherian rings $R \to S$ is *regular* (meaning flat with geometrically regular fibers) if and only if the relative Frobenius map $F_R \otimes_R S \to F_*S$ is flat. This is sometimes called the Radu-André Theorem; see [**MP20**, 10.1] for a nice proof.

is an isomorphism. Here, the notation $\cdot g$ means the map "multiplication by g."

Exercise 2.8. Likewise consider $I_{\text{perf}}R'_{\text{perf}} = \bigcup f^{1/p^e}R'_{\text{perf}}$, the ideal generated by the image of f^{1/p^e} in R'_{perf} and thus the direct system

$$\{R'_{\text{perf}}, \cdot f^{\frac{p-1}{p^n}}\} = R'_{\text{perf}} \xrightarrow{\cdot f^{\frac{p-1}{p^2}}} \cdots \xrightarrow{\cdot f^{\frac{p-1}{p^{n-1}}}} R'_{\text{perf}} \xrightarrow{\cdot f^{\frac{p-1}{p^n}}} R'_{\text{perf}} \xrightarrow{\cdot f^{\frac{p-1}{p^{n+1}}}} \cdots$$

As in the previous exercise, show that the induced map

$$\nu : \varinjlim\{R'_{\text{perf}}, \cdot f^{\frac{p-1}{p^n}}\} \longrightarrow I_{\text{perf}}R'_{\text{perf}}.$$

is an isomorphism.

Exercise 2.9. Using the previous exercises, show that

$$I_{\text{perf}} \otimes^{\mathbf{L}}_{S_{\text{perf}}} R'_{\text{perf}} \cong I_{\text{perf}}R'_{\text{perf}}.$$

Use this to deduce that $R_{\text{perf}} \otimes^{\mathbf{L}}_{S_{\text{perf}}} R'_{\text{perf}} \cong R_{\text{perf}} \otimes_{S_{\text{perf}}} R'_{\text{perf}}$ which proves Lemma 2.24.

3. Local Frobenius splitting

We have seen that the Frobenius map can be used to identify when a Noetherian scheme of prime characteristic is regular (or non-singular): regularity is equivalent to the flatness of Frobenius.

Our next goal is to relax the flatness assumption on Frobenius slightly. This leads to several classes of "F-singularities" that, as we will see, are robust classes of "mild" singularities with many nice properties.

In this section, we introduce the *local* property of prime characteristic rings called *Frobenius splitting.* Later in this chapter, we will introduce the related local notions of strongly F-regular, F-rational, F-pure, and F-injective singularities. Global versions of Frobenius splitting (and other F-singularities, at least in certain settings) can also be defined, but we caution the reader that global Frobenius splitting is typically quite a bit stronger than local Frobenius splitting in the non-affine case. All discussion of the global issues will be postponed until Chapter 3.

3.1. Frobenius splitting. Let R be a commutative ring of characteristic $p > 0$.

Definition 3.1. We say that R is **Frobenius split** (or simply F**-split**) if the Frobenius map $R \xrightarrow{F} F_*R$ splits in the category of R-modules. Explicitly, this means that there exists an R-module map $F_*R \xrightarrow{\pi} R$ such that the composition

$$R \xrightarrow{F} F_*R \xrightarrow{\pi} R$$

is the identity map for R.

Example 3.2. The polynomial ring $S = k[x_1, \ldots, x_d]$ over an arbitrary field of characteristic $p > 0$ is Frobenius split. For example, if k is perfect, then F_*S is a free S-module on the basis

$$\{F_*x_1^{a_1} \cdots x_d^{a_d} \mid 0 \leq a_i < p\},$$

so the S-module map $F_*S \xrightarrow{\pi} S$ defined by sending the basis element F_*1 to 1 and all other basis elements to zero is a splitting of Frobenius. The map π is sometimes called the *standard monomial* Frobenius splitting of S or the *toric* splitting of S (see also Example 1.4 in Chapter 3). See Exercise 3.3 for arbitrary k.

Proposition 3.3. *A Frobenius split ring is reduced.*

PROOF. Let R be a Frobenius split ring. Because the composition

$$R \xrightarrow{F} F_*R \xrightarrow{\pi} R$$

is the identity map, we see that the p-th power map F must be injective. This implies that R is reduced. □

Lemma 3.4. *The localization of a Frobenius split ring at any multiplicative set is Frobenius split.*

PROOF. Let $W \subseteq R$ be a multiplicative set. If $\pi \in \mathrm{Hom}_R(F_*R, R)$ is a splitting of Frobenius for R, then tensor the composition

$$R \xrightarrow{F} F_*R \xrightarrow{\pi} R$$

with $W^{-1}R$ to see that π induces a Frobenius splitting for $W^{-1}R$. □

The next proposition provides non-regular examples of Frobenius split rings:

Proposition 3.5. *Let $S \xrightarrow{\iota} R$ be any homomorphism of rings which splits as a map of S-modules. If R is Frobenius split, then so is S.*

PROOF. The point is that a composition of splittings is a splitting. Indeed, the hypothesis means that there is an S-linear map $R \xrightarrow{\phi} S$ such that $\phi \circ \iota$ is the identity map on S. Fix a Frobenius splitting $F_*R \xrightarrow{\pi} R$, and consider the composition

$$S \xrightarrow{F} F_*S \xrightarrow{F_*\iota} F_*R \xrightarrow{\pi} R \xrightarrow{\phi} S,$$

where the notation $F^*\iota$ denotes functor F_* applied to the S module map ι (see Subsection 1.4). All these maps are S-module maps (since any R-linear map is automatically linear over S), and the composition is the identity map on S. This means that the Frobenius map for S is split by the composition $\phi \circ \pi \circ F_*\iota$. □

Example 3.6. Any Veronese subring

$$k[x_1, \ldots, x_d]^{(n)} := k[\{\text{monomials of degree}\, n\}] \subseteq k[x_1, \ldots, x_d]$$

of a polynomial ring over a field of characteristic $p > 0$ is Frobenius split by Proposition 3.5. Indeed, Veronese subrings split off the polynomial ring (simply by mapping any homogenous element whose degree is a multiple of n to itself and killing all others).

Example 3.7 (Invariant Rings)**.** Let G be a finite group whose order is not divisible by p. Suppose that G acts on a ring R of characteristic p by ring automorphisms. If R is Frobenius split (e.g., a polynomial ring), then the ring of invariants

$$R^G := \{f \in R \mid g \cdot f = f \text{ for all } \ g \in G\}$$

is Frobenius split as well. This is because we have a splitting of $R^G \hookrightarrow R$ defined by "averaging the orbit" of each element r:

$$R \to R^G \qquad r \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot r.$$

Example 3.8. The homogeneous coordinate rings of Grassmannians $\mathbb{G}(d, n)$ of any size over an F-finite field are Frobenius split [**BK05**, Ch 2], as are generic determinantal rings [**HH94a**]. Likewise, *upper cluster algebras* are always Frobenius split when defined over an F-finite field, even when they they are not finitely generated [**BMRS15**].

As another application of the fact that a composition of split maps is split, we have the following characterizations of Frobenius split rings.

Proposition 3.9. *Let R be a commutative ring of prime characteristic p. Then the following are equivalent:*

(a) R is Frobenius split;
(b) For every $e > 0$, the iterated Frobenius $R \xrightarrow{F^e} F^e_* R$ defined by $r \mapsto F^e_* r^{p^e}$ splits;
(c) There exists $e > 0$ such that $R \xrightarrow{F^e} F^e_* R$ splits.

Proof. Because a composition of split ring maps is split, (a) easily implies (b), which obviously implies (c). It remains to show (c) implies (a). Assume that for some e, the R-module map $R \xrightarrow{F^e} F^e_* R$ splits, and let $F^e_* R \xrightarrow{\phi} R$ be splitting. This means that composition

$$R \xrightarrow{F^e} F^e_* R \xrightarrow{\phi} R$$

is the identity map on R. But now we can factor F^e as $F^{e-1} \circ F$, so that the identity map on R factors further as

$$R \xrightarrow{F} F_* R \xrightarrow{F_* F^{e-1}} F^e_* R \xrightarrow{\phi} R.$$

In particular, the composition $\phi \circ F_* F^{e-1}$ is a splitting of Frobenius. $\square$

Definition 3.10. Maps $\phi : F^e_* R \longrightarrow R$ can be viewed as Abelian group maps $\phi : R \longrightarrow R$ by forgetting the R-module structure on the source. In such a case they are called **p^{-e}-linear maps** as they satisfy the relation $\phi(r^{p^e} x) = r\phi(x)$. We will adopt this notation later in Chapter 8.

3.2. Examples of non Frobenius split rings. Frobenius splitting is a fairly restrictive condition. For example, the following lemma is useful in identifying rings that are *not* Frobenius split.

Lemma 3.11. *Let $R \to S$ be a ring homomorphism that splits in the category of R-modules. For every $I \subseteq R$, we have $IS \cap R = I$. In particular, for an ideal I in a Frobenius split ring, if $z^p \in I^{[p]}$, then*[7] *$z \in I$.*

Proof. Suppose that $z \in IS \cap R$, it suffices to prove that $z \in I$. Write $z = s_1 x_1 + \cdots + s_n x_n$ where $s_i \in S$ and $x_i \in I \subseteq R$. Let $\phi : S \to R$ be the splitting of the ring map $R \to S$. Applying ϕ to z we have

$$z = \phi(z) = \phi(s_1)x_1 + \cdots + \phi(s_n)x_n \in I.$$

So $IS \cap R \subseteq I$. The final statement follows from the first by considering the split homomorphism $R \xrightarrow{F} F_* R$. Note that $z \in IF_* R \cap R$ means that $z^p \in I^{[p]}$. $\square$

Example 3.12. Consider the subring $R = \mathbb{F}_p[x^2, x^3]$ of $\mathbb{F}_p[x]$, where p is any prime. The ring R is not Frobenius split. To see this, note that $x^3 \notin (x^2)$ in R. However, $x^{3p} = x^{2p} x^p \in (x^2)^{[p]}$. So $x^3 \in IF_* R \cap R$ but not in I for the ideal $I = (x^2)$ in R. By Lemma 3.11, the ring R is not Frobenius split.

Example 3.13. For $d \geq 4$ (and p not dividing d), the normal ring

$$R = \mathbb{F}_p[x, y, z]/(x^d + y^d + z^d)$$

is *not* Frobenius split. This follows from Lemma 3.11, as we will show that $(z^{d-1})^p \in (x^p, y^p)R$ but $z^{d-1} \notin (x, y)R$. Indeed, R is a free module over its subring $A = \mathbb{F}_p[x, y]$, with basis $1, z, \ldots, z^{d-1}$, so $z^{d-1} \notin (x, y)R$. On the

[7] The notation $I^{[p]}$ is defined in Notation 1.9.

other hand, $(z^{d-1})^p$ has degree $dp-p$, so we can write $(z^{d-1})^p$ as an A-linear combination

$$a_0 + a_1 z + \cdots + a_{d-1} z^{d-1},$$

where the coefficients a_i are polynomials in $\mathbb{F}_p[x, y]$ of degrees $(d-1)p - i$. We claim that each such $a_i \in (x^p, y^p)$, which of course implies $z^p \in (x^p, y^p)R$. To check this claim, it suffices to show that each monomial $x^\alpha y^\beta$ of degree at least $(d-1)p - (d-1)$ is in (x^p, y^p). But if not, we'd have $\alpha, \beta \leq p-1$ such that

$$(d-1)p - (d-1) \leq \alpha + \beta \leq 2p - 2;$$

rearranging,

$$(d-3)p \leq (d-3),$$

a contradiction. So R is not Frobenius split by Lemma 3.11. See also Exercise 3.10.

Remark 3.14. Frobenius splitting is closely related to ***F*-purity**, a slightly weaker condition equivalent to Frobenius splitting in the F-finite case; see Definition 7.18.

Remark 3.15. The converse of Lemma 3.11 holds under mild hypothesis, as well. Indeed, the property of a ring map $R \to S$ that *for every* $I \subseteq R$, $IS \cap R = I$ is often called **cyclic purity**. Hochster showed in [**Hoc77**] that cyclic purity is equivalent to *purity* for maps $R \to S$ whenever R is a local Noetherian ring which is *approximately Gorenstein*, a mild condition satisfied, for example, by any reduced excellent[8] local ring. On the other hand, purity of a finite extension of Noetherian rings is the same as splitting by Proposition 2.3 in Appendix A.

3.3. Frobenius splitting for F-finite rings.

Frobenius splitting is best behaved in the F-finite setting, where it can be thought of as a weakening of flatness of Frobenius (i.e. regularity):

Proposition 3.16. *An F-finite regular local ring is Frobenius split.*

PROOF. Suppose that $(R, \mathfrak{m})$ is an F-finite regular local ring. Then F_*R is both finitely generated and flat over the local ring R, hence *free.* Any minimal set of generators will be a free basis; we can find one[9] by lifting an $R/\mathfrak{m}$-basis for $F_*R/\mathfrak{m}F_*R = F_*(R/\mathfrak{m}^{[p]})$. In particular, since $1 \notin \mathfrak{m}^{[p]}$, we can take F_*1 to be among a free basis for F_*R over R. Any projection onto the R-submodule spanned by F_*1 is a Frobenius splitting. $\square$

[8] for example, F-finite, by [**Kun76**]

[9] by Nakayama's Lemma; [**Sta19**, Tag 07RC]

In F-finite Noetherian rings, Frobenius splitting is an open condition, which can be checked locally analytically[10]. This follows from the next two propositions:

Proposition 3.17. *For a Noetherian F-finite ring R of prime characteristic:*

(a) The locus of points Q in $\operatorname{Spec} R$ such that R_Q is Frobenius split is a Zariski open set of $\operatorname{Spec} R$.
(b) R is Frobenius split if and only if for every prime (or equivalently, maximal) ideal $P \in \operatorname{Spec} R$, the local ring R_P is Frobenius split.

Proposition 3.18. *Suppose R is a Noetherian F-finite ring. If R is F-split, then the completion of R at any ideal is also F-split. Furthermore, if $(R, \mathfrak{m})$ local, then R is Frobenius split if and only if its completion $\widehat{R}$ at the maximal ideal is Frobenius split.*

Both propositions follow from the following general lemma:

Lemma 3.19. *Let R be an arbitrary commutative ring, and $R \xrightarrow{\iota} M$ be any R-module homomorphism.*

(a) The map ι splits if and only if the natural R-module map

$$\operatorname{Hom}_R(M, R) \xrightarrow{\Psi} R \qquad \phi \mapsto \phi(\iota(1)).$$

is surjective.

(b) If M is finitely presented, then the locus of points

$$\{P \in \operatorname{Spec} R \mid \iota_P : R_P \to M_P \text{ splits}\}$$

where ι splits is the open set complementary to the closed set $\mathbb{V}(\operatorname{im} \Psi)$ of prime ideals containing the image of Ψ.

Proof of Lemma 3.19. For (a), observe that to find a splitting of $R \xrightarrow{\iota} M$ is to find an R-module map $\phi \in \operatorname{Hom}_R(M, R)$ such that $\phi(\iota(1)) = 1$. Clearly this is equivalent to the surjectivity of Ψ.

For (b), observe that (a) implies that the R-module $\frac{R}{\operatorname{im} \Psi}$ is zero if and only if the map $R \xrightarrow{\iota} M$ splits. Now if M is finitely presented, then there is a natural isomorphism[11]

$$\operatorname{Hom}_{R_P}(M_P, R_P) = R_P \otimes_R \operatorname{Hom}_R(M, R), \tag{3.19.1}$$

so that $R_P \otimes_R \frac{R}{\operatorname{im}\Psi} \cong \frac{R_P}{\operatorname{im}\Psi_P}$. So ι splits locally at P if and only if P is not in the support of $\frac{R}{\operatorname{im}\Psi}$, that is, if and only if $P \not\supset \operatorname{im} \Psi$. $\square$

[10]**Analytically** means "up to completion” in this context.
[11][**Sta19**, Tag 0583]

Proofs of Proposition 3.17 and Proposition 3.18 . Both propositions follow from Lemma 3.19, taking ι to be the Frobenius map $R \to F_*R$. In particular, the closed set defining the non-Frobenius split locus is $\mathbb{V}(\operatorname{im}\Psi) \subseteq \operatorname{Spec} R$ where Ψ is the "evaluation at F_*1" map

$$\operatorname{Hom}_R(F_*R, R) \xrightarrow{\Psi} R \qquad \phi \mapsto \phi(F_*1). \tag{3.19.2}$$

For the statements about completion, we see that (3.19.2) remains surjective after tensoring with $\widehat{R}$. Since $\widehat{R}$ is flat:

$$\operatorname{Hom}_R(F_*R, R) \otimes_R \widehat{R} \cong \operatorname{Hom}_{\widehat{R}}(F_*R \otimes \widehat{R}, \widehat{R}) \cong \operatorname{Hom}_{\widehat{R}}(F_*\widehat{R}, \widehat{R}),$$

where the first isomorphism is from Lemma 1.25 and the second from Corollary 1.15. The map $\operatorname{Hom}_{\widehat{R}}(F_*\widehat{R}, \widehat{R}) \longrightarrow \widehat{R}$ induced via this isomorphism and tensor product is still evaluation-at-1. This proves the first part of Proposition 3.18.

In the second part, completion at $\mathfrak{m}$ is faithfully flat, and so the map (3.19.2) is surjective if and only if it is surjective after tensoring with $\widehat{R}$. $\square$

Remark 3.20. From the proof, we see that Proposition 3.17 holds when R is not Noetherian, provided F_*R is a *finitely presented* R-module.

Caution 3.21. Propositions 3.17 and Proposition 3.18 can fail for non-F-finite rings. For example, Caution 2.4 constructed a one-dimensional Noetherian domain R with infinitely many maximal ideals whose Frobenius split locus consists only of the generic point, showing that 3.17(a) need not hold. And in [**DS18**, §3], examples of discrete valuation rings in the field $\overline{\mathbb{F}_p}(x, y)$ are constructed which are not Frobenius split; their completions, however, are power series rings over $\overline{\mathbb{F}_p}$, hence always Frobenius split, showing that Proposition 3.18 can fail. Those discrete valuation rings, however, are not F-finite.

Caution 3.22. For a non-F-finite ring R, the module $\operatorname{Hom}_R(F_*R, R)$ can be the zero module, even for relatively nice R. Clearly in this case, R has no hope to be Frobenius split. This can happen even if R is a one-dimensional regular local ring [**DS18**, §4] and even for the Tate algebra, which is an *excellent* regular ring [**DM20b**].

Remark 3.23. As Caution 3.21 and Caution 3.22 indicate, Frobenius splitting is not always well behaved in the non-F-finite setting. The property of *F-purity* is a often good substitute; see Subsection 7.6.

3.4. Local Frobenius splitting for schemes.

It is natural to extend the local condition of Frobenius splitting at each point to schemes:

Definition 3.24. Let X be a scheme of characteristic p. We say that X is **Frobenius split** at $x \in X$ if the local ring $\mathcal{O}_{X,x}$ (the stalk at x) is Frobenius split. We say X is **locally Frobenius split** (or simply locally F-split) if X is Frobenius split at every point.

Example 3.25. Any F-finite regular scheme is locally Frobenius split by Proposition 3.16. In particular, smooth varieties (over an algebraically closed field) of prime characteristic are always locally Frobenius split.

Caution 3.26. Splitting of the Frobenius map $\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X$ at each point $x \in X$ is a *much weaker* condition, for general X, than the splitting of Frobenius map $\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X$ in the category of $\mathcal{O}_X$-modules on X. The latter condition is called **global** Frobenius splitting, and will be discussed in detail in Chapter 3. Unlike flatness or smoothness, *splitting a map of sheaves on a scheme is not a local condition.*

Summarizing previously proved facts in the context of schemes, we have the following:

Corollary 3.27. *(C.f.Proposition 3.17(a)) Let X be a Noetherian F-finite scheme of prime characteristic. The locus of points of X at which X is locally Frobenius split is an open subset of X.*

Corollary 3.28. *(C.f.Proposition 3.17(b), Proposition 3.9) Let X be a Noetherian F-finite scheme of prime characteristic. The following are equivalent:*

(a) X is locally Frobenius split at every point;
(b) X has an open affine cover by spectra of Frobenius split rings;
(c) For every open affine set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is Frobenius split;
(d) The iterated Frobenius map $\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X$ splits at each point $x \in X$ for some (equivalently, every) $e \in \mathbb{N}$.

3.5. An application to lifting rings modulo p^2. The following is an interesting application of Frobenius splitting, but will not be used later.

Definition 3.29. Suppose $R = \mathbb{F}_p[x_1, \ldots, x_d]/I$ is a finite type $\mathbb{F}_p = \mathbb{Z}/(p)$ algebra. A **lift of R modulo p^2** is a finite type $\mathbb{Z}/(p^2)$ algebra R' such that:

(a) R' is flat over $\mathbb{Z}/(p^2)$ and,
(b) $R'/(p) \cong R$.

Not all finite type rings of characteristic p admit liftings modulo p^2, although it is not difficult to see that complete intersections do; see Exercise 3.14. The point of this section is to show that *Frobenius split rings always lift modulo p^2.* In fact, following [**Zda18**, Definition 3.15, Theorem 3.16], we explicitly construct a lifting to $\mathbb{Z}/(p^2)$ of R from a Frobenius splitting of R.

Fix a ring R as in Definition 3.29, and suppose that $\phi : F_*R \longrightarrow R$ is a fixed splitting of Frobenius. Consider the set $R' = R \times R$. Define a multiplication on R' by

$$(a_0, a_1) \cdot (b_0, b_1) = (a_0 b_0, a_1 b_0 + a_0 b_1), \tag{3.29.1}$$

and an addition by

$$(a_0, a_1) + (b_0, b_1) = \Big(a_0 + b_0, a_1 + b_1 - \phi\big(F_* \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} a_0^i b_0^{p-i}\big)\Big). \tag{3.29.2}$$

Here each coefficient $\frac{(p-1)!}{i!(p-i)!}$ is an integer which might be better thought of as the quotient of the binomial coefficient $\binom{p}{i}$ (which is always divisible by p for $0 < i < p$) by p. In particular, the term $\phi\big(F_* \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} a_0^i b_0^{p-i}\big)$ in (3.29.2) can be interpreted, informally, as

$$\frac{\big(a_0 + b_0\big)^p - a_0^p - b_0^p}{p}. \tag{3.29.3}$$

We claim that with these formulas for addition and multiplication, the set R' becomes a commutative ring with additive identity $(0, 0)$ and multiplicative identity $(1, 0)$. There are various properties to check, most of which we leave to the reader. For example, we verify distributivity:

$$\begin{array}{rl} & (a_0, a_1)\big((b_0, b_1) + (c_0, c_1)\big) \\ = & (a_0, a_1)\Big(b_0 + c_0, b_1 + c_1 - \phi\big(F_* \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} b_0^i c_0^{p-i}\big)\Big) \\ = & \Big(a_0 b_0 + a_0 c_0, a_1(b_0 + c_0) + a_0 b_1 + a_0 c_1 - a_0 \phi\big(F_* \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} b_0^i c_0^{p-i}\big)\Big) \\ = & \Big(a_0 b_0 + a_0 c_0, a_1 b_0 + a_0 b_1 + a_1 c_0 + a_0 c_1 - \phi\big(F_* \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} (a_0^i b_0^i)(a_0^{p-i} c_0^{p-i})\big)\Big) \\ = & \Big(a_0 b_0, a_1 b_0 + a_0 b_1\Big) + \Big(a_0 c_0, a_1 c_0 + a_0 c_1\Big) \\ = & (a_0, a_1)(b_0, b_1) + (a_0, a_1)(c_0, c_1). \end{array}$$

Notice the fact that $a\phi(F_* r) = \phi(F_* a^p r)$ for all $a, r \in R$ was key. Associativity is left to the reader in Exercise 3.11.

Since $\phi(F_* n) = n$ for all $n \in \mathbb{F}_p$, the reader can check that

$$\sum_{j=1}^{p} (1, 0) = (0, 1).$$

in R'; see Exercise 3.12. In particular, the element $(0, 1)$ is the image of the integer p under the canonical ring map $\mathbb{Z} \to R'$, and the second factor $\{(0, z) \mid z \in R\} \subseteq R'$ corresponds to the ideal of R' generated by p. In particular, $R'/(p) = R$. Furthermore,

$$\sum_{j=1}^{p} (0, 1) = (0, 0)$$

in R', so that $p^2 = (0,0)$ and R' contains $\mathbb{Z}/(p^2)$ as a subring.

To see that R' is flat over $\mathbb{Z}/p^2\mathbb{Z}$, it suffices to check that the natural multiplication map $p\mathbb{Z}/p^2\mathbb{Z} \otimes R' \longrightarrow R'$ is injective [**Sta19**, Tag 00ML]. But $p\mathbb{Z}/p^2\mathbb{Z} \otimes R' \longrightarrow R'$ maps bijectively onto $\{(0,t) \mid t \in R\}$, which is a subset of R'.

Putting this together, we see that:

Theorem 3.30 ([**Zda18**, Definition 3.15, Theorem 3.16])**.** *The set R', with addition and multiplication defined above, is a lift of R to $\mathbb{Z}/(p^2)$.*

Remark 3.31 (Witt vectors)**.** More generally, if k is an arbitrary perfect field, a lift of a finite type k-algebra is a $W_2(k)$-algebra, where $W_2(k)$ is the ring of second Witt vectors, satisfying the corresponding conditions (a) and (b). When $k = \mathbb{F}_p$, $W_2(\mathbb{F}_p) \cong \mathbb{Z}/(p^2)$. The proof of Theorem 3.30 carries over to this case with no essential change; see [**Zda18**, Theorem 3.16].

Remark 3.32 (Global versions)**.** A global nonconstructive version of Theorem 3.30 was known earlier. Indeed, it follows from [**Sri90**, Introduction, (iii)], *cf.* [**MS87**, Appendix], that if a smooth variety X is *globally* Frobenius split[12], then X has a lift to $\mathbb{Z}/(p^2)$, meaning that there exists a scheme X' flat over $\mathbb{Z}/(p^2)$, such that $X' \times_{\mathbb{Z}/(p^2)} \mathbb{F}_p = X$. For related results in the singular setting, see [**Bha14**].

3.6. Exercises.

Exercise 3.1. Prove the following are equivalent for a ring R of characteristic $p > 0$:

(a) The ring R is Frobenius split.
(b) There exists $\phi \in \mathrm{Hom}_R(F_*R, R)$ such that $\phi(F_*1) = 1$.
(c) There is a surjective map of R-modules $\pi : F_*R \longrightarrow R$.
(d) There is an R-module isomorphism $F_*R \cong R \oplus M$ for some R-module M.
(e) The "evaluation at 1" map

$$\mathrm{Hom}_R(F_*R, R) \to R \qquad \phi \mapsto \phi(F_*1)$$

is *surjective*.

Exercise 3.2. Show that a Noetherian zero dimensional Frobenius split ring is a product of fields.

Exercise 3.3. Let k be an arbitrary field of characteristic $p > 0$. Show that the polynomial ring $k[x_1, \dots, x_n]$ is Frobenius split.

[12] Meaning that $\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$ splits, see Chapter 3.

Exercise 3.4. Show that the non-Frobenius split locus of an F-finite Noetherian normal ring has codimension at least two.

Exercise 3.5. Show that $k[\![x_1, \ldots, x_d]\!]$ is Frobenius split, where k is an arbitrary field of characteristic p.

Exercise 3.6. Show that the ring $R = \mathbb{F}_p[x, y, z]/(xyz)$ is Frobenius split.

Exercise 3.7. Suppose R is a reduced Noetherian ring. The localization map to the total ring of fractions $R \longrightarrow \mathcal{K}(R)$ induces

$$\nu_R : \mathrm{Hom}_R(F_* R, R) \longrightarrow \mathrm{Hom}_R(F_* \mathcal{K}(R), \mathcal{K}(R))$$

Now consider the ring $R = \mathbb{F}_p[x^2, x^3]$ with normalization $R' = \mathbb{F}_p[x]$, so R and R' have the same field of fractions. Find an explicit map $\phi \in \mathrm{Image}(\nu_{R'})$ that is not in the image of ν_R.

Hint: Can there be any map $F_* R \longrightarrow R$ that sends $F_* 1 \mapsto 1$?

Exercise 3.8. Suppose that k is a field of characteristic $p > 0$ and $x \in k$ is an element without a pth root. Let $L = k(x^{1/p})$. Show that if a k-linear map $\phi : F_* k \longrightarrow k$ extends to an L-linear map $\phi_L : F_* L \longrightarrow L$, then ϕ is the zero map.

Exercise 3.9. Consider the ring $R = \mathbb{F}_p[x, y, z]/(xy - z^2)$ where $p \neq 2$. Show that there is a Frobenius splitting $F_* R \longrightarrow R$ that sends the ideal $F_*(x, z)$ to (x, z). Show that this splitting then induces a Frobenius splitting of $R/(x, z)$.

Hint: It might help to realize that $R \cong \mathbb{F}_p[a^2, b^2, ab] \subseteq \mathbb{F}_p[a, b]$.

Exercise 3.10. Let $f \in k[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d > n$ over an an F-finite field k. Show that the quotient $k[x_1, \ldots, x_n]/(f)$ is not Frobenius split.

Hint: Change coordinates so that $f = x_1^d + a_1 x^{d-1} + \cdots + a_d$ where a_i is homogeneous of degree i in $x_2, \ldots, x_n$; now use the method of Example 3.13.

Exercise 3.11. Suppose R' is the set with binary operations multiplication and addition defined as in (3.29.1) and (3.29.2). Show that these operations are commutative and associative, proving R' is in fact a commutative ring.

Exercise 3.12. Suppose R is a Frobenius split ring of characteristic $p > 0$ with Frobenius splitting $\phi : F_* R \longrightarrow R$. Show that in the ring R' constructed in Subsection 3.5 has the property that

$$p = \sum_{j=1}^{p} (1, 0) = (0, 1).$$

Hint:

Exercise 3.13 ([**Zda18**, Theorem 3.16])**.** Suppose $g : R \longrightarrow S$ is a map of Frobenius split rings of finite type over $\mathbb{F}_p = \mathbb{Z}/(p)$. Further suppose that $\phi_R : F_*R \longrightarrow R$ and $\phi_S : F_S \longrightarrow S$ are compatible splittings of Frobenius in the sense that the following diagram commutes:

$$\begin{tikzcd} F_*R \arrow[r, "\phi_R"] \arrow[d, "F_*g"'] & R \arrow[d, "g"] \\ F_*S \arrow[r, "\phi_S"'] & S \end{tikzcd}$$

Show that there is an induced map $R' \longrightarrow S'$ where R' and S' are lifts of R and S modulo p^2 constructed as in Subsection 3.5.

Exercise 3.14.

4. Frobenius splitting along elements and strong F-regularity

Strong F-regularity is a stronger form of Frobenius splitting. Strongly F-regular rings, as we will see, have *many* splittings of Frobenius—they are eventually Frobenius split along every non-zerodivisor.

4.1. Frobenius splitting along a non-zerodivisor.

Frobenius splittings are R-linear maps (in $\mathrm{Hom}_R(F_*R, R)$) sending F_*1 to 1. It turns out to be useful to consider R-linear maps sending other elements F^e_*c to 1.

Definition 4.1. Let R be a ring of prime characteristic, and let $c \in R$. We say that F^e **splits along** c or that R is e-**Frobenius split along** c if the R-module map

$$R \to F^e_*R \quad \text{determined by} \quad 1 \mapsto F^e_*c \tag{4.1.1}$$

splits in the category of R-modules. We say that R is **eventually Frobenius split along** c if there exists an $e > 0$ such that F^e splits along c.

Equivalently, R is e-Frobenius split along c if there exists $e > 0$ and $\phi \in \mathrm{Hom}_R(F^e_*R, R)$ such that $\phi(F^e_*c) = 1$. Note that 1-Frobenius splitting along 1 is simply Frobenius splitting.

Example 4.2. A ring of prime characteristic is eventually Frobenius split along 1 if and only if it is Frobenius split, by Proposition 3.9.

Caution 4.3. The map (4.1.1) is *not* the Frobenius map, nor indeed, any ring map (unless $c = 1$). Rather it is the R-module map obtained as the composition

$$
\begin{aligned}
R &\xrightarrow{F^e} F^e_*R \xrightarrow{\text{mult by } F^e_*c} F^e_*R \\
r &\longmapsto F^e_*r^{p^e} \longmapsto F^e_*cr^{p^e}.
\end{aligned}
\tag{4.3.1}
$$

Of course, being R-linear, the map (4.1.1) is determined by where it sends 1. Because we've specified that 1 goes to F^e_*c, it follows that each r must be sent to $rF^e_*c = F^e_*r^{p^e}c$.

Example 4.4. An F-finite *regular* local ring is eventually Frobenius split along *every non-zero* element. Indeed, let $(R, \mathfrak{m})$ be such a ring, and $c \in R$ any non-zero element. Choose $e \gg 0$ so that $c \notin \mathfrak{m}^{[p^e]}$ (see Exercise 4.1). Then the image of F^e_*c in $F^e_*R/\mathfrak{m}F^e_*R = F^e_*(R/\mathfrak{m}^{[p^e]})$ is non-zero, so that by Nakayama's Lemma, F^e_*c is part of a minimal generating set for the free R-module F^e_*R. So the R-submodule of F^e_*R generated by F^e_*c splits off F^e_*R.

Eventual Frobenius splitting along c places restrictions on both the ring and the element c, as the next two results show:

Proposition 4.5. *If R is eventually Frobenius split along c, then c is not a zero divisor.*

Proof of Proposition 4.5. Suppose that $rc = 0$ for some $r \neq 0$. Then the map $R \xrightarrow{\iota} F^e_*R$ sending $1 \mapsto F^e_*c$ is not injective: $\iota(r) = rF^e_*c = F^e_*r^{p^e}c = F^e_*\, 0 = 0$. So ι can not split: there is no $\phi \in \mathrm{Hom}_R(F^e_*R, R)$ such that $\phi \circ \iota$ is the identity. $\square$

Proposition 4.6. *If R is e-Frobenius split along a product cd, for some elements c and d in R, then R is e-Frobenius split along both c and d. In particular, if R is eventually Frobenius split along some $c \in R$, then R is Frobenius split.*

Proof of Proposition 4.6. Assume that the R-module map

$$R \to F^e_*R \qquad 1 \mapsto F^e_*cd$$

splits. Fix $\phi \in \mathrm{Hom}_R(F^e_*R, R)$ such that $\phi(F^e_*cd) = 1$. Consider the composition

$$F^e_*R \xrightarrow{F^e_*d} F^e_*R \xrightarrow{\phi} R$$

where the first map is multiplication by F^e_*d. It is easy to check this composition is an R-module map sending F^e_*c to $\phi(F^e_*cd) = 1$. That is, R is e-Frobenius split along c.

For the second statement, consider c as the product of c and 1. Eventual Frobenius splitting along c then implies eventual Frobenius splitting along 1, so R is Frobenius split by Proposition 3.9. $\square$

One splitting often means many more, as the next two results show:

Lemma 4.7. *Let R be a ring of prime characteristic p. Assume that R-module map*

$$R \xrightarrow{\iota_e} F^e_* R \qquad 1 \mapsto F^e_* c$$

splits in the category of R-modules for some $e = e_0$. Then it splits for all $e \geq e_0$ as well.

Proof of Lemma 4.7. Because R is eventually Frobenius split along c, we know R is Frobenius split by Proposition 4.6. Let $\pi : F_* R \longrightarrow R$ be a Frobenius splitting, and let $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ be a splitting of the map ι_e. Then the composition

$$R \xrightarrow{F} F_* R \xrightarrow{F_* \iota_e} F^{e+1}_* R \qquad 1 \mapsto F^{e+1}_* c$$

is split by $\pi \circ F_* \phi$, as one easily verifies. That is, ι_{e+1} splits as a map of R-modules as well. By induction, we have the desired splittings for all larger e as well. $\square$

Caution 4.8. If F^e splits along c, it is not usually the case that $F^{e'}$ splits along c for $e' < e$, although we saw that this *is* the case when $c = 1$ (Proposition 3.9). For example, if $c \in \mathfrak{m}^{[p]}$ in a local ring $(R, \mathfrak{m})$, then $F_* c \in F_* \mathfrak{m}^{[p]} = \mathfrak{m} F_* R$, so that F never splits along c, though a higher iterate F^e may; see Example 4.4.

The converse of Proposition 4.6 also holds:

Proposition 4.9. *Suppose that R is e-Frobenius split along c and f-Frobenius split along d. Then R is $(e+f)$-Frobenius split along $c^{p^e} d$.*

In particular, R is eventually split along cd if and only if it is eventually Frobenius split along both c and d.

Proof. Our hypothesis implies that the R-module maps

$$R \xrightarrow{\iota_c} F^e_* R \qquad \text{and} \qquad R \xrightarrow{\iota_d} F^f_* R$$

sending 1 to $F^e_* c$ and $F^f_* d$, respectively, are both split. Then the composition of split maps

$$\begin{array}{ccccc} R & \xrightarrow{\iota_c} & F^e_* R & \xrightarrow{F^e_* \iota_d} & F^{e+f}_* R \\ 1 & \longmapsto & F^e_* c & \longmapsto & F^{e+f}_* c^{p^e} d \end{array} \tag{4.9.1}$$

is split as well, proving the first sentence. The second follows from Proposition 4.6. $\square$

Remark 4.10. With notation as in the proof of Proposition 4.9, we can describe the splitting along $c^{p^e}d$ explicitly as

$$\phi \circ F^e_* \psi \in \mathrm{Hom}_R(F^{e+f}_* R, R)$$

where $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ and $\psi \in \mathrm{Hom}_R(F^f_* R, R)$ are eventual Frobenius splittings along c and d, respectively.

The following form of Proposition 4.9 is especially useful:

Corollary 4.11. *Suppose that R is e-Frobenius split along c. Then R is ne-Frobenius split along $c^{1+p^e+p^{2e}+\cdots p^{e(n-1)}}$ for every integer $n > 0$.*

In particular, if R is eventually Frobenius split along c, then R is eventually Frobenius split along c^m for every integer $m > 0$.

Remark 4.12. The idea of "Frobenius splitting along a divisor" was first used in [**RR85**] although it was not named such until [**Ram91**]. The extension to iterates of Frobenius and "eventual" splitting along c were introduced in [**Smi00a**] (although the latter is called "stable Frobenius splitting along c" there).

4.2. The Cartier Algebra. We digress to formalize a powerful conceptual tool that has so far been mostly implicit.

Take $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ and $\psi \in \mathrm{Hom}_R(F^d_* R, R)$. As we have seen, it can be useful to "compose" these maps, by which we mean the composition[13]

$$F^{e+d}_* R \xrightarrow{F^e_*\psi} F^e_* R \xrightarrow{\phi} R, \tag{4.12.1}$$

or equivalently, the R-linear map

$$\phi \star \psi := \phi \circ F^e_* \psi \in \mathrm{Hom}_R(F^{d+e}_* R, R), \tag{4.12.2}$$

defined by

$$\phi \star \psi (F^{e+d}_* r) = \phi(F^e_*(\psi(F^d_* r))$$

for all $r \in R$. The map $\phi \star \psi$ is literally composition of ϕ and ψ, if we remember that as a set, $F^e_* R$ is simply R.

The composition $\phi \star \psi$ defines a natural multiplication on the abelian group

$$\mathcal{C}_R = \bigoplus_{e \in \mathbb{N}} \mathrm{Hom}_R(F^e_* R, R), \tag{4.12.3}$$

[13]Here the notation $F^e_*\psi$ denotes the functor F^e_* applied to the map ψ as discussed in Subsection 1.4 of Chapter 1.

making it into graded (non-commutative) ring. Note that although R sits inside $\mathcal{C}_R$ as the subring of degree zero elements, $\mathcal{C}_R$ is *not an R-algebra* because R is *not central*:

$$r \star \phi = \phi \star r^{p^e} \tag{4.12.4}$$

for $r \in R$ and $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ in $\mathcal{C}_R$. The ring $\mathcal{C}_R$ is called the **(full) Cartier Algebra**; we will revisit it and some generalizations in depth in Chapter 8.

The Cartier Algebra point of view often simplifies notation and adds insight. For example, in the proof of Corollary 4.11, if we let ϕ denote a splitting of the map

$$R \xrightarrow{\iota} F^e_* R \quad \text{sending} \quad 1 \mapsto F^e_* c,$$

then the composition[14]

$$\phi^{\star n} := \phi \star \phi \star \cdots \phi \star \phi \in \operatorname{Hom}_R(F^{ne}_* R, R) \tag{4.12.5}$$

is a splitting of the composition

$$R \xrightarrow{\ \iota\ } F^e_* R \xrightarrow{F^e_* \iota} F^{2e}_* R \xrightarrow{F^{2e}_* \iota} \cdots \longrightarrow F^{ne}_* R,$$

sending 1 to $c^{1+p^e+\cdots p^{(n-1)e}}$. Note that notationally $\phi^{\star n}$ is much simpler than writing

$$\phi \circ F^e_* \phi \circ F^{2e}_* \phi \circ \cdots \circ F^{e(n-1)}_* \phi,$$

although they both denote the same mapping in $\operatorname{Hom}_R(F^{ne}_* R, R)$.

As another example of the power of this notation, we can rephrase the key idea of Corollary 4.11 as follows:

Proposition 4.13. *Let R be a commutative ring of characteristic $p > 0$ and $g \in R$ arbitrary. For any $\phi \in \operatorname{Hom}_R(F^e_* R, R)$, consider the map $\psi = \phi \circ F^e_* g \in \operatorname{Hom}_R(F^e_* R, R)$. Then*

$$\psi^{\star n} = \phi^{\star n} \circ F^{ne}_* g^{1+p^e+\cdots+p^{e(n-1)}}$$

as maps in $\operatorname{Hom}_R(F^{ne}_* R, R)$. Equivalently, in the Cartier algebra $\mathcal{C}_R$, we have $(\phi \star g)^{\star n} = \phi^{\star n} \star g^{1+p^e+\cdots+p^{e(n-1)}}$.

Proof. Working in the Cartier Algebra, this follows immediately from repeated applications of the relation $r \star \phi = \phi \star r^{p^e}$ (4.12.4). $\square$

[14]We sometimes also use the notation ϕ^n when the risk of confusion is low.

4.3. Strong F-regularity. The nicest Frobenius split rings are those that are eventually split along *every* non-zerodivisor:

Definition 4.14. Let R be a Noetherian F-finite ring of prime characteristic. We say that R is **strongly F-regular**[15] if R is eventually Frobenius split along every non-zerodivisor.

Remark 4.15. We can define a version of strong F-regularity when R is not assumed F-finite or Noetherian, but it is less well-behaved. See Chapter 7, where this notion is called *F-pure regularity.*

Example 4.16. Every F-finite regular local ring is strongly F-regular; the proof is essentially the same as the proof in Example 4.4.

Caution 4.17. There are regular local rings that are *not* eventually Frobenius split along *any* non-zerodivisor. See Caution 5.6.

Proposition 4.18. *Suppose that $S \hookrightarrow R$ is a ring homomorphism between F-finite Noetherian rings that splits as a map of S-modules. Then if R is strongly F-regular, so is S.*

Proof. When R is a domain, it is clear that every non-zerodivisor in S is a non-zerodivisor in R, so the proof is similar to the analogous statement for Frobenius split rings (Proposition 3.5). The reduction to the domain case is left as Exercise 4.10. □

Example 4.19. For instance, Proposition 4.18 implies that the ring of invariants for a finite group (whose order is not divisible by p) acting on a strongly F-regular ring is strongly F-regular; see Example 3.7.

Example 4.20. The coordinate ring of an affine (normal) toric variety (over an F-finite field) of prime characteristic is strongly F-regular. Indeed, any normal semi-group ring over a field is a direct summand of a Laurent ring by [**BH93**, 6.1.10]. Laurent rings, being localizations of polynomial rings, are regular and hence strongly F-regular (when over an F-finite field).

Example 4.21. A major class of finitely generated cluster algebras—the *locally acyclic cluster algebras* introduced by Greg Muller [**Mul13**]—are strongly F-regular when defined over an F-finite field [**BMRS15**].

Remark 4.22. An analog of Proposition 4.18 is an open problem in characteristic zero for Kawamata log terminal singularities except in the finite case see [**Sho92**] or [**FG12**]. Relevant work on this open question can be found for instance in [**Sch05, BGLM21**].

[15] We sometimes say "F-regular" for short, especially in the context of schemes where these adjectives are frequently decorated with the adverbs "locally" and "globally." Be warned, however, that there are three flavors of F-regularity in the literature—weak F-regularity, strong F-regularity and F-regularity—all conjectured to be equivalent for F-finite rings. See Chapter 7 for more on this topic.

Strong F-regularity behaves well under localization:

Proposition 4.23. *Let R be a Noetherian F-finite ring, and let $W \subseteq R$ be a multiplicative set.*

(a) If R is strongly F-regular, then so is $W^{-1}R$;
(b) Conversely, if $R_{\mathfrak{m}}$ is strongly F-regular for each maximal $\mathfrak{m} \in \operatorname{Spec} R$, then R is strongly F-regular.

A deeper fact will be proved in Theorem 5.12: the strongly F-regular locus is open. The proof is postponed until we have developed a theory of "test elements" in Section 5. The proof of Proposition 4.23 follows immediately from the following lemma:

Lemma 4.24. *Let R be a Noetherian F-finite ring, and let $W \subseteq R$ be a multiplicative set.*

(a) If R is e-Frobenius split along $c \in R$, then $W^{-1}R$ is e-Frobenius split along the image of c in $W^{-1}R$;
(b) Conversely, if for some $c \in R$, the ring $R_{\mathfrak{m}}$ is eventually Frobenius split along the image of c in the local ring $R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R, then R is eventually Frobenius split along c.

Proof of Lemma 4.24. For (a): Any R-module map $F^e_*R \to R$ sending F^e_*c to 1 can be tensored with $W^{-1}R$ to produce an $W^{-1}R$-linear map

$$F^e_*(W^{-1}R) \to W^{-1}R$$

sending $F^e_*\frac{c}{1}$ to $\frac{1}{1}$. So if R is e-Frobenius split along c, then $W^{-1}R$ is e-Frobenius split along $\frac{c}{1}$.

For (b): Consider the "evaluation at F^e_*c map"

$$\operatorname{Hom}(F^e_*R, R) \xrightarrow{\text{eval at } F^e_*c} R \qquad \phi \mapsto \phi(F^e_*c). \tag{4.24.1}$$

By Lemma 3.19, R is eventually Frobenius split along c if and only if there exists an $e \in \mathbb{N}$ such that the map (4.24.1) is surjective. By hypothesis, for each $\mathfrak{m} \in \operatorname{Spec} R$, the R-module map

$$\operatorname{Hom}_{R_{\mathfrak{m}}}(F^e_*R_{\mathfrak{m}}, R_{\mathfrak{m}}) \xrightarrow{\text{eval at } F^e_*\frac{c}{1}} R_{\mathfrak{m}} \qquad \phi \mapsto \phi(F^e_*\frac{c}{1}) \tag{4.24.2}$$

is surjective for some $e_0 > 0$ and hence for all $e \geq e_0$ by Lemma 4.7. But because R is Noetherian and F-finite, we know (from Lemma 1.25 (a)) that

$$\operatorname{Hom}_{R_{\mathfrak{m}}}(F^e_*R_{\mathfrak{m}}, R_{\mathfrak{m}}) \cong \operatorname{Hom}_R(F^e_*R, R) \otimes_R R_{\mathfrak{m}}$$

and that the map (4.24.2) is natural localization of the map (4.24.1).

Because the surjectivity of (4.24.1) is an open condition, there exist, for each $\mathfrak{m}$, some $e = e_{\mathfrak{m}} \in \mathbb{N}$ and a neighborhood $\mathcal{U}_{\mathfrak{m}}$ of $\mathfrak{m}$ such that $\mathrm{Hom}_R(F^e_* R, R)_{\mathfrak{n}} \xrightarrow{\text{eval at } F^e_* \frac{c}{1}} R_{\mathfrak{n}}$ is surjective for all $\mathfrak{n} \in \mathcal{U}_{\mathfrak{m}}$. These $U_{\mathfrak{m}}$ cover $\operatorname{Spec} R$. Now because $\operatorname{Spec} R$ is quasi-compact, we may pick a finite subcover, and then choose $e \geq e_{\mathfrak{m}}$ for each of the finitely many $\mathfrak{m}$ indexing the finite subcover. But then the R-module map (4.24.1) is surjective at every point of R, and hence surjective. That is, R is eventually Frobenius split along c. $\square$

Remark 4.25. The Noetherian F-finite hypothesis in Lemma 4.24 was used only to prove (b). In fact, the argument for (b) works without the Noetherian assumption provided $F_* R$ is a finitely presented R-module; this implies $F^e_* R$ is finitely presented for all $e > 0$ (see [**Sta19**, Tag 00F4]).

Remark 4.26. Strong F-regularity was originally defined by Hochster and Huneke in [**HH89**]. Their definition demanded that R be eventually Frobenius split along every *c not in any minimal prime of R,* rather than every non-zerodivisor c, but this is equivalent. Indeed, either formulation implies that R is eventually Frobenius split along 1, so that the ring R is reduced. But in a reduced ring, the set of zero-divisors is precisely the union of the minimal primes [**Sta19**, Tag 02LV].

Because strong F-regularity localizes well, it also globalizes well, and we can define F-regularity for schemes:

Definition 4.27. An F-finite Noetherian scheme X is **locally F-regular** (or **locally strongly F-regular**[16]) if any of the following equivalent conditions is satisfied

(a) The scheme X has an open cover by affine schemes $\operatorname{Spec} R_\lambda$ with each R_λ strongly F-regular;
(b) For every open affine set U in $\operatorname{Spec} R \subseteq X$, the ring $\mathcal{O}_X(U)$ is strongly F-regular.
(c) For every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is strongly F-regular.
(d) For every closed point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is strongly F-regular.

Caution 4.28. Like Frobenius splitting, we can also define a *global* form of F-regularity for a scheme X, which is a *much stronger condition* than *local* F-regularity if X is not affine. The global forms of both Frobenius splitting and strong F-regularity place very strong restrictions on a projective variety, which we consider carefully in Chapter 3.

[16]We often drop the adverb "strongly" when it is clear from the context. There are contexts where it is needed, including any discussion about tight closure. See Chapter 7.

Remark 4.29. A non-singular Noetherian F-finite scheme is locally F-regular (Example 4.16) and a locally F-regular scheme is locally Frobenius split (Proposition 4.6).

4.4. Normality of F-regular Rings. An important property of F-regular rings is *normality.* For domains, normal is synonymous with *integrally closed in its field of fractions.* See [**Sta19**, Tag 037B] for a review of the relevant facts about normality.

Theorem 4.30. [**HH90**] *A strongly F-regular ring is integrally closed in its total fraction ring. In particular, any strongly F-regular ring is a finite product of strongly F-regular domains, and every locally F-regular scheme is a disjoint union of irreducible locally F-regular schemes.*

Remark 4.31. A related fact is that Frobenius split rings are weakly normal (and hence semi-normal). See Exercise 4.14.

Proof of Theorem 4.30. Assume that R is strongly F-regular. Then R is Frobenius split, and hence reduced. Thus R is a subring of its total fraction ring $\mathcal{K}(R)$ obtained by inverting all non-zerodivisors of R.

Fix an element x/y in the total fraction ring $\mathcal{K}(R)$, and assume x/y is integral over R. We must show that y divides x in R. Since x/y is integral over R, the ring $T = R[x/y] \subseteq \mathcal{K}(R)$ is a finite integral extension of R—indeed, as an R-module, T is generated by $\{1, \frac{x}{y}, (\frac{x}{y})^2, \ldots, (\frac{x}{y})^{m-1}\}$ where m is the degree of the monic polynomial of integral dependence of x/y on R. Hence there is a non-zerodivisor $c \in R$ such that $cT \subseteq R$ (for example, we can take $c = y^{m-1}$).

Now, since $(\frac{x}{y})^{p^e} \in T$ for all e, we have $c(\frac{x}{y})^{p^e} \in R$ for all $e \in \mathbb{N}$. That is, $cx^{p^e} \in (y^{p^e})$ in R for all $e \geq 1$. Therefore, for *all* e,

$$cx^{p^e} = r_e y^{p^e} \tag{4.31.1}$$

for some $r_e \in R$ which depends on e. Since R is strongly F-regular, we can find $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* c) = 1$. Viewing (4.31.1) in the ring $F^e_* R$, we have

$$F^e_* c x^{p^e} = F^e_* r_e y^{p^e},$$

which simplifies to

$$x F^e_* c = y F^e_* r_e. \tag{4.31.2}$$

Applying the R-linear map ϕ to the equation (4.31.2), we get

$$x = x\phi(F^e_* c) = y\phi(F^e_* r_e) \in (y)$$

in R. This shows that $x/y \in R$, proving that R is integrally closed in $\mathcal{K}(R)$. The second statement follows from Exercise 4.6 and the fact that a reduced ring with finitely many minimal primes which is integrally closed in its total quotient ring is a finite product of normal domains; see *e.g.* [**Sta19**, Tag 030C]. ☐

Remark 4.32. We don't need R to be F-finite or Noetherian in the proof of Theorem 4.30 provided that R has finitely many minimal primes and is eventually Frobenius split along every non-zerodivisor.

4.5. Exercises.

Exercise 4.1. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of characteristic p. Suppose $0 \neq c \in R$. Show that there exists an e such that $c \notin \mathfrak{m}^{[p^e]}$.

Hint: Use Krull's Intersection Theorem, which says that if $(R, \mathfrak{m})$ is a Noetherian local ring, then $\cap_{n \in \mathbb{N}} \mathfrak{m}^n = 0$.

Exercise 4.2. Let R be an arbitrary ring of characteristic $p > 0$. Let $f \in R$. Show that there is a splitting of

$$R \to F^e_* R \qquad 1 \mapsto F^e_* f$$

if and only if there is a splitting of

$$R \to F^{e+1}_* R \qquad 1 \mapsto F^{e+1}_* f^p.$$

Hint: Either condition implies that R is Frobenius split. Now compose relevant splittings to get the desired ones.

Exercise 4.3. To practice the notion introduced in Subsection 4.2, show that in the Cartier Algebra $\mathcal{C}_R$, if $\pi \in \operatorname{Hom}_R(F_* R, R)$ is a Frobenius splitting, then $\pi^{\star n}$ is a splitting of the n-th iterate of Frobenius F^n.

Exercise 4.4. (Dual Cartier Algebra) Let R be a ring of characteristic p and consider the abelian group

$$\bigoplus_{e \geq 0} \operatorname{Hom}_R(R, F^e_* R). \tag{4.32.1}$$

Define for $\phi \in \operatorname{Hom}_R(R, F^e_* R)$ and $\psi \in \operatorname{Hom}_R(R, F^d_* R)$, define $\phi \star \psi \in \operatorname{Hom}_R(R, F^{d+e}_* R)$ by $F^d_* \phi \circ \psi$.

(a) Prove that this composition can be used to define a graded ring structure on the abelian group (4.32.1).
(b) Show that $\phi \star r = r^p \star \phi$ for all $r \in R$.

Exercise 4.5. (Cartier Modules) Let R be a ring of characteristic p and let M be an R-module. Consider the abelian group

$$\mathcal{C}_R(M) := \bigoplus_{e \geq 0} \operatorname{Hom}_R(F^e_* M, M). \tag{4.32.2}$$

Show that $\mathcal{C}_R(M)$ has a natural ring structure that is non-commutative.

Exercise 4.6. Prove that a product ring $R_1 \times R_2$ is strongly F-regular if and only if both R_1 and R_2 are strongly F-regular.

Exercise 4.7. Prove Proposition 4.23.

Hint: Show that a non-zerodivisor of $W^{-1}R$ can be assumed of the form $\frac{c}{w}$ where c is a non-zerodivisor of R. Then use Proposition 4.6.

Exercise 4.8. Let R be a ring of prime characteristic. Show that the following are equivalent:

(a) R is eventually Frobenius split along some unit;
(b) R is eventually Frobenius split along every unit;
(c) For every $e > 0$, R is e-Frobenius split along every unit;
(d) R is Frobenius split.

Exercise 4.9. Prove that if $R \longrightarrow S$ is a faithfully flat extension of reduced F-finite Noetherian rings, and S is strongly F-regular, then so is R.

Exercise 4.10. Prove Proposition 4.18.

Hint: Reduce to domain case using Theorem 4.30 and Exercise 4.6.

Exercise 4.11. Fix a ring R of characteristic $p > 0$ and a non-zerodivisor $c \in R$. Let $\frac{1}{c}R$ denote the cyclic R submodule of the total ring of fractions[17] of R generated by the element $\frac{1}{c}$. Prove that R is eventually Frobenius split along c if and only if there exists an $e \in \mathbb{N}$ such that the map

$$\begin{aligned} R &\longrightarrow F^e_* \tfrac{1}{c} R \\ 1 &\longmapsto F^e_* 1 \end{aligned}$$

splits as a map of R-modules.

Exercise 4.12. Let R be a reduced Noetherian ring. Suppose that $\phi : F^e_* R \longrightarrow R$ is an R-linear map. Fix I an ideal and consider $J := \Gamma_I(R) \subseteq R$, the ideal of elements annihilated by a power of I. Prove that $\phi(F^e_* J) \subseteq J$. Conclude that if $Q \subseteq R$ is a minimal prime, then $\phi(F^e_* Q) \subseteq Q$. (We will study this condition—called *compatibility* of J and ϕ— in detail in Section 6).

Exercise 4.13. Consider the Frobenius split ring $R = \mathbb{F}_p[x, y, z]/(xyz)$ from Exercise 3.6. Show that R is not eventually Frobenius split along any element in the maximal ideal $\mathfrak{m} = (x, y, z)$. In particular, conclude the local ring $R_\mathfrak{m}$ is not Frobenius split along any non-unit.

[17] If this level of generality is unfamiliar, it is valuable to keep the example of an integral domain in mind. See [**Sta19**, Tag 02LV] for basics on total rings of fractions.

Hint: One way to do this is to use Exercise 4.12. Notice that if $\phi(F^e_* Q_1) \subseteq Q_1$ and $\phi(F^e_* Q_2) \subseteq Q_2$ then $\phi(F^e_*(Q_1 + Q_2)) \subseteq Q_1 + Q_2$.

Exercise 4.14. A reduced Noetherian ring R of characteristic $p > 0$ is called **weakly normal** if $x \in \mathcal{K}(R)$ and $x^p \in R$ implies that $x \in R$. Any ring is called **seminormal** if $x \in \mathcal{K}(R)$, and $x^2, x^3 \in R$ implies that $x \in R$. Show that Frobenius split rings are weakly normal, and that weakly normal rings are seminormal. For a more thorough introduction to weak normality, see Chapter 2 Subsection 4.5 in the exercises of that section.

Exercise 4.15. Fix $p > 2$. Consider the injective ring homomorphism $R = \mathbb{F}_p[x] \xrightarrow{x \mapsto y^2} \mathbb{F}_p[y] = S$. For any i in the range $0 \leq i \leq p-1$, let ϕ_i be the R-module map $F_* R \longrightarrow R$ sending $F_* x^i \mapsto 1$ and the other basis elements $F_* x^j$ (with $j \neq i$, $0 \leq j \leq p-1$) of the free R module $F_* R$ to zero.

(a) Show that ϕ_1 extends to an S-module map $F_* S \longrightarrow S$.
(b) Show that ϕ_{p-1} does not extend to an S-module map $F_* S \longrightarrow S$.
(c) More generally, show that ϕ_i extends to an S-module map $F_* S \longrightarrow S$ if and only if $i \leq (p-1)/2$.

5. Test elements and the test ideal

The *test ideal* on a Noetherian F-finite scheme is a canonical sheaf of ideals defining the closed locus of non-strongly F-regular points. As such, the test ideal endows the non-strongly F-regular locus with a natural scheme structure enjoying very special properties with respect to Frobenius.

Test elements—essentially just elements of the test ideal—provide a useful technical tool for working with Frobenius splitting in various settings. For example, in this section, we will use test elements to prove that strongly F-regularity is preserved by completion (Proposition 5.5) and by any étale extension (Proposition 5.8), and that the strongly F-regular locus is open (Theorem 5.12).

While test ideals first arose in tight closure theory (see Chapter 7), they have since come to be viewed as "prime characteristic analogs" of multiplier ideals in complex algebraic geometry (see Chapter 6). The theory of test ideals will be substantially generalized in Chapters 4, 5, 7, and 8.

5.1. Testing for F-regularity.

One difficulty in verifying strong F-regularity is the need to check eventual Frobenius splitting along *every* non-zerodivisor. The next theorem, Theorem 5.1, streamlines this process by allowing us to check just *one* well-chosen non-zerodivisor:

Theorem 5.1. [**HH89**, Thm 3.3] *Let R be an F-finite Noetherian ring, and suppose that $d \in R$ has the property that $R[d^{-1}]$ is strongly F-regular. Then if R is eventually Frobenius split along d, then R is strongly F-regular.*

Remark 5.2. The element d of Theorem 5.1 can be informally called a *"test element"* since we can "test" for eventual splitting along one such d in order to conclude R is strongly F-regular. This is closely related to (but not exactly the same as) a *strong test element* (Definition 5.14) and a *test element for tight closure* (see Chapter 7).

Remark 5.3. In practice, it is easy to find non-zerodivisors d satisfying the first sentence of Theorem 5.1. For example, suppose that R is reduced, Noetherian and F-finite. For each minimal prime P of R, the stalk R_P is a field, and hence regular, so P does not contain the defining ideal I of the non-regular locus of $\operatorname{Spec} R$ (this locus is closed by Corollary 2.3). So by Prime Avoidance, there exist $d \in I$ but not in any minimal prime[18] of R. Such an element d is a non-zerodivisor with $R[d^{-1}]$ *regular* and hence strongly F-regular.

Proof of Theorem 5.1. Take any non-zerodivisor $c \in R$. We need to show that R is eventually split along c. Consider the "evaluation at c map"

$$\operatorname{Hom}_R(F^f_* R, R) \xrightarrow{\text{eval at } F^f_* c} R \qquad \phi \mapsto \phi(F^f_* c). \tag{5.3.1}$$

Because $R[d^{-1}]$ is strongly F-regular, for large enough f the map (5.3.1) becomes surjective after tensoring with $R[d^{-1}]$. This means that d^m is in the image of the map (5.3.1) for some (possibly large) $f > 0$ and some $m > 0$. Without loss of generality, we may assume that $m = p^\ell$ for some integer ℓ, and fix f and $\psi \in \operatorname{Hom}_R(F^f_* R, R)$ such that

$$\psi(F^f_* c) = d^{p^\ell}. \tag{5.3.2}$$

Now, because R is eventually Frobenius split along d, R is in particular Frobenius split, so we can find a splitting $\pi \in \operatorname{Hom}_R(F^\ell_* R, R)$ of F^ℓ. For this map,

$$\pi(F^\ell_* d^{p^\ell}) = d\pi(F^\ell_* 1) = d. \tag{5.3.3}$$

Also fixing an eventual Frobenius splitting along d, we have $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that

$$\phi(F^e_* d) = 1. \tag{5.3.4}$$

[18]See [**Sta19**, Tag 00DS] for the Prime Avoidance Lemma. However, for rings R containing an infinite field, it is worth noting that prime avoidance is a simple fact about vector spaces; in this case viewing ideals in R as sub vector spaces, an ideal I can not be contained in a finite union of ideals unless it is contained in one of them, so a general element of I satisfies the prime avoidance lemma.

Finally, consider the composition $\alpha = \phi \star \pi \star \psi \in \mathrm{Hom}_R(F_*^{f+\ell+e}R, R)$. We claim that $\alpha(F_*^{f+\ell+e}c) = 1$. This will complete the proof, as then α is an eventual Frobenius splitting along c. To check the claim, we follow $F_*^{f+\ell+e}c$ through the composition α:

$$F_*^{f+\ell+e}R \xrightarrow{F_*^{\ell+e}\psi} F_*^{\ell+e}R \xrightarrow{F_*^e\pi} F_*^eR \xrightarrow{\phi} R.$$

Making use of the equations (5.3.2), (5.3.3) and (5.3.4), we see

$$F_*^{\ell+e}F_*^f c \overset{F_*^{\ell+e}\psi}{\longmapsto} F_*^{e+\ell}d^{p^\ell} = F_*^e(dF_*^\ell 1) \overset{F_*^e\pi}{\longmapsto} F_*^e d \overset{\phi}{\longmapsto} 1,$$

completing the proof. $\square$

Remark 5.4. The proof of Theorem 5.1 cleans up nicely if remember that *as sets* $F_*^eR = R$, so we can (mostly) ignore the R-module structure when we compute $\phi \star \pi \star \psi$ applied to c. We know ψ takes c to d^{p^ℓ}, and π takes d^{p^ℓ} to d. Finally, ϕ takes d to 1.

Armed with this new tool for "testing" for F-regularity by splitting along just *one* element, we establish several basic properties of F-regularity.

Proposition 5.5. *Let $(R, \mathfrak{m})$ be an F-finite Noetherian local ring. Then R is strongly F-regular if and only if its completion $\widehat{R}$ at the maximal ideal is strongly F-regular.*

More generally, if R is F-finite Noetherian and strongly F-regular. Then completion of R along any ideal is strongly F-regular.

Caution 5.6. There are examples of discrete valuation rings in $\overline{\mathbb{F}_p}(x, y)$ that are *not* F-finite but whose completions are F-finite [**DS18**]. Such discrete valuation rings provide examples of non strongly F-regular Noetherian local rings whose completions are strongly F-regular, showing that the F-finite hypothesis in Proposition 5.5 is necessary. Likewise, such discrete valuation rings are *regular* but not strongly F-regular.

Proof. Fix a non-zerodivisor c of R. Because completion is flat, the image of c in $\widehat{R}$ is also a non-zerodivisor. Consider the R-module maps

$$\mathrm{Hom}_R(F_*^eR, R) \xrightarrow{\text{eval at } F_*^e c} R \qquad \phi \mapsto \phi(F_*^e c). \tag{5.6.1}$$

The formation of $\mathrm{Hom}_R(F_*^eR, R)$ commutes with completion (Lemma 1.25(b)), so the map (5.6.1) becomes

$$\mathrm{Hom}_{\widehat{R}}(F_*^e\widehat{R}, \widehat{R}) \longrightarrow \widehat{R} \qquad \phi \mapsto \phi(F_*^e c) \tag{5.6.2}$$

after tensoring with $\widehat{R}$. The map (5.6.1) is surjective if and only if (5.6.2) is surjective, by the faithful flatness of completion. So R is eventually Frobenius split along c if and only if $\widehat{R}$ is eventually Frobenius split along (the image of)

c. Thus clearly if $\widehat{R}$ is strongly F-regular, then also R is strongly F-regular, as c was an arbitrary non-zerodivisor of R.

For the converse, we assume that R is strongly F-regular but stop assuming R is local. In this case $\widehat{R}$ is the completion of R along an arbitrary ideal J.

To show that $\widehat{R}$ is strongly F-regular, we may check that $\widehat{R}$ is eventually Frobenius split along *one* c such that $\widehat{R}[c^{-1}]$ is regular (Theorem 5.1). Thus, by the previous paragraph, it suffices to find a non-zerodivisor $c \in R$ such that the localization of $\widehat{R}$ at (the image of) c is regular.

To find such c, note that because R is reduced, Noetherian and F-finite, we can find a non-zerodivisor $c \in R$ such that $R[c^{-1}]$ is regular (see Remark 5.3). Furthermore, since $F_*R[c^{-1}]$ is a locally free $R[c^{-1}]$-module, tensoring with $\widehat{R}$, we see

$$\widehat{R} \otimes_R F_*R[c^{-1}] \cong F_*(\widehat{R}[c^{-1}])$$

is locally free over $\widehat{R}[c^{-1}]$ as well. Thus $\widehat{R}[c^{-1}]$ is regular by Kunz's Theorem (Theorem 2.1), completing the proof. □

5.2. Étale extensions. Test elements can be applied to prove the "permanence" of strong F-regularity under étale extensions.[19] For a field K, an extension $K \to L$ is étale means simply that L is a finite product of finite separable field extensions. In general, an étale map is simply a flat and finitely presented map whose fibers are étale field extensions. More precisely,

Definition 5.7. An extension of rings $A \longrightarrow B$ is **étale** if it is finitely presented, flat, and for all $P \in \operatorname{Spec} A$, the ring $\frac{A_P}{PA_P} \otimes_A B$ is a product of finitely many finite *separable* field extensions of the residue field $\frac{A_P}{PA_P}$ at P.

Regularity is preserved by étale maps in the sense that if $A \to B$ is an étale map with A regular, then B is also regular [**Sta19**, Tag 025L]. Similarly, strong F-regularity is preserved by étale maps:

Proposition 5.8. *Let $R \longrightarrow S$ be an étale map of F-finite Noetherian rings. If R is strongly F-regular, then S is strongly F-regular.*

The point of the proof is the following fact (see Exercise 5.13 for a proof) about how Frobenius interacts with étale maps:

[19] The reader can consult [**Sta19**, Tag 00U0] or [**Mil80**] for a careful development of the theory of étale morphisms.

Proposition 5.9. [**Sta19**, Tag 0EBS] *Let $A \to B$ be an étale map of Noetherian rings of positive characteristic $p > 0$. Then the natural map $F^e_* A \otimes_A B \longrightarrow F^e_* B$ is an isomorphism.*

Proof of Proposition 5.8. Choose $c \in R$ such that $R[c^{-1}]$ is regular. Because étale maps are preserved by arbitrary base change [**Sta19**, Tag 00U0], the map $R[c^{-1}] \longrightarrow R[c^{-1}] \otimes_R S = S[c^{-1}]$ is étale, and so also $S[c^{-1}]$ is regular, since it is étale over the regular ring $R[c^{-1}]$. Thus, by Theorem 5.1, it suffices to show that S is eventually Frobenius split along c. We know there is an e such that

$$R \xrightarrow{1 \mapsto F^e_* c} F^e_* R$$

splits. Tensoring over R with S, then also

$$S \xrightarrow{1 \mapsto F^e_* c \otimes 1} F^e_* R \otimes_R S \cong F^e_* S,$$

splits as well, with the last isomorphism following from Proposition 5.9. In particular, S is strongly F-regular. $\square$

Remark 5.10. Conversely, if $R \longrightarrow S$ is a *faithfully flat* map of F-finite Noetherian rings, and S is strongly F-regular, then so is R. See Exercise 4.9. So for a local étale map of local Noetherian F-finite rings, the source is strongly F-regular if and only if the target is.

Remark 5.11. It is worth comparing Proposition 5.9 with Corollary 1.15, which says that for a Noetherian local *F-finite* ring, there is a natural isomorphism $F_* R \otimes_R \widehat{R} \cong F_* \widehat{R}$, where $\widehat{R}$ denotes the completion of R at its maximal ideal. Similarly, compare Proposition 5.5 with Proposition 5.8 (and Remark 5.10).

5.3. The openness of the strongly F-regular locus. Test elements allow us to establish the openness of the strongly F-regular locus:

Theorem 5.12. *The locus of strongly F-regular points on any Noetherian F-finite scheme is open.*

Proof. The statement reduces immediately to the affine case because both local F-regularity and openness can be checked on an affine open cover. So without loss of generality, assume R is a Noetherian F-finite ring. We need to prove the openness of the locus of points $Q \in \operatorname{Spec} R$ such that R_Q is strongly F-regular.

Suppose $Q \in \operatorname{Spec} R$ is a strongly F-regular point. We need to find an open neighborhood $\mathcal{U}$ of Q such that for every $P \in \mathcal{U}$, R_P is strongly F-regular. Note that the reduced locus is open (as its complement is the closed set defined by the annihilator of the nilradical) and non-empty (as it contains Q). So, replacing $\operatorname{Spec} R$ by an affine neighborhood of Q contained in the open set of reduced points, we may assume that R is reduced.

By prime avoidance, we can fix a non-zerodivisor d such that $R[d^{-1}]$ is regular (see Remark 5.3). Let J_e be the image of the map

$$\operatorname{Hom}_R(F_*^e R, R) \xrightarrow{\text{eval at } F_*^e d} R \qquad \phi \mapsto \phi(F_*^e d), \tag{5.12.1}$$

and note that the closed set $\mathbb{V}(J_e) \subseteq \operatorname{Spec} R$ is the locus of points where the map (5.12.1) fails to be surjective.

Now by Theorem 5.1, the local ring R_P is strongly F-regular if and only if there exists $e > 0$ such that the map (5.12.1) is surjective after localizing at P. Put differently, R_P is *not* strongly F-regular if and only if $P \in \bigcap_{e \in \mathbb{N}} \mathbb{V}(J_e)$. Thus the non-strongly F-regular locus is the closed set of $\operatorname{Spec} R$ defined by the ideal $J = \sum_{e \in \mathbb{N}} J_e$. □

Remark 5.13. The ideal J constructed in the proof of Theorem 5.12 defines the non-strongly F-regular locus, but *noncanonically* so, as it depends on the choice of a "test element" d; see, however, Exercise 5.2.

5.4. Strong test elements. To get a *canonically* defined ideal defining the non-strongly F-regular locus, we should look at the images of the maps (5.12.1) for *all* non-zerodivisors d:

Definition 5.14. Let R be ring of prime characteristic. A **strong test element** for R is an element c with the property that for all non-zerodivisors d, there exists an $e_0 > 0$ such that c is in the image of the map

$$\operatorname{Hom}_R(F_*^e R, R) \xrightarrow{\text{eval at } F_*^e d} R \qquad \phi \mapsto \phi(F_*^e d) \tag{5.14.1}$$

for all $e \geq e_0$.

Remark 5.15. A Noetherian F-finite ring is strongly F-regular, essentially by definition, if and only if the element 1 is a strong test element. In this case, all elements of R are strong test elements. See Exercise 5.3.

Remark 5.16. For any ring of characteristic p, clearly 0 is a strong test element[20] but it is not yet obvious, in the non-strongly F-regular case, whether any others exist. We'll address this in Theorem 5.21.

Lemma 5.17. *Let R be a ring of prime characteristic. The set of all strong test elements forms an ideal of R.*

PROOF. For each e, the image of the R-module map (5.14.1) is an ideal of R. So for each i, the intersection

$$J_i(d) =: \bigcap_{e \geq i} \operatorname{im}\left(\operatorname{Hom}_R(F_*^e R, R) \xrightarrow{\text{eval at } F_*^e d} R\right) \tag{5.17.1}$$

[20] But see Caution 5.26!

is an ideal of R. Clearly $J_i(d) \subseteq J_j(d)$ when $j > i$, since the latter is an intersection over fewer ideals. Thus as i increases, the ideals (5.17.1) form an increasing chain of ideals; let $J(d)$ be the union $\bigcup_{i \in \mathbb{N}} J_i(d)$. Clearly $J(d)$ consists of all $c \in R$ for which there exists some e_0 such that c is in the image of (5.14.1). Now, the set of strong test elements is the intersection, over all non-zerodivisors d, of the ideals $J(d)$, hence clearly an ideal of R. □

Armed with Lemma 5.17, we make the following definition:

Definition 5.18. Let R be a reduced Noetherian F-finite ring. The **test ideal** of R, denoted $\tau(R)$, is the ideal of all strong test elements of R.

The test ideal of a reduced Noetherian F-finite ring behaves well under localization, completion, and étale maps, but we postpone the proof until the next section.

Remark 5.19. Because the test ideal localizes well (Proposition 6.17), it also globalizes, so we get a sheaf of ideals $\tau(X)$ on any reduced Noetherian F-finite scheme X; this sheaf of ideals cuts out the non-strongly F-regular locus by Remark 5.15.

Remark 5.20. Even if R is not reduced, the **ideal of all strong test elements** is the entire ring if and only if R is strongly F-regular. We'll explore this ideal and a few other variants in the exercises, but we will not call this the *test ideal* in the non-reduced case, for technical reasons we discuss in Chapter 8.

5.5. Test elements and non-zerodivisors. Test elements are most useful when they are *non-zerodivisors*. Yet from the definition, it is not even clear whether or not the ideal of strong test elements is non-zero! Fortunately, we have the following important result, essentially due to Hochster and Huneke (*C.f.* [**HH89**, Thm 3.4]):

Theorem 5.21. *Let R be a Noetherian F-finite ring, and let $b \in R$ be such that $R[b^{-1}]$ is strongly F-regular. Then b has a power that is a strong test element for R.*

Before proving Theorem 5.21, we record some immediate consequences:

Corollary 5.22. *Let R be a Noetherian reduced F-finite ring. Then R has a strong test element that is a non-zerodivisor.*

In particular, the test ideal $\tau(R)$ has positive height.

Proof. This follows from Remark 5.3, so is left as an exercise. □

Corollary 5.23. *The test ideal $\tau(R)$ of a reduced Noetherian F-finite ring is generated by non-zerodivisors.*

PROOF. This is a general fact about any ideal of positive height in a reduced Noetherian ring. See Exercise 5.10. ☐

PROOF OF THEOREM 5.21. Suppose $R[b^{-1}]$ is strongly F-regular. In particular, $R[b^{-1}]$ is Frobenius split. Since

$$\operatorname{Hom}_{R[b^{-1}]}(F_*R[b^{-1}], R[b^{-1}]) = \operatorname{Hom}_R(F_*R, R) \otimes_R R[b^{-1}],$$

any splitting of Frobenius for $R[b^{-1}]$ is induced by some $\pi \in \operatorname{Hom}_R(F_*R, R)$ after localization at b. In particular, we may assume there exists some $\pi \in \operatorname{Hom}_R(F_*R, R)$ where $\pi(F_*1) = b^m$. Multiplying by b if needed, we may further assume that $m = pm'$ for some $m' \in \mathbb{N}$.

We claim that b^{2m} is in the image of $\pi^\ell \in \operatorname{Hom}_R(F_*^\ell R, R)$ for all $\ell \in \mathbb{N}$, where by π^ℓ we mean the ℓ-fold composition $\pi^{\star\ell}$ in the Cartier Algebra. Indeed, this is clear for $\ell = 1$, so assume inductively, that $\pi^{\ell-1}(F_*^{\ell-1}r) = b^{2m}$ for some $r \in R$. Then

$$\pi^\ell(F_*^\ell r) = \pi(F_*(\pi^{\ell-1}(F_*^{\ell-1}r))) = \pi(F_*b^{2m}) = b^{2m'}\pi(F_*1) = b^{2m'}b^m,$$

so that

$$b^{2m} = b^{m-2m'}\pi^\ell(F_*^\ell r) \in \operatorname{im}(\pi^\ell). \tag{5.23.1}$$

Finally, we claim that b^{2m+1} is a strong test element. To prove this, we must show that for every non-zerodivisor $d \in R$,

$$b^{2m+1} \in \bigcap_{e \gg 0} \operatorname{im}\left(\operatorname{Hom}_R(F_*^e R, R) \xrightarrow{\text{eval at } F_*^e d} R\right). \tag{5.23.2}$$

Because $\frac{d}{1} \in R[b^{-1}]$ is a non-zerodivisor in $R[b^{-1}]$ and $R[b^{-1}]$ is strongly F-regular, there exists $\psi \in \operatorname{Hom}_R(F_*^e R, R)$ such that $\psi(F_*^e d) = b^{m_e}$ for some $m_e > 0$ (using an argument as in the opening paragraph). Multiplying by a power of b if needed, we may assume that

$$\psi(F_*^e d) = b^{p^n}$$

for any sufficiently large $n \in \mathbb{N}$. Also, since $b^{2m} \in \operatorname{im}\pi^\ell$ for all ℓ, we can find $r \in R$ such that

$$b^{2m} = \pi^n(F_*^n r).$$

Now, we claim that the composition

$$\beta = \pi^n \star r \star \psi := \pi^n \circ F_*^n r \circ F_*^n \psi \in \operatorname{Hom}_R(F_*^{n+e} R, R)$$

has the property that $\beta(F_*^{n+e}d) = b^{2m+1}$. Since d was an arbitrary non-zero-divisor in R and this works for all $n \gg 0$, this will show that b^{2m+1} is a strong test element, completing the proof.

To check this claim, we follow $F_*^{n+e}d$ through the composition β

$$F_*^{n+e}n \xrightarrow{F_*^n\psi} F_*^n R \xrightarrow{F_*^n r} F_*^n R \xrightarrow{\pi^n} R$$

to see that

$$F_*^{n+e}d \xmapsto{F_*^n\psi} F_*^n b^{p^n} = bF_*^n 1 \xmapsto{F_*^n r} bF_*^n r \xmapsto{\pi^n} bb^{2m}.$$

So $\beta(F_*^{n+e}d) = b^{2m+1}$, as needed. □

Proposition 5.24. *The test ideal of a Noetherian F-finite Frobenius split ring is radical.*

Proof. Suppose $b^n \in \tau(R)$. Then $b^{p^n} \in \tau(R)$ as well. Let $\pi \in \mathrm{Hom}_R(F_*^n R, R)$ be a splitting of (the n-th iterate of) Frobenius. In particular, $\pi(F_*^n b^{p^n}) = b\pi(F_*^n 1) = b$. Now, fix any non-zerodivisor $d \in R$. By definition of strong test element, for all $e \gg 0$, we can find $\phi_e \in \mathrm{Hom}_R(F_*^e R, R)$ such that $\phi_e(F_*^e d) = b^{p^n}$. Applying π, we have

$$\pi(F_*^n \phi_e(F_*^e d)) = \pi_*(b^{p^n}) = b$$

so the element $\pi \star \phi_e \in \mathrm{Hom}_R(F_*^{e+n}R, R)$ takes $F_*^{e+n}d$ to b for all $e \gg 0$. This shows $b \in \tau(R)$. See also Exercise 6.24 for a different proof. □

Another immediate consequence is left as Exercise 5.7:

Corollary 5.25. *Let R be a Noetherian F-finite ring. Then the ideal of strong test elements*[21] *defines the locus of non strongly F-regular points of* $\operatorname{Spec} R$.

Caution 5.26. Test elements were first defined by Hochster and Huneke in their theory of tight closure; their definition is *different*, but closely related to ours, as we will discuss later in Chapter 7. One important difference is worth highlighting immediately: while the tight closure literature defines test elements exclusively as *elements not in any minimal prime* with certain features, we do *not* make that restriction here; see, for example, Remark 5.16. Otherwise, our strong test elements—at least for a reduced Noetherian F-finite local ring—turn out to be the same as *non-finitistic test elements* or *big* test elements in tight closure theory. These are slightly "stronger," *a priori*, than *completely stable test elements*, although there is evidence for a conjecture predicting that all these types of test elements are equivalent under mild hypothesis.

Remark 5.27. One may also consider a variant of Definition 5.14 in which the elements d are required only to be *not in any minimal prime of R*. Of course, for reduced rings, there is no difference, but for non-reduced rings, this leads to an interesting parallel theory developed in the exercises, beginning with Exercise 5.17.

[21] By definition, this is the test ideal if R is reduced.

Caution 5.28. The reader is cautioned not to confuse our notion of a strong test element (Definition 5.14) with "an element in a strong test ideal" in the sense of Huneke [**Hun97**], a notion refining the test ideal for tight closure. Vraciu showed [**Vra02**] that the test ideal for tight closure of a complete local reduced ring is a strong test ideal in Huneke's sense. In particular, it is expected that under mild hypothesis, the test ideal (in any equivalent sense) will be the largest strong test ideal (in Huneke's sense).

5.6. Exercises.

Exercise 5.1. Let $R = \mathbb{F}_p[x_1, \ldots, x_n]/(x_1^d + \cdots + x_n^d)$ for some $d > n$ and p not dividing d. Prove that the test ideal of R is primary to $(x_1, \ldots, x_n)$.

Hint: Use the method of Example 3.13 to show that R is not strongly F-regular.

Exercise 5.2. Let R be a Noetherian F-finite ring. Show that the non-strongly F-regular locus of $\operatorname{Spec} R$ is the closed set defined by the ideal J_e for $e \gg 0$, where J_e is the image of the "evaluation at $F^e_* d$" map from the proof of Theorem 5.12.

Hint: Show that the *radicals* of the ideals J_e are increasing by making use of Lemma 4.7.

Exercise 5.3. Let R be a Noetherian F-finite ring. Prove that R is strongly F-regular if and only if $1 \in R$ is a strong test element.

Exercise 5.4. Let R be a Noetherian ring of prime characteristic. Prove that if R has a strong test element that is a non-zerodivisor, then R is reduced.

Hint: If Q is an associated prime of R, show that R_Q is eventually Frobenius split along a unit, and hence reduced.

Exercise 5.5. Let R be a Noetherian F-finite Frobenius split ring. Prove that the test ideal $\tau(R)$ is precisely the set of all elements $b \in R$ such that $R[b^{-1}]$ is strongly F-regular[22].

Exercise 5.6. Let R be a Frobenius split ring. Show that if c has the property that for every non-zero-divisor $d \in R$, there exists an e and $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* d) = c$, then c is a strong test element.

Exercise 5.7. Prove Corollary 5.25.

[22]Here, we make the convention that the zero ring is vacuously strongly F-regular.

Exercise 5.8. Let R be a ring a prime characteristic. Show that if c is a strong test element, then so is $\phi(F^e_* c)$ for any $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. We will study this property in some depth in the next section.

Exercise 5.9. Let R be a ring a prime characteristic. Prove that the nilradical of R kills the ideal of all strong test elements. That is, if n is nilpotent and c is a strong test element, then $nc = 0$.

Exercise 5.10. Let R be a reduced Noetherian ring. If an ideal $J \subseteq R$ has positive height, then J is generated by non-zerodivisors.

Hint: Use prime avoidance ([**Sta19**, Tag 00DS] and the fact that J is not contained in any minimal prime of R.

Exercise 5.11. Let $A \longrightarrow B$ be an étale map of F-finite Noetherian rings. Show that if A is Frobenius split, then so is B.

Hint: Use Proposition 5.9.

Exercise 5.12. Let K be a field of characteristic $p > 0$. For any finite separable field extension $K \hookrightarrow L$, show that $K^{1/p^e} \otimes_K L \cong L^{1/p^e}$. This is a special case of Proposition 5.9.

Exercise 5.13. Let $g : A \to B$ be an étale map of Noetherian F-finite rings. Prove Proposition 5.9 using the following steps and the diagram below.

$$\begin{tikzcd}
 & & F_*B \\
F_*A \arrow[r] \arrow[urr, "F_*g"] & F_*A \otimes_A B \arrow[ur, dotted] & \\
A \arrow[u, "F"] \arrow[r, "g"] & B \arrow[u] \arrow[uur, "F"'] &
\end{tikzcd}$$

(a) Explain and label the diagram. In particular, identify three solid arrows representing étale maps.
(b) Prove that the dotted arrow[23] is étale by proving the following: If a composition $f \circ h$ and h are both étale, then also f is étale.
(c) Show that the dotted arrow induces an isomorphism on fibers—that is, it becomes an isomorphism after tensoring with $\kappa(x)$ for any $x \in \operatorname{Spec} F_*A \otimes_A B$.
(d) Show that the dotted arrow is an isomorphism.

Hint: For (b), use the fact that a finite type map of rings is unramified if and only if the corresponding diagonal map is open [**Mil80**, Prop 3.5 in Chap 1]

[23] which, by the way, is the relative Frobenius map $F_{B/A}$ of Exercise 1.8.

For (c), show that the fiber over x is both separable and purely inseparable over $\kappa(x)$. For (d), show that a finite flat map of Noetherian rings with isomorphic fibers is an isomorphism.

5.6.1. *Variants of test ideals.*

Exercise 5.14. Suppose R is a Noetherian F-finite reduced ring with minimal primes $Q_1, \dots, Q_t, \dots, Q_n$. We define $\tau_{\not\subseteq Q_1,\dots Q_t}(R)$ to be the set of elements $c \in R$ such that for every $d \in R \setminus (Q_1 \cup \cdots \cup Q_t)$ there exists an $e_0 \geq 0$ such that for every $e \geq e_0$ there exists $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that

$$c \in \operatorname{Image}\big(\operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\text{eval at } F^e_* d} R\big).$$

Prove that $\tau_{\not\subseteq Q_1,\dots Q_t}(R)$ is an ideal of R whose expansions in $R_{Q_{t+1}}, \dots, R_{Q_n}$ are all zero.

Hint: Use prime avoidance and multiplication of elements to show that we may restrict to ds in the above definition whose images $d/1$ in each of $R_{Q_{t+1}}, \dots, R_{Q_n}$ are all zero.

Exercise 5.15. Suppose R is a Noetherian F-finite reduced ring with minimal primes $Q_1, \dots, Q_t, \dots, Q_n$. Suppose that $c \in R \setminus (Q_1 \cup \cdots \cup Q_t)$ but $c \in Q_{t+1}, \dots, Q_n$. Suppose that R_c is strongly F-regular. Prove that

$$c^N \in \tau_{\not\subseteq Q_1,\dots,Q_t}(R)$$

for some integer $N \gg 0$. In particular conclude that $\tau_{\not\subseteq Q_1,\dots,Q_t}(R) \not\subseteq Q_1, \dots, Q_t$.

Hint: Mimic the argument of Theorem 5.21.

Exercise 5.16. Suppose R is a Noetherian F-finite reduced ring with minimal primes $Q_1, \dots, Q_t, \dots, Q_n$. Show that $\tau_{\not\subseteq Q_1,\dots,Q_t}(R)$ can be generated by elements which are not contained in any of $Q_1, \dots, Q_t$.

Hint: Modify the proof of Corollary 5.23.

Exercise 5.17. Consider the following variant of a strong test element in which we replace the phrase "all non-zerodivisors d" by the phrase "all d not in any minimal prime": Define c to be a **height test element** if for all d *not in any minimal prime of* R, there exists an $e_0 > 0$ such that c is in the image of the map

$$\operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\text{eval at } F^e_* d} R \tag{5.28.1}$$

for all $e \geq e_0$. Prove that the set of all height test elements forms an ideal of R, which is trivial (in the Noetherian F-finite setting) if and only if R is strongly F-regular.

Hint: The argument is the same as in Lemma 5.17.

Exercise 5.18. Prove that if R is an F-finite Noetherian ring and $b \in R$ is such that $R[b^{-1}]$ is strongly F-regular, then b has a power that is a *height test element* (as defined in Exercise 5.17). Deduce that the ideal of height test elements has positive height if R is a Noetherian F-finite generically reduced ring

Hint: See Theorem 5.21.

Exercise 5.19. Prove analogs of Exercises 5.8 and 5.9 for the height test ideal (in the sense of Exercise 5.17).

6. Compatibility of ideals and maps

We now introduce the important idea of *compatibility* between ideals of R and maps in $\mathrm{Hom}_R(F^e_* R, R)$. Essentially, an ideal J is compatible with a map ϕ whenever ϕ induces a map for the quotient ring R/J. This leads to the well-known notion of *compatibly split* subschemes of Frobenius split schemes. Such schemes (when irreducible) can be viewed as prime characteristic analogs of *centers of log-canonicity* in complex geometry; see Chapter 6.

In this section, we will characterize the test ideal $\tau(R)$ of a reduced Noetherian F-finite ring as the *unique smallest* ideal of positive height that is *uniformly compatible*—meaning compatible with *every* map in $\mathrm{Hom}_R(F^e_* R, R)$ for *every* $e \geq 0$. This characterization allows us to easily establish basic properties of the test ideal, such as its good behavior under localization, completion, and étale maps.

6.1. Compatible ideals.

Fix a ring R of prime characteristic p, and an *arbitrary* R-linear map $F^e_* R \longrightarrow R$ (*not* necessarily a splitting of Frobenius). Given some quotient ring $\overline{R} = R/J$, we would like to understand when our map descends to a well-defined map $F^e_* \overline{R} \longrightarrow \overline{R}$.

Definition 6.1. Given a map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ and an ideal $J \subseteq R$, we say that J is **compatible with** ϕ, or symmetrically, that ϕ **is compatible with** J, if

$$\phi(F^e_* J) \subseteq J.$$

Equivalently, ϕ and J are **compatible** if there exists an $\overline{R}$-module map $\overline{\phi}$ making the diagram

$$\begin{tikzcd} F^e_* R \arrow[r, "\phi"] \arrow[d, two heads] & R \arrow[d, two heads] \\ F^e_* \overline{R} \arrow[r, dotted, "\overline{\phi}"] & \overline{R} \end{tikzcd} \tag{6.1.1}$$

commute. Here, the vertical arrows are the natural quotient maps, so $\overline{\phi}$, when it exists, is uniquely defined by

$$F^e_*(x \bmod J) \mapsto \phi(F^e_* x) \bmod J.$$

We also say J **is ϕ-compatible,** or simply that J **is compatible** when the map ϕ clear from the context.

Example 6.2. For a trivial example, note that every ideal of R is compatible with the zero map in $\operatorname{Hom}_R(F^e_* R, R)$. Likewise, every map in $\operatorname{Hom}_R(F^e_* R, R)$ is compatible with both the zero ideal and the trivial ideal R.

Example 6.3. For a less trivial example, let π be the standard monomial splitting Frobenius for $\mathbb{F}_p[x, y]$ described in Example 3.2. Then the principal ideal (x) is compatible with π. Indeed, for any monomial $x^a y^b \in \mathbb{F}_p[x, y]$, if we use the division algorithm to write the exponents as

$$a = n_1 p + r_1 \qquad \text{and} \qquad b = n_2 p + r_2 \qquad\qquad \text{where} \qquad r_i < p, \tag{6.3.1}$$

then we can check that

$$\pi(F_* x^a y^b) = x^{n_1} y^{n_2} \phi(F_* x^{r_1} y^{r_2}) = \begin{cases} x^{n_1} y^{n_2} & r_1 = r_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now, for a monomial $x^a y^b \in (x)$, we have $a \geq 1$, so either $r_1 > 0$ or $n_1 \geq 1$ in (6.3.1). Thus $\pi(F_* x^a y^b) \in (x)$, and (x) is π compatible.

Compatible ideals are closed under sums and intersections:

Proposition 6.4. *Let $F^e_* R \xrightarrow{\phi} R$ be an R-linear map, where R is an arbitrary ring of prime characteristic. Then*

(a) Arbitrary sums of ϕ-compatible ideals are ϕ-compatible.
(b) Arbitrary intersections ϕ-compatible ideals are ϕ-compatible.

Proof. This is immediate from the definition. $\square$

Example 6.5. Let $S = \mathbb{F}_p[x, y]$. The principal ideal (x) is compatible with the standard monomial splitting π (Example 6.3), and symmetrically, so is (y). Thus both $(xy) = (x) \cap (y)$ and $(x, y) = (x) + (y)$ are π-compatible, by Proposition 6.4.

Example 6.6. Let R be a ring of characteristic $p > 0$, and let $f \in R$. Fix any $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, and consider the composition ϕ' :

$$F^e_* R \xrightarrow{\text{mult by } F^e_* f^{p^e-1}} F^e_* R \xrightarrow{\phi} R.$$

Then the map ϕ' is compatible with the principal ideal (f). Indeed, for an arbitrary $fr \in (f)$, we have

$$\phi'(F^e_* fr) = \phi(F^e_* f^{p^e-1} fr) = \phi(F^e_* f^{p^e} r) = f\phi(F^e_* r) \in (f).$$

In other words, every map in the submodule

$$\{\phi \circ F^e_* f^{p^e-1} \mid \phi \in \mathrm{Hom}_R(F^e_* R, R)\} \subseteq \mathrm{Hom}_R(F^e_* R, R)$$

is compatible with (f).

Compatibility behaves well under localization and completion:

Proposition 6.7. *Let R be a ring of characteristic $p > 0$ and let $W \subseteq R$ be an arbitrary multiplicative set in R. For any ideal $J \subseteq R$, let $W^{-1}J$ denote its image in the localization $W^{-1}R$, and for any map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, let $\frac{\phi}{1}$ denote the naturally induced map $F^e_* W^{-1}R \longrightarrow W^{-1}R$.*

(a) If J and ϕ are compatible, then their localizations are compatible—that is, the ideal $W^{-1}J$ is compatible with the map $\frac{\phi}{1}$.
(b) If the localization $W^{-1}J$ is compatible with $\frac{\phi}{1}$, then the contraction[24] of $W^{-1}J$ to R is compatible with ϕ.
(c) In particular, a prime ideal $P \subseteq R$ is ϕ compatible if and only if $PR_P \subseteq R_P$ is $\frac{\phi}{1}$ compatible.

Proof. We leave (a) and (b) as straightforward exercises; see Exercise 6.9. Note that (c) follows from (a) and (b), since $PR_P \cap R = P$. □

Proposition 6.8. *Let $(R, \mathfrak{m})$ be an F-finite Noetherian local ring of characteristic $p > 0$, and let $(\widehat{R}, \widehat{\mathfrak{m}})$ be its completion at the maximal ideal. For any ideal $J \subseteq R$, let $\hat{J}$ be its $\mathfrak{m}$-adic completion, and for any map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, let $\widehat{\phi}$ be the induced map in $\mathrm{Hom}_{\widehat{R}}(\widehat{F^e_* R}, \widehat{R})$.*

Then J is compatible with ϕ if and only if $\widehat{J}$ is compatible with $\widehat{\phi}$.

Proof. Recall that the $\mathfrak{m}$-adic completion $\widehat{F^e_* R}$ of $F^e_* R$ is canonically identified with $F^e_* \widehat{R}$ by Lemma 1.14. Now if J is ϕ-compatible, we have a

[24] Recall that for any ring map $f : R \longrightarrow S$, the contraction of an ideal $J \subseteq S$ is the ideal $f^{-1}(J) \subseteq R$. This is often (abusively) written $J \cap R$ though it is not literally intersection unless f is injective.

commutative diagram

$$\begin{tikzcd} F^e_* R \arrow[r, "\phi"] \arrow[d] & R \arrow[d] \\ F^e_* R/J \arrow[r, "\overline{\phi}"'] & R/J, \end{tikzcd}$$

so completing in the $\mathfrak{m}$-adic topology produces a commutative diagram

$$\begin{tikzcd} F^e_* \widehat{R} = \widehat{F^e_* R} \arrow[r, "\widehat{\phi}"] \arrow[d] & \widehat{R} \arrow[d] \\ F^e_* \widehat{R/J} = \widehat{F^e_* R/J} \arrow[r, "\widehat{\overline{\phi}}"] & \widehat{R/J}. \end{tikzcd} \tag{6.8.1}$$

Because the completion functor is exact on finitely generated modules, we have $\widehat{R/J} = \widehat{R}/\widehat{J}$, and so the diagram (6.8.1) ensures $\widehat{J}$ is $\widehat{\phi}$ compatible. For this direction we do not need F-finite.

For the other direction, note that since $\widehat{R}$ is faithfully flat, $\widehat{J} \cap R = J\widehat{R} \cap R = J$. The desired result now follows easily from the fact that $\operatorname{Hom}_{\widehat{R}}(F^e_* \widehat{R}, \widehat{R}) \cong \operatorname{Hom}_R(F^e_* R, R) \otimes_R \widehat{R}$ (Lemma 1.25). $\square$

Remark 6.9. Suppose R is F-finite Noetherian but not necessarily local and $\widehat{R}$ is the completion of R along an arbitrary ideal. Then we still have that if J is compatible with ϕ, that $\widehat{J}$ is compatible with $\widehat{\phi}$. The proof is unchanged.

6.2. Uniform compatibility.

6.2. Uniform compatibility. A distinguished class of ideals are those compatible with *every* map:

Definition 6.10. An ideal J in a ring R of characteristic p is said to be **uniformly compatible** if, for every $e \geq 0$, J is compatible with *every* $\phi \in \operatorname{Hom}_R(F^e_* R, R)$.

The next proposition gives many examples:

Proposition 6.11. *Each associated prime of a ring of prime characteristic is uniformly compatible.*

PROOF. When $Q \in \operatorname{Spec} R$ is an associated prime of R, there is an R-module injection $R/Q \hookrightarrow R$ identifying Q with

$$Q = \operatorname{ann}_R x = \{r \in R \mid xr = 0\}$$

for some[25] $x \in R$. Take any $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. We need to show that $\phi(F^e_* Q) \subseteq Q$. To this end, take any $r \in Q$. To see that $\phi(F^e_* r) \in Q$, we

[25]specifically, take x to be the image of $\overline{1}$ under the embedding $R/Q \hookrightarrow R$

check
$$x\phi(F^e_* r) = \phi(F^e_* x^{p^e} r) = \phi(0) = 0.$$
So $\phi(F^e_* r) \in \operatorname{ann}_R x = Q$. □

Example 6.12. Let $R = \mathbb{F}_p[x,y]/(xy, x^2)$. Then the ideals (x) and (x,y) are both associated primes of R, so they are uniformly compatible by Proposition 6.11.

At the opposite extreme, we have

Proposition 6.13. *The only uniformly compatible ideals in a strongly F-regular domain are the zero ideal and the whole ring.*

Proof. Let J be a non-zero ideal in a strongly F-regular domain R. Take any non-zero $c \in R$. By definition of strong F-regularity, there exists $e > 0$ and $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such $\phi(F^e_* c) = 1$. So if J is compatible with ϕ, then $J = R$. □

Remark 6.14. Uniform compatibility can be expressed in terms of the Cartier algebra $\mathcal{C}_R$ introduced in Subsection 4.2. By construction, R is a naturally a module over $\mathcal{C}_R$. Now it follows easily that an ideal $J \subseteq R$ is uniformly compatible if and only if J is a $\mathcal{C}_R$-submodule of R; see Exercise 6.10.

6.3. The test ideal is uniformly compatible. The test ideal admits an important characterization in terms of compatibility:

Theorem 6.15. *Let R be a reduced Noetherian F-finite ring. The test ideal is the unique smallest uniformly compatible ideal containing a non-zerodivisor.*

Proof. We first show that $\tau(R)$ is uniformly compatible. Take any $c \in \tau(R)$. We must show that for any $\phi \in \operatorname{Hom}_R(F^{e'}_* R, R)$ (for arbitrary e'), $\phi(F^{e'}_* c) \in \tau(R)$.

By definition of strong test element, we know that for all non-zerodivisors $d \in R$, there exists e_0 such that for all $e \geq e_0$, there is some $\psi \in \operatorname{Hom}_R(F^e_* R, R)$ such that
$$c = \psi(F^e_* d). \tag{6.15.1}$$
Applying ϕ to (6.15.1), we have
$$\phi(F^{e'}_* c) = \phi(F^{e'}_* (\psi(F^e_* d)). \tag{6.15.2}$$
In other words,
$$\phi(F^{e'}_* c) = (\phi \star \psi)(F^{e'+e}_* d)$$

where $\phi \star \psi \in \operatorname{Hom}_R(F^{e+e'}_* R, R)$. Thus clearly for all $f \geq e_0 + e'$, we can find an element in $\operatorname{Hom}_R(F^f_* R, R)$ such that $\phi(F^{e'}_* c)$ is in the image of

$$\operatorname{Hom}_R(F^f_* R, R) \xrightarrow{\text{eval at } d} R.$$

That is, $\phi(F^{e'}_* c) \in \tau(R)$.

Next, note that $\tau(R)$ contains a non-zerodivisor by Corollary 5.22. To show that $\tau(R)$ is *minimal* among uniformly compatible ideals J containing a non-zerodivisor, take any uniformly compatible ideal J, and let $d \in J$ be a non-zerodivisor. We need to show that $\tau(R) \subseteq J$. Take an arbitrary $c \in \tau(R)$. By definition of strong test element, there exists some $\psi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\psi(F^e_* d) = c$. Since J is ψ-compatible and $d \in J$, we know $\psi(F^e_* d) \in J$. That is, $c \in J$. This shows $\tau(R) \subseteq J$. □

There is a sense in which any strong test element which is a non-zerodivisor generates *all* the strong test elements.

Corollary 6.16 (*cf.* [**HT04**, Lemma 2.1])**.** *] Let R be a reduced Noetherian F-finite ring. Let $d \in \tau(R)$ be any non-zerodivisor in the test ideal. Then*

$$\tau(R) = \sum_{e \geq 0} \sum_{\phi} \phi(F^e_* d)$$

where ϕ runs over elements of $\operatorname{Hom}_R(F^e_* R, R)$ Put differently, the test ideal $\tau(R)$ is generated by any non-zerodivisor in $\tau(R)$ as a (left-)module over the Cartier Algebra $\mathcal{C}_R$.

Proof. The ideal on the right is compatible with all maps $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ for all e, and is clearly the smallest such ideal containing d. So since $d \in \tau(R)$, it must be contained in $\tau(R)$ by Theorem 6.15. By minimality of $\tau(R)$, we get equality. □

Several basic properties of the test ideal follow easily (see Exercise 6.15):

Proposition 6.17. *Suppose R is a reduced F-finite Noetherian ring. Then:*

(a) *For any multiplicative set $W \subseteq R$, we have that $\tau(W^{-1}R) = W^{-1}\tau(R)$.*

(b) *If R is local and $\widehat{R}$ denotes the completion at an arbitrary ideal, then $\widehat{\tau(R)} = \tau(R)\widehat{R}$.*

(c) *If S is any F-finite étale R-algebra, then $\tau(S) = \tau(R)S$.*

Proof. See Exercise 6.15. □

6.4. Compatible Frobenius splitting. Compatibility is of particular interest for Frobenius splittings:

Definition 6.18. Let R be a Frobenius split ring. We say an ideal $J \subseteq R$ is **compatibly split**[26] (by ϕ) if there exists a splitting of Frobenius $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ compatible with J.

We see immediately that if J is a compatibly split ideal in a Frobenius split ring R, then R/J is Frobenius split. It follows that compatibly split ideals are always radical.

Example 6.19. Consider the polynomial ring $S = \mathbb{F}_p[x_1, \ldots, x_n]$, with its standard monomial splitting of Frobenius $\pi : F_* S \longrightarrow S$ sending the basis element $F_* 1$ to 1 and all other monomial basis elements in the basis $\{F_* x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i \leq p-1\}$ to zero. Generalizing Example 6.3, the reader will quickly verify that each ideal (x_i) is compatibly split with π. In light of Proposition 6.4, it follows that all ideals generated by square-free monomials are compatibly split by π. See Exercise 6.17.

Remark 6.20. Looking more closely at the *prime* compatibly split ideals we found in Example 6.19, we see that there are exactly $\binom{n}{d}$ of each height d:

$$\{(x_{i_1}, x_{i_2} \ldots, x_{i_d}) \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}.$$

In fact, according to [**ST10b**] (*cf.* [**HW15**]), there are at most $\binom{n}{d}$ primes compatibly split with respect to a fixed splitting in *any* Frobenius split ring R. So we have in fact found *all* ideals compatibly split with the standard monomial splitting of Frobenius in $\mathbb{F}_p[x_1, \ldots, x_n]$. See Exercise 6.18.

Caution 6.21. If R is Frobenius split, there can be Frobenius split quotients R/J that are not compatibly split for *any* $\phi \in \operatorname{Hom}_R(F^e_* R, R)$.

Remark 6.22. Compatible Frobenius splitting was first defined by Mehta and Ramanathan in [**MR85**], although it was also considered implicitly a few years earlier by Fedder [**Fed83**]. Later, ideals compatible with *all* $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ were identified as an important class in Schwede's work *centers of F-purity* [**Sch10a**].

Remark 6.23. In a normal $\mathbb{Q}$-Gorenstein ring R, the set of uniformly compatible ideals are closely related to the "log canonical centers" (or "non-klt centers") of $X = \operatorname{Spec} R$. For additional discussion see [**Sch10a**].

6.5. Exercises.

Exercise 6.1. Let $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ and $J \subseteq R$ be a compatible pair. and let $\overline{\phi}$ be the induced map $F^e_* R/J \xrightarrow{\phi} R/J$. Show that

[26] More precisely, one might say "compatibly *Frobenius* split" or "compatibly F-split", since we require that J be compatible with a splitting of *Frobenius* and not some other splitting. But historically, the "Frobenius" has been dropped from this phrase.

(a) If ϕ is surjective, then so is $\overline{\phi}$.
(b) If R is local and $\overline{\phi}$ is surjective, then so is ϕ.
(c) If R is local, show $\overline{\phi} \circ F$ is an automorphism of R/J if and only if $\phi \circ F$ is an automorphism for R.

Exercise 6.2. Let S be an F-finite *regular* ring, and R any quotient. Prove that *every* map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ lifts to some map in $\mathrm{Hom}_S(F^e_* S, S)$–that is, there exist $\psi \in \mathrm{Hom}_S(F^e_* S, S)$ compatible with the kernel of $S \twoheadrightarrow R$ and inducing ϕ.

Hint: Kunz's Theorem implies that $F^e_* S$ is a projective S-module.

Exercise 6.3. Show that for any $x \in R$ and for all $e \geq 0$, the ideal $\mathrm{ann}_R\, x \subseteq R$ is compatible with every $\phi \in \mathrm{Hom}_R(F^e_* R, R)$.

Exercise 6.4. Show that if J is compatible with $\phi \in \mathrm{Hom}(F^e_* R, R)$, then J is $\phi^{\star n}$-compatible for all n.

Exercise 6.5. Show that if $\phi \in \mathrm{Hom}(F^e_* R, R)$ is surjective, then every ϕ-compatible ideal J is radical and $\phi(F^e_* J) = J$.

Exercise 6.6. Show that if R is strongly F-regular, then the only proper uniformly compatible ideals are the minimal primes of R and intersections of them.

Hint: Use the fact that a strongly F-regular ring is a product of strongly F-regular domains.

Exercise 6.7. Prove that an F-finite Noetherian integral domain R is strongly F-regular if and only if the only uniformly compatible ideals are (0) and R. This proves a converse to Proposition 6.13.

Exercise 6.8. Let R be a Noetherian F-finite ring. Show that a prime P is uniformly compatible if and only if PR_P is uniformly compatible in R_P.

Exercise 6.9. Prove Proposition 6.7.

Exercise 6.10. Let R be any ring of characteristic $p > 0$, and let $J \subseteq R$ be any ideal. Show that J is uniformly compatible if and only if J is a $\mathcal{C}_R$-submodule of R. Here $\mathcal{C}_R$ denotes the Cartier algebra, as defined in Subsection 4.2.

Exercise 6.11. Find an example of a ring R and quotient ring R/J, both of which are Frobenius split, but not compatibly for any $\phi \in \mathrm{Hom}_R(F_* R, R)$.

Exercise 6.12. Let $R \xrightarrow{f} S$ be a homomorphism of rings of characteristic p and assume $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ and $\psi \in \mathrm{Hom}_S(F^e_* S, S)$ commute with f in the sense that the diagram

$$\begin{tikzcd} F^e_* S \arrow[r, "\psi"] & S \\ F^e_* R \arrow[u, "F^e_* f"] \arrow[r, "\phi"'] & R. \arrow[u, "f"'] \end{tikzcd}$$

commutes. Prove that if $J \subseteq S$ is a ψ-compatible ideal, then its contraction $f^{-1}(J) \subseteq R$ is ϕ-compatible.

Exercise 6.13. Suppose S is a ring of characteristic $p > 0$ and $Q \subseteq S$ is prime. Let $\phi \in \mathrm{Hom}_S(F^e_* S, S)$ and let $\phi_Q : F^e_* S_Q \longrightarrow S_Q$ be the induced map. Show that ϕ is compatible with Q if and only if ϕ_Q is compatible with QS_Q.

Exercise 6.14. Suppose S is a ring of characteristic $p > 0$ and $Q_1, \dots, Q_t \in \operatorname{Spec} S$ are *incomparable* prime ideals and $\phi \in \mathrm{Hom}_S(F^e_* S, S)$. Prove that each Q_i is compatible with ϕ if and only if $Q_1 \cap Q_2 \cap \cdots \cap Q_t$ is compatible with ϕ.

Hint: Use Exercise 6.13 and its proof.

Exercise 6.15. Use Corollary 6.16 to deduce that the test ideal commutes with localization, completion and étale maps (see Proposition 6.17).

Hint: In each case, show that there exists $d \in R$ that is also a strong test element for $W^{-1}R, \widehat{R}, S$. For (c), you'll also need to use Proposition 5.9 to show that for an étale map $R \to S$, $\mathrm{Hom}_R(F^e_* R, R) \otimes_R S \cong \mathrm{Hom}_S(F^e_* S, S)$ for all $e \geq 0$.

Exercise 6.16. Suppose R is a ring of characteristic $p > 0$ and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$. Let J be an ideal. Prove that

$$J + \phi(F^e_* J) + \phi^2(F^{2e}_* J) + \cdots = \sum_{i=0}^{\infty} \phi^i(F^{ie}_* J)$$

is the smallest ideal containing J and compatible with ϕ. Note if R is Noetherian then this sum eventually stabilizes. In fact, as soon as the nth sum agrees with the $(n+1)$st sum, it stabilizes. This observation is useful in computing test ideals, see Corollary 6.16 and **[Kat08]**.

Exercise 6.17. Prove that all radical monomial ideals in $\mathbb{F}_p[x_1, \dots, x_n]$ are compatible with the standard monomial Frobenius splitting of $\mathbb{F}_p[x_1, \dots, x_n]$ defined in Example 3.2. This generalizes Example 6.3.

Hint: A radical monomial ideal is the intersection of monomial *prime* ideals—that is, ideals generated by subsets of the variables.

Exercise 6.18. Show that any ideal of $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]$ compatible with the standard monomial splitting of Frobenius is a radical monomial ideal.

Hint: Use the fact that the monomials are the elements homogeneous with respect to the standard multi-grading.

Exercise 6.19. Generalizing the previous exercise, let R be a finitely generated algebra over an F-finite field that is *graded* by some torsion free abelian semi-group G. For example, R might be a polynomial algebra with its standard $\mathbb{N}$-grading, or with its standard multi-grading (which we can view as its standard $\mathbb{N}^n$-grading).

(a) Show that $F^e_* R$ has a natural $\frac{1}{p^e}G$ grading defined by $\deg(F^e_* r) = \frac{\deg r}{p^e}$ and that the Frobenius map $R \to F^e_* R$ is degree preserving map of graded R-modules with this grading.
(b) Show that if $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ is a homogeneous mapping, then any ideal compatible with ϕ is homogeneous.
(c) Show that if $J \subseteq R$ is homogeneous, then the submodule $\mathcal{M}_J \subseteq \mathrm{Hom}_R(F^e_* R, R)$ of maps compatible with J is a graded submodule.

Exercise 6.20. Suppose R is a Noetherian F-finite domain and $R \subseteq S$ is a finite extension of rings. Prove that

$$\tau(R) \subseteq \mathrm{Image}(\mathrm{Hom}_R(S, R) \xrightarrow{\text{eval at } 1_S} R).$$

Hint: Localize to show that $0 \neq \mathrm{Image}(\mathrm{Hom}_R(S, R) \xrightarrow{\text{eval at } 1_S} R)$ and show that the image is uniformly compatible.

Exercise 6.21. Suppose that R is strongly F-regular. Use Exercise 6.20 to prove that every finite ring extension $R \subseteq S$ splits as a map of R-modules. That is, prove that strongly F-regular rings are **splinters.**[27]

Exercise 6.22. Let R be a Noetherian F-finite ring, *not necessarily reduced.* Show that if the ideal of all height test elements contains some d not in any minimal prime, then it is the unique smallest uniformly compatible ideal containing of positive height, and equal to

$$\sum_{e \geq 0} \sum_{\phi} \phi(F^e_* d)$$

[27] It is an open question whether the converse is true for Noetherian F-finite rings. This is known, for example, for $\mathbb{Q}$-Gorenstein rings [**Sin99a**], and rings with finitely generated anti-canonical rings [**CEMS18**] (also proven independently by Singh, but not published).

where ϕ runs over all elements of $\mathrm{Hom}_R(F^e_* R, R)$. Show, furthermore in this case, that the ideal of height test elements commutes with localization in this case.

Hint: See Exercise 5.17.

Exercise 6.23. Suppose R is an F-finite reduced ring. Fix $e_0 \geq 0$. Show that $\tau(R)$ is the smallest ideal, not contained in any minimal prime of R, such that $\phi\big(F^e_* \tau(R)\big) \subseteq \tau(R)$ for all $e \geq e_0$. Additionally show that

$$\tau(R) = \sum_{e \geq e_0} \sum_{\phi} \phi(F^e_* \tau(R))$$

where ϕ runs over all elements of $\mathrm{Hom}_R(F^e_* R, R)$.

Exercise 6.24. Let R be a Noetherian F-finite reduced ring, and let d be a non-zerodivisor that is also a strong test element. Show that

$$\tau(R) := \bigcap_{e \gg 0} \mathrm{im}\left(\mathrm{Hom}_R(F^e_* R, R) \xrightarrow{\text{eval at } F^e_* d} R\right).$$

6.5.1. *More on variants of test ideals.* Refer to the exercises above in Subsection 5.6.1 for notation.

Exercise 6.25. Fix R a Noetherian F-finite reduced ring with minimal primes $Q_1, \ldots, Q_t, \ldots, Q_n$. With notation as in Exercise 5.14, show that $\tau_{\not\subseteq Q_1,\ldots,Q_t}(R)$ is the smallest ideal, not contained in any of $Q_1, \ldots, Q_t$, that is uniformly compatible. Conclude that

$$\tau_{\not\subseteq Q_1,\ldots,Q_t}(R) = \sum_{e \geq 0} \sum_{\phi \in \mathrm{Hom}_R(F^e_* R, R)} \phi(F^e_* dR)$$

for any $d \in \tau_{\not\subseteq Q_1,\ldots,Q_t}(R)$ not in $Q_1, \ldots, Q_t$.

Exercise 6.26. Fix R a Noetherian F-finite reduced ring with minimal primes $Q_1, \ldots, Q_t, \ldots, Q_n$. Show that

$$\tau_{\not\subseteq Q_1,\ldots,Q_t}(R) = \sum_{i=1}^{t} \tau_{\not\subseteq Q_i}(R) \cong \bigoplus_{i=1}^{t} \tau_{\not\subseteq Q_i}(R).$$

In particular, $\tau(R) = \sum_{i=1}^{n} \tau_{\not\subseteq Q_i}(R)$. Conclude that $\tau_{\not\subseteq Q_1,\ldots,Q_t}(R) = \tau(R) \cap Q_{t+1} \cap \cdots \cap Q_m$.

Hint: Show first that each $\tau_{\not\subseteq Q_i}(R) \subseteq \tau_{\not\subseteq Q_1,\ldots,Q_t}(R)$ for $i = 1, \ldots, t$. Then use minimality for the first equality. For the isomorphism, use Exercise 5.14.

Exercise 6.27. Suppose R is a Noetherian F-finite reduced ring with minimal primes $Q_1, \ldots, Q_t, \ldots, Q_n$. Suppose c is a strong test element such that $c \in Q_{t+1} \cap Q_n$ (that is, $c/1$ is zero in $R_{Q_{t+1}}, \dots, R_{Q_n}$). Show for every element $d \in R \setminus (Q_1 \cup \dots \cup Q_t)$ we have that there exists $e_0 > 0$ such that

$$c \in \mathrm{Image}(\mathrm{Hom}_R(F^e_* R, R) \xrightarrow{\text{eval at } F^e_* d} R)$$

for all $e \geq e_0$.

Hint: Note it is harmless to replace d with a multiple. Also use Exercise 6.26 to show that $c \in \tau_{\not\subseteq Q_1, \dots, Q_t}(R)$.

6.5.2. *Exercises on conductors and normalization.* We next recall the definition of the conductor.

Definition 6.24 (The conductor)**.** Suppose that R is a reduced Noetherian ring with normalization in its total ring of fractions $R^{\mathrm{N}} \subseteq \mathcal{K}(R)$. For any $R \subseteq R' \subseteq R^{\mathrm{N}}$ we define the **conductor ideal of R in R'** to be

$$\mathfrak{c}_{R'/R} := R :_{\mathcal{K}(R)} R'.$$

If $R' = R^{\mathrm{N}}$, then this is just called the **conductor of** R as we've seen before, and simply denoted by $\mathfrak{c}$. This ideal gives a scheme structure to the locus where $\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R$ is not an isomorphism.

Exercise 6.28. Show that also $\mathfrak{c}_{R'/R} = R :_{R'} R' = R :_R R'$. Next prove that the conductor is the unique largest ideal of R that is also an ideal of R'.

Exercise 6.29. In the notation of Exercise 6.20, assume S is the normalization of the domain R. Show also that $\mathrm{Image}(\mathrm{Hom}_R(S, R) \xrightarrow{\text{eval at } 1_S} R) = \mathfrak{c}$, where $\mathfrak{c} = R :_R S$ is the **conductor** of R. Conclude that $\tau(R) \subseteq \mathfrak{c}$.

Exercise 6.30. With notation as in Exercise 6.28, show that for any map $\psi \in \mathrm{Hom}_R(F^e_* R, R)$, we have that $\psi(F^e_* \mathfrak{c}_{R'/R}) \subseteq \mathfrak{c}_{R'/R}$, that is $\mathfrak{c}_{R'/R}$ is compatible with ψ.

Hint: Tensor ψ with $\mathcal{K}(R)$ to obtain a map $F^e_* \mathcal{K}(R) \longrightarrow \mathcal{K}(R)$. Apply that map to $x \in \mathfrak{c}_{R'/R}$ and use that $\mathfrak{c}_{R'/R}$ is an ideal of R'.

Exercise 6.31. Let R be a Noetherian F-finite domain whose normalization S is strongly F-regular. Prove that $\tau(R)$ is the conductor $\mathfrak{c} = (R :_R S)$.

Hint: For $c \in \mathfrak{c}$ and non-zerodivisor $d \in \tau(R)$, prove there is a map in $\mathrm{Hom}_R(F^e_* R, R)$ sending $F^e_* d$ to c by modifying an appropriate map in $\mathrm{Hom}_S(F^e_* S, S)$.

6.5.3. *Test ideals in non-reduced rings.*

Exercise 6.32. Let R be a Noetherian F-finite ring, *not necessarily reduced.* Prove that the ideal of strong test elements is uniformly compatible. Show, furthermore, that the ideal of strong test elements is contained in every uniformly compatible ideal containing a non-zerodivisor.

Exercise 6.33. Let R be a Noetherian F-finite ring, *not necessarily reduced.* Prove that if there is a strong test element that is a non-zerodivisor, then the ideal of strong test elements is the unique smallest uniformly compatible ideal containing a non-zerodivisor.

Exercise 6.34. Let R be a Noetherian F-finite ring, *not necessarily reduced.* With the definition of height test element introduced in Exercise 5.17, show that the ideal of height test elements is uniformly compatible and contained in every uniformly compatible of positive height.

Exercise 6.35. Let R be a Noetherian F-finite ring, *not necessarily reduced.* Show that if the ideal of all strong test elements contains a non-zerodivisor d, then this ideal is

$$\sum_{e \geq 0} \sum_{\phi} \phi(F^e_* d)$$

where ϕ runs over all elements of $\mathrm{Hom}_R(F^e_* R, R)$. Use this to show that the ideal of strong test elements commutes with localization in this case.

7. The Frobenius action on local cohomology

Like any map of schemes, the Frobenius map induces natural maps on cohomology in various settings. In Chapter 3, we will see the tremendous applications of Frobenius to proving vanishing theorems for line bundles on projective varieties.

In this section, we look at the Frobenius action on the *local* cohomology modules of a local ring $(R, \mathfrak{m})$. This will allow us to prove that strongly F-regular rings are Cohen-Macaulay. It also leads naturally to two new classes of singularities: F-injective and F-rational singularities, and allows us to prove that Frobenius splitting behaves well under deformation in the Gorenstein setting.

7.1. Frobenius Action on Local Cohomology. Local cohomology is a standard tool in commutative algebra.[28] Given a ring R and an ideal $J \subseteq R$, recall that the *local cohomology with respect to J* is the collection of

[28]See any of the standard sources [**Har67**], [**BS98**], [**Sta19**, Tag 0952], [**BH93**] or [**Hoc11**] for the basic theory.

derived functors $H^i_J(-)$ on the category of R-modules for the functor $H^0_J(-)$ defined by

$$H^0_J(M) = \{m \in M \mid m_{|\mathcal{U}} = 0 \text{ as a section of } \widetilde{M}, \text{ where } \mathcal{U} = \operatorname{Spec} R \setminus \mathbb{V}(J)\},$$

where $\widetilde{M}$ denotes the quasicoherent sheaf on $\operatorname{Spec} R$ determined by M. If J is a finitely generated ideal of R, we can also write

$$H^0_J(M) = \bigcup_{t\in\mathbb{N}} \{m \in M \mid J^t m = 0\}.$$

The main case for us is the case where $(R, \mathfrak{m})$ is local and $J = \mathfrak{m}$.

The Frobenius map $R \xrightarrow{F^e} F^e_* R$ naturally induces R-linear maps of local cohomology

$$H^i_J(R) \longrightarrow H^i_J(F^e_* R) \cong F^e_* H^i_J(R) \tag{7.0.1}$$

called the *Frobenius action* on local cohomology. For a cohomology class $\eta \in H^i_J(R)$, we will write $F^e_* \eta^{p^e}$ for its image under this induced Frobenius map. The map (7.0.1) itself will be denoted by F^e (somewhat abusively, since this is also the name of the map $R \to F^e_* R$ that induces it).

Remark 7.1. The isomorphism $H^i_J(F^e_* R) \cong F^e_* H^i_J(R)$ in (7.0.1) is a natural isomorphism of R-modules; see Exercise 7.8.

Caution 7.2. Remembering that $R = F^e_* R$ as an abelian group, the Frobenius action can be written

$$H^i_J(R) \xrightarrow{F^e} H^i_J(R) \qquad \eta \mapsto \eta^{p^e}, \tag{7.2.1}$$

which is typical in the literature. Written this way, however, (7.2.1) is *not* R-linear, as

$$F^e(r\eta) = r^{p^e} \eta^{p^e} = r^{p^e} F^e(\eta). \tag{7.2.2}$$

This emphasizes the usefulness of the notation introduced in Subsection 1.4: if we instead view the target of (7.0.1) as the R-module $F^e_* H^i_J(R)$, the equation (7.2.2) precisely says that the Frobenius map is R-linear:

$$F^e(r\eta) = F^e_*(r^{p^e} \eta^{p^e}) = r F^e_*(\eta^{p^e}).$$

7.2. Cohen-Macaulayness of strongly F-regular Rings. Cohen-Macaulayness is a property of Noetherian schemes that is *defined locally*—meaning that a scheme X is defined to be Cohen-Macaulay if all its local rings at closed points are Cohen-Macaulay.[29] A local ring is Cohen-Macaulay if some (equivalently, every) system of parameters is a regular sequence. Cohen-Macaulayness can also be characterized cohomologically:

[29]There are many good basic sources for background on Cohen-Macaulay rings; we suggest [**Har67**], [**BH93**, Chapter 3], [**BS98**] and [**Hoc11**], for example. See also Appendix C for a discussion in terms of the dualizing complex from Grothendieck duality.

Proposition 7.3 ([**Har67**, Theorem 3.8], [**Sta19**, Tag 0AVZ])**.** *A Noetherian local ring $(R, \mathfrak{m})$ is Cohen-Macaulay if and only if the local cohomology modules $H^i_{\mathfrak{m}}(R) = 0$ for all $i < \dim R$.*

Using this local cohomological characterization, we easily prove

Theorem 7.4. [**HH90**] *Every strongly F-regular ring is Cohen-Macaulay.*

Corollary 7.5 (The Hochster Roberts Theorem)**.** [**HR74**] *A direct summand of a regular ring of characteristic $p > 0$ is Cohen-Macaulay.*

Proof of Corollary. We prove the corollary here under the additional assumption that both rings are F-finite; see [**HR74**] for the general case. Recall that F-finite regular rings are strongly F-regular (Example 4.16), and direct summands of strongly F-regular rings are strongly F-regular (Proposition 4.18). So Theorem 7.4 guarantees they are Cohen-Macaulay. $\square$

Remark 7.6. Hochster and Roberts actually proved a stronger statement—that a *pure* subring of a regular ring of characteristic p is Cohen-Macaulay [**HR74**]. *Purity* is a weakening of splitting which is often better behaved for non-finite maps; see Subsection 7.6. Indeed, their motivation was to establish the Cohen-Macaulayness of rings of invariants of linearly reductive groups acting linearly on polynomial rings over an *arbitrary field*, but their proof reduces to characteristic p, a technique we will discuss in Chapter 6. As Hochster and Roberts themselves say "the study of pure subrings is forced upon us, because [in certain steps of the argument], we lose the direct summand property but retain purity."

Remark 7.7. The *mixed characteristic* case of Corollary 7.5—meaning for rings that do not contain any field— had been open for decades, until recently proved by Heitmann and Ma [**HM18**] using Scholze's perfectoid spaces [**Sch12**] and building upon recent work of André, Bhatt, Gabber, and others [**And18, Bha18**].

Proof of Theorem 7.4. Let $(R, \mathfrak{m})$ be a strongly F-regular ring. In particular, R is a normal domain by Theorem 4.30. To prove R is Cohen-Macaulay, it suffices to show that $H^i_{\mathfrak{m}}(R) = 0$ for each $i < \dim R$ (Proposition 7.3). For this, there is no loss of generality in assuming that $(R, \mathfrak{m})$ is complete (Proposition 5.5).

Now, there exists non-zero $c \in R$ such that $cH^i_{\mathfrak{m}}(R) = 0$ for all $i < \dim R$ (by Exercise 7.5, or see Appendix C Corollary 6.5). Since R is eventually Frobenius split along c, by definition, there is an $e > 0$ such that the composition

$$R \xrightarrow{F^e} F^e_* R \xrightarrow{F^e_* c} F^e_* R$$

splits, where the second arrow is just multiplication by $F^e_* c$. So take a splitting $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, and note that this means the composition

$$R \xrightarrow{F^e} F^e_* R \xrightarrow{F^e_* c} F^e_* R \xrightarrow{\phi} R$$

is the identity map on R. Applying the functor $H^i_{\mathfrak{m}}(-)$ we see that the composition

$$H^i_{\mathfrak{m}}(R) \xrightarrow{F^e} H^i_{\mathfrak{m}}(F^e_* R) \xrightarrow{F^e_* c} H^i_{\mathfrak{m}}(F^e_* R) \xrightarrow{\phi} H^i_{\mathfrak{m}}(R) \tag{7.7.1}$$

is also an isomorphism. But because c kills $H^i_{\mathfrak{m}}(R)$, of course $F^e_* c$ kills $F^e_* H^i_{\mathfrak{m}}(R) \cong H^i_{\mathfrak{m}}(F^e_* R)$ as well. So the isomorphism (7.7.1) is the zero map! This means that $H^i_{\mathfrak{m}}(R)$ is zero for all $i < \dim R$, proving that R is Cohen-Macaulay. □

7.3. F-injective and F-rational rings. We are led naturally to define F-injective and F-rational singularities, two classes of singularities that can be viewed as weakenings of Frobenius split and strongly F-regular singularities, respectively.

The key point in the proof of Theorem 7.4 was that, when R is strongly F-regular and c is a non-zerodivisor, the map

$$H^i_{\mathfrak{m}}(R) \xrightarrow{F^e_* c \,\circ\, F^e} F^e_* H^i_{\mathfrak{m}}(R) \tag{7.7.2}$$

is *injective* for each $i \geq 0$ and each $e > 0$. Focusing on this property leads to the following natural definitions:

Definition 7.8. A local ring $(R, \mathfrak{m})$ of prime characteristic is F**-injective** if the natural Frobenius action on local cohomology $H^i_{\mathfrak{m}}(R) \xrightarrow{F^e} F^e_* H^i_{\mathfrak{m}}(R)$ is injective for all $i \in \mathbb{N}$ and all $e > 0$.

Definition 7.9. A Noetherian local ring $(R, \mathfrak{m})$ of prime characteristic and dimension d is F**-rational** if R is Cohen-Macaulay and for all non-zerodivisors c, the map on local cohomology

$$H^d_{\mathfrak{m}}(R) \xrightarrow{F^e_* c \,\circ\, F^e} F^e_* H^d_{\mathfrak{m}}(R) \tag{7.9.1}$$

is injective for some $e > 0$ (equivalently, for all $e \gg 0$, Exercise 7.3)

While these definitions make clear that a local strongly F-regular ring is both F-rational and F-injective, they have some disadvantages. For example, it is not immediately clear whether these definitions of F-rationality or F-injectivity pass to localizations. We will return to these singularities with a dual approach in Chapter 2 which will have the advantage of globalizing easily to non-local rings and schemes in the F-finite setting.

None-the-less the local-analytic approach has advantages, such as making certain properties quite obvious:

Proposition 7.10. *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic. Then*

(a) *If R is Frobenius split, then R is F-injective.*
(b) *If R is strongly F-regular, then R is F-rational.*
(c) *An F-rational ring is F-injective.*
(d) *The ring R is F-injective if and only if its completion $\widehat{R}$ is F-injective.*
(e) *If $\widehat{R}$ is F-rational, then so is R.*

PROOF. Left as Exercise 7.2. □

Remark 7.11. While we did not assume any finiteness conditions in the definition of F-injective, nor F-finiteness in the definition of F-rational, these notions are better behaved when we do. For instance, when R is Noetherian and F-finite, both the F-injective and the F-rational loci are open (Corollary 4.9 and Lemma 5.16, respectively, in Chapter 2).[30] In addition, the converse of (e) holds for F-finite local rings (Corollary 5.18 in Chapter 2).

Remark 7.12. We will prove later that F-rational rings are normal while F-injective rings weakly normal (Lemma 5.19 and Lemma 4.11 in Chapter 2). We do this in the F-finite case although the statements hold more generally.

Remark 7.13. The definition of F-rationality here is different from (but equivalent to) the original tight closure definition of Hochster and Huneke in the case of an excellent local ring [**Smi94**]; see Chapter 7.

7.4. Deformation of F-singularities. We now prove that, in Cohen-Macaulay rings, F-injectivity and F-rationality behave well under deformation, and then deduce the same for Frobenius splitting and F-regularity in Gorenstein rings. The main result is:

Theorem 7.14. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and let $f \in \mathfrak{m}$ be a non-zerodivisor.*

(a) *If $R/(f)$ is F-injective, then also R is F-injective.*
(b) *If $R/(f)$ is F-rational, then also R is F-rational.*

Remark 7.15. We digress to explain the phrase "behaves well under deformation." Suppose $\mathcal{P}$ is a local property of schemes— such as regularity, Cohen-Macaulayness, or strong F-regularity. The phrase "*$\mathcal{P}$ behaves well under deformation*" means that, given a local ring $(R, \mathfrak{m})$ and non-zerodivisor $f \in R$, if $R/(f)$ has property $\mathcal{P}$, then also R has property $\mathcal{P}$. For example, both the Cohen-Macaulay property and regularity (smoothness for varieties) behave well under deformation.

[30]Recently, the openness of the F-injective locus was established for rings of finite type over a local ring that is sufficiently close to excellent [**DM20a**].

So what does this have to do "deformation" in a geometric sense? Given a proper flat family of varieties $X \longrightarrow \mathbb{A}^1_k = \operatorname{Spec} k[t]$, we'd often like to know that if some fiber, say $X_0 = X \times_{\operatorname{Spec} k[t]} \operatorname{Spec} k[t]/(t)$, has property $\mathcal{P}$, then all "nearby" fibers X_λ—deformations of X_0— also have property $\mathcal{P}$. The key step is to show that property $\mathcal{P}$ "deforms" from the fiber X_0 to the *total* space X—that is, that X has property $\mathcal{P}$ at each point x of $X_0 \subseteq X$. Looking at the corresponding surjection of local rings $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X_0,x} = \mathcal{O}_{X,x}/(t)$, this amounts to proving that if $\mathcal{O}_{X,x}/(t)$ satisfies $\mathcal{P}$, then also $\mathcal{O}_{X,x}$ satisfies $\mathcal{P}$.

Proof of Theorem 7.14. Set $T = R/(f)$, and $\dim R = d$. Consider the commutative diagram with exact rows and the two rightmost downward arrows the Frobenius on R and T, respectively:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "f"] \arrow[d] & R \arrow[r] \arrow[d, "F^e"] & T \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & F^e_* R \arrow[r, "F^e_* f"] & F^e_* R \arrow[r] & F^e_* T \arrow[r] & 0.
\end{tikzcd}$$

Note that to make the diagram commute, the leftmost downward arrow must be $F^e_* f^{p^e-1} \circ F^e$, or equivalently, the R-module map sending 1 to $F^e_* f^{p^e-1}$. Applying the local cohomology functor, we have a map of long exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(T) \arrow[r] \arrow[d, hook, "F^e"] & H^d_{\mathfrak{m}}(R) \arrow[r, "f"] \arrow[d, "F^e_* f^{p^e-1} \circ F^e"] & H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(F^e_* T) \arrow[r] & H^d_{\mathfrak{m}}(F^e_* R) \arrow[r] & H^d_{\mathfrak{m}}(F^e_* R) \arrow[r] & 0,
\end{tikzcd}$$

with the exactness of the rows coming from the Cohen-Macaulay assumption (Proposition 7.3).

Now, to prove (a), assume that T is F-injective, so that the left-most downward arrow above is injective. To show R is F-injective, we must show the rightmost downward arrow is injective. For this, it suffices to show that middle vertical arrow is injective (as if $F^e(\eta) = 0$ for some $\eta \in H^d_{\mathfrak{m}}(R)$, then certainly also $F^e_* c F^e(\eta) = 0$). So suppose that $\eta \in H^d_{\mathfrak{m}}(R)$ is in the kernel of the middle map. Without loss of generality, we may assume $f\eta = 0$; indeed, like all elements of $H^d_{\mathfrak{m}}(R)$, η is killed by a power of $\mathfrak{m}$, so we can replace η by $f^t \eta$ where t is maximal such that $f^t \eta \neq 0$. Now, chasing the diagram we see that η must come from the submodule $H^{d-1}_{\mathfrak{m}}(T)$, and using the F-injectivity of T, ultimately, that $\eta = 0$. This proves the deformation of F-injectivity in the Cohen-Macaulay case.

Statement (b) is left as a somewhat tricky exercise; see Exercise 7.19. We will prove it for F-finite (quasi-)Gorenstein rings later using a dual approach (see Remark 5.22 in Chapter 2). We will give a slick proof later in Chapter 7 Theorem 5.12 which is essentially the same as our proof of F-injectivity above. $\square$

It is an open question whether F-injectivity deforms without the Cohen-Macaulay hypothesis.

Conjecture 7.16. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of characteristic $p > 0$ and $f \in \mathfrak{m}$ is a non-zerodivisor such that $R/(f)$ is F-injective. Then R is F-injective.*

Some results in this direction can be found in [**HMS14**], including the fact that Frobenius splitting of $R/(f)$ implies F-injectivity of R. We will prove this and more in Chapter 8, Theorem 1.28.

7.5. The Gorenstein case. There are many equivalent formulations of the (quasi-)Gorenstein property, see Appendix A, Appendix C, [**Hoc11**], or [**BH93**, Chapter 3]. For us the most crucial characterization is the following: A Noetherian local ring $(R, \mathfrak{m})$ is **Gorenstein** if it has a dualizing complex (which is automatic if R is F-finite), and is

(1) Cohen-Macaulay and

(2) the local cohomology module $H^{\dim R}_{\mathfrak{m}}(R)$ is an injective hull of the residue field $R/\mathfrak{m}$.

Condition (2) is equivalent to the condition:

(2′) The canonical module $\omega_R \cong R$.

If condition (2) holds (or equivalently (2′)) without the Cohen-Macaulay hypothesis, then the ring is called **quasi-Gorenstein**. For more discussion see Appendix A Section 8 and Appendix C Lemma 3.14, Corollary 6.6, and Corollary 6.6.

Although Frobenius splitting is generally a *stronger* condition than F-injectivity, these concepts agree for Gorenstein rings:

Proposition 7.17. *An F-finite* quasi-Gorenstein *Noetherian local ring is*

(a) F-injective if and only if it is Frobenius split
(b) F-rational if and only if it is strongly F-regular (in which case it is also Gorenstein since it is Cohen-Macaulay).

To prove Proposition 7.17, we find that it is actually simpler to prove a more general statement. Namely, we can stop insisting that R is F-finite and replace splitting by *purity*, which we now digress to review.

Without the Gorenstein hypothesis, F-rational rings need not be strongly F-regular (or even F-split) and F-injective rings need not be F-split. See Chapter 4 Example 2.16 and Example 2.17.

7.6. F-purity. Recall that an R-module map $M \to N$ is *pure*[31] if, for every R-module P, the induced map $M \otimes_R P \to N \otimes_R P$ is injective. In particular,

Definition 7.18. A ring R of prime characteristic is **F-pure** if the Frobenius map $R \xrightarrow{F} F_*R$ is pure.

Example 7.19. Any faithfully flat map is pure [**Sta19**, Tag 08WP]. In particular, the Frobenius map is pure in a regular ring, by Kunz's theorem. That is, regular rings are F-pure.

F-purity is closely related to Frobenius splitting:

Proposition 7.20. *Every Frobenius split ring is F-pure. Conversely, every Noetherian F-finite F-pure ring is Frobenius split.*

Proof. The first statement is obvious: if the Frobenius map splits, then it also splits (so is injective) after tensoring with any R-module. The second statement is an immediate consequence of Proposition 2.3 in Appendix A. Indeed, that proposition implies the converse even without the Noetherian assumption, provided F_*R is finitely presented as an R-module. $\square$

Remark 7.21. Proposition 7.20 is not surprising, because in general, a pure map is essentially the same as a direct limit of split maps [**Sta19**, Tag 058K].

Now, purity has a useful local analytic characterization, from which we can easily deduce Proposition 7.17:

Lemma 7.22 ([**HR74**, Prop 6.11], Lemma 2.4 in Appendix A)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let E denote an injective hull of its residue field. Then the R-module map $R \xrightarrow{\phi} M$ is pure if and only if the map induced by tensoring with E,*

$$E \xrightarrow{1_E \otimes \phi} E \otimes_R M$$

is injective.

[31] See [**Sta19**, Tag 058H] for a detailed discussion of purity, or Section 1 of Appendix A for the basic facts.

Proof of Proposition 7.17. For (a), we need only check that F-injective implies F-pure, by Proposition 7.20. Now for a (quasi-)Gorenstein ring $(R, \mathfrak{m})$, the local cohomology module $H_{\mathfrak{m}}^{\dim R}(R)$ is an injective hull E of the residue field $R/\mathfrak{m}$ (Appendix C Remark 8.4), so the proof is immediate from Lemma 7.22 once one recalls the natural isomorphism $H_{\mathfrak{m}}^{\dim R}(R) \otimes N \cong H_{\mathfrak{m}}^{\dim R}(N)$ for any R-module N, for us $N = F_*^e R$; see Appendix A Proposition 10.1. The analogous result for F-regularity follows similarly. $\square$

Remark 7.23. The proof of Proposition 7.17 actually shows that a Noetherian quasi-gorenstein ring that is F-injective is also F-pure—without assuming the finiteness of Frobenius or Cohen-Macaulayness.

The next statement follows immediately from Theorem 7.14:

Corollary 7.24. *Let $(R, \mathfrak{m})$ be an F-finite Gorenstein local ring and let $f \in \mathfrak{m}$ be a non-zerodivisor.*

(a) If $R/(f)$ is Frobenius split, then so is R.
(b) If $R/(f)$ is strongly F-regular, then so is R.

Remark 7.25. See Remark 5.22 in Chapter 2 for another proof of Corollary 7.24 which is dual to the one here. That proof extends, by working with a dualizing complex and using Grothendieck duality, to include the case where R is not necessarily Cohen-Macaulay. Thus Corollary 7.24 holds for *quasi-Gorenstein* rings. In fact, it also generalizes to $\mathbb{Q}$-Gorenstein rings in general by [**PS22**].

Remark 7.26. We can not weaken the hypothesis of Corollary 7.24 to include non-Gorenstein Cohen-Macaulay rings: it was known as early as [**Fed83**] that the F-pure property does not deform in general, as we will see in Chapter 4 Example 2.16. At this time, the strongest positive results on deformation of F-purity can be found in [**PS22**] where deformation for F-purity is proved in the $\mathbb{Q}$-Gorenstein case; also see Section 4 in Chapter 5.

7.7. The story so far. We summarize the main classes of F-singularities introduced so far in the following diagram. For an F-finite Noetherian local ring $(R, \mathfrak{m})$, we have proven that the following implications hold:

$$\begin{tikzcd} \text{Strongly } F\text{-Regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-Rational} \arrow[d, Rightarrow] \arrow[l, Rightarrow, bend right, "+\text{Gor.}"'] \\ F\text{-Split} \arrow[r, Rightarrow] & F\text{-Injective} \arrow[l, Rightarrow, bend left, "+\text{quasi-Gor.}"] \end{tikzcd}$$

In addition, the singularities in the top row of the diagram are Cohen-Macaulay and normal, while those on the bottom are weakly normal, although we have not yet proven the normality conditions for the singularities on the right side of the diagram.

Exercises.

Exercise 7.1. Let R be a Frobenius split ring and let $J \subseteq R$ be an arbitrary ideal. Then show that the annihilator, in R, of the local cohomology module $H^i_J(R)$ is radical for all $i \geq 0$.

Hint: If x^2 is in the annihilator, so is x^p. Now use the Frobenius splitting.

Exercise 7.2. Prove Proposition 7.10.

Hint: For (d), observe that $H^i_{\mathfrak{m}}(R) \cong H^i_{\mathfrak{m}}(\widehat{R})$, and for (e), observe that if $c \in R$ is a non-zerodivisor, then its image in $\widehat{R}$ is too.

Exercise 7.3. Suppose that $(R, \mathfrak{m})$ is F-rational and $c \in R$ is a nonzero divisor. Show that $H^d_{\mathfrak{m}}(R) \xrightarrow{F^e_* c \,\circ\, F^e} F^e_* H^d_{\mathfrak{m}}(R)$ injects for all $e \gg 0$.

Hint: Note if $c = 1$, we may take $e = 1$ as well. Then compose injective maps.

Exercise 7.4. Let $\iota : R \to S$ be a finite extension of F-finite Noetherian local domains. Prove that if ι splits, then F-injectivity of S implies F-injectivity of R, and similarly for F-rationality.

Exercise 7.5. Let $(R, \mathfrak{m})$ be a complete local equidimensional ring. Prove that there exists a $c \in R$, not in any minimal prime, such that c annihilates the local cohomology modules $H^i_{\mathfrak{m}}(R)$ for all $i < \dim R$.

Hint: Use the Cohen Structure theorem to write R as a homomorphic image of a regular local ring S. Now use local duality on S to find c.

Exercise 7.6. Suppose R is a Noetherian F-finite ring eventually Frobenius split along some $c \in R$ such that $R[c^{-1}]$ is Cohen-Macaulay. Show that R is Cohen-Macaulay, and in particular, R is locally equidimensional.

Hint: Reduce to the complete local case and first prove that R is equidimensional using Exercise 4.12 (note that if Q_1, Q_2 are minimal primes whose quotients have different dimensions, then $c^n \in Q_1 + Q_2$ for $n \gg 0$).

Exercise 7.7. Suppose that $(R, \mathfrak{m})$ is an F-finite equidimensional reduced local ring. Suppose that $c \in R$ is a strong test element. Prove that $cH^i_{\mathfrak{m}}(R) = 0$ for all $i < d = \dim R$.

Hint: Pick $b \in R$ a non-zerodivisor such that R_b is regular and $bH^i_{\mathfrak{m}}(R) = 0$. Use that there exists $\phi \in \operatorname{Hom} -R(F^e_* R, R)$ such that $\phi(F^e_* b) = c$.

Exercise 7.8. Let J be an ideal in a Noetherian ring R of prime characteristic, and let M be an R-module. Show that for every $i \in \mathbb{N}$ and every $e \in \mathbb{N}$, there is a natural isomorphism of R-modules $H^i_J(F^e_* M) \cong F^e_* H^i_J(M)$.

Hint: Show that if $M \longrightarrow I^\bullet$ is an injective resolution of M, then $F_* M \longrightarrow F_* I^\bullet$ is a flasque[32] resolution of $F_* M$.

Exercise 7.9. Suppose that $R \longrightarrow S$ is an arbitrary map of rings that is pure as an R-module map (for example, split). Show that for every ideal $J \subseteq R$, $J = JS \cap R$. Compare to Lemma 3.11.

Hint: Tensor $R \to S$ with R/J and consider the kernel.

Exercise 7.10. Prove that if $B \hookrightarrow R$ is a finite extension of Noetherian domains, then there exists $\phi \in \operatorname{Hom}_B(R, B)$ such that $\phi(1_R) \neq 0$.

Exercise 7.11. Let $(R, \mathfrak{m})$ be a complete local domain containing a copy of its residue field, and let $x_1, \ldots, x_d$ be a system of parameters. Given $z \in \mathfrak{m}$, prove that R is a finite extension of a Cohen-Macaulay ring B that contains the elements $\{x_1, \ldots, x_d, z\}$.

Hint: Use the Cohen Structure Theorem to represent R as a finite extension of a power series ring in $\{x_1, \ldots, x_d\}$. Now adjoin z.

Exercise 7.12. Let $(R, \mathfrak{m})$ be a complete local domain containing a copy of its residue field, and let $x_1, \ldots, x_d$ be a system of parameters. Suppose there is some z such that $zx_i \in (x_1, \ldots, x_{i-1})$. Prove that there is a non-zerodivisor $c \in R$ such that $cz^{p^e} \in (x_1^{p^e}, \ldots, x_{i-1}^{p^e})$ for all $e > 0$.

Hint: Note that $z^{p^e} x_i^{p^e} \in (x_1^{p^e}, \ldots, x_{i-1}^{p^e})$ for all $e \geq 1$. Now, use Exercises 7.11 and 7.10 to construct a Cohen-Macaulay subring B of R where $cz^{p^e} x_i^{p^e} \in (x_1^{p^e}, \ldots, x_{i-1}^{p^e})$.

[32]By definition, an R module N is **flasque** if the localization maps $N \longrightarrow N_P$ are surjective for all $P \in \operatorname{Spec} R$; see [**Sta19**, Tag 09SV].

Exercise 7.13. Use Exercise 7.12 to show directly that, in a strongly F-regular ring, every system of parameters is a regular sequence—that is, strongly F-regular rings are Cohen-Macaulay.

Exercise 7.14. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay Noetherian local ring of dimension d. Suppose $x_1, \dots, x_d$ is a system of parameters. Because local cohomology can be computed with the Čech complex ([**Sta19**, Tag 0A6R] or [**Hoc11**]), there is an exact sequence

$$\bigoplus_{i=1}^{d} R[(x_1 x_2 \cdots \hat{x}_i \cdots x_d)^{-1}] \longrightarrow R[(x_1 x_2 \cdots x_{d-1} x_d)^{-1}] \longrightarrow H^d_{\mathfrak{m}}(R) \longrightarrow 0,$$

where the first map is induced by the natural localization maps. Representing $\eta \in H^d_{\mathfrak{m}}(R)$ by a fraction $\frac{z}{(x_1^{t_1} x_2^{t_2} \cdots x_d^{t_d})} \in R[(x_1 x_2 \cdots x_{d-1} x_d)^{-1}]$, prove

$$\eta = 0 \quad \text{if and only if} \quad z \in (x_1^{t_1}, x_2^{t_2}, \ \cdots, \ x_d^{t_d}). \tag{7.26.1}$$

Hint: Use the fact that the system of parameters is a regular sequence.

Exercise 7.15. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension d. Let η be a non-zero element in the socle of $H^d_{\mathfrak{m}}(R)$—that is, such that $\mathfrak{m}\eta = 0$. Suppose $x_1, \dots, x_d$ is a system of parameters. Prove that η can be represented by a fraction of the form $\frac{z}{x_1 x_2 \cdots x_d}$ in the notation of Exercise 7.14

Exercise 7.16. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring. Prove that the Frobenius action on $H^d_{\mathfrak{m}}(R)$ is injective if and only if for every ideal $I \subseteq R$ generated by a system of parameters, $I = IF_*R \cap R$.

Exercise 7.17. Give an alternate proof of Theorem 7.14(a) using the representation of elements in local cohomology given in Exercise 7.14.

Hint: Exercise 7.15 may be of use.

Exercise 7.18. An alternate approach to defining F-rationality suggested by the proof of Theorem 7.4 would be to require the injectivity of (7.7.2) for *all* i (and drop the Cohen-Macaulay requirement). Show that if R is a complete, reduced and equidimensional, this definition agrees with the one given in Definition 7.9.

Hint: Use Exercise 7.5 and the proof of Theorem 7.4.

Exercise 7.19. Prove Theorem 7.14(b) under the hypothesis that $R/(f)$ is a domain.

Hint: One approach is outlined in Exercise 7.17; step (c) below might be helpful as well. Another follows our proof for F-injectivity using the following the steps:

(a) Use the same strategy as the proof of deformation for F-injectivity, to show that $F^e f^{p^e-1} c \circ F^e$ is injective on $H^d_{\mathfrak{m}}(R)$ whenever the image $\overline{c} \in R/(f)$ is a non-zerodivisor.

(b) Show that if $d, d' \in R$, and the induced map $F^e dd' \circ F^e$ on $H^d_{\mathfrak{m}}(R)$ is injective for some e, then also the induced map $F^e d \circ F^e$ is injective. Conclude that, given any non-zerodivisor in R, it suffices to find $c \in R$ whose image in T is a non-zerodivisor, and such that $cf^m \in (d)$ for some $m > 0$.

(c) Given an arbitrary non-zerodivisor d on R, show we can write $d = cf^m$ where c is a non-zerodivisor on $R/(f)$.

CHAPTER 2

An intermezzo on Frobenius and canonical modules

Warning, this chapter is likely to be substantially revised.

The goal of this intermezzo is to introduce *the Groethendieck dual to Frobenius* for F-finite schemes. To do so, we must carefully define canonical modules (or sheaves), which as we will see, can be done in a natural way for any Noetherian F-finite scheme of positive characteristic.

Let R be a Noetherian F-finite ring of prime characteristic $p > 0$ and let ω_R be a canonical module for R (which we will define below). As we will explain, applying the functor $\mathrm{Hom}_R(-, \omega_R)$ to the Frobenius map $R \longrightarrow F_*R$, produces an R-module map

$$T : F_*\omega_R \longrightarrow \omega_R, \tag{0.0.1}$$

the *Grothendieck dual of Frobenius* or the *Grothendieck trace of Frobenius*.[1] Such a dual to Frobenius map $T : F_*\omega_X \longrightarrow \omega_X$ exists for any Noetherian F-finite scheme X.

This dual-to-Frobenius map T is important for both the global and local theory of F-singularities. For example, it allows us to globalize the definitions of F-rational and F-injective singularities, which had previously been defined only in the local case (Section 7 of Chapter 1). Likewise, the theory of compatible ideals (Section 6 of Chapter 1) can be generalized to T-compatible submodules of ω_X, which leads to an analog of the *test ideal* called the *test module* living inside the canonical module. The test module provides an obstruction to F-rationality in the same way the test ideal provides an obstruction for strong F-regularity; in particular, we will see that the locus of F-rational points is open for F-finite Noetherian rings.

We hope to bring the dual-to-Frobenius map T down to earth by discussing it in several concrete settings before defining it in general. We will

[1] Technically, the Grothendieck trace map is a map in the derived category of bounded $\mathcal{O}_X$-modules; our map (0.0.1) is the induced map on the homology at a particular spot.

explain how and why the dual-to-Frobenius morphism exists for any Noetherian F-finite scheme, using the recent work of Bhatt, Blickle, Schwede and Tucker guaranteeing the existence of a *canonically defined* dualizing complex in this setting.

Simply defining canonical modules (or dualizing complexes) is a laborious process that has been done only in certain settings, most notably, for proper schemes over a complete local ring.[2] In fact, there are Noetherian schemes—even Cohen-Macaulay local rings—that do not admit any canonical module. Moreover, despite their name, canonical modules are not even unique up to isomorphism! None-the-less, the reader has likely encountered duality and canonical modules in *some* context before. Familiar settings include Serre duality for a smooth projective variety X over k, where the canonical module is $\omega_X = \wedge^{\dim X}\Omega_{X/k}$, and Matlis duality for a complete local equidimensional ring $(R, \mathfrak{m})$, where the canonical module ω_R is the Matlis dual of $H^{\dim R}_{\mathfrak{m}}(R)$. While it is unclear how these definition interact in general, there is a unified approach for Noetherian F-finite schemes of positive characteristic: there is a natural choice—or "*canonical* canonical module" ω_X—in this setting, and applying the dualizing functor $\mathscr{H}om_{\mathcal{O}_X}(-, \omega_X)$, we get a functorial map $T : F_*\omega_X \longrightarrow \omega_X$. This dual-to-Frobenius map plays a starring role in the theory of F-singularities, as we will see in subsequent chapters.

Section 1 begins by discussing the dual to Frobenius for *polynomial rings over a perfect field*, building from there the general story for finitely generated algebras over an F-finite field. In Section 2, we discuss a natural construction of the canonical module and the dual of Frobenius for any *normal* variety, building from the familiar setting of Serre duality on smooth varieties. For local rings, we discuss canonical modules and duality in Section 3 using Matlis duality, culminating with a discussion of the dualizing complex and Grothendieck duality more generally. In Section 4 we will describe the general picture for arbitrary F-finite rings and schemes. Finally, in Section 5, we will discuss an application to F-rational singularities.

While we attempt to keep the discussion elementary, duality is a difficult topic; many technical details—for example, Matlis duality and derived functors—are relegated to the appendices and original references, which dedicated readers may wish to consult.

1. The dual to Frobenius for finite type k-algebras

Let $A \to B$ be any map of commutative rings. Considering $A \longrightarrow B$ as a map of A-modules, we can apply the functor $\operatorname{Hom}_A(-, A)$ to get an A-linear

[2] Add references to Residues and Duality and Conrad's book.

map

$$\begin{aligned} \operatorname{Hom}_A(B, A) &\to \operatorname{Hom}_A(A, A) \cong A \\ \phi \ &\mapsto \quad \phi(1_B). \end{aligned} \tag{1.0.1}$$

We are interested in understanding this map when A is a ring of prime characteristic and $A \longrightarrow B$ is the Frobenius map $A \to F^e_* A$. We have already encountered this "evaluation at 1" map in this context in our proof of the openness of the Frobenius split locus for F-finite rings (Proposition 3.17 in Chapter 1).

Ultimately, for a satisfying duality theory, the functor $\operatorname{Hom}_A(-, A)$ will need to be replaced by $\operatorname{Hom}_A(-, \omega_A)$ for some suitable A-module ω_A. In this section, we explain how to do this for finite type algebras over a field.

1.1. The dual to Frobenius for polynomial rings. We begin by examining this dual to Frobenius map,

$$\operatorname{Hom}_A(F^e_* A, A) \to A \qquad \phi \mapsto \phi(F^e_* 1), \tag{1.0.2}$$

carefully in the case where A is the polynomial ring $k[x_1, \dots, x_d]$ over a perfect field k of characteristic $p > 0$. The story for finite type k-algebras will follow by Noether normalization.

Our first step is a closer look at the module $\operatorname{Hom}_A(F_* A, A)$. Quite generally, if B is an algebra over some ring A and M is some A-module, then $\operatorname{Hom}_A(B, M)$ has a (right) B-module structure induced by the action of B on the *source*. That is, $b \in B$ acts on $\phi \in \operatorname{Hom}_A(B, M)$ by $\phi \circ b$, where we view $b \in B$ as the "multiplication by b" map $B \to B$.

Now fixing any ring R of prime characteristic, we give $\operatorname{Hom}_R(F^e_* R, R)$ the structure of an $F^e_* R$-module in exactly this way. A ring element $F^e_* r$ acts on $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ to produce the composition $\phi \circ F^e_* r$ (or equivalently, $\phi \star r$ in the notation of Subsection 4.2 of Chapter 1).

Example 1.1. Let k be an F-finite field. Then $\operatorname{Hom}_k(F^e_* k, k)$ is isomorphic to $F^e_* k$ as a $F^e_* k$-vector space. Indeed, fixing any non-zero $\psi \in \operatorname{Hom}_k(F^e_* k, k)$, the $F^e_* k$-linear map

$$F^e_* k \longrightarrow \operatorname{Hom}_k(F^e_* k, k) \qquad F^e_* \lambda \mapsto \psi \circ F^e_* \lambda$$

is $F^e_* k$-linear and non-zero, hence injective. On the other hand, it is also a map of k-vector spaces of the same finite dimension over k, so it must be surjective as well.

Caution 1.2. The isomorphism in Example 1.1 is *non-canonical*: it depends on the choice of some non-zero ψ. This non-canonicity is the heart of the difficulty in duality theory.

Example 1.1 generalizes to polynomial rings over k:

Proposition 1.3. *Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field k of characteristic $p > 0$. Then there exists $\Phi^e \in \operatorname{Hom}_A(F^e_* A, A)$ such that the map*

$$F^e_* A \to \operatorname{Hom}_A(F^e_* A, A) \qquad F^e_* a \mapsto \Phi^e \circ F^e_* a \tag{1.3.1}$$

is an isomorphism of $F^e_* A$-modules. Explicitly, Φ^e can be taken to be the A-module map defined on the standard monomial A-module basis

$$\{F^e_* \mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad 0 \leq a_i \leq p^e - 1\} \tag{1.3.2}$$

by

$$\Phi^e(F^e_* \mathbf{x}^{\mathbf{a}}) = \begin{cases} 1 & \text{if } a_1 = \cdots = a_n = p^e - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.3}$$

That is, the map Φ^e (freely) generates $\operatorname{Hom}_A(F^e_* A, A)$ as an $F^e_* A$-module.[3]

Proof of Proposition 1.3. The dual A-module $\operatorname{Hom}_A(F^e_* A, A)$ is freely generated (over A) by the dual basis to (1.3.2):

$$\{\rho_{\mathbf{x}^{\mathbf{b}}} \mid 0 \leq b_i \leq p^e - 1 \text{ for all } i = 1, \ldots n\}, \tag{1.3.4}$$

where $\rho_{\mathbf{x}^{\mathbf{b}}} \in \operatorname{Hom}_A(F^e_* A, A)$ is defined on the basis (1.3.2) by

$$\rho_{\mathbf{x}^{\mathbf{b}}}(F^e_* \mathbf{x}^{\mathbf{a}}) = \begin{cases} 1 & \text{if } a_i = b_i \text{ for } i = 1, 2, \ldots n \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\rho_{\mathbf{x}^{\mathbf{p^e-1}}}$ is the generating map Φ^e, where $\mathbf{x}^{\mathbf{p^e-1}}$ is the monomial $(x_1 x_2 \ldots x_n)^{p^e-1}$.

The map (1.3.1) is clearly $F^e_* A$-linear. To see that it is surjective, it suffices to show that each projection $\rho_{\mathbf{x}^{\mathbf{b}}}$ can be obtained from Φ^e by pre-composition with some element in $F^e_* A$. This is clear: $\rho_{\mathbf{x}^{\mathbf{b}}} = \Phi^e \circ F^e_* \mathbf{x}^{\mathbf{p^e-1-b}}$, because the composition

$$F^e_* A \xrightarrow{\text{mult by } F^e_* \mathbf{x}^{\mathbf{p^e-1-b}}} F^e_* A \xrightarrow{\Phi^e} A \tag{1.3.5}$$

has the exact same effect on each of the free basis elements $F^e_* \mathbf{x}^{\mathbf{a}}$ for $F^e_* A$ as does the map $\rho_{\mathbf{x}^{\mathbf{b}}}$. On the other hand, the fact that the map (1.3.1) is injective can be deduced from Example 1.1, since Φ^e is not a torsion element. □

Remark 1.4. The map Φ^e is called the **standard monomial generating map** for the polynomial ring. Of course, any other $F^e_* A$-module generator $\Psi \in \operatorname{Hom}_A(F^e_* A, A)$ differs from Φ^e by pre-multiplication by a unit in $F^e_* A$.

Definition 1.5. For a Noetherian ring of prime characteristic, an element $\Psi \in \operatorname{Hom}_R(F^e_* R, R)$ that generates as an $F^e_* R$-module will be called a **generating map**.

[3]The map Φ^e is *not* a Frobenius splitting; note that it sends $F^e_* 1$ to 0.

Remark 1.6. There are prime characteristic rings R, other than polynomial rings, which admit generating maps—that is, for which $\mathrm{Hom}_R(F^e_*R, R)$ is isomorphic to F^e_*R. However, this is not the case in general, even for finitely generated algebras over finite fields. A key goal of this chapter is to construct, for all Noetherian F-finite rings, an R-module ω_R for which $\mathrm{Hom}_R(F^e_*R, \omega_R)$ is isomorphic to $F^e_*\omega_R$ as an F^e_*R-module. Such a *functorial* isomorphism is a crucial component of a satisfying duality theory.

Remark 1.7. Although the symbol e in the notation Φ^e in Proposition 1.3 can be read as a simple index, the reader can check that in fact $\Phi^e = (\Phi^1)^{\star e}$, the e-fold composition of Φ^1 with itself in the Cartier algebra; also see Appendix A Proposition 5.3.

Remark 1.8. The proof of Proposition 1.3 works also for the power series ring $k[\![x_1, \dots, x_n]\!]$, or the localization of the polynomial ring at the maximal ideal $(x_1, \dots, x_n)$. Furthermore, one can extend to the case where k is F-finite (rather than perfect) with only slightly more fuss; see Exercise 1.3.

We return to the "evaluation at F^e_*1" map (1.0.2) dual to Frobenius in the polynomial case. Identifying the source $\mathrm{Hom}_A(F^e_*A, A)$ with F^e_*A using Proposition 1.3, the composition

$$F^e_*A \xrightarrow{F^e_*a \mapsto \Phi^e \circ F^e_*a} \mathrm{Hom}_A(F^e_*A, A) \xrightarrow{\text{eval at } F^e_*1} A, \tag{1.8.1}$$

recovers precisely our choice of generating map

$$F^e_*A \to A \qquad \text{sending} \qquad F^e_*a \mapsto \Phi^e_*(F^e_*a).$$

Thus, for a polynomial ring, if we *define* ω_A to be A itself, the generating map (1.8.1) becomes

$$T^e_A : F^e_*\omega_A \to \omega_A, \tag{1.8.2}$$

which we can view as the **dual to Frobenius** for the polynomial ring A. Of course, the map **??** depends on that choice of generator Φ^e, which we see is a generator for $\mathrm{Hom}_A(F^e_*\omega_A, \omega_A)$ as an F^e_*A-module in this case.

Alternatively, the map

$$T^e_A : \mathrm{Hom}_A(F^e_*A, \omega_A) \xrightarrow{\text{eval at } F^e_*1} \omega_A \tag{1.8.3}$$

is defined with no choices, so might be considered a more canonical "dual-to-Frobenius", provided we have a natural isomorphism $F^e_*\omega_A \cong \mathrm{Hom}_A(F^e_*A, \omega_A)$. We will eventually discuss a way to construct ω_A to eliminate all such choices.

1.2. Finitely generated k-algebras. In general, if $A \hookrightarrow B$ is a *finite* inclusion of domains[4] and ω_A is a canonical module for A, then

$$\mathrm{Hom}_A(B, \omega_A) \tag{1.8.4}$$

[4] or more generally, locally equidimensional rings

is a canonical module for B; see Proposition 5.6 in Appendix C.

Now if R is domain of finite type over a field k, then R admits a Noether normalization—that is, a polynomial subalgebra $A = k[x_1, \dots, x_d]$ over which R is finite. So, since A is a canonical module for itself, we can *define* the **canonical module for** R to be

$$\omega_R := \operatorname{Hom}_A(R, A). \tag{1.8.5}$$

This canonical module is independent of the choice of Noether normalization, up to isomorphism, as we will see below in Remark 1.11. Even a smaller ground field $k' \subseteq k$ produces an isomorphic canonical module; see Exercise 1.7.

The following lemma is critical:

Lemma 1.9. *If R is a finitely generated domain over a perfect field k of prime characteristic, then (defining ω_R as in (1.8.5)) there is an $F^e_* R$-module isomorphism*

$$\operatorname{Hom}_R(F^e_* R, \omega_R) \cong F^e_* \omega_R.$$

Remark 1.10. In fact, Lemma 1.9 holds more generally, assuming only that k is F-finite, rather than perfect. See Exercise 1.8.

Proof. Because k is perfect, $F^e_* A \hookrightarrow F^e_* R$ is a Noether Normalization for the finitely generated k-algebra $F^e_* R$, so by definition

$$\omega_{F^e_* R} := \operatorname{Hom}_{F^e_* A}(F^e_* R, F^e_* A) = F^e_* \operatorname{Hom}_A(R, A) = F^e_* \omega_R. \tag{1.10.1}$$

Because $R \to F^e_* R$ is finite, $\operatorname{Hom}_R(F^e_* R, \omega_R)$ is also a canonical module for $F^e_* R$. In fact, it is isomorphic to the canonical module defined in (1.10.1), via the adjunction of tensor and Hom:

$$\operatorname{Hom}_R(F^e_* R, \omega_R) = \operatorname{Hom}_R(F^e_* R, \operatorname{Hom}_A(R, A)) \cong \operatorname{Hom}_A(F^e_* R, A) \cong \omega_{F^e_* R},$$

with the last isomorphism a consequence of the fact that $A \to F^e_* R$ is also a Noether normalization for $F^e_* R$. $\square$

Now applying the functor $\operatorname{Hom}_R(-, \omega_R)$ to the Frobenius map $R \to F^e_* R$, we get an R-module map

$$\operatorname{Hom}_R(F^e_* R, \omega_R) \to \operatorname{Hom}_R(R, \omega_R), \tag{1.10.2}$$

or equivalently, the **dual-to-Frobenius** map

$$T^e_R : F^e_* \omega_R \to \omega_R. \tag{1.10.3}$$

Remark 1.11. The canonical module ω_R, and hence the map above, is independent of the choice of Noether normalization up to isomorphism. This follows from the general theory of Grothendieck duality, *cf.* [**Har66, Con00**] or [**Sta19**, Tag 0DWE]. The point is if $f : \operatorname{Spec} R \longrightarrow \operatorname{Spec} k$ is the induced map, then ω_R is the first non-zero cohomology of $f^!k$. On the other hand, for a Noether normalization $k[x_1, \ldots, x_d] = A \subseteq R$ and induced

$$\operatorname{Spec} R \xrightarrow{g} \operatorname{Spec} A \xrightarrow{h} \operatorname{Spec} k,$$

then $f^!k \cong g^!h^!k$. On the other hand, $h^!k := \wedge^d \Omega_{A/k}[d] \cong A[d]$ and $g^!A = \mathbf{R}\operatorname{Hom}_A(R, A)$, and the bottom cohomology of $\mathbf{R}\operatorname{Hom}_A(R, A)$ is $\operatorname{Hom}_A(R, A)$. Putting this together we see that

$$\omega_R := \operatorname{Hom}_R(R, A)$$

is independent of A. The map $F_*F^!(-) \longrightarrow (-)$ is the trace map in Grothendieck duality.

Remark 1.12. Like in the polynomial ring case, the map T^e_R is a F^e_*R-module generator for $\operatorname{Hom}_R(F^e_*\omega_R, \omega_R)$ under mild hypothesis on R; see Proposition 3.14.

1.3. Affine Varieties. If X is an affine variety whose coordinate ring is the finitely generated k-algebra R, then we can define ω_X to be the coherent sheaf $\widetilde{\omega_R}$ given by the module ω_R. Furthermore, when k is an F-finite field, we define the **dual-to-Frobenius** map

$$T^e_X : F^e_*\omega_X \longrightarrow \omega_X$$

to be the one induced by T^e_R. While this works for the affine variety $X = \operatorname{Spec} R$, more effort is needed to correctly patch the ω_R together to define the canonical module ω_X (and hence the dual to Frobenius T^e_X) on a non-affine variety. To avoid this, for now, we take a more geometric approach to the construction in the next section.

1.4. Exercises.

Exercise 1.1. Let $\Phi^e \in \operatorname{Hom}_A(F^e_*A, A)$ be the standard monomial generating map as defined in Proposition 1.3. Prove that for an arbitrary monomial $x_1^{b_1} \cdots x_n^{b_n}$, we have

$$\Phi^e(F^e_* x_1^{b_1} \cdots\ x_n^{b_n}) = \begin{cases} x_1^{\frac{b_1-(p^e-1)}{p^e}} \cdots\ x_n^{\frac{b_n-(p^e-1)}{p^e}} & \text{if } b_i \equiv p^e - 1 \,(\operatorname{mod} p^e)\ \forall i \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 1.2. Fix a polynomial ring $A = k[x_1, \ldots, x_n]$ over an field k of characteristic $p > 0$. Let $F_{R/k} : F_*k \otimes_k R \to F_*R$ be the relative Frobenius map (see Exercise 1.8 in Chapter 1). Prove a version of Proposition 1.3 in this setting, and use it to derive a "dual to relative Frobenius" for finitely generated domains over k such that the map ${F_{R/k}}_*\omega_R \to F_*k \otimes_k \omega_R$ is F_*k-linear.

Hint: It might be helpful to think of the relative Frobenius as the F_*k-algebra map $(F_*k)[x_1,\ldots,x_n] \hookrightarrow F_*(k[x_1,\ldots,x_n])$ defined by sending each x_i to $F^e_* x_i^{p^e}$.

Exercise 1.3. Fix a polynomial ring $A = k[x_1,\ldots,x_n]$ over an F-finite field k. Fix a basis $\{F^e_*\lambda_i, \mid i = 1\ldots,t\}$ for F^e_*k over k such that $F^e_*\lambda_1 = F^e_*1$. Define $\Phi^e \in \mathrm{Hom}_A(F^e_*A, A)$ by

$$\Phi^e(F^e_*\lambda_i\mathbf{x}^{\mathbf{a}}) = \begin{cases} 1 & \text{if } a_1 = \cdots = a_n = p^e - 1,\ \ \lambda_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that the map

$$F^e_*A \to \mathrm{Hom}_A(F^e_*A, A) \quad F^e_*a \mapsto \Phi^e \circ F^e_*a$$

is an isomorphism of F^e_*A-modules.

Hint: Fix any splitting $\nu \in \mathrm{Hom}_k(F^e_*k, k)$ of Frobenius on k. Consider the composition

$$F^e_*A \to (F^e_*k)[x_1,\ldots,x_n] \xrightarrow{\nu} k[x_1,\ldots,x_n]$$

where the first map is given by formula (1.3.3).

Exercise 1.4. Let R be an F-finite Noetherian ring, and assume that $\phi \in \mathrm{Hom}_R(F^e_*R, R)$ is a generating map.[5] For any multiplicative set $W \subseteq R$, prove that $\frac{\phi}{1} \in \mathrm{Hom}_{W^{-1}R}(F^e_*W^{-1}R, W^{-1}R)$ is an $F^e_*W^{-1}R$-module generator.

Exercise 1.5. Let $(R,\mathfrak{m})$ be an F-finite Noetherian local ring. For any ϕ in $\mathrm{Hom}_R(F^e_*R, R)$, prove that ϕ is an F^e_*R-generator if and only if $\widehat{\phi} \in \mathrm{Hom}_{\widehat{R}}(F^e_*\widehat{R}, \widehat{R})$ is a $F^e_*\hat{R}$-generator.

Exercise 1.6. Let k be a perfect field, and let A be the polynomial ring $k[x_1,\ldots,x_n]$. Show that the dual to Frobenius T^e_A (as defined in (1.8.2)) satisfies

$$T_A\Big(F_*\frac{1}{x_1\cdots x_d}\Big) = \frac{1}{x_1\cdots x_d}T_A\Big(F_*(x_1\cdots x_d)^{p-1}\Big) = \frac{1}{x_1\cdots x_d}$$

when extended to the fraction field of A.

Exercise 1.7. Let R be a domain finitely generated over k. Fix a Noether Normalization $A = k[x_1,\ldots,x_n] \subseteq R$. Now consider a subfield $k' \subseteq k$ such that $[k:k']$ is finite, and the polynomial ring $A' = k'[x_1,\ldots,x_n] \subseteq R$. Notice that $A' \subseteq R$ is a Noether normalization of R with respect to k'. Show that there is an isomorphism of R-modules $\mathrm{Hom}_A(R,A) \cong \mathrm{Hom}_{A'}(R,A')$.

Hint: Adjunction of tensor and Hom

[5] If R is Gorenstein and local, such a ϕ will exist, see Corollary 3.16.

Exercise 1.8. Prove Lemma 1.9 in the case where k is only F-finite, making use of Exercise 1.7 and Exercise 1.3.

Exercise 1.9. Let R be a finitely generated algebra over a perfect field k, and let T_R^1 be the dual of Frobenius as defined in (1.10.3). Explain how the Cartier algebra composition $\star$ on $\bigoplus_e \operatorname{Hom}_R(F_*^e\omega_R, \omega_R)$ gives that $T_R^e = (T_R^1)^{\star e}$ for all $e \geq 1$.

Hint: See Subsection 4.2 of Chapter 1.

2. The dual to Frobenius for varieties

We now examine the dual-to-Frobenius map in a less coordinate-dependent way, starting with the case of smooth varieties over a perfect field. In local coordinates, of course, such a dual-to-Frobenius map will look like the map T_A constructed for polynomial rings in (1.8.2), but by working with differential forms we get a globally defined dual of Frobenius (induced by the *Cartier operator*). Using standard tricks, we can then construct the dual-to-Frobenius map T_X in a coordinate-free way for any *normal* variety X by defining it first on the nonsingular locus and then extending to all of X. For non-normal varieties, a different approach is needed, which we explain concretely for *quasi-projective* varieties over a perfect field, and more generally by making use of the **dualizing complex.**

2.1. Canonical modules on smooth varieties. Recall that a variety X is **smooth over** k if the coherent sheaf of Kähler differentials $\Omega_{X/k}$ is locally free of rank $\dim X$.

Definition 2.1. Let X be an n-dimensional variety *smooth* over a field k. The **canonical module** of X is the invertible sheaf $\omega_X = \bigwedge^n \Omega_{X/k}$.

Example 2.2. If X is $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \dots, x_n]$, then $\Omega_{X/k}$ is the free $\mathcal{O}_X$-module of rank n generated by the differentials dx_i. In particular, $\omega_{\mathbb{A}^n}$ is the free $\mathcal{O}_X$-module generated by $dx_1 \wedge \cdots \wedge dx_n$. On the other hand, if X is $\mathbb{P}^n$, then ω_X is *not* trivial. The n-th exterior power of the locally free sheaf sheaf $\Omega_{X/k}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-n-1)$ [**Har77**, Ch II, Example 8.20.1].

Remark 2.3. This definition of ω_X depends on the ground field k; but see Exercise 2.1.

As before, the adjunction of tensor and Hom produces the following key insight in prime characteristic:

Lemma 2.4. *Let X be a smooth variety over a perfect field of positive characteristic, and let ω_X be as defined in Definition 2.1. Then there is an* *isomorphism of $F^e_*\mathcal{O}_X$-modules*

$$\mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \omega_X) \cong F^e_*\omega_X.$$

PROOF. For those not familiar with Grothendieck duality, the following argument will not be very useful (but perhaps see the references cited). Note that $X \xrightarrow{F} X \xrightarrow{\pi} \operatorname{Spec} k$ gives X another structure as a variety over k. Since $k = k^p$, we see that $\Omega_{(F_*O_X)/k} = \Omega_{(F_*\mathcal{O}_X)/F_*k}$ and so X is also smooth over k with this structure as well. The result then follows from properties (9) and (6) in [**Sta19**, Tag 0AU3] as well as [**Sta19**, Tag 0ATX] applied to the factorization above. □

Now fix a smooth variety X over a perfect field, and consider the Frobenius map $\mathcal{O}_X \to F^e_*\mathcal{O}_X$. Applying the Serre dualizing functor $\mathscr{H}om_{\mathcal{O}_X}(-, \omega_X)$ we get a natural map of sheaves of $\mathcal{O}_X$-modules

$$F^e_*\omega_X \cong \mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \omega_X) \xrightarrow{\text{eval at } 1} \omega_X, \tag{2.4.1}$$

which is our **dual to Frobenius**

$$T_X : F^e_*\omega_X \to \omega_X. \tag{2.4.2}$$

Remark 2.5. Indeed, this turns out to be equivalent to the **Cartier map**[6] which, in local coordinates $\{x_1, \ldots, x_d\}$ at a point z on X, has the property that

$$F^e_* x_1^{p^e-1} \cdots x_d^{p^e-1} dx_1 \wedge \cdots \wedge dx_d \mapsto u dx_1 \wedge \cdots \wedge dx_d,$$

(where u is a unit). Note the similarity with the dual-to-Frobenius map defined in Subsection 1.2 in the case $\mathbb{A}^n_k$.

2.2. Extending to normal varieties. There is a standard trick for working with the canonical module and duality on a *normal* variety. The point is that *reflexive sheaves* on a normal Noetherian scheme, as well as maps between them, are completely determined by their restriction to any open set $\mathcal{U}$ whose complement has codimension two or more: the functor i_*, where $i : \mathcal{U} \hookrightarrow X$ is the natural inclusion, defines an equivalence of categories of reflexive sheaves on $\mathcal{U}$ and X. See [**Sta19**, Tag 0EBJ] and Appendix B Theorem 4.9.

In particular, for normal varieties over a perfect field k, the smooth locus of X/k is an open set whose complement has codimension two or more,[7] so we can define:

[6] because it is induced by the *Cartier isomorphism*. See [**BK05**, Def 1.3.5, Lemma 1.3.6], where T_X is called the "trace map", or [**EV92**, §9.13] for more details on the construction of the Cartier map.

[7] since normal implies regular in codimension 1 [**Sta19**, Tag 031S], which in our setting implies smooth in codimension 1 over k, as k is perfect

Definition 2.6. If X is a normal variety over a perfect field k, then its **canonical module** is defined to be $\omega_X := i_*\omega_{\mathcal{U}}$, where $i : \mathcal{U} \hookrightarrow X$ is the natural inclusion of the smooth locus of X/k in X.

Equivalently, ω_X is the unique reflexive sheaf on X whose restriction to the smooth locus agrees with $\bigwedge^{\dim X} \Omega_{\mathcal{U}/k}$.

Remark 2.7. In terms of divisors, if $K_{\mathcal{U}}$ is a Weil divisor on $\mathcal{U}$ such that $\omega_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}(K_{\mathcal{U}})$, then $\omega_X \cong \mathcal{O}_X(K_X)$ where K_X is the unique Weil divisor on X such that $K_X|_{\mathcal{U}} = K_{\mathcal{U}}$. (Here $\mathcal{U}$ is as in Definition 2.6, or any open set smooth over k whose complement has codimension two or more.)

Remark 2.8. The sheaf ω_X can also be defined as the "double dual"

$$\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\bigwedge^{\dim X} \Omega_{X/k}, \mathcal{O}_X), \mathcal{O}_X). \tag{2.8.1}$$

Indeed, the sheaf (2.8.1) is reflexive and the natural map

$$\bigwedge^{\dim X} \Omega_{X/k} \longrightarrow \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\bigwedge^{\dim X} \Omega_{X/k}, \mathcal{O}_X), \mathcal{O}_X)$$

is an isomorphism on the smooth locus of X/k. The sheaf (2.8.1) is called the *"reflexive hull"* or *"S_2-ification"* of $\bigwedge^{\dim X} \Omega_{X/k}$, so we denote it $(\bigwedge^{\dim X} \Omega_{X/k})^{\mathrm{S}_2}$. See Appendix C.

Now to define a "dual-to-Frobenius" map $F^e_*\omega_X \longrightarrow \omega_X$, we apply the functor i_* to the map $T_{\mathcal{U}} : F^e_*\omega_{\mathcal{U}} \longrightarrow \omega_{\mathcal{U}}$ described in (2.4.2), where $\mathcal{U} \subseteq X$ is the smooth locus over k. The resulting map

$$T_{X/k} : i_*F^e_*\omega_{\mathcal{U}} = F^e_* i_*\omega_{\mathcal{U}} = F^e_*\omega_X \longrightarrow i_*\omega_{\mathcal{U}} = \omega_X$$

is the **dual of Frobenius.** In particular, the dual of Frobenius for a normal variety can be viewed as the Cartier map in a neighborhood of any smooth point.

2.3. The dual to Frobenius for quasi-projective varieties.

For a non-normal variety, a different approach is needed to define ω_X. We now describe a generalization of the approach to finitely generated algebras in Section 1 that works for all quasi-projective varieties.

A quasi-projective variety X is an open subset of a closed subvariety in some projective space $\mathbb{P}^n$ over k. So to understand the canonical module (and the dual to Frobenius for X), it is enough to understand these constructions for closed projective varieties in the smooth variety $\mathbb{P}^n_k$ (as we can then restrict to the open set X). The next definition of canonical module works in this, and somewhat more general, situations:

Definition 2.9. Let X be a variety finite type over a field k and admitting a finite k-morphism $j : X \longrightarrow W$ to a smooth variety W of dimension n over k (for instance, a quasi-projective variety). Then we can define a **canonical module** ω_X for X (over k) to be the unique coherent[8] $\mathcal{O}_X$-module ω_X such that

$$j_*\omega_X \cong \mathscr{E}\mathrm{xt}^{n-d}_{\mathcal{O}_W}(j_*\mathcal{O}_X, \omega_W)$$

as $j_*\mathcal{O}_X$-modules. Here ω_W is as defined in Section 2. This canonical module ω_X is independent of the map j, up to isomorphism; see [**Sta19**, Tag 0AA3] or [**Har66**]. See also [**Har77**, Chapter III, Section 7].

Remark 2.10. Examples of maps $j : X \longrightarrow W$ satisfying the hypothesis of Definition 2.9 include closed embeddings, generic projections of a d-dimensional variety in $\mathbb{P}^n$ to a d-dimensional projective subspace, or the Noether Normalizations considered in Section 1 for affine varieties. In particular, for an affine variety over k, Definition 2.9 agrees with our construction of the canonical module in Subsection 1.2. Furthermore, if X is smooth, we see that Definition 2.9 agrees with the construction in Section 2 by taking j to be the identity map.

Remark 2.11. Given a scheme X over a perfect k, we can view it as a variety in many different ways. Of course, we can view it as $X \longrightarrow \operatorname{Spec} k$, but we can also view it as a variety over k via the composition $X \xrightarrow{F_X} X \longrightarrow \operatorname{Spec} k$ (equivalently $X \longrightarrow \operatorname{Spec} k \xrightarrow{F_k} \operatorname{Spec} k$), and via iterates of Frobenius in the same way. If $j : X \longrightarrow W$ is finite with W smooth over k. However, if k is perfect, so that $\operatorname{Spec} k \xrightarrow{F_k} \operatorname{Spec} k$ is an isomorphism, a scheme W is smooth over k no matter which way we view it as a variety. Furthermore, $\Omega_{W/k}$, and hence the top wedge ω_W is independent of the variety structure we give it via iterates of Frobenius. It follows then that ω_X is independent of its variety structure via Frobenius as well. This, and in fact everything in this section,also holds over F-finite fields k as well, using an isomorphism $F_*k = \operatorname{Hom}_k(F_*k, k)$.

Lemma 2.12. *Let X be a quasi-projective variety over a perfect field k. Then with ω_X defined as in Definition 2.9, we have an isomorphism of $F^e_*\mathcal{O}_X$-modules*

$$\omega_{F^e_*\mathcal{O}_X} = F^e_*\omega_X \cong \mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X, \omega_X). \tag{2.12.1}$$

Proof. See Appendix C Remark 4.5 and the surrounding discussion which even does it over a possibly imperfect, but still F-finite, field.

∎

[8] Such a sheaf exists because j is affine, so the category of coherent $j_*\mathcal{O}_X$-modules is equivalent to the category of coherent $\mathcal{O}_X$-modules, see [**Sta19**, Tag 01SB].

In particular, with this definition of the canonical module, we can now apply the functor $\mathscr{H}om_{\mathcal{O}_X}(-, \omega_X)$ to the Frobenius map

$$\mathcal{O}_X \xrightarrow{F} F^e_* \mathcal{O}_X$$

on our quasi-projective variety to get a dual-to-Frobenius map

$$T^e_X : F^e_* \omega_X \to \omega_X.$$

Again, in a neighborhood of a smooth point, this will look like the Cartier map (2.4.2) or the map (1.10.3) described in Subsection 1.2. If X is additionally S_2, then we will see that

$$T^e_X \in \mathscr{H}om_{\mathcal{O}_X}(F^e_* \omega_X, \omega_X)$$

generates the $\mathscr{H}$om-sheaf as a $F^e_* \mathcal{O}_X$-module. The point is the generation holds locally by Proposition 3.14 below, and so it holds globally as well.

2.4. Dualizing complexes. To understand the dual-to-Frobenius map more deeply, and to generalize beyond quasi-projective varieties, we must look at the dualizing functor in the *derived category*[9], where the coherent sheaf ω_X is replaced by the full *dualizing complex* $\omega^\bullet_X$.

Instead of working with coherent $\mathcal{O}_X$-modules, we now work in the bounded derived category $D^b_{\mathrm{coh}}(X)$ of $\mathcal{O}_X$-modules with coherent cohomology. Each object in this category is represented, up to quasi-isomorphism, by a complex of $\mathcal{O}_X$-modules whose cohomology is trivial for all but finitely many cohomological degrees, where it is coherent. A map of complexes of $\mathcal{O}_X$-modules induces a morphism in $D^b_{\mathrm{coh}}(X)$; if the induced map on cohomology is an isomorphism in each cohomological degree, this induced map is a *quasi-isomorphism.*[10] We think of $D^b_{\mathrm{coh}}(X)$ as an enlargement of the category of coherent $\mathcal{O}_X$-modules by thinking of a coherent sheaf $\mathcal{M}$ as a complex of sheaves with $\mathcal{M}$ in cohomological degree zero, and zeros elsewhere. In the derived category, the module $\mathcal{M}$ can be identified with an injective resolution (or any other resolution of $\mathcal{M}$).

Definition 2.13 (Dualizing complexes for quasi-projective schemes)**.** Suppose that X is a scheme of finite type over a field k admitting a finite k-morphism $j : X \longrightarrow W$ to a smooth scheme of pure[11] dimension n over k.

[9]The reader who is not comfortable with derived categories can skip ahead to Section 3 as dualizing complexes will not be needed immediately. On the other hand, the "derived" picture mimics (and generalizes) the non-derived world, so this section can be approached as a crash course in Grothendieck duality as used in this text. For more details, see Appendix C.

[10]Caution: there are other morphisms in the category $D^b_{\mathrm{coh}}(X)$; formally we "invert" all quasi-isomorphisms. See for instance [**Har66**], [**Wei94**], or [**Sta19**, Tag 031S].

[11]If W has components of different dimensions, one must replace $\omega_W[n]$ with an appropriate complex $\omega^\bullet_W$ which will be the appropriate shift of ω_W on each component.

Then the *dualizing complex of X (over k)* is defined to be the object $\omega_X^\bullet$ in $D^b_{\mathrm{coh}}(X)$ such that

$$j_*\omega_X^\bullet := \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_W}(j_*\mathcal{O}_X, \omega_W[n]).$$

Here ω_W denotes the canonical sheaf as defined in Definition 2.1, viewed as a complex, and $\omega_W[n]$ is the "shifted" complex.[12]

The dualizing complex is independent of the map j (up to quasi-isomorphism), this is discussed in this setting in Appendix C Remark 4.4.

Definition 2.14. Let X be an equidimensional[13] scheme of finite type over a field k admitting a finite k-morphism $j : X \longrightarrow W$ to a smooth variety of equidimension n over k. Then the canonical module of X (over k) is $\omega_X := \mathcal{H}^{-\dim X}(\omega_X^\bullet)$.

Since the dualizing complex is independent of the map j up to quasi-isomorphism, the canonical module ω_X is independent of j up to $\mathcal{O}_X$-module isomorphism.

2.5. Hom'ing into a dualizing complex. The dualizing complex $\omega_X^\bullet$ has the property that the functor $\mathbf{R}\mathscr{H}\mathrm{om}(-, \omega_X^\bullet)$ defines a natural involution on the category $D^b_{\mathrm{coh}}(X)$. In particular, for any object $\mathcal{A}^\bullet \in D^b_{\mathrm{coh}}(X)$, there is a natural quasi-isomorphism

$$\mathcal{A}^\bullet \to \mathbf{R}\mathscr{H}\mathrm{om}(\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{A}^\bullet, \omega_X^\bullet), \omega_X^\bullet). \tag{2.14.1}$$

This can be thought of as a massive generalization of the natural isomorphism

$$V \to \mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k) \qquad v \mapsto \text{"eval at } v \text{ map"} \tag{2.14.2}$$

for finite dimensional vector spaces over a field.

Remark 2.15. When the dualizing complex is concentrated in one cohomological degree, the canonical module ω_X is also called a **dualizing module**. This is the case if and only if X is Cohen-Macaulay and equidimensional (see Appendix C Lemma 3.20 for a generalization).

Again, we have the fundamental fact about how the dualizing complex interacts with Frobenius:

Lemma 2.16. *With hypothesis and definition of $\omega_X^\bullet$ as in Definition 2.13, assume furthermore that k is a perfect field of prime characteristic. Then*

$$F^e_*\omega_X^\bullet \cong \mathbf{R}\mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X, \omega_X^\bullet). \tag{2.16.1}$$

[12] In general, for a complex $\mathcal{A}^\bullet$, the shifted complex $\mathcal{A}^\bullet[n]$ is the same as $\mathcal{A}^\bullet$ except that its cohomological degree i spot is the $i+n$ spot of the complex $\mathcal{A}^\bullet$.

[13] The dualizing complex is defined without any equidimensionality assumption; to derive a canonical module from it, however, we need a well-defined dimension at which to to compute the cohomology.

PROOF. This is a direct consequence of Grothendieck duality for the finite map F, but we sketch it below (which essentially proves a special case of Grothendieck duality). We argue similarly as in the proof of (2.12.1), using a version of Tensor-Hom adjointness in the derived category. To wit, there are natural isomorphisms of functors from $D^b_{\mathrm{coh}}(X)$ to itself:

$$\begin{aligned} & j_*\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(-,\omega_X^\bullet) \\ \cong\ & \mathbf{R}\mathscr{H}\mathrm{om}_{j_*\mathcal{O}_X}(j_*-, j_*\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_W}(j_*\mathcal{O}_X,\omega_W[n])) \\ \cong\ & \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_W}(j_*-\otimes^{\mathbf{L}}_{j_*\mathcal{O}_X} j_*\mathcal{O}_X,\omega_W[n]) \\ \cong\ & \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_W}(j_*-,\omega_W[n]) \end{aligned}$$

Thus, loosely speaking, $\mathbf{R}\mathscr{H}$om-ing into $\omega_X^\bullet$ is really the same as $\mathbf{R}\mathscr{H}$om-ing into $\omega_W[n]$. Now applying this functor to $F^e_*\mathcal{O}_X$, we get an isomorphism in the derived category

$$\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X,\omega_X^\bullet)\cong\omega^\bullet_{F^e_*\mathcal{O}_X}\cong F^e_*\omega_X^\bullet. \tag{2.16.2}$$

$\square$

Remark 2.17. More generally, any scheme of finite type over an F-finite field—and indeed, any Noetherian F-finite scheme— admits a dualizing complex such that (2.16.2) holds; see Section 4. In particular, taking cohomology, any locally equidimensional connected F-finite scheme X admits a canonical module, $\omega_X=\mathcal{H}^{-n}(\omega_X^\bullet)$ for some appropriate $n\in\mathbb{Z}$ (n is not always the dimension of the scheme X). This ω_X satisfies $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X,\omega_X)=F^e_*\omega_X$.

Again, we get a *dual-to-Frobenius* map:

Definition 2.18 (Dual of Frobenius, on the dualizing complex)**.** With hypothesis and definition of $\omega_X^\bullet$ as in Definition 2.13 over a perfect field k, applying the functor

$$\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(-,\omega_X^\bullet) \tag{2.18.1}$$

to the e-iterated Frobenius map $\mathcal{O}_X\longrightarrow F^e_*\mathcal{O}_X$, we obtain a $\mathcal{O}_X$-linear map

$$F^e_*\omega_X^\bullet\longrightarrow\omega_X^\bullet. \tag{2.18.2}$$

This map of complexes is the **(Grothendieck) dual to Frobenius**.

When X is locally equidimensional, we obtain the dual-to-Frobenius map $F^e_*\omega_X\to\omega_X$ by taking the first nonzero cohomology of (2.18.2).

Remark 2.19. Sometimes it is useful to consider the maps induced by (2.18.1) by taking cohomology at different degrees, as we shall see when we study F-injectivity later in Subsection 4.3.

2.6. Exercises.

Exercise 2.1. Let X be a smooth variety over a field k. For any subfield $L \subseteq k$ such that $[k : L]$ is finite, explain why X can also be viewed as a variety over L. Show that X is smooth over L if and only if $L \subseteq k$ is *separable*. In this case, show that the canonical module ω_X is independent of whether we compute it as $\wedge^d \Omega_{X/k}$ or $\wedge^d \Omega_{X/L}$. What happens when $L \subseteq k$ is *not* separable?

Exercise 2.2. Show that for any separated d-dimensional scheme X finite type over a field, that $\mathcal{H}^{-d}\omega_X^\bullet \neq 0$.

Hint: Take the map j from Chapter 2 Definition 2.13 to be a finite surjective map (for instance, a generic projection).

Exercise 2.3. Suppose X is a reduced separated equidimensional d-dimensional scheme finite type of field K. Suppose further that $X = Y \cup Z$ where Y and Z are distinct irreducible components of X. Let $c = \dim Y \cap Z$. If $c < d - 1$ then show that $\mathcal{H}^{-c-1}\omega_X^\bullet \neq 0$.

3. The dual to Frobenius for F-finite local rings

We begin with a brief description of where canonical modules and dualizing complexes come from. A summary of more facts about dualizing complexes and canonical modules can be found in the appendix Appendix C Section 3 and Section 5 respectively.

Definition 3.1 (Canonical modules for local rings)**.** Suppose $(R, \mathfrak{m})$ is a Noetherian local ring which admits a finite ring map $S \longrightarrow R$ [14] from a Gorenstein local ring $(S, \mathfrak{n})$. If R is equidimensional, the **canonical module of** R is the defined to be

$$\omega_R = \operatorname{Ext}^{\dim S - \dim R}(R, S).$$

Canonical modules for local Noetherian rings are independent of the choice of S, up to isomorphism[15].

The dualizing complex is constructed analogously:

[14] For example, this assumption holds when R is essentially of finite type over a field, complete (by the Cohen structure theorem) or F-finite (by **[Gab04]**).

[15] In full generality, for either rings or schemes, canonical modules are only unique up to tensoring with rank-1 locally free modules, but over a local ring this does nothing. There are canonical choices one can make to make them more canonical for F-finite schemes though as we will discuss.

Definition 3.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring, and assume that there is a finite ring map $(S, \mathfrak{n}) \to (R, \mathfrak{m})$, with S Gorenstein of dimension n. A **normalized dualizing complex of** R is defined to be

$$\omega_R^{\bullet} := \mathbf{R}\operatorname{Hom}_S(R, S[n]). \tag{3.2.1}$$

Dualizing complexes for local Noetherian rings are independent of the choice of S, up to isomorphism in the derived category.

If $\omega_R^{\bullet}$ is a dualizing complex for R, then so is $\omega_R^{\bullet}[i]$ for any integer i. It is with respect to this shift that the dualizing complexes we defined above are *normalized*.

Remark 3.3. If a Noetherian local ring has a dualizing complex then it is a quotient of a Gorenstein ring [**Kaw02**], so our hypothesis in Definition 3.2 or Definition 3.1 is not a substantive restriction. Furthermore, if a Noetherian ring has a dualizing complex then it is universally catenary and in particular, dimension behaves geometrically [**Sta19**, Tag 0A80].

Remark 3.4. If $(R, \mathfrak{m})$ is *equidimensional* with a dualizing complex, it follows from the above that we can define the canonical module as the first nonzero cohomology of the dualizing complex. However, we do not define the canonical module of a ring that fails to be locally equidimensional, see Exercise 3.2.

As a corollary of uniqueness of canonical modules and normalized dualizing complexes for local rings, we immediately obtain the following.

Corollary 3.5. *Suppose $(S, \mathfrak{m}) \longrightarrow (S', \mathfrak{m}')$ is a finite map of Gorenstein local rings of the same dimension n. Then we have an S'-module isomorphism*

$$\operatorname{Hom}_S(S', S) \cong S'$$

and

$$\operatorname{Ext}^i_S(S', S) = 0$$

for every $i > 0$.

In particular, if S is an F-finite Gorenstein local ring, then $F^e_* S \cong \operatorname{Hom}_S(F^e_* S, S) \cong \mathbf{R}\operatorname{Hom}_S(F^e_* S, S)$.

Proof. The point is that S' is a canonical module and $S'[n]$ a normalized dualizing complex for S'. But $\operatorname{Hom}_S(S', S)$ is also a canonical module and $\mathbf{R}\operatorname{Hom}_S(S', S[n])$ is a normalized dualizing complex. A simple computation gives the Ext vanishing. $\square$

For a generalization, see also Corollary 3.16 below.

By **[Gab04]**, every F-finite local Noetherian ring is a quotient of an F-finite regular ring. In particular, each such ring has a dualizing complex and hence it has a canonical module if it is equidimensional.

We record a few easy properties of canonical modules and dualizing complexes for future reference.

Proposition 3.6. *Let R with a canonical module ω_R and dualizing complex $\omega_R^{\bullet}$. Then the following properties are satisfied.*

(a) For every multiplicative set W in R, the module $W^{-1}R \otimes_R \omega_R$ is a canonical module for $W^{-1}R$. Likewise, $W^{-1}R \otimes_R \omega_R^{\bullet}$ is a dualizing complex for R.
(b) The canonical module ω_R is supported on all of $\operatorname{Spec} R$; that is, for all $P \in \operatorname{Spec} R$, $(\omega_R)_P \neq 0$.
(c) For reduced R, the canonical module is torsion-free; that is, the natural map $\omega_R \longrightarrow \omega_R \otimes_R \mathcal{K}(R)$ is injective, where $\mathcal{K}(R)$ is the total ring of fractions.
(d) For R-modules M satisfying Serre's S_2 condition,[16] the natural map
$$M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$$
is an isomorphism.

Proof. These follow from Appendix C Lemma 3.9, Lemma 5.8, Lemma 5.4, and Proposition 6.9. □

Lemma 3.7. *Let $(R, \mathfrak{m})$ be an F-finite Noetherian local ring with a canonical module ω_R. there is an isomorphism of $F^e_* R$-modules*
$$F^e_* \omega_R \cong \operatorname{Hom}_R(F^e_* R, \omega_R).$$

Proof. By **[Gab04]** we know that $R = S/I$ where S is a local regular (and hence Gorenstein) F-finite ring. We set $c = \dim S - \dim R$. Since S is F-finite, then $S \longrightarrow R$ and the composition $S \longrightarrow R \xrightarrow{F^e} F^e_* R$ are both finite. So we have an isomorphism of $F^e_* R$-modules
$$\begin{array}{rl} & \omega_{F^e_* R} \\ = & \operatorname{Ext}^c_S(F^e_* R, S) \\ \cong & \operatorname{Ext}^c_{F^e_* S}(F^e_* R, \operatorname{Hom}_S(F^e_* S, S)) \\ \cong & \operatorname{Ext}^c_{F^e_* S}(F^e_* R, F^e_* S) \\ = & F^e_* \omega_R. \end{array}$$
The third line comes from a derived variant of the adjunction of tensor and Hom, plus the Ext vanishing of Corollary 3.5 or a spectral sequence argument. The fourth lines come from Corollary 3.5. □

[16] If R is normal, all reflexive R modules satisfy this condition; see Appendix C Proposition 6.9 for more discussion.

Remark 3.8. More generally, a slight modification of the proof of Lemma 3.7 shows that for an F-finite Noetherian local ring,

$$\omega^\bullet_{F^e_* R} = \mathbf{R}\operatorname{Hom}_R(F^e_* R, \omega^\bullet_R) \cong F^e_* \omega^\bullet_R$$

in the derived category as well, using the derived version of tensor-Hom adjunction (see also the proof of Lemma 2.16).

We are now ready to construct the *dual-to-Frobenius map* for a Noetherian F-finite local ring. Applying the contravariant functor $\operatorname{Ext}^{n-d}_S(-, S)$ to the Frobenius map $R \longrightarrow F^e_* R$, we get a natural R-module map

$$\operatorname{Ext}^{n-d}_S(F^e_* R, S) \longrightarrow \operatorname{Ext}^{n-d}_S(R, S)$$

which becomes our desired **dual-to-Frobenius**

$$T^e_R : F^e_* \omega_R \longrightarrow \omega_R, \tag{3.8.1}$$

using the identification Lemma 3.7.

Because the functors $\mathbf{R}\operatorname{Hom}_S(-, S[n])$ and $\mathbf{R}\operatorname{Hom}_R(-, \omega^\bullet_R)$ are naturally isomorphic,[17] the map (3.8.1) can also be described as applying the functor $\operatorname{Hom}_R(-, \omega_R)$ to the Frobenius map:

$$T^e_R : F^e_* \omega_R \cong \operatorname{Hom}_R(F^e_* R, \omega_R) \longrightarrow \omega_R. \tag{3.8.2}$$

Again, the maps (3.8.1) and (3.8.2) agree with our previous descriptions of the dual to Frobenius (up to isomorphism of ω_R).

Remark 3.9. We can also apply the full dualizing functor $\mathbf{R}\operatorname{Hom}_R(-, \omega^\bullet_R)$ (or equivalently, $\mathbf{R}\operatorname{Hom}_S(-, S[n])$) to the Frobenius map $R \longrightarrow F^e_* R$ to get a dual-to-Frobenius map in the derived category:

$$F^e_* \omega^\bullet_R \cong \mathbf{R}\operatorname{Hom}_R(F^e_* R, \omega^\bullet_R) \longrightarrow \mathbf{R}\operatorname{Hom}_R(R, \omega^\bullet_R) = \omega^\bullet_R. \tag{3.9.1}$$

The first isomorphism here comes from Remark 3.8. Our dual-to-Frobenius map (3.8.1) can be obtained by taking the $-d$-th cohomology of (3.9.1).

Remark 3.10. We can also take other cohomologies of (3.9.1) to obtain maps:

$$F^e_* \mathcal{H}^i \omega^\bullet_R \longrightarrow \mathcal{H}^i \omega^\bullet_R$$

for every integer i. We will see another interpretation of these maps via local cohomology in Proposition 3.13 below.

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[17] by an application of derived Hom-tensor adjointness, using that $\omega^\bullet_R = \mathbf{R}\operatorname{Hom}_S(R, S[n])$

3.1. Matlis duality and Frobenius on local cohomology. Fix an F-finite Noetherian local ring $(R, \mathfrak{m})$, of equidimension d. We now explain how the dual-to-Frobenius map $F^e_*\omega_R \longrightarrow \omega_R$ is related to the Frobenius action on local cohomology

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(F^e_* R)$$

discussed in Section 7 in Chapter 1. The point is that they are *Matlis Dual.*

Definition 3.11 (The Matlis duality functor)**.** Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and E an injective hull of its residue field. The Matlis duality functor is the exact functor

$$M \longmapsto M^\vee := \operatorname{Hom}_R(M, E)$$

taking the category of R-modules to itself.

The Matlis duality functor[18] takes Noetherian R-modules to Artinian R-modules, and Artinian R-modules to Noetherian $\widehat{R}$-modules where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R. In particular, over a complete ring, $(M^\vee)^\vee \cong M$ for all Noetherian and all Artinian R-modules.

Proposition 3.12. *Suppose $(R, \mathfrak{m})$ is an F-finite local Noetherian ring of equidimension d. Applying Matlis duality to the dual to Frobenius map (3.8.1)*

$$T^e : F^e_*\omega_R \longrightarrow \omega_R$$

produces the Frobenius action on top local cohomology

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(F^e_* R)$$

as defined in Chapter 1, Section 7. Furthermore, Matlis duality applied to the Frobenius action on top local cohomology produces the dual to Frobenius on $\widehat{R}$, the $\mathfrak{m}$-adic completion of R:

$$F^e_*\omega_{\widehat{R}} \longrightarrow \omega_{\widehat{R}}.$$

Proof. Write $R = S/I$ where S is Gorenstein, local, and n-dimensional. Then the dual to Frobenius may be identified with

$$\operatorname{Ext}^{n-d}_S(F^e_* R, S) \longrightarrow \operatorname{Ext}^{n-d}_S(R, S).$$

Local duality (see Appendix C Corollary 6.2) tells us that the Matlis dual of this map is precisely $H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(F^e_* R)$. The final statement follows because applying Matlis duality to a finitely generated module twice simply completes the module (Appendix C Theorem 1.5). $\square$

More generally:

[18]See Appendix C Section 1" for more about Matlis duality.

Proposition 3.13. *Suppose $(R, \mathfrak{m})$ is an F-finite Noetherian local ring. By taking cohomology of the dual to Frobenius map (3.9.1), we obtain maps*

$$\mathcal{H}^{-i} F^e_* \omega^\bullet_R \longrightarrow \mathcal{H}^{-i} \omega^\bullet_R$$

whose Matlis duals are the maps

$$H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(F^e_* R) \tag{3.13.1}$$

induced by Frobenius on R. Furthermore, the Matlis dual of the maps (3.13.1) are dual to Frobenius on the completion of R.

$$\mathcal{H}^{-i} F^e_* \omega^\bullet_{\widehat{R}} \longrightarrow \mathcal{H}^{-i} \omega^\bullet_{\widehat{R}}.$$

Proof. The proof is essentially the same as that of Proposition 3.12, but we use the derived category notation and take cohomology besides the dth. Note $\mathcal{H}^{-i}\omega^\bullet_R = \mathcal{H}^{-i}\mathbf{R}\operatorname{Hom}_S(R, S[n]) = \operatorname{Ext}^{n-i}_S(R, S)$. $\square$

An an application, we prove the following useful fact that we have already observed in the polynomial ring case:

Proposition 3.14. *Let $(R, \mathfrak{m})$ be an F-finite Noetherian local ring satisfying Serre's S_2-condition*[19]*. Let ω_R be the canonical module for R. For $e \in \mathbb{N}$, let $T^e \in \operatorname{Hom}_R(F^e_*\omega_R, \omega_R)$ be the dual to Frobenius as defined in (3.8.1). Then the natural map*

$$F^e_* R \longrightarrow \operatorname{Hom}_R(F^e_*\omega_R, \omega_R) \qquad F^e_* r \longmapsto T^e \circ F^e_* r \tag{3.14.1}$$

is an isomorphism of F^e_*R-modules. In particular, $\operatorname{Hom}_R(F^e_*\omega_R, \omega_R)$ is freely generated as an F^e_*R-module by the dual of Frobenius T^e.

Remark 3.15. We can even take $e = 0$ in Proposition 3.14. The "0-th Frobenius map" $F^0 : R \to R$ is the identity map $R \mapsto r^{p^0} = r$. The map (3.14.1) becomes the isomorphism

$$R \to \operatorname{Hom}_R(\omega_R, \omega_R) \qquad r \mapsto \cdot r,$$

of Proposition 3.6(d).

Proof. The map (3.14.1) is clearly F^e_*R-linear. To see that it is surjective, take arbitrary $\phi \in \operatorname{Hom}_R(F^e_*\omega_R, \omega_R)$. We need to show that ϕ factors as $T^e_* \circ F^e_* r = T^e \star r$ for some $c \in R$. Applying the contravariant functor

[19] For example, all normal rings satisfy this condition.

$\mathrm{Hom}_R(-, \omega_R)$ to ϕ we obtain:

$$\begin{tikzcd}
\mathrm{Hom}_R(\omega_R, \omega_R) \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & \mathrm{Hom}_R(F^e_* \omega_R, \omega_R) \arrow[d, leftrightarrow, "\cong"] \\
\mathrm{Hom}_R(\mathrm{Hom}_R(R, \omega_R), \omega_R) \arrow[r] & \mathrm{Hom}_R(\mathrm{Hom}_R(F^e_* R, \omega_R), \omega_R) \\
R \arrow[r, "\phi^{\vee_\omega}"] \arrow[u, "\cong"] & F^e_* R. \arrow[u, "\cong"']
\end{tikzcd}$$

Here, the lower two vertical isomorphisms follow from Proposition 3.6(d) since both R and $F^e_* R$ satisfy Serre's S_2 condition.

To see that ϕ factors as desired, we observe that its dual $\phi^{\vee_\omega}$ does. Indeed, $\phi^{\vee_\omega}$ is determined by where it sends 1: if $\phi^{\vee_\omega}(1) = F^e_* r$, then we may factor $\phi^{\vee_\omega}$ as $F^e_* r \circ F^e$. Applying the contravariant functor $(-)^{\vee_\omega} = \mathrm{Hom}_R(-, \omega_R)$ therefore,

$$\phi = (\phi^{\vee_\omega})^{\vee_\omega} = (F^e_* r \circ F^e)^{\vee_\omega} = \circ (F^e)^{\vee_\omega} \circ (F^e_* r)^{\vee_\omega} = T^e \circ F^e_* r,$$

which is exactly what we wanted to show to see that (3.14.1) is surjective. Finally, the map is injective, since no element of $F^e_* R$ acts trivially on $\mathrm{Hom}_R(F^e_* \omega_R, \omega_R)$. $\square$

Corollary 3.16. *With notation as above, suppose R is quasi-Gorenstein (which means that $\omega_R \cong R$ and R is S_2, see Appendix C Lemma 6.10) then*

$$\mathrm{Hom}_R(F^e_* R, R) \cong F^e_* R$$

as an $F^e_* R$-module. In particular, if R is Gorenstein, then the dual of Frobenius is a generating map for $\mathrm{Hom}_R(F^e_* R, R)$.

Dually, even without the F-finite hypothesis, we also have the following result.

Proposition 3.17 ([**LS01**, Example 3.7])**.** *Suppose $(R, \mathfrak{m})$ is a complete local ring satisfying Serre's S_2 condition. Then the Frobenius action on local cohomology $F^e \in \mathrm{Hom}_R(H^d_\mathfrak{m}(R), H^d_\mathfrak{m}(F^e_* R))$ is an $F^e_* R$-module generator for this module.*

Proof. In the F-finite case this is simply the Matlis dual of Proposition 3.14. The general case is left as an exercise to the reader below in Exercise 3.5 or see the reference above. $\square$

3.2. Exercises.

Exercise 3.1. Suppose that $(R, \mathfrak{m})$ is a complete Noetherian F-finite local ring and $I \subseteq R$ is an ideal. Then I is uniformly F-compatible[20] if and only

[20] This means that $\phi(F^e_* I) \subseteq I$ for all $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ and all $e \geq 0$.

if the image of

$$E_{R/I} \subseteq E \longrightarrow E \otimes_R F^e_* R$$

is annihilated by $F^e_* I$. *cf.* [**LS01, Sch10a**].

Exercise 3.2 (Non-locally equidimensional rings)**.** Let S be the power series ring $k[\![x,y,z]\!]$ over a field k. For $I = (x,y) \cap (z) = (xz, yz)$, set $R = S/I$, and consider its dualizing complex $\omega_R^\bullet = \mathbf{R}\operatorname{Hom}_S(R, S[\dim S])$. Investigate what happens if we define a "canonical module" $\omega_R = \mathcal{H}^{-\dim R}\omega_R^\bullet$. For example, show that this module is non-zero at one minimal prime of R and is zero at the other. Explain why this implies that the "canonical module" doesn't behave well under localization.

Hint: You did a related computation in Exercise 2.3.

Exercise 3.3. Show that an equidimensional ring of finite type over a field is locally equidimensional. In particular, our definition of canonical module in this section agrees with that in Subsection 1.2.

Exercise 3.4. Suppose $(R, \mathfrak{m})$ is a complete Noetherian F-finite local ring and E is an injective hull of the residue field. Show that the Matlis dual of some $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ can be identified with a map $E \longrightarrow F^e_* E$, in other words, a Frobenius action on E.

Hint: Use Appendix C Lemma 1.8 to see that $\operatorname{Hom}_R(F^e_* R, E) \cong F^e_* E$.

Exercise 3.5. Prove Proposition 3.17, that is show that if $(R, \mathfrak{m})$ is a d-dimensional complete local Noetherian S_2-local ring, then every R-module map $\phi : H^d_\mathfrak{m}(R) \longrightarrow F^e_* H^d_\mathfrak{m}(R)$ is a post-multiple of the canonical e-iterated Frobenius on local cohomology.

Hint: Use the isomorphism coming from the adjointness of Hom and tensor:

$$\operatorname{Hom}_R\left(H^d_\mathfrak{m}(R), F^e_* H^d_\mathfrak{m}(R)\right) \cong \operatorname{Hom}_{F^e_* R}\left(H^d_\mathfrak{m}(R) \otimes_R (F^e_* R), H^d_\mathfrak{m}(R)\right),$$

combined with the isomorphism

$$H^d_\mathfrak{m}(N) \otimes M \cong H^d_\mathfrak{m}(N \otimes M),$$

see Appendix A Proposition 10.1. Finally, show that $\operatorname{Hom}_R\left(H^d_\mathfrak{m}(R), H^d_\mathfrak{m}(R)\right) \cong R$ since R is complete and S_2. See also [**LS01**, Example 3.7].

Exercise 3.6. Suppose that R is an F-finite equidimensional local ring ring. Show that $T^e = T^{\star e}$.

Hint: Observe that F^2 is obtained by composing Frobenius with itself. Now apply the *functor* $\operatorname{Hom}_R(-, \omega_R)$.

Exercise 3.7. Suppose that $(R, \mathfrak{m})$ is an F-finite strongly F-regular Gorenstein local ring and that $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ such that ϕ generates $\mathrm{Hom}_R(F^e_* R, R)$ as an $F^e_* R$-module. Prove that R has no ideals compatible with ϕ.

Exercise 3.8. Suppose $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring and $J \subseteq R$ is an ideal with $J \cong \omega_R$. Suppose that R/J is F-split (respectively strongly F-regular), prove that then R is also. For generalizations see [**Ma14**].

Hint: Dualize the sequence $0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$, use that to observe that R/J is Gorenstein. Note a local ring is Gorenstein if its dualizing complex looks like a shift of the ring itself.

4. The dual to Frobenius for F-finite rings and schemes

We now discuss the canonical module and Grothendieck duality for *F-finite Noetherian rings* more generally. This section black boxes the existence of *canonical* canonical modules and *canonical* dualizing complexes, which spring from a result of Gabber.

We begin with a brief description of where general dualizing complexes come from. A summary of more facts about dualizing complexes can be found in the appendix, Appendix C Section 3.

4.1. Rings with canonical modules and dualizing complexes.

We begin with some cautionary tales on dimension in this generality.

Example 4.1 (Codimension is not the difference of dimensions)**.** Even for "nice" excellent domains R, different maximal ideals can have different heights. For instance, suppose k is any field, and $R = k[\![t]\!][x]$. Then $I = (t, x)$ is a maximal ideal of height 2 and $J = (tx - 1)$ is a maximal ideal of height 1. In particular, both $Y = \mathbb{V}(I) = \mathrm{Spec}(R/I)$ and $Z = \mathbb{V}(J) = \mathrm{Spec}(R/J)$ are subschemes of $X = \mathrm{Spec}\, R$ of dimension 0. However, Y has codimension 2 in X while Z has codimension 1.

Example 4.2 (Equidimensionality vs local equidimensionality)**.** Let $R = k[\![t]\!]$, $S = k(\!(t)\!)[u, v]$ and define a new ring A to be the pullback of the diagram $\{R \longrightarrow k(\!(t)\!) = S/(u, v) \leftarrow S\}$. Then A is not normal and has normalization $R \times S$. In fact, $\mathrm{Spec}\, A$ is the gluing of $\mathrm{Spec}\, R$ to $\mathrm{Spec}\, S$ along the closed subscheme $\mathbb{V}_R(J) = \mathbb{V}_S(u, v)$, see for instance [**Fer03**]. Furthermore, even though A is equidimensional of dimension 2 (both irreducible components have dimension 2), it is not locally equidimensional since at the point where we glued $\mathrm{Spec}\, R$ to $\mathrm{Spec}\, S$, the two schemes have different dimensions 1 and 2 respectively.

Dualizing complexes in fact provide a sort of dimension on projective schemes in general, see [**Sta19**, Tag 0A7W]. But due to gluings like we saw above, this dimension may not be the one we expect. If R is locally equidimensional, connected and catenary (the latter of which follows if R is F-finite), then we can treat R as if it is equidimensional.

4.2. F-finite rings and schemes. A theorem of Gabber guarantees that any F-finite Noetherian ring R always admits a dualizing complex $\omega_R^{\bullet}$ because it is a homomorphic image of an F-finite *regular* ring S.

We summarize this now (we do not prove it, although parts are sketched in the exercises, see Exercise 4.6).

Theorem 4.3 (Gabber, [**Gab04, BBST13**]). *If R is an F-finite ring, then there exists an F-finite Noetherian regular ring S surjecting onto R ($\pi : S \twoheadrightarrow R$) so that:*

(a) $F^e_* S$ is a free S-module. Furthermore, $\Omega_{S/S^p} = \Omega_{S/\mathbb{F}_p}$ is a free S-module (see Exercise 4.1).
(b) S is complete with respect to $\ker \pi$.
(c) If R has connected Spec, *then so does S.*

Notice that for any ring S of characteristic $p > 0$, $\Omega_{S/S^p} = \Omega_{S/\mathbb{F}_p}$ as you can see Exercise 4.1. Thus even though a-priori, $\Omega_{S/\mathbb{F}_p}$ need not be a finite type S-module, if S is F-finite it is. Indeed, if S is regular, then a computation shows that $\Omega_{S/\mathbb{F}_p}$ is a locally free S-module, see Exercise 4.2. Following Bhatt-Blickle-Schwede-Tucker, we use this module to define our canonical dualizing complex and canonical canonical module on S. Then we use the canonical module on S to produce a canonical module on R.

Definition 4.4 ([**BBST13**]). Suppose S is an F-finite regular ring with connected Spec. We define the **canonical canonical module of** S to be

$$\omega_S := \wedge^n \Omega_{S/\mathbb{F}_p}.$$

Here $n = \operatorname{rank}_S \Omega_{S/\mathbb{F}_p}$ is the rank of the free module (which must be constant since Spec is connected).

We likewise define the **canonical dualizing complex of** S to be

$$\omega_S^{\bullet} = \omega_S[n].$$

We can also define the dualizing complex on a general regular ring without connected Spec by working one connected component at a time.

One nice aspect of this construction is it clearly commutes with localization. Regardless, we can now define our canonical canonical module (and dualizing complex) in general.

Definition-Proposition 4.5 (Canonical canonical modules)**.** *Suppose R is an F-finite Noetherian locally equidimensional ring with connected* Spec*. Let $S \twoheadrightarrow R$ be a surjection from an F-finite regular ring with connected* Spec*. Let m denote the codimension of* $\operatorname{Spec} R$ *in* $\operatorname{Spec} S$*. Then we define the* **canonical canonical module of** *R to be:*

$$\omega_R = \operatorname{Ext}^m_S(R, \omega_S)$$

where ω_S is defined as in Definition 4.4. ω_R is independent of the choice of $S \twoheadrightarrow R$ by [**BBST13**]*. This canonical module satisfies the property that*

$$F^e_* \omega_R \cong \operatorname{Hom}_R(F^e_* R, \omega_R)$$

for every $e > 0$.

More generally, suppose X is an F-finite Noetherian scheme that is locally equidimensional and connected. Then the canonical modules implicitly defined on affine charts above glue (in a canonical way) to produce a **canonical canonical module on** *X, ω_X, such that for every $e > 0$,*

$$F^e_* \omega_X = \mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \omega_X).$$

Due to the fact that the canonical canonical module above is independent of the choice of S, one particularly convenient way of constructing canonical canonical modules for normal rings and varieties is as follows.

Corollary 4.6. *Suppose X is an F-finite normal integral scheme with regular locus $i : U \hookrightarrow X$. Then the canonical canonical module of X is*

$$\omega_X = i_* \wedge^n \Omega_{U/\mathbb{F}_p} = (\wedge^n \Omega_{X/\mathbb{F}_p})^{\mathrm{S}_2}$$

where $n = \operatorname{rank}_{\mathcal{O}_U} \Omega_{U/\mathbb{F}_p}$.

In particular, if R is an F-finite normal integral domain, then

$$\omega_R = (\wedge^n \Omega_{R/\mathbb{F}_p})^{\mathrm{S}_2} = \operatorname{Hom}_R(\operatorname{Hom}_R(\Omega_{R/\mathbb{F}_p}, R), R)$$

is the canonical canonical module for R.

Definition-Proposition 4.7 (Canonical dualizing complexes, [**BBST13**])**.** *Suppose R is an F-finite Noetherian ring. Let $S \twoheadrightarrow R$ be a surjection from an F-finite regular ring. Then we define the canonical dualizing complex of R to be*

$$\mathbf{R}\operatorname{Hom}_S(R, \omega^{\bullet}_S) = \mathbf{R}\operatorname{Hom}_S(R, \wedge^n \Omega_{S/\mathbb{F}_p})$$

where $n = \operatorname{rank}_S \Omega_{S/\mathbb{F}_p}$. This is independent of the choice of S.

More generally, for any F-finite Noetherian scheme X, the dualizing complexes above glue[21] *in such a way to create a canonical dualizing complex*

$$\omega^{\bullet}_X.$$

[21] via the Beilinson-Bernstein-Deligne gluing lemma, see [**Sta19**, Tag 0D6C] or [**BBD82**]

Furthermore, for any finite type morphism of F-finite Noetherian excellent schemes $\pi : Y \longrightarrow X$ with canonical dualizing complex $\omega_X^{\bullet}$ and $\omega_Y^{\bullet}$ we have that

$$\pi^! \omega_X^{\bullet} = \omega_Y^{\bullet}.$$

In particular, specializing π to be the Frobenius on X, we have that $(F^e)^! \omega_X^{\bullet} = \omega_X^{\bullet}$. In other words

$$F^e_* \omega_X^{\bullet} \cong \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \omega_X^{\bullet}) = F^e_*(F^e)^! \omega_X^{\bullet}$$

for every $e > 0$.

4.3. Applications to F-injective rings. Our work above lets us handle our *global* notions of F-injective singularities.

Recall the definition from Chapter 1 Definition 7.8. We say that a Noetherian ring is **F-injective** if for every maximal ideal $\mathfrak{m} \subseteq R$, the map on local cohomology induced by Frobenius

$$H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(F^* R)$$

is injective for every i. Throughout the rest of the chapter, when working with F-injectivity, we work with only F-finite rings (an assumption we did not make in Chapter 1).

If R is local, by local duality, $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(F^* R)$ is Matlis dual to

$$\mathcal{H}^{-i} F_* \omega_R^{\bullet} \cong \mathcal{H}^{-i} \mathbf{R}\operatorname{Hom}_R(F_* R, \omega_R^{\bullet}) \longrightarrow \mathcal{H}^{-i} \mathbf{R}\operatorname{Hom}_R(R, \omega_R^{\bullet}) \cong \mathcal{H}^{-i} \omega_R^{\bullet}.$$

Thus we obtain that:

Corollary 4.8. *Suppose R is an F-finite Noetherian ring of characteristic $p > 0$ and dualizing complex $\omega_R^{\bullet}$. Then R is F-injective if and only if the maps (dual to Frobenius)*

$$\mathcal{H}^{-i} F_* \omega_R^{\bullet} \longrightarrow \mathcal{H}^{-i} \omega_R^{\bullet}$$

surjects for every i.

Proof. Surjection between modules can be checked locally. Hence we may assume that R is local. Localizing the dualizing complex may yield a non-normalized dualizing complex, but we may shift it so it is normalized. The result then follows from the argument above. □

We thus obtain the following.

Corollary 4.9. *Suppose that R is an F-finite ring and $W \subseteq R$ is a multiplicative set. If R is F-injective, so is $W^{-1}R$. Furthermore R is F-injective if and only if $R_{\mathfrak{m}}$ is F-injective for all maximal ideals $\mathfrak{m} \subseteq R$. Finally, the locus where R is F-injective is open.*

PROOF. The maps $F_*\mathcal{H}^{-i}\omega_R^\bullet \to \mathcal{H}^{-i}\omega_R^\bullet$ (for all i) are surjective exactly where R is F-injective, and so the F-injective locus is open (since there are only finitely many non-zero $\mathcal{H}^{-i}\omega_R^\bullet$). The formation of these maps also commutes with localization (as we have seen before) which proves the remaining results. $\square$

For a partial generalization of the above result when R is not necessarily F-finite, see Exercise 4.7 and for more general results, [**DM20a**].

We saw in Chapter 1 Proposition 7.17 that an F-finite F-split ring is always F-injective and a quasi-Gorenstein F-finite F-injective ring is always F-split. We also saw that F-split rings are weakly normal in Chapter 1 Exercise 4.14. In fact, F-injective singularities are also weakly normal.[22] See the exercises of Subsection 4.5 for a series of exercises developing some properties of weak normality and weak normalization.

First though, let us show that F-injective rings are reduced.

Lemma 4.10. *If a Noetherian local ring $(R, \mathfrak{m})$ is F-injective, then R is reduced.*

PROOF. A ring is reduced if and only if it is R0 and S1 [**Sta19**, Tag 031R]. Thus suppose first that R is not reduced after localizing at a minimal prime $Q \subseteq R$. Then $0 \neq H^0_Q(R_Q) = QR_Q$, and so Frobenius is not injective on $H^0_Q(R_Q)$, a contradiction.

Now suppose that R is F-injective but not reduced and so not S1. Then there exists a embedded associated prime $Q \subseteq R$ such that $\operatorname{depth}(R_Q) = 0$ but where $\dim(R_Q) = h > 1$, see [**Sta19**, Tag 031Q]. We consider Frobenius acting on $H^0_Q(QR_Q) \subseteq QR_Q$. But R_Q is also non-reduced and hence Frobenius does not act injectively on QR_Q. $\square$

We now provide an alternate characterization of weak normality involving local cohomology. We rely on some facts about weakly normal rings left to the exercises.

Lemma 4.11. *Suppose $(R, \mathfrak{m})$ is a reduced ring of characteristic $p > 0$ such that $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ is weakly normal. Then R is weakly normal if and only if the Frobenius action is injective on $H^1_{\mathfrak{m}}(R)$.*

PROOF. We may assume that $\dim R > 0$. Let R^{WN} denote the weak normalization of R which exists by Exercise 4.9. Set $X = \operatorname{Spec} R$, $U =$

[22] Recall a reduced ring $\mathcal{K}(R)$ is weakly normal if and only if for each $x \in \mathcal{K}(R)$ such that $x^p \in R$, we then have that $x \in R$.

$X \setminus \{\mathfrak{m}\}$ and notice also that $\nu : X^{\mathrm{WN}} = \operatorname{Spec} R^{WN} \longrightarrow X$ is a bijection, see Definition 4.14.

By [**Har77**, Chapter III, Exercise 2.3(e)] we have the following diagram of modules.

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, hook] \arrow[d, hook] & \Gamma(U, \mathcal{O}_X) \arrow[r, two heads] \arrow[d, "\cong"] & H^1_{\mathfrak{m}}(R) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & R^{\mathrm{WN}} \arrow[r, hook] & \Gamma(U, \nu_* \mathcal{O}_{X^{\mathrm{WN}}}) \arrow[r, two heads] & H^1_{\mathfrak{m}}(R^{\mathrm{WN}}) \arrow[r] & 0 \end{tikzcd}$$

The left horizontal maps are injective because R and R^{WN} are reduced. Now, R is weakly normal if and only if every $r \in R^{\mathrm{WN}}$ with $r^p \in R$ also satisfies $r \in R$.

First assume that the action of Frobenius is injective on $H^1_{\mathfrak{m}}(R)$. So suppose that there is such an $r \in R^{\mathrm{WN}}$ with $r^p \in R$. Then r has an image in $\Gamma(U, \mathcal{O}_X)$ and therefore an image in $H^1_{\mathfrak{m}}(R)$. But r^p has a zero image in $H^1_{\mathfrak{m}}(R)$, which means r has zero image in $H^1_{\mathfrak{m}}(R)$ which guarantees that $r \in R$ as desired.

Conversely, suppose that R is weakly normal. Let $r \in \Gamma(U, \mathcal{O}_X)$ be an element such that the action of Frobenius annihilates its image $\overline{r}$ in $H^1_{\mathfrak{m}}(R)$. Since $r \in \Gamma(U, \mathcal{O}_X)$ we identify r with a unique element of the total field of fractions of R. On the other hand, $r^p \in R$ so $r \in R^{\mathrm{WN}} = R$. Thus we obtain that $r \in R$ and so $\overline{r}$ is zero as desired. $\square$

Theorem 4.12. *An F-finite F-injective ring R is weakly normal.*

Proof. By Corollary 4.9 and Exercise 4.11, both F-injectivity and weak normalization localize. We may thus assume that $(R, \mathfrak{m})$ is local and $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ is weakly normal. But now the result follows from Lemma 4.11. $\square$

In fact, the same result holds without the F-finite hypothesis [**DM20a**, Corollary 3.5].

4.4. Exercises.

Exercise 4.1. Suppose R is a ring of characteristic $p > 0$. Prove that $\Omega_{R/\mathbb{F}_p} = \Omega_{R/R^p}$ where $R^p \subseteq R$ is the image of the Frobenius map. In the case that R is F-finite and Noetherian, conclude that $\Omega_{R/\mathbb{F}_p}$ is a finitely generated R-module.

Exercise 4.2. Suppose $(S, \mathfrak{n}, k)$ is an F-finite local regular ring. Prove that $\Omega_{S/\mathbb{F}_p} = \Omega_{S/S^p}$ is a free S-module.

Hint: We can write $S = S^p[x_1, \ldots, x_n, y_1, \ldots, y_m]$ where the x_i are minimal generators of the maximal ideal and the y_i form a differential basis for k/k^p.

Exercise 4.3. Let A be a DVR and set $R = A[x]$. Show that there is no dualizing complex $\omega_R^{\bullet}$ on R such that for every maximal ideal $\mathfrak{m} \subseteq R$, that $(\omega_R^{\bullet})_{\mathfrak{m}}$ is a normalized dualizing complex for $R_{\mathfrak{m}}$.

Definition 4.13. Suppose S is a Noetherian domain of characteristic $p > 0$. A set of elements $x_1, \ldots, x_d$ is called **p-independent**if the set of monomials $\Lambda := \{x_1^{a_1} \cdots x_d^{a_d} \mid 0 \leq a_i \leq p\}$ is linear independent over S^p. It is called a **p-generating set** if $S = S^p[x_1, \ldots, x_d]$. A p-independent p-generating set is called a **p-basis**.

It turns out that $x_1, \ldots, x_d$ is a p-basis if and only if it is a differential basis. See [**Mat80**, 38.A, page 269] and also [**Tyc88**, Theorem 1] or [**KN84**].

Exercise 4.4. Suppose S is a regular ring with a p-basis $x_1, \ldots, x_d$ (equivalently differential basis). Observe that $\wedge^d \Omega_{S/\mathbb{F}_p} \cong S$. Prove by direct computation that $\mathrm{Hom}_S(F_*S, S) \cong F_*S$ as F_*S-modules. Conclude that for any F-finite Noetherian ring R there exists a dualizing complex $\omega_R^{\bullet}$ such that

$$\mathrm{Hom}_R(F^e_* R, \omega_R^{\bullet}) \cong F^e_* \omega_R^{\bullet}.$$

Hint: Every F-finite ring is a quotient a regular ring with a p-basis by [**Gab04**], see the theorems we stated this section.

Exercise 4.5. Suppose R is a Noetherian F-finite ring of characteristic $p > 0$ and $x_1, \ldots, x_n \in R$ is a p-generating set (by definition, this means that $R = R^p[x_1, \ldots, x_n]$). Consider the rings

$$R_e := R[X_1, \ldots, X_n]/(X_1^{p^e} - x_1, \ldots, X_n^{p^e} - x_n).$$

Show that:

(a) For each $e > 0$, there is a surjective map $\pi_e : R_e \twoheadrightarrow R$ which acts as F^e on R and sends $X_i \mapsto x_i$.
(b) For each $e > 0$, there is a map $f_e : R_{e+1} \longrightarrow R_e$ sending $X_i \mapsto X_i$ and which acts as Frobenius on R so that the following diagram commutes:

$$\begin{tikzcd} R_{e+1} \arrow[r, "f_e"] \arrow[dr, "\pi_{e+1}"'] & R_e \arrow[d, "\pi_e"] \\ & R. \end{tikzcd}$$

Exercise 4.6. With notation as in Exercise 4.5, prove that

$$R_\infty := \varprojlim_{f_e} R_e$$

is a reduced ring with a surjective map $R_\infty \longrightarrow R$. Further show that the $\overline{X_i} = (X_i, X_i, \dots) \in R_\infty$ satisfy the property that R_∞ is a free R_∞^p-module with basis

$$\overline{X_1}^{a_1} \cdots \overline{X_n}^{a_n} \quad 0 \leq a_i \leq p-1.$$

In particular, conclude that if one knew that R is Noetherian (which is true, see [**Gab04, MP20, BBST13**]) then R is regular. This is how Gabber constructs the regular rings $S \twoheadrightarrow R$ we described above.

Exercise 4.7. Suppose $(R, \mathfrak{m})$ is an F-injective Noetherian local ring with a normalized dualizing complex $\omega_R^\bullet$. Suppose $Q \in \operatorname{Spec} R$. Prove that R_Q is also F-injective. One can find a more general result (without even assuming R has a dualizing complex) in [**DM20a**].

Hint: This is slightly easier in the F-finite case. For the non-F-finite case, show that R is F-injective if and only if $H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(S)$ injects for every *finite* generically purely inseparable ring extension $R \subseteq S$.

4.5. Exercises on weak normality. The following definition will be used in the exercises that follow. Also see Subsection 5.5 in Chapter 6 for a related discussion on seminormality.

Definition 4.14 (Weakly subintegral extensions and weakly normal rings)**.** An integral extension of reduced rings $R \subseteq S$ is called **weakly subintegral** if

(a) $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is a bijection and,
(b) For every prime $Q \in \operatorname{Spec} R$ with corresponding $Q' \in \operatorname{Spec} S$, the inclusion of residue fields $k(Q) \subseteq k(Q')$ is purely inseparable (for instance, an isomorphism).

We say R is **weakly normal in an overring** B if every weakly subintegral extension $R \subseteq S \subseteq B$ has the property that $R = S$. We say that R is **weakly normal** if it is weakly normal in its total ring of fractions $\mathcal{K}(R)$.

Exercise 4.8. Show that an integral extension $R \subseteq S$ is weakly normal if and only if $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is a universal homeomorphism by using [**Sta19**, Tag 04DF].

Exercise 4.9. Suppose R is a reduced ring and let $R \subseteq S$ be a ring extension. The **weak normalization of** R **in S** is the largest weakly subintegral extension of R that is contained in S. In the case that $S = \mathcal{K}(R)$ it is denoted R^{WN}. Show that R^{WN} exists.

Hint: Consider the category of weakly subintegral extensions of R inside S with morphisms the induced inclusions. Show that any two objects of this category are contained in another object in this category. See [**Sta19**, Tag 0EUK] to show that the weak normalization exists.

Exercise 4.10. Now suppose that R is a reduced Noetherian ring of characteristic $p > 0$ and $R \subseteq R^{\mathrm{N}}$ is finite[23] (for generalizations see for instance [**Yan85**]). In Chapter 1 Exercise 4.14 we said that R is weakly normal if and only if the following condition was satisfied:

$$(\star) \qquad \text{any } r \in \mathcal{K}(R) \text{ satisfying } r^p \in R \text{ also satisfies } r \in R.$$

We will show that this definition is equivalent to the definition of weak normality given above in Definition 4.14.

(a) Suppose that $r \in \mathcal{K}(R)$ is such that $r^p \in R$. Show that $R[r]$ is a weakly subintegral extension of R. This shows that if R is weakly normal, then it satisfies condition $\star$.
(b) Suppose that R is a ring satisfying $\star$. Show that any localization of R also satisfies $\star$.
(c) Suppose that R is a ring satisfying $\star$ but that $R \subseteq R' \subseteq \mathcal{K}(R)$ is weakly subintegral. Show that $R' = R$.

    Hint: Let $\mathfrak{c} = \mathrm{Ann}_R(R'/R)$ denote the conductor of R' over R. Localizing at a minimal prime of $\mathfrak{c}$ we may assume that $(R, \mathfrak{m})$ and $(R', \mathfrak{m}')$ are local and $\mathfrak{c}$ is $\mathfrak{m}'$-primary. Choose $r \in R'$. Consider two cases: If $r \in \mathfrak{m}'$ then show that a p^eth power must be in $\mathfrak{c}$. If $r \notin \mathfrak{m}'$, consider its image in the residue field $k(\mathfrak{m}') \supseteq k(\mathfrak{m})$ and use Definition 4.14 (a) to replace r by an element in $\mathfrak{m}'$.

Exercise 4.11. In the setting of Exercise 4.10, show that the formation of weak normalization commutes with localization. For more general statements see for instance [**Yan85**].

5. Test submodules and F-rational rings

The theory of *compatibility* for ideals and maps developed in Section 6 of Chapter 1 can be generalized to modules. This idea becomes especially powerful when applied to the *canonical module*. Indeed, using it we get a parallel theory to test ideals in which we consider compatibility of submodules within the canonical module ω_R. For example, for a Noetherian F-finite domain, we'll see that there is a unique non-zero uniformly compatible submodule of ω_R—an analog of the *test ideal* called the *test module* $\tau(\omega_R)$. The test module provides an obstruction to *F-rationality*[24] in much the same way that the test ideal itself is an obstruction to strong F-regularity. In particular, using the test module we can work easily with F-rationality in non-local rings.

[23] This holds for instance if R is excellent so it holds if R is F-finite.

[24] Chapter 1 Section 7

5.1. Compatible submodules. Suppose that R is an F-finite Noetherian ring. For any R-module M, we have a (typically non-commutative) ring

$$\mathcal{C}_R(M) = \bigoplus_{e \in \mathbb{N}} \mathrm{Hom}_R(F^e_* M, M)$$

whose multiplication is defined, for $\phi \in \mathrm{Hom}_R(F^e_* M, M)$ and $\psi \in \mathrm{Hom}_R(F^f_* M, M)$, by

$$\phi \star \psi = \phi \circ F^e_* \psi \in \mathrm{Hom}_R(F^{e+f}_* M, M),$$

just as we defined it in the case where $M = R$ in Subsection 4.2 in Chapter 1.

Definition 5.1. Given a submodule $N \subseteq M$ and a map $\phi \in \mathrm{Hom}_R(F^e_* M, M)$, we say that N and ϕ are **compatible** if $\phi(F^e_* N) \subseteq N$. In this case, we also say that N **is ϕ-compatible**.

Example 5.2. Suppose that M is an R-module and $\phi \in \mathrm{Hom}_R(F^e_* M, M)$. Suppose J is any ideal. Then the submodule:

$$0 :_M J = \{x \in M \mid Jx = 0\}$$

is compatible with ϕ. Indeed, let us write $N = 0 :_M J$ and suppose $x \in N$. We must show that $\phi(F^e_* x) \in N$. But observe that

$$J\phi(F^e_* x) = \phi(F^e_* J^{[p^e]} x) \subseteq \phi(F^e_* Jx) = \phi(F^e_* 0) = 0.$$

It follows that

$$\Gamma_J(M) = \bigcup_n 0 :_M J^n$$

is also compatible with *every* $\phi \in \mathrm{Hom}_R(F^e_* M, M)$.

In particular, if Q is a minimal prime in the support of M, then we see that $\Gamma_Q(M)$ is compatible with every $\phi \in \mathrm{Hom}_R(F^e_* M, M)$.

Sometimes modules are compatible with all possible maps.

Definition 5.3. Given a submodule $N \subseteq M$, we say that N is **uniformly compatible** if for all $e \in \mathbb{N}$, N is compatible with all $\phi \in \mathrm{Hom}_R(F^e_* M, M)$.

The case of interest in this section is when M is the canonical module ω_R (recall that for us if R has a canonical module, then R also has a dualizing complex and R is locally equidimensional). In this case, the canonical Cartier algebra $\mathcal{C}_R(\omega_R)$ has the following interesting property which is essentially a restatement of Proposition 3.14:

Proposition 5.4. *Let R be an F-finite Noetherian reduced ring, satisfying Serre's S_2-condition.*[25] *Then the dual-to-Frobenius map $T \in \mathrm{Hom}_R(F_* \omega_R, \omega_R)$*

[25]For example, all normal rings and all Cohen-Macaulay rings satisfy this condition; see [**Sta19**, Tag 031S].

generates

$$\mathcal{C}_R(\omega_R) = \bigoplus_{e \in \mathbb{N}} \operatorname{Hom}_R(F^e_* \omega_R, \omega_R)$$

over R in the sense that every element of $\mathcal{C}_R(\omega_R)$ is a sum of maps of the form $T^{\star e} \star r$ for some $e \in \mathbb{N}$ and $r \in R$.

Proof. First note that an excellent S_2 ring is locally equidimensional ([**Sta19**, Tag 0FIW]), so that the F-finite ring R admits a canonical module ω_R, see also Chapter 1 Theorem 1.23. Now the statement follows from the facts that the dual to Frobenius map

$$T^e_R \in \operatorname{Hom}_R(F^e_* \omega_R, \omega_R)$$

generates $\operatorname{Hom}_R(F^e_* \omega_R, \omega_R)$ as a $F^e_* R$-module (Proposition 3.14), and that $T^{\star e} = T^e$ (Exercise 3.6). □

As an immediate corollary we obtain:

Corollary 5.5. *Let R be an F-finite Noetherian reduced ring satisfying Serre's S_2 condition. A submodule $N \subseteq \omega_R$ is uniformly compatible if and only if N is compatible with the dual-to-Frobenius map $T : F_* \omega_R \longrightarrow \omega_R$. In other words, N is compatible with all maps if and only if*

$$T(F_* N) \subseteq N.$$

5.2. The test module. We can now define the test module inside the canonical module, also called the *parameter test module* (see Remark 5.11). To do so, we need the following analog of Corollary 6.16 in Chapter 1:

Proposition 5.6. *Let R be a reduced F-finite Noetherian ring with canonical module ω_R. Fix any non-zero divisor c that is a strong test element for R. Then*

$$\sum_{e>0} T^e(F^e_*(c\, \omega_R)) \subseteq \omega_R$$

is contained in every *T-compatible submodule of ω_R supported on* $\operatorname{Spec} R$.

Proposition 5.6 implies that there is a *unique smallest* submodule of the canonical module ω_R compatible with T and supported on $\operatorname{Spec} R$. We can therefore make the following definition:

Definition 5.7. [*cf.* [**Smi95**]] Let R be a reduced F-finite Noetherian ring with canonical module ω_R. The **test module** in ω_R, denoted $\tau(\omega_R)$, is defined to be the *smallest* submodule $M \subseteq \omega_R$

(a) supported on all of $\operatorname{Spec} R$ (that is, not zero at any minimal prime of R) and,

(b) compatible with T.

In particular, $\tau(\omega_R)$ is the sum in Proposition 5.6.

We now turn to the proof of Proposition 5.6.

PROOF OF PROPOSITION 5.6. Little is lost if one simply assumes that R is a domain, so for a first reading we encourage the reader to make this simplifying assumption.

Fix any $M \subseteq \omega_R$, supported on $\operatorname{Spec} R$, and compatible with T. For any minimal prime $P \in \operatorname{Spec} R$, $(\omega_R)_P = \omega_{R_P}$ is the field R_P. So the inclusion $M \subseteq \omega_R$ becomes an isomorphism after localizing at any minimal prime. By prime avoidance, then, we can find a nonzero-divisor $d \in \operatorname{Ann}_R(\omega_R/M)$. For this d, we have $d\omega_R \subseteq M$.

Now, by the definition of a strong test element, there exists some $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $c = \phi(F^e_* d)$ for some $e > 0$. It follows that we have a factorization:

$$R \xrightarrow{F^e} F^e_* R \xrightarrow{F^e_* d} F^e_* R \xrightarrow{\phi} R$$

so that the composition is multiplication by c. Applying the contravariant functor $\operatorname{Hom}_R(-, \omega_R)$, we have a map

$$\omega_R \xrightarrow{\phi^\vee} F^e_* \omega_R \xrightarrow{F^e_* d} F^e_* \omega_R \xrightarrow{T^e} \omega_R$$

that is also multiplication by c. The image of the right two maps is

$$T^e(F^e_*(d\, \omega_R)) \subseteq T^e(F^e_* M) \subseteq M.$$

Since the total composition has even smaller image, we see that $c\, \omega_R \subseteq M$. Thus for all $f \in \mathbb{N}$,

$$T^f(F^f_*(c\, \omega_R)) \subseteq T^f(F^f_* M) \subseteq M.$$

Now, clearly the submodule of ω_R defined by

$$\sum_{e>0} T^e(F^e_*(c\, \omega_R)) \tag{5.7.1}$$

is compatible with T and contained in M. We only need to check that its support is all of $\operatorname{Spec} R$. For this, we must show that it is non-zero after localizing at any minimal prime $Q \in \operatorname{Spec} R$. At Q, the Frobenius map becomes a split injection of the field R_Q, and its dual T_Q a split surjection. Likewise, $T^e_Q \circ F^e_* c$ is a split surjection because c (being a non-zerodivisor of R) becomes a unit in R_Q. In particular, $T^e_Q(F^e_*(c\, \omega_R)) = (T^e(F^e_*(c\, \omega_R)))_Q$ is nonzero, and the module (5.7.1) is supported at Q. $\square$

Proposition 5.6 says that the test module in ω_R is essentially generated by any non-zerodivisor that is a strong test element plus the action of T. This makes the following results easy to prove:

Corollary 5.8. *Suppose that R is F-finite with a canonical module ω_R.*

(a) *Suppose $W \subseteq R$ is a multiplicative set. Then $W^{-1}\tau(\omega_R) = \tau(\omega_{W^{-1}R})$.*
(b) *Suppose $(R, \mathfrak{m})$ is local. Then $\widehat{\tau(\omega_R)} = \tau(\omega_{\widehat{R}})$ where $\widehat{(-)}$ denotes $\mathfrak{m}$-adic completion.*

Proof. Recall that $W^{-1}\omega_R = \omega_{W^{-1}R}$ by Appendix C Lemma 5.8. Part (a) follows because we can choose $c \in R$ a strong test element and non-zerodivisor for both R and $W^{-1}R$, and then apply Proposition 5.6. Noting that the dual to Frobenius map $F^e_*\omega_R \longrightarrow \omega_R$ localizes since it is identified with $\operatorname{Hom}_R(F^e_*R, \omega_R) \longrightarrow \operatorname{Hom}_R(R, \omega_R)$.

For part (b), we need to know that the formation of ω_R commutes with completion Appendix C Lemma 5.8. We also need that $T : F_*\omega_R \longrightarrow \omega_R$ commutes with completion but that follows since the map $\operatorname{Hom}_R(F_*R, \omega_R) \longrightarrow \operatorname{Hom}_R(R, \omega_R)$ commutes with completion. $\square$

Now that we know the formation of $\tau(\omega_R)$ commutes with localization we can make the following definition.

Definition 5.9. Suppose that X is an F-finite Noetherian locally equidimensional scheme with canonical module ω_X such that $\mathscr{H}om(F^e_*\mathcal{O}_X, \omega_X) \cong F^e_*\omega_X$. The test submodule $\tau(\omega_X) \subseteq \omega_X$ is the coherent subsheaf that agrees on affine charts with $\tau(\omega_R)$.

Remark 5.10. One can weaken the hypothesis that c is a strong test element in Proposition 5.6; see Exercise 5.2.

Remark 5.11. The test module was first constructed in [**Smi95**], in the case of local rings, where it was called the *parameter test module*. The construction there was in Matlis dual form (*cf.* Chapter 7), and so was defined as the annihilator, in ω_R, of the maximal proper Frobenius stable submodule of the local cohomology module $H^d_{\mathfrak{m}}(R)$. Originally, this parameter test module was of interest because it annihilated tight closures of *parameter ideals*. We will discuss the connections with tight closure in Chapter 7.

Remark 5.12. More generally, for any module M with map $\phi : F^e_*M \longrightarrow M$, we could likewise try to make a similar definition (although being non-zero at the minimal primes like we did for ω_R is not quite the right thing to do). This is the topic of Cartier modules, which we'll discuss in Chapter 8. This will also provide a definition of $\tau(\omega_R)$ even when R is non-reduced.

5.3. F-rational rings. In Chapter 1 Definition 7.9 we defined a local ring $(R, \mathfrak{m})$ to be F-rational if it was Cohen-Macaulay and if for every nonzero divisor $c \in R$ the map $H^d_{\mathfrak{m}}(R) \xrightarrow{F^e_* c \circ F^e} F^e_* H^d_{\mathfrak{m}}(R)$ is injective.

By Matlis duality, we immediately see that if $(R, \mathfrak{m})$ is F-rational and F-finite, then for every nonzero divisor $c \in R$ there exists $e > 0$ such that

$$T^e(F^e_* c\, \omega_R) = \omega_R.$$

This certainly implies that $\tau(\omega_R) = \omega_R$. The converse is also true:

Lemma 5.13. *Suppose $(R, \mathfrak{m})$ is a F-finite reduced Noetherian local ring with canonical module ω_R. If $\tau(\omega_R) = \omega_R$, then for every non-zerodivisor $c \in R$ we have that $H^d_{\mathfrak{m}}(R) \xrightarrow{F^e_* c \circ F^e} F^e_* H^d_{\mathfrak{m}}(R)$ injects for some $e > 0$.*

PROOF. It suffices to check this for c a strong test element for R. First notice that $T(F_* \omega_R)$ is a T-compatible submodule, and hence $T(F_* \omega_R) = \omega_R$. This implies that for any $e > 0$:

$$T^{e+1}(F^{e+1}_* c\, \omega_R) \supseteq T^{e+1}(F^{e+1}_* c^p \omega_R) = T^e(F^e_* c\, T(F_* \omega_R)) = T^e(F^e_* c\, \omega_R).$$

In particular

$$\dots \subseteq T^e(F^e_* c\, \omega_R) \subseteq T^{e+1}(F^{e+1}_* c\, \omega_R) \subseteq \dots$$

is ascending, and by definition the sum is $\tau(\omega_R) = \omega_R$. Hence $T^e(F^e_* c\, \omega_R) = \omega_R$. □

This immediately yields the following.

Corollary 5.14. *Suppose R is a Noetherian F-finite reduced ring with canonical module ω_R. Then R is F-rational (meaning all localizations at maximal ideals are F-rational) if and only if*

(a) R is Cohen-Macaulay[26] *and*
(b) $\tau(\omega_R) = \omega_R$.

Remark 5.15. This is *still* not the classical definition of an F-rational ring, which involves tight closure, see Chapter 7. There is also a well-behaved notion of the test module outside the reduced case due to M. Blickle and G. Böckle, see Chapter 8 Definition 3.17 and [**BB11**].

Lemma 5.16 ([**Vél95**])**.** *If an F-finite ring R is F-rational and W is any multiplicative set, then $W^{-1}R$ is also F-rational. Furthermore R is F-rational if and only if $R_{\mathfrak{m}}$ is F-rational for every maximal ideal $\mathfrak{m} \subseteq R$.*

PROOF. This follows since the formation of the finitely generated module $\tau(\omega_R)$ commutes with localization by Corollary 5.8. □

[26] which implies that R is locally equidimensional

Corollary 5.17 ([**Vél95**]). *If R is an F-finite Noetherian locally equidimensional ring then the F-rational locus of* $\operatorname{Spec} R$ *is open.*

Proof. Since R is F-finite, it is a quotient of a regular and ring hence is excellent and has a dualizing complex as we've observed before. Next observe that Cohen-Macaulay and reduced are open conditions. Therefore, it suffices to observe that $\omega_R/\tau(\omega_R)$ is a finitely generated module, and hence supported on a closed set. $\square$

Corollary 5.18. *If $(R, \mathfrak{m})$ is an F-finite local ring. Then R is if F-rational if and only if $\widehat{R}$ is F-rational.*

Proof. This is a direct consequence of Corollary 5.8 (b) and that fact that a Noetherian ring is Cohen-Macaulay if and only if its completion is. $\square$

While F-rational rings are Cohen-Macaulay by hypothesis, they are also normal. Note this proof foreshadows the proof that F-rational singularities are pseudo-rational, that we will study chapter Chapter 6.

Lemma 5.19. *If an F-finite ring R is F-rational, then R is normal.*

Proof. We may assume that R is local. Let $R' \subseteq \mathcal{K}(R)$ denote a finite extension (for instance, we could take the normalization since R is F-finite and hence excellent) and consider the map $R \longrightarrow R'$. We will show $R \longrightarrow R'$ is an isomorphism. Applying the functor $\operatorname{Hom}_R(-, \omega_R)$ gives us a map

$$\psi : \omega_{R'} \cong \operatorname{Hom}_R(R', \omega_R) \longrightarrow \omega_R. \tag{5.19.1}$$

This is generically an isomorphism (since every reduced 0-dimensional ring is normal), and so that map is also injective since these modules are torsion free. In fact, this map ψ (and $F^e_*\psi$) sits in the following diagram obtained by applying Frobenius and then duality to R and $R^{'}$ simultaneously:

$$\begin{tikzcd} F^e_*\omega_{R'} \arrow[r, "T^e_{R'}"] \arrow[d, "F^e_*\psi"'] & \omega_{R'} \arrow[d, "\psi"] \\ F^e_*\omega_R \arrow[r, "T^e_R"] & \omega_R \end{tikzcd}$$

Thus the image of ψ is compatible with T^e and so ψ is surjective since R is F-rational. But now ψ is an isomorphism, and so by applying $\operatorname{Hom}_R(-, \omega_R)$ again to ψ and using by Appendix C Proposition 6.9 we obtain that

$$R \longrightarrow R''$$

is also an isomorphism where R'' is the S2-ification of R'. $\square$

We show that F-rationality satisfies the following deformation-type result after some preliminary setup. In particular, we need to explain the origin of a map $\pi : \omega_R \longrightarrow \omega_{R/(f)}$. For those uncomfortable with derived categories, we invite you to simply assume that R is Cohen-Macaulay and observe that $\omega_R/f\omega_R \cong \omega_{R/(f)}$ by [**BH93**, Exercise 3.3.23], in which case the map π is simply the map which mods out by f.

Suppose R Noetherian locally equidimensional ring with connected Spec and a dualizing complex and that $f \in \mathfrak{m} \subseteq R$ is a non-zerodivisor. Consider the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot f} R \longrightarrow R/(f) \longrightarrow 0.$$

Suppose $\omega_R^{\bullet}$ is zero in degree $< -d$ (if R is local then we may take $\omega_R^{\bullet}$ to be normalized with $d = \dim R$). Applying the functor $\mathbf{R}\operatorname{Hom}_R(-, \omega_R^{\bullet})$ and recalling that $\mathbf{R}\operatorname{Hom}_R(R/(f), \omega_R^{\bullet}) \cong \omega_{R/(f)}^{\bullet}$ is a dualizing complex for $R/(f)$ (so that $\mathcal{H}^{-d}\omega_{R/(f)}^{\bullet} = 0$), we obtain a long exact sequence.

$$0 \longrightarrow \omega_R \xrightarrow{\cdot f} \omega_R \xrightarrow{\pi} \omega_{R/(f)} \longrightarrow \mathcal{H}^{-d+1}\omega_R^{\bullet} \longrightarrow \dots$$

That sequence produced a map

$$\pi : \omega_R \longrightarrow \omega_{R/(f)} \tag{5.19.2}$$

which is surjective sometimes (for instance, when R is Cohen-Macaulay so that $\mathcal{H}^{-d+1}\omega_R^{\bullet} = 0$). Not for a scheme X with a Cartier divisor $D > 0$, this map becomes $\omega_X \otimes \mathcal{O}_X(D) \longrightarrow \omega_D$ which is sometimes called the **adjunction map**.

Lemma 5.20. *Suppose R is an F-finite reduced ring and $f \in R$ is a non-zero divisor such that both R and $R/(f)$ are locally equidimensional. Suppose also that $\pi : \omega_R \longrightarrow \omega_{R/(f)}$ is the map described above. Then the following diagram commutes:*

$$\begin{tikzcd} \omega_R \arrow[r, "\pi"] & \omega_{R/(f)} \\ F^e_*\omega_R \arrow[u, "T^e \star f^{p^e-1}"] \arrow[r, "F^e_*\pi"'] & F^e_*\omega_{R/(f)} \arrow[u, "T^e"'] \end{tikzcd} \tag{5.20.1}$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "\cdot f"] \arrow[d, "1 \mapsto F^e_* f^{p^e-1}"'] & R \arrow[r] \arrow[d, "F^e"] & R/(f) \arrow[r] \arrow[d, "F^e"] & 0 \\ 0 \arrow[r] & F^e_*R \arrow[r, "F^e_*\cdot f"'] & F^e_*R \arrow[r] & F^e_*R/(f) \arrow[r] & 0. \end{tikzcd}$$

We apply the functor $\mathbf{R}\operatorname{Hom}_R(-,\omega_R^{\bullet})$ and take cohomology to obtain the following:

$$\begin{tikzcd} 0 \arrow[r] & \omega_R \arrow[r, "\cdot f"] & \omega_R \arrow[r, "\pi"] & \omega_{R/(f)} \arrow[r] & \mathcal{H}^{-d+1}\omega_R^{\bullet} \\ 0 \arrow[r] & F^e_*\omega_R \arrow[r, "F^e_*\cdot f"'] \arrow[u, "T^e"] & F^e_*\omega_R \arrow[r] \arrow[u, "T^e \star f^{p^e-1}"'] & F^e_*\omega_{R/(f)} \arrow[r] \arrow[u, "T^e"'] & F^e_*\mathcal{H}^{-d+1}\omega_R^{\bullet} \arrow[u] \end{tikzcd} \tag{5.20.2}$$

The middle square is exactly what we wanted. $\square$

Theorem 5.21. *Suppose that $(R,\mathfrak{m})$ is an F-finite reduced local ring and that $f \in \mathfrak{m} \subseteq R$ is a non-zerodivisor so that $R/(f)$ is also reduced and that both R and $R/(f)$ are equidimensional. Suppose that $\pi : \omega_R \longrightarrow \omega_{R/(f)}$ is the from (5.19.2) above. Then*

$$\tau(\omega_{R/(f)}) \subseteq \pi(\tau(\omega_R)).$$

Proof. First notice that since $T(F_*\tau(\omega_R)) \subseteq \tau(\omega_R)$ we have that

$$T(F_* f^{p-1}\tau(\omega_R)) \subseteq \tau(\omega_R)$$

as well. It follows from Lemma 5.20 that $\pi(\tau(\omega_R))$ is compatible with T (on $\omega_{R/(f)}$). We want to show that $\pi(\tau(\omega_R))$ is nonzero at each minimal prime of $R/(f)$. However, at each minimal prime $\overline{Q}$ of $R/(f)$, since f is a non-zerodivisor and $R/(f)$ is reduced, we see that R_Q is regular where $Q \in \operatorname{Spec} R$ is the corresponding prime (a minimal associated prime of the ideal (f)). Hence $\tau(\omega_R)_Q = (\omega_R)_Q$. Since R_Q is also Cohen-Macaulay, $\pi_Q : (\omega_R)_Q \longrightarrow (\omega_{R/(f)})_{\overline{Q}}$ surjects and so $\pi(\tau(\omega_R))_{\overline{Q}}$ is nonzero. Thus $\tau(\omega_{R/(f)}) \subseteq \pi(\tau(\omega_R))$ by its definition of the test module as the smallest module satisfying these properties, see Definition 5.7. $\square$

Remark 5.22. It would be nice if the above easily implied the following. That if $(R,\mathfrak{m})$ is local and $R/(f)$ is F-rational, then R is F-rational since in that case both $R/(f)$ and R are Cohen-Macaulay (and so $\pi : \omega_R \longrightarrow \omega_{R/(f)}$ is surjective). There's a slight complication however, just because $\pi(\tau(\omega_R)) = \frac{\tau(\omega_R)}{f\omega_R \cap \tau(\omega_R)} \supseteq \omega_{R/(f)}$ does not immediately imply that $\tau(\omega_R) = \omega_R$. If R is Gorenstein, so that $\omega_R \cong R$ this consequence does follow. Indeed, in that case π is identified with the quotient map $R \longrightarrow R/(f)$ and a submodule of a ring is the entire ring if and only if it contains a unit, and that can be checked modulo $f \in \mathfrak{m}$.

In particular, we have shown that if $(R,\mathfrak{m})$ is local, F-finite and Gorenstein, and $R/(f)$ is F-rational, then R is F-rational.

Removing the Gorenstein hypothesis can be handled by modifying the above argument, but it requires more work that we have chosen to omit (this result was already deduced in an exercise in Chapter 1 and will be proven in a different and particularly slick way in Chapter 7).

5.4. Exercises.

Exercise 5.1. Suppose R is a Noetherian F-finite reduced ring with canonical module ω_R. Prove that $T(F_*\tau(\omega_R)) = \tau(\omega_R)$.

Hint: Note the containment $\subseteq$ is definitional.

Exercise 5.2. With R an F-finite ring, suppose $c \in R$ is not in any minimal prime of R and that $c\,\omega_R \subseteq \tau(\omega_R)$. Prove that

$$\tau(\omega_R) = \sum_{e>0} T^e(F^e_*(c\,\omega_R)) \subseteq \omega_R.$$

If R is Cohen-Macaulay, $c \in R$ is not in any minimal prime, and $c\omega_R \subseteq \tau(\omega_R)$, then c is called a **parameter test element**.

Exercise 5.3. With notation as in Exercise 5.2, show that every strong test element is a parameter test element.

Exercise 5.4. Prove Theorem 6.12.

Exercise 5.5. Suppose that X is a projective variety of finite type over a field k. Show that X has a dualizing complex $\omega_X^{\bullet}$ so that for every closed point $x \in X$, we have that $(\omega_X^{\bullet})_x$ is a normalized dualizing complex for $\mathcal{O}_{X,x}$.

Exercise 5.6. **[Smi97a]** Let $(R, \mathfrak{m})$ be a d-dimensional local ring. We say a submodule $N \subseteq H^d_{\mathfrak{m}}(R)$ is F-stable if $F : H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(F_*R) = F_*H^d_{\mathfrak{m}}(R)$ is the map induced by Frobenius, then $F(N) \subseteq F_*N$. Suppose now that R is F-finite and Cohen-Macaulay. Show that R is F-rational if and only if the only F-stable submodules of $H^d_{\mathfrak{m}}(R)$ are 0 and $H^d_{\mathfrak{m}}(R)$.

Exercise 5.7. Suppose that $(R, \mathfrak{m})$ is a local ring and $f \in \mathfrak{m} \subseteq R$ is a non-zerodivisor. Suppose that $R/(f)$ is normal, prove that R is normal.

Hint: Use the fact that R is normal if and only if it is R_1 and S_2.

Exercise 5.8. A weakly normal ring R is called WN1 if for each height one prime Q of R, we have that the normalization morphism $R_Q \longrightarrow R_Q^{\mathrm{N}}$ is unramified[27]. Show that Frobenius split rings are WN1.

Hint: The conductor ideal is compatible with a Frobenius splitting by Chapter 1 Exercise 6.30.

[27]This means that QR_Q^{N} is a radical ideal.

Exercise 5.9. Suppose that $(R, \mathfrak{m})$ is an F-finite local Noetherian equidimensional ring and $f \in \mathfrak{m}$ is a non-zerodivisor such that $R/(f)$ is reduced. Show that there exists a unique smallest submodule $M \subseteq \omega_R$, which agrees with ω_R at every minimal associated prime of (f) and which is compatible with $T \star f^{p-1} : F_* \omega_R \longrightarrow \omega_R$.

Further show that if $\pi : \omega_R \longrightarrow \omega_{R/(f)}$ is the map of (5.19.2), then $\pi(M) = \tau(\omega_{R/(f)})$.

CHAPTER 3

Introduction to the global theory of Frobenius splitting

In this chapter, we look at global implications of Frobenius splitting for a scheme X of characteristic $p > 0$. The natural Frobenius map

$$\mathcal{O}_X \to F_*\mathcal{O}_X$$

may split *globally*, a much stronger condition than splitting locally at each stalk. In this case, the scheme X is said to be **globally Frobenius split.**

A projective scheme with a *global* splitting of Frobenius has all the local restrictions on its singularities implied by local Frobenius splitting, but also very strong vanishing theorems that force restrictions on its *global geometry.* For example, Frobenius splittings for a smooth projective X—that is, particular maps in $\mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$—give rise to particular kinds of effective *anti-canonical divisors* on X. In particular, smooth varieties with many splittings of Frobenius—globally F-regular varieties— have *big* anti-canonical bundles. Thus while *local Frobenius splitting* can be thought of as a restriction on the singularities at each point of a variety, *global Frobenius splitting* tells us something further about its positivity.

In this chapter, we discuss vanishing theorems for globally Frobenius split and globally F-regular varieties, and restrictions these conditions impose on global geometry. We also examine compatible splitting in the global setting, and connect global Frobenius splitting for a projective variety X to local Frobenius splitting at the vertex of the affine cone over X.

1. Global Frobenius splitting

Let X be an arbitrary scheme of prime characteristic $p > 0$.

Definition 1.1. We say that X is **globally Frobenius split** if the Frobenius map $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits in the category of $\mathcal{O}_X$-modules. Explicitly, this means that there is a $\mathcal{O}_X$-module map $\phi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that the composition

$$\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$$

is the identity map of the sheaf $\mathcal{O}_X$. Equivalently, X is globally Frobenius split if there exists a global section

$$\phi \in \operatorname{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X) = H^0(X, \mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X))$$

such that $\phi(F_*1) = 1$ on every open set.

Global Frobenius splitting always implies local Frobenius splitting:

Lemma 1.2. *Let X be a globally Frobenius split scheme.*

(a) For every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ is Frobenius split.
(b) For every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is Frobenius split.

Proof. Both statements follow by considering that if a composition

$$\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_X$$

is the identity map, then this is true on every open set U, and, by taking a direct limit, at every stalk. But because Frobenius commutes with localization, we know that $(F_*\mathcal{O}_X)(U)$ is $F_*(\mathcal{O}_X(U))$ and also that the stalk $(F_*\mathcal{O}_X)_x$ is $F_*\mathcal{O}_{X,x}$. $\square$

Remark 1.3. For an *affine* scheme that is both Noetherian and F-finite, global and local Frobenius splitting are equivalent by Proposition 3.17 in Chapter 1.

Example 1.4. The scheme $\mathbb{P}^n_k$ is globally Frobenius split, for any field k of prime characteristic p.

Proof of Example 1.4. We prove the case when k is perfect, and leave the reduction to this case as Exercise 1.2.

Fix homogeneous coordinates $x_0, x_1, \ldots, x_n$ for $\mathbb{P}^n_k$. Recall that $\mathbb{P}^n_k$ is covered by affine coordinate charts $D_+(x_i) = \operatorname{Spec} k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{xi}]$. On each such chart, we define a Frobenius splitting of the polynomial ring $\mathcal{O}_{\mathbb{P}^n_k}(D_+(x_i)) = k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{xi}]$ by sending

$$F_*(\lambda(\frac{x_0}{x_i})^{t_0} \ldots (\frac{x_n}{xi})^{t_n}) \mapsto \begin{cases} \lambda^{1/p}(\frac{x_0}{x_i})^{t_0/p} \ldots (\frac{x_n}{xi})^{t_n/p} & \text{if } t_j/p \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

We claim that these local splittings can be glued together to give a global splitting of Frobenius for $\mathbb{P}^n$. Indeed, it is easy to check that these splittings agree on the intersections

$$\begin{aligned} & D_+(x_i) \cap D_+(x_j) \\ & = D_+(x_i x_j) \\ = \quad & \operatorname{Spec} k[\tfrac{x_0}{x_i}, \ldots, \tfrac{x_n}{xi}, \tfrac{x_n}{xi}] \\ = \quad & \operatorname{Spec} k[\tfrac{x_n}{x_j}, \ldots, \tfrac{x_n}{x_j}, \tfrac{x_j}{xi}]. \end{aligned}$$

because the monomials of one chart also are monomials of the other chart. $\square$

Remark 1.5. The splitting of Frobenius constructed in Example 1.4 is sometimes called the *canonical toric splitting* of $\mathbb{P}^n$. Indeed, this map is the *unique* Frobenius splitting of $\mathbb{P}^n$ which respects the standard toric variety structure on $\mathbb{P}^n$. More generally, *upper cluster varieties*—which can be viewed as a union of affine toric varieties glued together by a collection of *cluster mutations*— admit a canonical splitting of Frobenius generalizing the splitting in Example 1.4. See [**BMRS15**, Thm 3.7]

Example 1.6. Many familiar varieties are Frobenius split, including all G/Q where G is a reductive group and Q a parabolic subgroup, and all their Schubert subvarieties; see [**MR85**].

Global Frobenius splitting is much stronger, in general, than local Frobenius splitting at each point: splitting a map of $\mathcal{O}_X$-modules is *a priori* much stronger than splitting at each stalk. The next example illustrates the dramatic difference:

Example 1.7. Let X be a smooth projective curve over an algebraically closed field of characteristic p. Then X is *locally* Frobenius split because the stalk of $\mathcal{O}_X$ at each point is an F-finite *regular* local ring (by Proposition 3.16 in Chapter 1). However, global Frobenius splitting depends, among other things, on the *genus* of our curve:

(i). If X has genus zero, then X is globally Frobenius split; this follows from Example 1.4.
(ii). If the genus of X exceeds one, then X is *not* globally Frobenius split; this will be proved in Corollary 1.13.
(iii). If X has genus one, then X is globally Frobenius split if and only if it is an ordinary (non-super-singular) elliptic curve; see Example 1.29.

Remark 1.8. In Section 4, we explain how the local and global points of view converge by translating global splittings of a projective variety X into local splittings "at the vertex of the cone" over X.

We record the following fact, which can be proved similarly to the affine case (see Proposition 3.9 in Chapter 1):

Proposition 1.9. *Let X be a scheme of prime characteristic p. Then the following are equivalent:*

(a) X is globally Frobenius split;

(b) *There exists an* $e > 0$ *such that the iterated Frobenius*

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X$$

splits;

(c) *For every* $e > 0$*, the iterated Frobenius* $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ *splits.*

Remark 1.10. Global Frobenius splitting is often simply called *Frobenius Splitting* in the literature dealing primarily with smooth projective varieties, including [**BK05**]. When the context is clear, or when a statement is true for *both local* and *global* Frobenius splitting, we sometimes do the same. Usually, however, we decorate the term "Frobenius split" with the adverbs "globally" or "locally" in the non-affine case in order to avoid confusion.

1.1. Vanishing Theorems for Frobenius Split Varieties. Global Frobenius splitting places strong restrictions on the geometry of a projective variety. This is a result of the strong vanishing theorems implied by Frobenius splitting, such as the following prototypical example:

Theorem 1.11. *Let* X *be a Noetherian scheme of prime characteristic and let* $\mathcal{L}$ *be an invertible sheaf on* X*. If* X *is globally Frobenius split and* $H^i(X, \mathscr{L}^t) = 0$ *for some* $i \geq 0$ *and all* $t \gg 0$*, then* $H^i(X, \mathscr{L}) = 0$*.*

Before proving Theorem 1.11, we deduce some easy consequences.

Corollary 1.12. *Let* X *be a projective variety over a field of prime characteristic. If* X *is globally Frobenius split and* $\mathcal{L}$ *is an ample invertible sheaf on* X*, then* $H^i(X, \mathscr{L}) = 0$ *for all* $i \geq 1$*.*

Proof of Corollary. The corollary follows immediately by combining Theorem 1.11 with Serre Vanishing[1], which says that an ample invertible sheaf $\mathscr{L}$ on a projective scheme X always satisfies $H^i(X, \mathscr{L}^t) = 0$ for sufficiently large t and all $i \geq 1$. $\square$

Corollary 1.13. *A smooth projective curve of genus two or more can not be globally Frobenius split.*

Proof of Corollary 1.13. Suppose a smooth projective curve X has genus two or more. Then ω_X is ample (see [**Har77**, Chapter V]), and so $H^1(X, \omega_X) = 0$ by Corollary 1.12. But now by Serre duality,[2] $H^0(X, \mathcal{O}_X)$ is also zero. This is a contradiction, as $H^0(X, \mathcal{O}_X)$ includes, for example, all the constant functions on X. $\square$

[1][**Har77**, Chapter III, Prop 5.3]

[2]Serre Duality states that for an invertible sheaf $\mathscr{L}$ on a Cohen-Macaulay variety X proper over a field k, the k-vector space $H^i(X, \mathscr{L})$ is canonically identified with the k-vector space dual of $H^{\dim X - i}(X, \omega_X \otimes \mathscr{L}^{-1})$ [**Har77**, Chapter III, Thm 7.6].

Example 1.14. Corollary 1.13 generalizes to higher dimension as well: if the canonical sheaf ω_X of a smooth projective variety X is ample, then X is not globally Frobenius split; see Exercise 1.8.

Example 1.15. Suppose $d > n+1$. Then a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d in $\mathbb{P}^n$ is not globally Frobenius split. This follows from Example 1.14 because in this case, ω_X is ample. Indeed, by the adjunction formula,[3]

$$K_X = (K_{\mathbb{P}^n} + X)|_X = (-(n+1)H + dH)|_X = (d-n-1)H|_X,$$

so that ω_X is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(d-n-1)$ to X, and is ample when $d > n+1$. In fact, in it not necessary to assume X is smooth—the canonical module ω_X exists and is invertible because X is Gorenstein, so the same arguments apply.

The Kodaira Vanishing Theorem can fail for smooth varieties of characteristic $p > 0$; see [**Ray78**], [**LR97**]. But for *globally Frobenius split* varieties, the following weak version of Kodaira vanishing holds:

Corollary 1.16. *Let X be a Cohen-Macaulay projective scheme over a field of prime characteristic p. If X is globally Frobenius split and $\mathscr{L}$ is an ample invertible sheaf, then $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ for all $i \geq 1$.*

Proof. To show that $H^i(X, \omega_X \otimes \mathscr{L}) = 0$, it is equivalent to show that $H^{\dim X - i}(X, \mathscr{L}^{-1}) = 0$ by Serre duality. By Theorem 1.11, it suffices to show that $H^{\dim X - i}(X, \mathscr{L}^{-t}) = 0$ for large t. Dualizing again, this is equivalent to the vanishing of $H^i(X, \omega_X \otimes \mathscr{L}^t)$ for large t, which follows from the ampleness of $\mathscr{L}$ by Serre Vanishing. $\square$

Remark 1.17. Corollary 1.16 is true without the Cohen-Macaulay hypothesis; see Theorem 2.32.

Turning to the proof of Theorem 1.11, we need the following lemma:

Lemma 1.18. *Suppose X is a scheme of characteristic $p > 0$ and $\mathscr{L}$ is an invertible sheaf. Then*

$$\mathscr{L} \otimes F^e_* \mathcal{O}_X \cong F^e_* \mathscr{L}^{p^e}.$$

Proof. By the projection formula,[4] there is a natural isomorphism $F^e_*\mathcal{O}_X \otimes \mathscr{L} \cong F^e_*(F^{e*}\mathscr{L})$. But pulling back the invertible sheaf $\mathscr{L}$ via (the e-th iterate of) Frobenius produces $\mathscr{L}^{p^e}$: the transition functions pull back under F^e to their p^e-th powers, which are the transition functions for $\mathscr{L}^{p^e}$. $\square$

[3] Add reference to Hartshorne.

[4] [**Har77**, Chapter II, Exercise 5.1(d)]

PROOF OF THEOREM 1.11. Assume that X is globally Frobenius split. Then all iterates F^e of the Frobenius map split. Let π be a splitting of F^e, so that the composition

$$\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_X$$

is the identity map. Tensoring with the invertible sheaf $\mathscr{L}$, the composition is still the identity map:

$$\mathscr{L} \xrightarrow{F^e} F^e_*\mathcal{O}_X \otimes \mathscr{L} \xrightarrow{\pi} \mathscr{L}. \tag{1.18.1}$$

Applying Lemma 1.18, we see that the composition

$$\mathscr{L} \xrightarrow{F^e} F^e_*\mathscr{L}^{p^e} \xrightarrow{\pi} \mathscr{L}, \tag{1.18.2}$$

is again the identity map of sheaves. Taking cohomology, we have the identity map of abelian groups as well:

$$H^i(X, \mathscr{L}) \hookrightarrow H^i(X, F^e_*\mathscr{L}^{p^e}) \twoheadrightarrow H^i(X, \mathscr{L}).$$

Since F is affine, $H^i(X, F^e_*\mathscr{L}^{p^e}) = H^i(X, \mathscr{L}^{p^e})$ [**Har77**, Chapter III, Exercise 4.1 or Exercise 8.1]. Thus $H^i(X, \mathscr{L})$ is a direct summand of $H^i(X, \mathscr{L}^{p^e})$, and so it will vanish if $H^i(X, \mathscr{L}^{p^e})$ does. $\square$

Remark 1.19. The global consequences of splitting Frobenius, and indeed the term *Frobenius split,* were first treated systematically by Mehta and Ramanathan in [**MR85**]. While Hochster and Roberts' paper [**HR74**] ten years prior focused on Frobenius splitting for local (or graded) rings, Mehta and Ramanathan used *global* Frobenius splitting to study the global geometry of Schubert varieties and related objects in algebro-geometric representation theory. Note also the work of Haboush, and independently, Anderson, who used similar ideas in characteristic $p > 0$ to prove vanishing theorems [**Hab80**], [**And80**].

1.2. Detecting global Frobenius splitting. Having established the utility of global Frobenius splitting, we record some ways to prove a variety is globally Frobenius split. First, we point out that *contractions*[5] of Frobenius split varieties are Frobenius split:

Proposition 1.20. *Suppose that Y is globally Frobenius split scheme and $Y \xrightarrow{\pi} X$ is a map of schemes such that the induced map $\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y$ splits (for instance, we could have $\pi_*\mathcal{O}_X = \mathcal{O}_Y$). Then X is also globally Frobenius split.*

PROOF. Since the Frobenius map $\mathcal{O}_Y \longrightarrow F_*\mathcal{O}_Y$ on Y splits, pushing forward to X, the map of sheaves $\pi_*\mathcal{O}_Y \longrightarrow \pi_*F_*\mathcal{O}_Y$ splits as well. Hence

[5]By definition, a contraction is a map $\pi : Y \longrightarrow X$ such that $\pi_*\mathcal{O}_Y = \mathcal{O}_X$.

the composition

$$\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y \longrightarrow \pi_*F_*\mathcal{O}_Y \tag{1.20.1}$$

is a composition of split maps and hence split. Because Frobenius commutes with all scheme maps, including π, we know that $F_*\pi_*\mathcal{O}_Y = \pi_*F_*\mathcal{O}_Y$ (see Remark 1.6 in Chapter 1, especially the diagram (1.6.1), or Exercise 1.4). So the composition (1.20.1) can also be factored as

$$\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \longrightarrow \pi_*F_*\mathcal{O}_Y. \tag{1.20.2}$$

splits, and in particular $\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$ splits. Thus X is globally Frobenius split. □

Caution 1.21. Proposition 1.20 is wildly false if we replace "globally Frobenius split" by "locally Frobenius split." For example, suppose $Y \xrightarrow{\pi} X$ is a resolution of singularities of a normal affine variety X of characteristic p. Then $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ and the variety Y, being non-singular, is *locally* Frobenius split. But X can be far from Frobenius split.

Example 1.22. Let k be an algebraically closed field of characteristic $p > 0$. Consider the affine cone $X = \operatorname{Spec} k[x_0, \ldots, x_n]/(F)$ over the projective hypersurface $\operatorname{Proj} k[x_0, \ldots, x_n]/(F)$ of degree d in $\mathbb{P}^n$. The cone X has an an isolated singularity at its vertex, which can be resolved by blowing up the vertex to get a smooth variety Y and proper birational map $Y \xrightarrow{\pi} X$. The scheme Y is locally Frobenius split (because it is non-singular) and $\pi_*\mathcal{O}_Y = \mathcal{O}_X$. But the affine scheme X is *not* Frobenius split at the origin if $d > n$ (see Example 1.11 in Chapter 4). So Y can not be *globally* Frobenius split either, by Proposition 1.20.

Remark 1.23. To adapt Proposition 1.20 to a statement about *local* Frobenius splitting, we should assume that π is an *affine* map such that $\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y$ splits. In this case, local Frobenius splitting of Y does imply local Frobenius splitting of X. See Exercise 1.10

As a corollary, we see that to check a finite type scheme over a field k is globally (or locally) Frobenius split, we can always assume k is F-finite or algebraically closed:

Corollary 1.24. *Let X be a scheme of finite type over a field k prime characteristic, and let L denote any field extension. If $L \times_k X$ is globally (respectively, locally) Frobenius split, then so is X.*

Proof. The point is that the natural projection map $i : L \times_k X \longrightarrow X$ satisfies $\mathcal{O}_X \longrightarrow i_*\mathcal{O}_X$ is *split* (respectively, and i is affine). We leave the details as Exercise 1.3. □

Remark 1.25. The converse of Corollary 1.24 is false in general, but true with some additional restrictions. See Exercise 1.6 and Exercise 1.7.

Corollary 1.26. *Suppose X is an F-finite normal integral Noetherian scheme and $i : U \hookrightarrow X$ is an open set whose complement has codimension ≥ 2. Suppose that U is globally F-split, then X is also globally F-split.*

Proof. Let $i : U \hookrightarrow X$ denote the inclusion of the open set. Since $\mathcal{O}_X$ is S_2, $i_*\mathcal{O}_U = \mathcal{O}_X$ (see Appendix B Theorem 4.9), and so the result follows from Proposition 1.20. $\square$

We conclude the section with a variant of Lemma 7.22 in Chapter 1.

Theorem 1.27. *Let X be a smooth projective variety of dimension d and prime characteristic. Then X is globally Frobenius split if and only if the natural map induced by Frobenius*

$$H^d(X, \omega_X) \to H^d(X, \omega_X \otimes F_*\mathcal{O}_X) \cong H^d(X, F_*\omega_X^p) \tag{1.27.1}$$

is injective.

Remark 1.28. The map (1.27.1) in Theorem 1.27 is injective if and only if it is non-zero. This is because, by Serre duality, $H^d(X, \omega_X)$ is k-dual to $H^0(X, \mathcal{O}_X)$, which is a product of copies of k (one for each component of X).

Before proving Theorem 1.27, let's apply it to some examples.

Example 1.29. An elliptic curve is globally Frobenius split if and only if Frobenius acts injectively on $H^1(X, \mathcal{O}_X)$, that is, if and only if X is *ordinary*, see [**Har77**, Chapter IV, Section 4]. Indeed, for an elliptic curve, $\omega_X = \mathcal{O}_X$, so this statement is an immediate corollary of Theorem 1.27.

Example 1.30. Generalizing the previous application, assume that $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $n+1$. Then $\mathcal{O}_X \cong \omega_X$, and X is globally Frobenius split if and only if the Frobenius map acts injectively on the one dimensional vector space $H^{\dim X}(X, \mathcal{O}_X)$.

Example 1.31. An Abelian variety X of characteristic p is Frobenius split if and only if the Frobenius action $H^{\dim X}(X, \mathcal{O}_X) \xrightarrow{F} H^{\dim X}(X, \mathcal{O}_X)$ is injective (equivalently, non-zero). The point is that $\omega_X = \mathcal{O}_X$ for an Abelian variety.

Proof of Theorem 1.27. To split the Frobenius map $\mathcal{O}_X \to F_*\mathcal{O}_X$, we need to find some

$$\phi \in \operatorname{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$$

such that $\phi(F_*1_X) = 1_X$. For this, it suffices if the map

$$H^0(X, \mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)) \longrightarrow H^0(X, \mathcal{O}_X) \tag{1.31.1}$$

sending each ϕ to $\phi(F_*1)$ is surjective. By Serre duality [**Har77**, III, Thm 7.6], this is equivalent to the injectivity of

$$\mathrm{Ext}^d(\mathcal{O}_X, \omega_X) \longrightarrow \mathrm{Ext}^d(\mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X), \omega_X).$$

There is a natural isomorphism $\mathrm{Ext}^d(\mathcal{O}_X, \omega_X) \cong H^d(X, \omega_X)$, so we need only find another

$$\mathrm{Ext}^d(\mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X), \omega_X) \cong H^d(X, F_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \omega_X).$$

But because X is non-singular, the $\mathcal{O}_X$-module $F_*\mathcal{O}_X$ is locally free, and so there is a natural isomorphism

$$F_*\mathcal{O}_X \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X), \mathcal{O}_X). \tag{1.31.2}$$

Therefore,

$$\mathrm{Ext}^d(\mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X), \omega_X) \cong \mathrm{Ext}^d(\mathcal{O}_X, F_*\mathcal{O}_X \otimes \omega_X) \tag{1.31.3}$$

by [**Har77**, III, Prop 6.7], which is isomorphic to $H^d(X, F_*\mathcal{O}_X \otimes \omega_X)$ by [**Har77**, III, Prop 6.3]. Putting these isomorphisms together we see that global Frobenius splitting is equivalent to the injectivity of

$$H^d(X, \omega_X) \longrightarrow H^d(X, F_*\mathcal{O}_X \otimes \omega_X)$$

as desired. This completes the proof of Theorem 1.27. □

Remark 1.32. Theorem 1.27 holds when X is assumed only normal and proper over a field; see Theorem 2.16. Our proof goes through because, in this setting, Serre duality holds "at the top spot" (see Exercise 1.12) and the isomorphism (1.31.3) holds (see Exercise 1.12). Alternatively, for normal *projective* varieties, we can see that Theorem 1.27 holds by arguing on an affine cone over X; see **??**.

1.3. Exercises.

Exercise 1.1. Let X be an arbitrary scheme of prime characteristic. For any point $x \in X$, show that the stalk of the Frobenius map $\mathcal{O}_{X,x} \xrightarrow{F} (F_*\mathcal{O}_X)_x$ is the Frobenius map of the stalk $\mathcal{O}_{X,x} \xrightarrow{F} F_*(\mathcal{O}_{X,x})$.

Hint: See Lemma 1.11 in Chapter 1.

Exercise 1.2. Prove Example 1.4 in the general case when k is not necessarily perfect.

Hint: You can use the fact that k is Frobenius split to construct an explicit splitting, or make use of Corollary 1.24.

Exercise 1.3. Prove Corollary 1.24.

Hint: See Proposition 1.20 and Remark 1.23.

Exercise 1.4. Suppose $X \xrightarrow{\pi} Y$ is a map of schemes. Show that $\pi_* {F_X}^e_* \mathscr{M}$ is naturally isomorphic to ${F_Y}^e_* \pi_* \mathscr{M}$ for any sheaf of $\mathcal{O}_X$-modules $\mathscr{M}$.

Hint: Revisit Chapter 1 Remark 1.6, especially the diagram (1.6.1).

Exercise 1.5. Suppose that X is a globally Frobenius split scheme over a field k. Prove that $X \times_k \mathbb{A}^1_k$ is also globally Frobenius split.

Exercise 1.6. Let k be the field $\mathbb{F}_p(t)$. Show that finite type k-scheme $X = \operatorname{Spec} k[x]/(x^p - t)$ is Frobenius split, but $X \otimes \overline{k}$ is not, where $\overline{k}$ is the algebraic closure of k. C.f. Remark 1.25.

Exercise 1.7. Suppose that X is an F-split variety proper over an F-finite field k and $k \subseteq H^0(X, \mathcal{O}_X)$ is a separable field extension. Then for any algebraic extension $L \supseteq k$ we have that $X \times_k L$ is also F-split. For a hint see [**GLP**$^+$**15**, Lemma 2.4].

Exercise 1.8. Show that if a non-singular projective variety has ample canonical sheaf, then it is not globally Frobenius split.

Exercise 1.9. Suppose k is a field and let X denote the blowup of $\mathbb{A}^2_k$ at the origin. Show that X is globally Frobenius split.

Exercise 1.10. Let $Y \xrightarrow{\pi} X$ be an *affine* map of Noetherian F-finite schemes such that the induced map $\mathcal{O}_X \longrightarrow \pi_* \mathcal{O}_Y$ splits. Prove that if Y is locally Frobenius split, then X is too.

Hint: See Proposition 3.5 in Chapter 1.

Exercise 1.11. Suppose that X is a globally Frobenius split Cohen-Macaulay variety and $\mathscr{L}$ is an invertible sheaf. Prove that if $H^i(X, \omega_X \otimes \mathscr{L}^t) = 0$ for all $t \gg 0$ and some fixed i, then $H^i(X, \omega_X \otimes \mathscr{L}) = 0$.

Exercise 1.12. Suppose that X is a d-dimensional normal proper variety over a field k. Further suppose that $\mathscr{M}$ is coherent $\mathcal{O}_X$-module. We will use Grothendieck duality to prove that there is an natural isomorphism

$$H^0(X, \mathscr{H}om(\mathscr{M}, \mathcal{O}_X)) \cong H^d(X, \mathscr{M} \otimes \omega_X)^\vee, \tag{1.32.1}$$

where the symbol $(-)^\vee$ indicates k-vector space dual. This will prove Theorem 1.27 in the normal (possibly singular) case.

(a) First verify that we have a natural isomorphism

$$H^0(X, \mathscr{H}om(\mathscr{M}, \mathcal{O}_X)) \cong \operatorname{Hom}(\mathscr{M}, \mathcal{O}_X).$$

(b) Next, recall that X is normal so that ω_X is locally free in codimension one.[6] Use this to show that:

$$\operatorname{Hom}(\mathscr{M}, \mathcal{O}_X) \cong \operatorname{Hom}(\mathscr{M} \otimes \omega_X, \omega_X).$$

(c) Using a spectral sequence of low degree terms, show that

$$\operatorname{Hom}(\mathscr{M}\otimes\omega_X, \omega_X) \cong \mathcal{H}^0\mathbf{R}\operatorname{Hom}(\mathscr{M}\otimes\omega_X, \omega_X^{\bullet}[-d]) = \mathcal{H}^{-d}\mathbf{R}\operatorname{Hom}(\mathscr{M}\otimes\omega_X, \omega_X^{\bullet}).$$

(d) Use Grothendieck duality, Appendix C Theorem 4.1, to show that for any coherent sheaf $\mathscr{F}$ that

$$\mathcal{H}^{-d}\mathbf{R}\operatorname{Hom}(\mathscr{F}, \omega_X^{\bullet}) \cong H^d(X, \mathscr{F})^{\vee}$$

Conclude that (1.32.1) holds.

Exercise 1.13. A sheaf $\mathscr{F}$ on a projective variety X is said to be **globally generated at a point** $p \in X$ if there exist sections $s_1, \ldots, s_n \in H^0(X, \mathscr{F})$ such that the restriction of the s_i to the stalk $\mathscr{F}_p$ generate $\mathscr{F}_p$ as an $\mathcal{O}_{X,p}$-module. We say $\mathscr{F}$ is *globally generated* if it is globally generated at all $p \in X$.

Suppose X is a projective Cohen-Macaulay globally Frobenius split variety of dimension d and $\mathscr{L}$ is a globally generated ample line bundle. Prove that $\omega_X \otimes \mathscr{L}^{d+1}$ is globally generated.

Hint: Use the following criterion for a coherent sheaf $\mathcal{F}$ to be globally generated: Given a globally generated ample line bundle $\mathscr{L}$, if $H^i(X, \mathscr{F} \otimes \mathscr{L}^{-i}) = 0$ for all $i > 0$, then $\mathscr{F}$ is globally generated.[7]

2. Global Frobenius splitting along divisors

Globally F-regular varieties are special kinds of Frobenius split varieties with an abundance of splittings of Frobenius— they are eventually globally split along *every* effective divisor. They satisfy even stronger vanishing theorems, and hence have stronger restrictions on their geometry. For example, among smooth projective curves of characteristic p, only the genus zero curves—those isomorphic to $\mathbb{P}^1$—are globally F-regular. Many familiar classes of varieties are globally F-regular including projective toric varieties, Grassmannians and other homogeneous spaces, and certain moduli spaces.

[6] One really only needs that ω_X is locally free in codimension 1 and that X is S_2 for this step, so in fact the result holds if the normality hypothesis is weakened to S_2 and Gorenstein in codimension 1.

[7] This condition is called being Castelnuovo-Mumford 0-*regular*; see [**Laz04a**, Section 1.8].

2.1. Frobenius Splitting Along Divisors. Let X be a Noetherian normal scheme of characteristic $p > 0$. Because Noetherian normal schemes are a disjoint union of normal components, there is little loss of generality in assuming X is integral.

Now for a Weil divisor D on X, there is an associated reflexive[8] subsheaf $\mathcal{O}_X(D)$ of the (locally) constant sheaf $\mathcal{K}(X)$ of rational functions on X defined on each open set U by

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{K}(X) \mid \operatorname{div}_U f + D_{|U} \geq 0\}.$$

If D is effective, therefore, there is an inclusion of sheaves

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D) \tag{2.0.1}$$

which, on each open U, is determined by sending 1 to the section $1 \in \mathcal{O}_X(D)$. Thus for all $e \in \mathbb{N}$, there is a natural map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X \hookrightarrow F^e_* \mathcal{O}_X(D) \qquad 1 \mapsto F^e_* 1 \mapsto F^e_* 1 \tag{2.0.2}$$

obtained by composing the Frobenius map with the Frobenius pushdown of the natural inclusion (2.0.1).

Now, if X is globally Frobenius split, we can ask whether a splitting $F^e_* \mathcal{O}_X \to \mathcal{O}_X$ extends to the larger $\mathcal{O}_X$-module $F^e_* \mathcal{O}_X(D)$, or equivalently, whether the composition map (2.0.2) splits as a map of $\mathcal{O}_X$-modules. Such a splitting would be a map

$$\phi \in \operatorname{Hom}_X(F^e_* \mathcal{O}_X(D), \mathcal{O}_X)$$

such that $\phi(F^e_* 1) = 1$ on every open set $U \subset X$. This brings us to the important notion of *Frobenius splitting along a divisor:*

Definition 2.1. Let X be a normal Noetherian scheme of characteristic $p > 0$, and let D be an effective divisor on X. The scheme X is **globally e-Frobenius split along** D if the map

$$\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(D) \qquad 1 \mapsto F^e_* 1$$

splits as a map of $\mathcal{O}_X$-modules.

We say that X is **globally eventually Frobenius split along** D if there exists $e > 0$ such that X is e-Frobenius split along D.

Remark 2.2. Clearly, if X is globally eventually Frobenius split along some effective divisor D, then X is globally Frobenius split, by restricting the splitting to the subsheaf $F^e_* \mathcal{O}_X$.

Remark 2.3. Definition 2.1 generalizes the notion of *Frobenius splitting along an element* c from Definition 4.1 in Chapter 1. Indeed, a *principal* divisor on an affine scheme $X = \operatorname{Spec} R$ is nothing more than a divisor D of

[8] For background on reflexive sheaves, see [**Sta19**, Tag 0AVT].

the form div(c) for some nonzerodivisor c of R. Now, the reader will readily check that X is e-Frobenius split along D if and only if R is e-Frobenius split along c (Exercise 2.1).

Remark 2.4. Clearly, if X is globally Frobenius e-split along D, then any open set $U \subset X$ is globally Frobenius e-split along $D_{|U}$.

The following facts generalize analogous statements for rings; Compare with Proposition 4.6, Lemma 4.7, and Corollary 4.11 in Chapter 1:

Proposition 2.5. *Suppose X is a normal Noetherian F-finitescheme of characteristic $p > 0$. Let D and D' be effective divisors on X.*

(a) If $D' \geq D$ and X is globally e-Frobenius split along D', then X is globally e-Frobenius split along D.
(b) If X is globally e_0-Frobenius split along D, then X is globally e-Frobenius split along D for all $e \geq e_0$.
(c) If X is globally e-Frobenius split along D and globally e' Frobenius split along D', then X is globally $(e + e')$ Frobenius split along $D + p^e D'$.
(d) In particular, X is eventually globally Frobenius split along $D + D'$ if and only if X is eventually globally Frobenius split along D and along D'.

2.2. Digression on handling divisors on singular varieties. Proposition 2.5 can be proved using the same strategies we used for rings, after we have sorted out how to work with divisors on singular varieties.

On a non-singular scheme, all divisors are locally principal, so all sheaves $\mathcal{O}_X(D)$ are invertible. This is typically *not* the case on a singular scheme.[9] On a *normal* Noetherian scheme X, however, the sheaves $\mathcal{O}_X(D)$ are always rank one and *reflexive.* We saw this reflexive definition appear in a slightly different context

Reflexive sheaves on a Noetherian scheme X are those coherent sheaves $\mathscr{F}$ for which the natural map to the double dual

$$\mathscr{F} \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X), \mathcal{O}_X)$$

is an isomorphism; more generally, the double dual, also denoted $\mathscr{F}^{\mathrm{S}_2}$, is called the **reflexive hull** of $\mathscr{F}$.

Reflexive sheaves on a normal Noetherian scheme X are uniquely determined by their restrictions to any open set U whose complement has

[9]The reader mainly interested in non-singular varieties can skip this discussion, and simply assume X is non-singular for the rest of this section; the theorems in this section are still interesting in this case.

codimension two or more: if $i : U \hookrightarrow X$ is the inclusion, then i^* and i_* define an equivalence of categories[10] between the reflexive sheaves on X and the reflexive sheaves on U. Indeed, on a Noetherian integral scheme, a torsion-free sheaf $\mathscr{F}$ being reflexive is equivalent to there being an open set $U \subseteq X$ whose complement has codimension ≥ 2 and such that $\mathscr{F}$ is locally free of finite rank [**Sta19**, Tag 0AY6].

We apply this idea mainly when X is a normal variety over a field k (or an F-finite normal integral scheme, or an excellent normal integral scheme more generally), in which case we can take U to be the non-singular locus. In this case every divisor D restricts to a locally principal (or *Cartier*) divisor on U, and the sheaf $\mathcal{O}_X(D)$ is the unique reflexive extension of the invertible sheaf $\mathcal{O}_U(D_{|U})$. We record some basic facts as a lemma for future reference:

Lemma 2.6. *Let X be a normal excellent Noetherian scheme, and let D and D' be Weil divisors on X. Then*

(a) *The reflexive sheaf $\mathcal{O}_X(D + D')$ is naturally isomorphic to the reflexive hull of the sheaf $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$;*
(b) *In prime characteristic, the reflexive hull of $\mathcal{O}_X(D) \otimes F^e_*\mathcal{O}_X(D')$ is $F^e_*\mathcal{O}_X(D' + p^e D)$.*

PROOF. Statement (a) follows easily from the discussion above: since D and D' are Cartier on the non-singular locus, we have $\mathcal{O}_U(D_{|U}) \otimes_{\mathcal{O}_U} \mathcal{O}_U(D'_{|U}) \cong \mathcal{O}_U(D_{|U} + D'_{|U})$ which agrees with the restriction of $\mathcal{O}_X(D + D')$ to U. Similarly, (b) follows, using the projection formula [**Har77**, II, Exercise 5.1(d)] on U. See also [**Sta19**, Tag 0AVT] or Appendix B (6.3.1) for additional discussion. $\square$

PROOF OF PROPOSITION 2.5. For (a), our hypothesis ensures that the composition
$$\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D) \hookrightarrow F^e_*\mathcal{O}_X(D')$$
splits for some $e > 0$. Restricting a splitting to the subsheaf $F^e_*\mathcal{O}_X(D)$, we get a splitting of the composition $\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$ as well.

For (b), it suffices to show that if
$$\begin{aligned} \mathcal{O}_X &\longrightarrow F^e_*\mathcal{O}_X(D) \\ 1 &\longmapsto F^e_*1 \end{aligned}$$
splits, then so does the corresponding map with $e+1$. Applying the functor F_*, we have a split map
$$F_*\mathcal{O}_X \longrightarrow F^{e+1}_*\mathcal{O}_X(D).$$

[10] See the discussion in Subsection 2.2 of Chapter 2, or Appendix B.

Finally, by (a), we know that X is e-Frobenius split along the trivial divisor, and so globally Frobenius split. Hence composing the split maps

$$\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \longrightarrow F_*^{e+1}\mathcal{O}_X(D),$$

we see that X is also globally $(e+1)$-Frobenius split along D.

For (c), we are given that the natural maps

$$\mathcal{O}_X \longrightarrow F_*^e\mathcal{O}_X(D) \qquad \text{and} \qquad \mathcal{O}_X \longrightarrow F_*^{e'}\mathcal{O}_X(D')$$

are split. Now, tensoring the second map with $\mathcal{O}_X(D)$, we have a split map

$$\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D) \otimes F_*^{e'}\mathcal{O}_X(D').$$

Taking the reflexive hull of this map[11], we get a split map

$$\mathcal{O}_X(D) \longrightarrow F_*^{e'}\mathcal{O}_X(p^{e'}D + D'),$$

by Lemma 2.6 and Exercise 2.4. Applying the functor F_*^e, we have a split map

$$F_*^e\mathcal{O}_X(D) \longrightarrow F_*^{e+e'}\mathcal{O}_X(p^{e'}D + D').$$

Therefore, composing the split maps

$$\mathcal{O}_X \longrightarrow F_*^e\mathcal{O}_X(D) \to F_*^{e+e'}\mathcal{O}_X(p^{e'}D + D'),$$

we have a split map, completing the proof.

The final statement (d) follows from the previous three. □

2.3. Global F-regularity. We now introduce a global form of strong F-regularity.

Definition 2.7. A normal Noetherian F-finite scheme is **globally F-regular** if it is globally eventually Frobenius split along every effective divisor.

Proposition 2.8. *The following are equivalent for an affine scheme $X = \operatorname{Spec} R$:*

(a) The scheme X is globally F-regular;
(b) For all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is strongly F-regular;
(c) The ring R is strongly F-regular.

Proof. We may assume X is Noetherian, F-finite and normal since all three conditions imply these features, see Chapter 1 Theorem 4.30.[12] The equivalence of (b) and (c) were proved in Proposition 4.23 in Chapter 1.

[11] Equivalently, double dualizing, or equivalently, applying the functor $i_* \circ i^*$ where $i : U \hookrightarrow X$ is the inclusion of the non-singular set U of X

[12] Furthermore, we might as well assume R is a domain (equivalently, that X is integral), since it is a finite product of normal domains (respectively, disjoint union of normal components) and we can argue on each factor.

To see (a) implies (c), assume $X = \operatorname{Spec} R$ is globally F-regular. Take any nonzerodivisor c, and consider the principal Cartier $D = \operatorname{div}(c)$ on $X = \operatorname{Spec} R$. By assumption, X is eventually Frobenius split along $\operatorname{div}(c)$, which is equivalent to R being eventually Frobenius split along c (Remark 2.3).

For the converse, assume that R is strongly F-regular and let D be an effective divisor on $X = \operatorname{Spec} R$. Consider the ideal sheaf $\mathcal{O}_X(-D)$ of the subscheme D on X. Taking any nonzerodivisor $c \in \mathcal{O}_X(-D)$, we know that c vanishes everywhere on D, and so $\operatorname{div}(c) \geq D$. Now because R is eventually Frobenius split along $\operatorname{div}(c)$, it follows that X is eventually Frobenius split along D by Proposition 2.5 (a). This completes the proof. $\square$

Corollary 2.9. *If X is globally F-regular, then every local ring $\mathcal{O}_{X,x}$ is strongly F-regular, that is, X is locally F-regular.*[13]

PROOF. This follows easily from Remark 2.4 and Proposition 2.8, so is left as an exercise. $\square$

Remark 2.10. It is natural to ask, given Proposition 2.8, whether global F-regularity is equivalent to eventual Frobenius splitting along all effective *Cartier* (that is, locally principal) divisors. This is true for *quasi-projective* varieties (see Exercise 2.8) but false in general. Indeed, there are complete varieties that admit *no non-trivial* Cartier divisor—some globally F-regular and some not. See Example 2.12 and Exercise 2.12.

Remark 2.11. Following up with the idea in Remark 2.10, given an invertible sheaf $\mathscr{L}$ on a scheme X and a non-zero global section $c \in H^0(X, \mathscr{L})$, we can say that X is globally eventually Frobenius split along the section $c \in \mathscr{L}(X)$ if there exists an $e \in \mathbb{N}$ such that the composition

$$\mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X \longrightarrow F^e_* \mathscr{L} \qquad 1 \mapsto F^e_* 1 \mapsto F^e_* c$$

splits as a map of $\mathcal{O}_X$-modules. We can ask whether X is *globally F-regular* if and only if X is eventually globally split along all non-zero global sections of all invertible sheaves. Again, this is the case for quasi-projective X (see Exercise 2.8), but not in general.

Example 2.12.

Remark 2.13. Global F-regularity was first defined in [**Smi00a**] though the approach we take here is from [**SS10**].

2.4. Criterion for global F-regularity. To check a variety is globally F-regular, it is not necessary to check eventual Frobenius splitting along *every* effective divisor. As in the affine setting,[14] it is enough to check eventual Frobenius splitting along one well-chosen effective divisor:

[13] Technically, we should say X is locally strongly F-regular, but enough is enough.

[14] see Theorem 5.1 in Chapter 1

Theorem 2.14. *Suppose X is a normal F-finite scheme and $B \subseteq X$ is an effective Weil divisor such that the open set $X \setminus \mathrm{Supp}(B)$ is globally F-regular. Then X is globally F-regular if and only if X is eventually globally Frobenius split along B.*

Remark 2.15. If X is a smooth projective variety, we can "test" for global F-regularity along any ample divisor B on X, since in this case, $X \setminus \mathrm{Supp}(B)$ is affine and regular, hence globally F-regular (Proposition 2.8). More generally, if X is normal and quasiprojective, we can always find some ample B containing the closed locus of non-strongly F-regular points; again $X \setminus \mathrm{Supp}(B)$ is globally F-regular. See Exercise 2.7.

Proof of Theorem 2.14. Fix a Weil divisor $D \geq 0$ on X. We need to show that $\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(D)$ splits for some large e. Because X is normal, it is the disjoint union of its components, and we can check the splitting separately on each. So there is no loss of generality in assuming X is integral.

Let $U = X \setminus B$. Since U is globally F-regular, there exists a splitting $F^{e_1}_* \mathcal{O}_U(D) \longrightarrow \mathcal{O}_U$ for some $e_1 \in \mathbb{N}$. Letting $\mathcal{K}(X)$ denote the function field of X, we have an induced splitting

$$\psi_1 : F^{e_1}_* \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$$

by taking the stalk at the generic point. Restricting ψ_1 to the subsheaf $F^{e_1}_* \mathcal{O}_X(D)$, we have a map

$$\psi : F^{e_1}_* \mathcal{O}_X(D) \longrightarrow \mathcal{K}(X).$$

The image of ψ is coherent, and becomes $\mathcal{O}_U$ when restricted to $U = X \setminus B$, so the image of ψ is contained in $\mathcal{O}_X(nB)$ for some $n \gg 0$. Thus we have a map

$$\psi : F^{e_1}_* \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(nB).$$

By hypothesis, X is globally eventually Frobenius split along B, and hence along nB (by Proposition 2.5). So there exists $e > 0$ and a map $\phi \in \mathrm{Hom}(F^e_* \mathcal{O}_X(nB), \mathcal{O}_X)$ such that $\phi(F^e_* 1) = 1$. The composition $\phi \circ F^e_* \psi$ (or $\phi \star \psi$ in the notation of Subsection 4.2 in Chapter 1) is a global $(e + e_1)$-Frobenius splitting along D. $\square$

Our next goal is to generalize the criterion for global Frobenius splitting in Theorem 1.27 to global F-regularity.

Theorem 2.16. *Let X be a normal proper variety over a field of positive characteristic and dimension d. For an effective divisor D on X, the map $\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(D)$ splits if and only if the map induced by Frobenius*

$$H^d(X, \omega_X) \to H^d(X, \omega_X \otimes F^e_* \mathcal{O}_X(D)) \tag{2.16.1}$$

is injective.

PROOF. Splitting the map $\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X(D)$ is equivalent to the surjectivity of the natural map

$$H^0(X, \mathscr{H}om(F^e_*\mathcal{O}_X(D), \mathcal{O}_X)) = \mathrm{Hom}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X)$$

sending ϕ to $\phi(1_{\mathcal{O}_X(X)})$. By Serre-Grothendieck duality,[15] this is equivalent to the injectivity of

$$\mathrm{Ext}^d(\mathcal{O}_X, \omega_X) \longrightarrow \mathrm{Ext}^d(\mathscr{H}om(F^e_*\mathcal{O}_X(D), \mathcal{O}_X), \omega_X). \tag{2.16.2}$$

If X is non-singular, the sheaf $\mathscr{H}om(F^e_*\mathcal{O}_X(D), \mathcal{O}_X)$ is locally free, so the natural double map

$$F^e_*\mathcal{O}_X(D) \longrightarrow \mathscr{H}om(\mathscr{H}om(F^e_*\mathcal{O}_X(D), \mathcal{O}_X), \mathcal{O}_X)$$

is an isomorphism. In this case, the map (2.16.2) becomes

$$\mathrm{Ext}^d(\mathcal{O}_X, \omega_X) \longrightarrow \mathrm{Ext}^d(\mathcal{O}_X, \omega_X \otimes F^e_*\mathcal{O}_X(D))$$

by [**Har77**, III, Prop 6.7], which is naturally identified with

$$H^d(X, \omega_X) \longrightarrow H^d(X, \omega_X \otimes F^e_*\mathcal{O}_X(D))$$

by [**Har77**, III, Prop 6.3]. This completes the proof when X is non-singular. More generally, if X is normal, the same argument goes through using Exercise 1.12. ☐

To check whether or not a variety X is globally F-regular, therefore, we can check that the map (2.16.1) is injective for *every* effective divisor D on X. Alternatively, rephrasing of Theorem 2.14 we can check injectivity along *one, well-chosen* divisor:

Corollary 2.17. *Let X be a normal proper variety and let D be an effective divisor such that $X \setminus \mathrm{Supp}(D)$ is globally F-regular. If*

$$H^d(X, \omega_X) \to H^d(X, \omega_X \otimes F^e_*\mathcal{O}_X(D))$$

is injective for some $e > 0$ then X is globally F-regular.

As an application, we see easily that proper varieties with trivial canonical module are never globally F-regular:

Example 2.18. Any proper variety for which ω_X is trivial—such as a Calabi-Yau or Abelian variety—cannot be globally F-regular. Indeed, assuming $\omega_X \cong \mathcal{O}_X$, suppose that X is globally F-regular. For D effective, then, there is an e such that

$$k \cong H^d(X, \omega_X) \longrightarrow H^d(X, \omega_X \otimes F^e_*\mathcal{O}_X(D))$$

[15] This is proved for projective X in [**Har77**, Chap III, Thm 7.6], and stated in the proper case in the remarks there before Lemma 7.3 [**Har77**, Chap III]. See also REF TO APPENDIX.

is injective. This forces $H^d(X, F^e_*\mathcal{O}_X(D))$, and hence $H^d(X, \mathcal{O}_X(D))$, to be non-zero (the latter because F^e is affine). But now, dually,[16] $H^0(X, \mathcal{O}_X(-D))$ is non-zero, a contradiction because D is effective (see Exercise 2.19).

Example 2.19. A hypersurface of degree d in $\mathbb{P}^n$ is never globally F-regular if $d > n$. Indeed, for $d > n+1$, the hypersurface is not even Frobenius split (see Example 1.15), but when $d = n+1$, we have $\omega_X \cong \mathcal{O}_X$, so is not globally F-regular by Example 2.18.

Example 2.20. The only smooth projective curves that are globally F-regular have genus zero. Indeed, we have already seen that curves of genus two or more are not globally Frobenius split (Corollary 1.13), and a genus one curve has $\omega_X \cong \mathcal{O}_X$, so is not globally F-regular by Example 2.18.

2.5. Frobenius splitting and anti-canonical divisors. A canonical divisor on a normal Noetherian scheme is any divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$, where ω_X is a fixed canonical module. Global Frobenius splittings, and generalizations, are tied closely to canonical divisors.

Theorem 2.21. *Let X be a normal proper scheme over an F-finite field k. If X is Frobenius split along some effective divisor A, then there is an effective divisor D on X such that $A + D$ is $\mathbb{Q}$-linearly equivalent to $-K_X$.*

Proof. Fix a canonical divisor K_X. Because X is eventually Frobenius split along A, by Theorem 2.16, the map

$$H^d(X, \mathcal{O}_X(K_X)) \hookrightarrow H^d(X, \mathcal{O}_X(K_X) \otimes F^e_*\mathcal{O}_X(A)) \cong H^d(X, F^e_*\mathcal{O}_X(p^eK_X + A))$$

is a split injection (the isomorphism follows from Exercise 2.9 and Lemma 2.6). Because $H^d(X, \mathcal{O}_X(K_X)) \neq 0$, it follows that $H^d(X, F^e_*\mathcal{O}_X(p^eK_X + A)) \neq 0$, so also $H^d(X, \mathcal{O}_X(p^eK_X + A))$ is non-zero as the Frobenius map is affine. Now, dualizing over k, also

$$H^0(X, \mathcal{O}_X((1 - p^e)K_X - A)) \neq 0;$$

this is simply Serre duality if X is smooth; see Exercise 2.10 for the general case. This means there is some effective divisor $D \in |(1 - p^e)K_X - A|$. In other words, there is an effective D such that $A + D \sim (1 - p^e)K_X$. In other words, $-K_X$ is $\mathbb{Q}$-linearly equivalent to $A + D$. $\square$

Corollary 2.22. *A normal globally Frobenius split projective variety has non-positive Kodaira dimension.*

The Kodaira dimension of normal projective variety X over k is the integer δ such that the function

$$m \mapsto \frac{\dim_k H^0(X, \mathcal{O}_X(mK_X))}{m^\delta}$$

[16]See Exercise 2.10 for the singular case of this form of Serre duality.

is bounded away from both zero and ∞. In other words, $\dim_k H^0(X, \mathcal{O}_X(mK_X))$ grows like a polynomial of degree δ. Equivalently, assume K_X is ($\mathbb{Q}$-)Cartier, the Kodaira dimension is the dimension of the image of the rational map $X \dashrightarrow \mathbb{P}^n$ given by the global sections of $\mathcal{O}_X(mK_X)$ for $m \gg 0$. See [**KM98**].

Proof. By Theorem 2.21, if X is globally Frobenius split, then $-K_X$ is $\mathbb{Q}$-linearly equivalent to an effective divisor. But if the Kodaira dimension of X is positive, then there are m for which $H^0(X, \mathcal{O}_X(mK_X)) \neq 0$, so K_X is also $\mathbb{Q}$-linearly equivalent to an effective divisor. But on a normal projective variety, a divisor can't be linearly equivalent to both an effective and an anti-effective divisor; Exercise 2.19. $\square$

Recall that a divisor D on a normal integral variety is **big**[17] if for some $m > 0$, there are global sections $s_0, \ldots, s_n \in H^0(X, \mathcal{O}_X(mD))$ such that the corresponding rational map $X \dashrightarrow \mathbb{P}^n$ to projective space has image of dimension equal to $\dim X$. Equivalently, if X is proper over k, this says that the function $m \mapsto \dim_k H^0(X, \mathcal{O}_X(mD))$ grows likes a polynomial of degree $\dim X$ as m goes to infinity.

Corollary 2.23. *The anti-canonical divisor of a globally F-regular projective variety is big.*

Proof. Taking A to be *ample* in Theorem 2.21, we see that $-K_X$ is $\mathbb{Q}$-linearly equivalent to an ample divisor plus an effective divisor, which implies that $-K_X$ is big by [**KM98**, 2.60]; the extension to non-Cartier D is carefully checked in [**KP17**, 4.6]. $\square$

Remark 2.24. The bigness of $-K_X$ is akin to *positive curvature* of the projective variety X [].

Remark 2.25. Even if X is not projective, but only proper, Theorem 2.21 still produces an abundance of effective $\mathbb{Q}$-divisors $\mathbb{Q}$-linearly equivalent to $-K_X$: for *each* choice of effective divisor A, we get some effective D such that $A + D$ is $\mathbb{Q}$-linearly equivalent to $-K_X$. The union of the supports of all such divisors clearly covers X.

[17] While bigness is usually defined in the literature only for $\mathbb{Q}$-Cartier D, the definition makes sense for any divisor on a normal variety; the map to projective space won't be defined at any point where D is not Cartier, but the locus of such points is a closed set of codimension two or more in any case. Note also that restricting to the non-singular set U, where D is Cartier, $H^0(X, \mathcal{O}_X(mD)) = H^0(U, \mathcal{O}_U(mD_{|U}))$ for all m, using the equivalence of categories of reflexive sheaves on U and X given by i^* and i_* where $i : U \hookrightarrow X$ in the inclusion of the non-singular set of X; see also Exercise 2.9. See [**KM98**] or [**KP17**, §4] for more on big divisors.

The next result makes the relationship between Frobenius splitting and anti-canonical divisors even more explicit:

Theorem 2.26. *Suppose that X is a normal variety, or more generally any F-finite normal Noetherian scheme. Then there is a natural isomorphism*

$$\mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X, \mathcal{O}_X) \cong F^e_*\mathcal{O}_X((1-p^e)K_X). \tag{2.26.1}$$

PROOF. First, assuming X is non-singular, we observe that this follows easily from the projection formula and Lemma 2.12 or Definition-Proposition 4.5 from Chapter 2:

$$\begin{aligned} &\mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X, \mathcal{O}_X) \\ &\qquad \cong \mathscr{H}\mathrm{om}((F^e_*\mathcal{O}_X) \otimes \mathcal{O}_X(K_X), \mathcal{O}_X(K_X)) \\ &\qquad \cong \mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X(p^e K_X), \mathcal{O}_X(K_X)) \\ &\qquad = \mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X(p^e K_X), \omega_X) \\ &\qquad \cong F^e_* \mathscr{H}\mathrm{om}(\mathcal{O}_X(p^e K_X), \omega_X) \\ &\qquad \cong F^e_* \mathscr{H}\mathrm{om}(\mathcal{O}_X(p^e K_X), \mathcal{O}_X(K_X)) \\ &\qquad \cong F^e_*\mathcal{O}_X((1-p^e)K_X). \end{aligned}$$

More generally, when X is singular, the desired isomorphism holds on the non-singular locus U (whose complement has codimension ≥ 2), and because the sheaves

$$\mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X, \mathcal{O}_X) \quad \text{and} \quad F^e_*\mathcal{O}_X((1-p^e)K_X)$$

are reflexive (see Exercise 2.14), the isomorphism extends to an isomorphism on all of X. □

Remark 2.27. Using the same argument as in the proof of Theorem 2.26, the reader will easily verify that

$$\mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) = F^e_*\mathcal{O}_X((1-p^e)K_X - D) \tag{2.27.1}$$

for any divisor D on a normal F-finite Noetherian scheme; see Exercise 2.18.

2.6. Vanishing theorems for globally F-regular varieties. An invertible sheaf $\mathcal{L}$ on a variety X proper over some field k is *nef*[18] if the pull back of $\mathscr{L}$ to each closed irreducible curve in X has non-negative degree. Alternatively, a Cartier divisor D is nef on X if $D \cdot C \geq 0$ for all closed curves C in X.

Example 2.28. All ample invertible sheaves are nef, as is the trivial sheaf and all torsion invertible sheaves. More generally, any globally generated invertible sheaf is nef.

Theorem 2.29. *Let X be a globally F-regular projective variety. Then $H^i(X, \mathscr{L}) = 0$ for any nef invertible sheaf $\mathscr{L}$. In particular, $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

[18]See §1.4 of **[Laz04a]** for a detailed discussion of nefness.

Theorem 2.29 follows easily from the following more general vanishing theorem:

Theorem 2.30. *Suppose that a Noetherian normal scheme X is globally eventually Frobenius split along some effective divisor D. If $\mathscr{L}$ is an invertible sheaf on X such that $H^i(X, \mathscr{L}^n(D)) = 0$ for some fixed i and all $n \gg 0$, then $H^i(X, \mathscr{L}) = 0$.*

More generally, for any divisor D', if $H^i(X, \mathcal{O}_X(nD' + D)) = 0$ for $n \gg 0$, then $H^i(X, \mathcal{O}_X(D')) = 0$.

PROOF. Because X is globally eventually Frobenius split along D, the map

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(D) \qquad 1 \mapsto F^e_* 1 \tag{2.30.1}$$

splits for all large e by Proposition 2.5. Now tensoring with the reflexive sheaf $\mathcal{O}_X(D')$, the induced map

$$\mathcal{O}_X(D') \to \mathcal{O}_X(D') \otimes F^e_* \mathcal{O}_X(D)$$

splits, and so passing to the reflexive hull, also

$$\mathcal{O}_X(D') \to F^e_*(\mathcal{O}_X(p^e D' + D)) \tag{2.30.2}$$

splits (using Lemma 2.6 and Exercise 2.4). This induces a split inclusion of cohomology groups

$$H^i(X, \mathcal{O}_X(D')) \hookrightarrow H^i(X, F^e_*(\mathcal{O}_X(p^e D' + D))) \cong H^i(X, \mathcal{O}_X(p^e D' + D))$$

for all $e \gg 0$, with the isomorphism holding because the Frobenius map F^e is affine. Now it is immediate that if $H^i(X, \mathcal{O}_X(p^e D' + D))$ vanishes for any $e > 0$, then $H^i(X, \mathcal{O}_X(D'))$ vanishes as well. $\square$

PROOF OF THEOREM 2.29. Suppose $\mathscr{L}$ is nef. Fix an ample effective divisor H. Then $\mathscr{L}^n(H)$ is ample for all $n \geq 0$ [**Laz04a**, Thm 1.4.10]. Since X is globally Frobenius split, we know $H^i(X, \mathscr{L}^n(H)) = 0$ for all $n \geq 0$ and all $i \geq 1$ by Corollary 1.12. Now it follows from Theorem 2.30 that $H^i(X, \mathscr{L}) = 0$ for all $i \geq 1$. $\square$

Another Corollary of Theorem 2.29 is a version of the Kawamata–Viehweg Vanishing theorem for globally F-regular varieties.

Corollary 2.31. *Let X be a globally F-regular projective variety and let $\mathscr{L}$ be a big and nef invertible sheaf on X. Then $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ for all $i > 0$.*

PROOF. Since X is globally F-regular, it is locally F-regular, hence Cohen-Macaulay, so we can use Serre Duality. By Serre duality [**Har77**, III Thm 7.6], it suffices to show $H^i(X, \mathscr{L}^{-1}) = 0$ for all $i < \dim X$.

Because $\mathscr{L}$ is big and nef, we can find an effective Cartier divisor D such that $\mathscr{L}^m(-D)$ is ample for all $m \gg 0$ [**Laz04a**, Ex 2.2.19]. But then $H^i(X, \omega_X \otimes (\mathscr{L}^m(-D))^n) = 0$ for all $i > 0$ and all $n \gg 0$ by Serre Vanishing. So by Serre Duality, again, $H^i(X, (\mathscr{L}^{-m}(D))^n) = 0$ for all large n and all $i < \dim X$. Since X is Frobenius split, Theorem 1.11 implies that $H^i(X, \mathscr{L}^{-m}(D)) = 0$ for all $i < \dim X$. Now by Theorem 2.30, we conclude that $H^i(X, \mathscr{L}^{-1}) = 0$. $\square$

We now generalize Corollary 1.16 by removing the Cohen-Macaulay hypothesis. For this, we will use the dual-to-Frobenius map discussed in Chapter 2.

Theorem 2.32. *Let X be a locally equidimensional and connected globally Frobenius split projective scheme over an F-finite field of prime characteristic $p > 0$. For any ample invertible sheaf $\mathcal{L}$, we have $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for all $i \geq 1$.*

Proof. Choose an embedding $i : X \longrightarrow \mathbb{P}^n$. The split injection $\mathcal{O}_X \hookrightarrow F^e_* \mathcal{O}_X$ induces a split surjection $F^e_* \omega_X \twoheadrightarrow \omega_X$. This continues to be a split surjection after twisting by any ample invertible sheaf $\mathcal{L}$ and taking cohomology, and so we have surjections

$$H^i(F^e_*(\omega_X \otimes \mathcal{L}^{p^e})) \cong H^i((F^e_* \omega_X) \otimes \mathcal{L}) \twoheadrightarrow H^i(\omega_X \otimes \mathcal{L}).$$

The left side vanishes by Serre vanishing, [**Har77**, III, Prop. 5.3], and thus so does the right as claimed. $\square$

The proof of Theorem 2.32 did not use much about the canonical module ω_X: we just needed a module[19] $\mathscr{M}$ and a split surjection $F^e_* \mathscr{M} \longrightarrow \mathscr{M}$. Therefore, we have the following corollary of the proof of Theorem 2.32:

Corollary 2.33. *Suppose X is a projective scheme over an F-finite field of prime characteristic $p > 0$ and suppose $\mathscr{M}$ is any coherent sheaf on X that admits a split surjective map $F^e_* \mathscr{M} \longrightarrow \mathscr{M}$. Then for any ample line bundle $\mathcal{L}$, we have that $H^i(X, \mathscr{M} \otimes \mathcal{L}) = 0$ for $i > 0$.*

As an application, we can take the modules $\mathscr{M}$ to be the cohomologies of the dualizing complex of X. With X as in Theorem 2.32, for every integer i, we have a map of sheaves on X

$$F^e_* \mathcal{H}^i \omega^{\bullet}_X \longrightarrow \mathcal{H}^i \omega^{\bullet}_X,$$

with the interesting values of i in the range from $i = -\dim X$ to $i = 0$. If X is globally Frobenius split, these maps are split surjections. We then obtain the following corollary, which substantially generalizes Corollary 1.16:

[19]Modules with such a dual Frobenius action, split or not, are called *Cartier modules* and will be studied in detail in Chapter 8.

Corollary 2.34. *Let X be an equidimensional projective globally Frobenius split scheme over an F-finite field of prime characteristic $p > 0$. Let $\omega_X^\bullet$ be the dualizing complex on X as above. For any ample invertible sheaf $\mathcal{L}$, we have*

$$H^i(X, \mathcal{H}^j(\omega_X^\bullet) \otimes \mathcal{L}) = 0$$

for all $i \geq 1$ and all $j \in \mathbb{Z}$.

Proof. The proof is left as Exercise 2.17. $\square$

2.7. Exercises.

Exercise 2.1. Let c be a nonzerodivisor in the prime characteristic ring R. Let $D = \operatorname{div}(c)$ be the corresponding effective divisor on $X = \operatorname{Spec} R$. Show that X is e-Frobenius split along D if and only if R is e-Frobenius split along c.

Hint: Use Exercise 4.11 in Chapter 1.

Exercise 2.2. Show a scheme X is globally F-regular if and only if X is a disjoint union of finitely many globally F-regular schemes.

Hint: Remember globally F-regular implies Noetherian normal, and Noetherian and normal is equivalent to a disjoint union of finitely many Noetherian and normal schemes.

Exercise 2.3. Let X be a scheme of finite type over an F-finite field k. Then if $X \times_k \operatorname{Spec} L$ is globally (respectively, locally) F-regular, then so is X.

Hint: You proved a similar statement for Frobenius splitting.

Exercise 2.4. Let X be a normal Noetherian scheme, and let $\mathcal{M} \longrightarrow \mathcal{N}$ be a split map of coherent $\mathcal{O}_X$-modules. Prove that there is a naturally induced map of reflexive hulls $\mathcal{M}^{\mathrm{S}_2} \longrightarrow \mathcal{N}^{\mathrm{S}_2}$ which is also split.

Exercise 2.5. Suppose X is Frobenius e-split along a divisor $D \geq 0$. Prove that every coefficient of D is $\leq p^e - 1$.

Hint: Localize at a generic point of any prime component of D.

Exercise 2.6. Let D be an effective divisor on a normal Noetherian scheme X of prime characteristic. Prove that X is globally e-Frobenius split along D if and only if X is globally ne-Frobenius split along $\frac{p^{ne}-1}{p^e-1}D$ for some (equivalently every) integer $n > 0$.

Hint: Use Proposition 2.5 (c).

Exercise 2.7. Show that if X is a normal quasi-projective variety, we can always find B such that $X \setminus \operatorname{Supp} B$ is affine and globally F-regular.

Hint: Show that given a proper closed subset Z of a quasi projective variety $X \subseteq \mathbb{P}^n$, we can find a hypersurface H whose support contains Z.

Exercise 2.8. Let X be a normal quasi-projective variety over an F-finite field, and suppose that X is eventually Frobenius split along every effective Cartier divisor. Prove that X is globally F-regular.

Hint: Use the Hint to Exercise 2.7 and Theorem 2.14.

Exercise 2.9. Let X be a normal Noetherian separated scheme and $\mathcal{F}$ a reflexive sheaf on X. Prove that for all $i \geq 0$, there is a natural isomorphism $H^i(X, \mathcal{F}) \cong H^i(U, \mathcal{F})$ where $U \subset X$ is an open set whose compliment has codimension two or more.

Hint: Compute cohomology on X using the Cech complex for an affine cover $\{U_\lambda\}$ noting that $U_\lambda \cap U \subset U_\lambda$ has compliment of codimension two or more.

Exercise 2.10. Let X be a normal proper variety over k with canonical divisor K_X. For any Weil divisor D, show that $H^d(X, \mathcal{O}(K_X + D))$ is dual (over k) to $H^0(X, \mathcal{O}(-D))$.

Hint: Use Exercise 2.9 and Exercise 1.12.

Exercise 2.11. Generalize Exercise 1.13 in the following way. Suppose X is a globally F-regular projective variety of dimension d and $\mathscr{L}$ is a globally generated ample line bundle. Further suppose that $\mathscr{H}$ is a big and nef line bundle. Prove that $\omega_X \otimes \mathscr{L}^d \otimes \mathscr{H}$ is globally generated.

Exercise 2.12.

Exercise 2.13. Suppose that X is a globally F-regular scheme and that $\pi : X \longrightarrow Y$ is a map of schemes such that $\pi_* \mathcal{O}_X = \mathcal{O}_Y$. Assuming Y is quasi-projective, prove that Y is globally F-regular. See Exercise 2.15 for a stronger statement.

Exercise 2.14. Prove that if X is a normal Noetherian scheme, and $\mathcal{F}$ and $\mathcal{G}$ are reflexive coherent sheaves, then $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $F^e_* \mathcal{F}$ (in the case X has prime characteristic) are reflexive.

Exercise 2.15. Suppose that $\pi : Y \longrightarrow X$ is a morphism of normal integral F-finite schemes such that $\mathcal{O}_X \longrightarrow \pi_* \mathcal{O}_Y$ splits as a map of $\mathcal{O}_X$-modules.

Further assume that X is quasi-projective over an affine scheme. Prove that if Y is globally F-regular, then so is X.

Hint: We cannot check global F-regularity with Cartier divisors, since X is not assumed to be quasi-projective, see Remark 2.10. However, we can use normality to restrict to the non-singular locus of X by a similar argument to Corollary 1.26.

Exercise 2.16. Suppose k is an F-finite field. Show by direct computation on charts that

$$F^e_* \omega_{\mathbb{P}^n} \cong \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_{\mathbb{P}^n}, \omega_{\mathbb{P}^n}).$$

Exercise 2.17. Prove Corollary 2.34.

Exercise 2.18. Suppose that X is a normal projective variety over an F-finite field and D is a Weil divisor on X. Show that the formula,

$$\mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_X(D), \mathcal{O}_X) = F^e_* \mathcal{O}_X((1-p^e)K_X - D)$$

of (2.27.1) holds.

Exercise 2.19. Suppose that X is a normal projective variety over an F-finite field k. Show that there cannot exist a divisor B on X such that $nB \sim D_1 > 0$ and $-mB \sim D_2 \geq 0$ for some integers $n, m > 0$. This completes the proof of Corollary 2.22.

Hint: Replace n, m by multiples so that $m = n$. One has $\mathcal{O}_X(-D_2) \subseteq \mathcal{O}_X \subseteq \mathcal{O}_X(D_1)$.

3. Compatibly Frobenius split subschemes and ideals

Sometimes a global Frobenius splitting of a variety X also induces global Frobenius splittings of a subvariety. We saw this in the local setting in Chapter 1 Section 6.

We first define compatibility between closed subschemes Y of X and maps $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)$:

Definition 3.1. Let X be a scheme of characteristic p. Fix a closed subscheme Y and a map $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$. We say that ϕ **is compatible with Y (or that Y and ϕ are compatible**[20]**)** if $\phi(F^e_* \mathcal{I}_Y) \subseteq \mathcal{I}_Y$, where $\mathcal{I}_Y \subseteq \mathcal{O}_X$ is the ideal sheaf of Y. In the case that ϕ is a global Frobenius splitting, we say that ϕ **compatibly splits Y**.

We say that X **is compatibly (Frobenius) split with Y** if there exists a globally Frobenius splitting $\phi : F_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ that is compatible with Y.

[20]Or that Y is ϕ-compatible

Just as in the affine case, to say that X is compatibly split with a subscheme Y means that there is a global Frobenius splitting of X which restricts to a global Frobenius splitting for Y. That is, there is some global Frobenius splitting ϕ for X and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F_*\mathcal{I}_Y \arrow[r] \arrow[d, "\phi|_Y"] & F_*\mathcal{O}_X \arrow[r] \arrow[d, "\phi"] & i_*F_*\mathcal{O}_Y \arrow[r] \arrow[d, "\phi_Y"] & 0 \\ 0 \arrow[r] & \mathcal{I}_Y \arrow[r] & \mathcal{O}_X \arrow[r] & i_*\mathcal{O}_Y \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact and $i : Y \longrightarrow X$ is the closed embedding of Y as a subscheme of X. Since $\phi_Y(F_*1) = 1$ on every open set $U \in Y$, we see that in particular, Y is globally Frobenius split as well.

Example 3.2. Consider the canonical toric splitting ϕ on $\mathbb{P}^1_k$ as described in the proof of Example 1.4. Consider two points $0, \infty \in \mathbb{P}^1$, namely the "origins" $\operatorname{Spec} k[t]/(t)$ in the coordinate charts $\operatorname{Spec} k[t]$, where t stands for $\frac{x_1}{x_0}$ and $\frac{x_0}{x_1}$, respectively. We claim that these are the only two closed subschemes compatible with ϕ.

To see that these points are compatibly Frobenius split for ϕ, we need to show that $\phi(F_*\mathcal{I}_P) \subseteq \mathcal{I}_P$, where P stands for either point 0 or ∞. We first check the affine chart $\operatorname{Spec} k[t]$ where $\mathcal{I}_P$ corresponds to the ideal (t). The elements of this ideal are k-linear sums of t^m for integers m. All of those monomials are sent to other monomials (or zero) which are in (t). Hence $\mathcal{I}_P$ is compatible with ϕ on our first affine chart. However, on the other affine chart, $\mathcal{I}_P$ corresponds to the unit ideal (the whole ring), which is certainly compatible with ϕ. Thus $\mathcal{I}_P$ is compatible with ϕ globally. We leave it as an exercise to check that no other closed point of $\mathbb{P}^1$ is compatibly Frobenius split for ϕ.

Caution 3.3. It can happen that $Y \subseteq X$ are both individually globally Frobenius split, but they are not compatibly globally Frobenius split. For example, the only two closed points of $\mathbb{P}^1$ that are compatibly split by the canonical toric splitting of Example 1.4 are 0 and ∞. But every closed point of $\mathbb{P}^1$ (being Spec of a field) is Frobenius split in its own right. It can also happen that some Frobenius splittings on Y are obtained from X as in the diagram above, while others are not. We'll return to these issues later.

The subschemes compatible with a particular $\phi : F^e_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ satisfy some surprising properties.

Lemma 3.4 (*cf.* Proposition 6.4)**.** *Suppose X is a scheme of characteristic $p > 0$ and $\phi : F^e_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ is an $\mathcal{O}_X$-linear map. Then*

(a) *Arbitrary scheme-theoretic intersections of ϕ-compatible subschemes are ϕ-compatible. That is, if $\{\mathcal{I}_\gamma\}_{\gamma\in\Gamma}$ is a collection of ϕ-compatible ideal sheaves, then $\sum_{\gamma\in\Gamma}\mathcal{I}_\gamma$ is also ϕ-compatible.*
(b) *Finite scheme-theoretic unions of ϕ-compatible subschemes are ϕ-compatible. That is if $\mathcal{I}_1,\dots,\mathcal{I}_t$ are ϕ-compatible, so is $\mathcal{I}_1\cap\cdots\cap\mathcal{I}_t$.*

Proof. This is immediate from the definition; alternatively, one can work locally an apply Chapter 1 Proposition 6.4. ☐

Remark 3.5. Note infinite intersections of ϕ-compatible ideal sheaves are also ϕ-compatible, but one should note what category one is taking the intersection in. Indeed, an arbitrary intersection of ideal sheaves in the category of $\mathcal{O}_X$-modules need not be quasi-coherent. Alternately, one could take the intersection in the category of quasi-coherent sheaves, see [**Sta19**, Tag 077P] for some discussion.

Additionally, as a direct application of Chapter 1 Proposition 6.7 (c), we have the following.

Proposition 3.6. *Suppose that X is a Noetherian F-finite scheme and $\phi\in\operatorname{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X,\mathcal{O}_X)$. Let $Z\subseteq X$ be an* integral *subscheme with generic point η. Then Z is compatible with ϕ if and only if the stalk of the ideal sheaf of Z at η, $I_{Z,\eta}$, is compatible with $\phi_\eta: F^e_*\mathcal{O}_{X,\eta}\longrightarrow\mathcal{O}_{X,\eta}$.*

3.1. Splitting along a divisor vs compatibly splitting a divisor.

For normal schemes, splitting along a carefully chosen divisor also compatibly splits a certain divisor.

Lemma 3.7. *Suppose that X is a normal Noetherian scheme and D is a reduced*[21] *divisor on X. Then the following are equivalent:*

(a) *X is globally Frobenius split along the divisor $(p-1)D$;*
(b) *X is compatibly Frobenius split with* $\operatorname{Supp} D$.

Proof. First assume (b). Then there is a map of $\mathcal{O}_X$-modules

$$\phi: F_*\mathcal{O}_X((p-1)D)\longrightarrow\mathcal{O}_X$$

sending F_*1 to 1. Tensoring with $\mathcal{O}_X(-D)$ and reflexifying yields:

$$F_*\mathcal{O}_X(-D)\cong\mathcal{O}_X(-D)\otimes F_*\mathcal{O}_X((p-1)D)\longrightarrow\mathcal{O}_X(-D)\otimes\mathcal{O}_X\cong\mathcal{O}_X(-D).$$

In other words, ϕ takes $F_*\mathcal{O}_X(-D)$ to $\mathcal{O}_X(-D)$. Since $\mathcal{O}_X(-D)$ is the ideal sheaf of $\operatorname{Supp} D$, this shows (a).

[21] Meaning all coefficients are equal to 1

Conversely, suppose we have a global Frobenius splitting $\phi : F_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ that restricts to

$$F_* \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X(-D).$$

Tensoring with $\mathcal{O}_X(D)$ and taking the reflexive hull yields an $\mathcal{O}_X$-module map

$$F_* \mathcal{O}_X((p-1)D) = F_* \mathcal{O}_X(pD - D) \longrightarrow \mathcal{O}_X(D - D) = \mathcal{O}_X.$$

Since ϕ sent $F_* 1 \mapsto 1$, so does this map. Thus we have shown that X is globally Frobenius split along $(p-1)D$. $\square$

For more general results, see Chapter 5 Subsection 2.1.

Remark 3.8 (Affine local interpretation)**.** Interpreting the above proof affine locally is worth doing. By using the tool of reflexification / S_2-ification, the proof may be performed on the non-singular locus of X. Then, working on a chart $U = \operatorname{Spec} R$ we can write $D|_U = \operatorname{div}_U(f)$ for some $f \in R$. Now we are simply claiming that R is Frobenius split along f^{p-1} if and only if it is compatibly Frobenius split with the ideal (f). But now we notice that a non-zero map

$$\phi : F_* R \longrightarrow R$$

sends $F_* f^{p-1} \mapsto 1$ if and only if the composition

$$F_* R \xrightarrow{F_* f^{p-1}} F_* R \xrightarrow{\phi} R$$

sends 1 to 1. But that composition then sends the ideal $F_*(f)$ to the ideal $F_*(f^p)$ which is then sent into (f) by ϕ.

In many cases, a splitting of $\mathcal{I}_Z \longrightarrow F_* \mathcal{I}_Z$ in fact induces a splitting of X compatible with Z.

Proposition 3.9. *Suppose that X is a normal Noetherian scheme and $Z \subseteq X$ is a closed subscheme with ideal sheaf $\mathcal{I}_Z$. If the Frobenius map*

$$\mathcal{I}_Z \longrightarrow F^e_* \mathcal{I}_Z$$

splits in the category of $\mathcal{O}_X$-modules, then X is globally Frobenius split compatibly with Z.

Proof. We apply the reflexive hull functor to the split map $\mathcal{I}_Z \longrightarrow F^e_* \mathcal{I}_Z$. One way to understand this is as follows. Let Z' denote the union of components of Z that have codimension ≥ 2 in X. Let $U = X \setminus Z'$ with $i : U \longrightarrow X$ the inclusion. Then $i_* \mathcal{I}_Z|_U$ is an ideal sheaf of pure height 1, that is $i_* \mathcal{I}_Z|_U = \mathcal{O}_X(-D)$ for some divisor D. In particular, applying this to our splitting we obtain the following commutative diagram whose rows compose to be the identity:

$$\begin{tikzcd}
\mathcal{I}_Z \arrow[r] \arrow[d, hook] & F^e_*\mathcal{I}_Z \arrow[r, "\phi_Z"] \arrow[d, hook] & \mathcal{I}_Z \arrow[d] \\
\mathcal{O}_X(-D) \arrow[r] & F^e_*\mathcal{O}_X(-D) \arrow[r, "\phi_D"'] & \mathcal{O}_X(-D)
\end{tikzcd}$$

In particular, there exists $\phi : F^e_*\mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X(-D)$ which is surjective (and which sends $F^e_*1 \mapsto 1$ at the generic point of X). Arguing as in the proof of Lemma 3.7 we see that tensoring with $\mathcal{O}_X(D)$ and reflexifying produces a splitting of $\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X(-D + p^eD) = F^e_*\mathcal{O}_X((p^e-1)D)$ and hence a global Frobenius splitting of X compatible with both D and Z. $\square$

3.2. Compatible Frobenius splittings and vanishing theorems.

Theorem 3.10. *Suppose X is a projective variety of prime characteristic and $\mathscr{L}$ is an ample line bundle on X. Let $Z \subseteq X$ be a closed subscheme of X that is globally compatibly Frobenius split in X. If $\mathcal{I}_Z$ is the corresponding ideal sheaf, then*

$$H^i(X, \mathcal{I}_Z \otimes \mathscr{L}) = 0$$

for all $i > 0$.

Proof. $\mathcal{I}_Z \longrightarrow F^e_*\mathcal{I}_Z$ splits for every $e > 0$ and so we see that

$$H^i(X, \mathcal{I}_Z \otimes \mathscr{L}) \longrightarrow H^i(X, F^e_*(\mathcal{I}_Z \otimes \mathscr{L}^{p^e}))$$

splits as well, and hence is injective. But $H^i(X, \mathcal{I}_Z \otimes \mathscr{L}^{p^e})$ is zero for $e \gg 0$ by Serre vanishing. The result follows. $\square$

We show how this can be used in an example.

Example 3.11 (Compatibly splitting points in $\mathbb{P}^n$)**.** Let $k = \overline{k}$ be an algebraically closed field and suppose $Z \subseteq X = \mathbb{P}^n_k$ is a finite set of closed points. Suppose Z is globally compatibly Frobenius split in X. We will show that the number of points m in Z is $\leq n+1$.

Consider the exact sequence

$$H^0(X, \mathscr{L}) \longrightarrow H^0(X, \mathcal{O}_Z \otimes \mathscr{L}) \longrightarrow H^1(X, \mathcal{I}_Z \otimes \mathscr{L}) = 0$$

where the last term is zero by Theorem 3.10. Since $\mathcal{O}_Z$ is supported at finitely many points $\mathcal{O}_Z \otimes \mathscr{L} \cong \mathcal{O}_Z$, and in particular $H^0(X, \mathcal{O}_Z \otimes \mathscr{L})$ is a vector space of dimension $m = |Z|$. But $H^0(X, \mathscr{L}) = H^0(X, \mathcal{O}_X(1))$ the vector space spanned by $x_0, x_1, \ldots, x_n$, and so has dimension $n+1$. In other words we have a surjection from an $(n+1)$-dimensional vector space to an m-dimensional vector space and so $m \leq n+1$.

3.3. Exercises.

Exercise 3.1. Suppose that $\phi : F^e_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ is surjective as a map of sheaves (and not necessarily surjective on global sections) and that $Y \subseteq X$ is a ϕ-compatible subscheme. Prove that Y is reduced (or equivalently that $\mathcal{I}_Y$ is radical).

Hint: Restrict to an affine subscheme and suppose $I \subseteq R$ is the corresponding ideal. Use the fact that there exists $c \in R$ such that $\phi(F^e_* c) = 1$. What happens with $x^{p^e} \cdot c$?

Exercise 3.2. Show that if k is an algebraically closed field of characteristic p, then the only two closed points of $\mathbb{P}^1_k$ that are globally compatibly split for the canonical toric Frobenius splitting are 0 and ∞.

Exercise 3.3. Show that the coordinate linear spaces[22] in $\mathbb{P}^n$ are globally compatibly Frobenius split by the canonical toric Frobenius splitting we introduced in the proof of Example 1.4.

Exercise 3.4. Suppose that X is a globally F-regular scheme (for instance a strongly F-regular affine scheme). For each closed subscheme $Z \subseteq X$, show that there exists a global Frobenius splitting $\phi : F^e_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ such that $\phi(F^e_* \mathcal{I}_Z) \not\subseteq \mathcal{I}_Z$, in other words so that Z is not compatible with ϕ.

Hint: This is easier in the case that X is quasi-projective when one can choose D Cartier. In that case choose D such that $1 \in \mathcal{O}_X(D) \otimes \mathcal{I}_Z$. Now apply the compatibility condition of the global Frobenius splitting and obtain a contradiction. In the general case, fix an affine open set $U \subseteq X$ and find a global D which is Cartier on U and has a property similar to the one that worked in the quasi-projective case.

Exercise 3.5. Show that $n + 1$ closed points is the most that can be compatibly Frobenius split in $\mathbb{P}^n_k$.

Hint: Let Z be a set of n points. Apply the previous exercise to an appropriately twisting of the short exact sequence: $0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$.

Exercise 3.6. Suppose that a Cohen-Macaulay projective variety X is Frobenius split by $\phi : F_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$. Suppose that ϕ compatibly splits a Cartier divisor $D \subseteq X$ with ideal sheaf $\mathcal{O}_X(-D)$. Suppose further that $\mathscr{L}$ is any ample line bundle. Prove that

$$H^i(X, \omega_X \otimes \mathcal{O}_X(D) \otimes \mathscr{L}) = 0$$

for $i > 0$.

[22]The coordinate linear spaces of $\mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$ are those defined as the vanishing of the homogeneous monomial prime ideals, like (x_0, x_2, x_3).

Hint: First use Serre duality. Next observe that $\mathcal{O}_X(-D) \longrightarrow F^e_* \mathcal{O}_X(-D)$ is split injective and hence stays injective after tensoring with line bundles and taking cohomology.

Exercise 3.7. Suppose that X is globally Frobenius split combatibly with a subscheme $Z \subseteq X$. Prove that every irreducible component of Z is also globally Frobenius split compatibly with X.

Hint: This can be checked locally on affine charts, then use Exercise 4.12 in Chapter 1.

Exercise 3.8. Suppose that X is a non-singular projective variety and D is a Cartier divisor on X. Further suppose that for some closed point $x \in X$, we can write $D = D_1 + \cdots + D_{\dim X} + E$ where the D_i are Cartier divisors which are smooth at x and intersect transversally at x, and E is a divisor that does not contain x.

With notation as above, if X is compatibly F-split with D, then show that for any ample line bundle $\mathscr{L}$ on X, that

$$\omega_X \otimes \mathcal{O}_X(D) \otimes \mathscr{L}$$

is globally generated at $x \in X$.

Hint: Verify the statement first for a curve. Then restrict to some D_i and proceed by induction on dimension using (and twisting) the short exact "adjunction" sequence:

$$0 \longrightarrow \omega_X \longrightarrow \omega_X(D_i) \longrightarrow \omega_{D_i} \longrightarrow 0$$

4. Projective varieties and their affine cones

In this section, we connect global and local Frobenius splitting for *projective* varieties by showing that global Frobenius splitting of a projective variety X is essentially equivalent to local Frobenius splitting at the vertex of an affine cone over X. Specifically, we will see that for a finitely generated $\mathbb{N}$-graded ring S, if $\operatorname{Spec} S$ is Frobenius split, then $\operatorname{Proj} S$ is globally Frobenius split; furthermore, the converse holds if S is a **section ring**. Similar statements hold for Frobenius splitting along divisors and F-regularity.

4.1. From cones to projective varieties.

Consider an $\mathbb{N}$-graded ring

$$S = \bigoplus_{m \geq 0} S_m$$

of characteristic $p > 0$, assumed to be finitely generated over its degree zero subring S_0. The scheme $\operatorname{Proj} S$ is the associated projective scheme over $\operatorname{Spec} S_0$ whose points are the *homogeneous* prime ideals of S not contained in the **irrelevant ideal** $\bigoplus_{m\geq 1} S_m$. The scheme $\operatorname{Proj} S$ is covered by affine charts $D_+(f) := \operatorname{Spec} \left[S[\frac{1}{f}]\right]_0$, where f ranges through homogeneous elements of positive degree, and $\left[S[\frac{1}{f}]\right]_0$ denotes the degree zero subring of the $\mathbb{Z}$-graded ring obtained by inverting f. For background on this Proj construction, the reader should consult [**Gro61**, §2], [**Har77**, II §5], or [**Sta19**, Tag 01M3].

Quite generally, Frobenius splitting for a graded ring S implies Frobenius splitting for the projective scheme $\operatorname{Proj} S$ it defines:

Theorem 4.1. *Let S be an $\mathbb{N}$-graded F-finite ring of characteristic p, finitely generated over its degree zero subring S_0.*

(i) If S is locally Frobenius split, then the projective scheme $\operatorname{Proj} S$ *is globally Frobenius split.*
(ii) If S is strongly F-regular, then $\operatorname{Proj} S$ *is globally F-regular.*

Example 4.2. Theorem 4.1 gives a new way to see that projective space $\mathbb{P}^n$ (over an F-finite field k) is globally F-regular: the polynomial ring $k[x_0, \dots, x_n]$ is strongly F-regular, so $\operatorname{Proj} k[x_0, \dots, x_n] = \mathbb{P}^n_k$ is globally F-regular by Theorem 4.1.

Example 4.3. Let G be a finite group whose order is not divisible by p, and suppose G acts linearly on the projective space $\mathbb{P}^n_k$, where k is an F-finite field of characteristic p. Then the quotient scheme $\mathbb{P}^n/G := \operatorname{Proj} k[x_0, \dots, x_n]^G$, where $k[x_0, \dots, x_n]^G$ is the ring of invariants for this action is globally F-regular. This is immediate from Theorem 4.1 and Chapter 1 Example 3.7.

The converse of Theorem 4.1 fails:

Example 4.4. Consider Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^4$. Its homogeneous coordinate ring is $S = k[s^4, s^3t, s^2t^2, st^3, t^4]$. Now project from the point $[0 : 0 : 1 : 0 : 0] \in \mathbb{P}^4$ to embed $\mathbb{P}^1$ in $\mathbb{P}^3$ with coordinate ring $S' = k[s^4, s^3t, st^3, t^4]$. The inclusion $S' \hookrightarrow S$ of graded rings induces an isomorphism $\mathbb{P}^1 \cong \operatorname{Proj} S \to \operatorname{Proj} S'$, so $\operatorname{Proj} S'$ is globally F-regular. But S' is not normal; indeed, its normalization is S. Therefore the ring S' is not strongly F-regular even though $\operatorname{Proj} S'$ is globally F-regular.

Theorem 4.1 does have a converse if we restrict ourselves to the most natural rings defining the projective scheme X, the *section rings* with respect to ample invertible sheaves; see Subsection 4.3.

4.2. Frobenius and graded rings. To prove Theorem 4.1, we examine Serre's correspondence between finitely generated $\mathbb{Z}$-graded S-modules and coherent sheaves on $\operatorname{Proj} S$ in the context of Frobenius.

Fix an $\mathbb{N}$-graded ring

$$S = \bigoplus_{m \geq 0} S_m$$

of characteristic $p > 0$. The S-module $F^e_* S$ has a natural grading by $\frac{1}{p^e}\mathbb{N}$: namely,

$$\deg F^e_* s = \frac{1}{p^e} \deg s$$

for homogeneous $s \in S$. With this grading, the Frobenius map

$$S \to F^e_* S \qquad s \mapsto F^e_* s^{p^e}$$

is a degree-preserving map of rings. For any $\mathbb{Z}$-graded S-module M, we can define a natural $\frac{1}{p^e}\mathbb{Z}$ grading on $F^e_* M$ in the same way:

$$\deg F^e_* m = \frac{1}{p^e} \deg m$$

for homogeneous $m \in M$. This grading is compatible with the natural action of S on $F^e_* M$: $s \in S_i$ acts on $F^e_* m \in [F^e_* M]_{\frac{n}{p^e}}$ to produce

$$sF^e_*(m) = F^e_*(s^{p^e} m) \in [F^e_* M]_{\frac{n}{p^e}+i},$$

so $\deg(sF^e_* m) = \deg s + \deg(F^e_* m)$.

Each $\mathbb{Z}$-graded S-module M determines a unique quasi-coherent sheaf $\widetilde{M}$ on $\operatorname{Proj} S$ whose sections over the affine chart $D_+(f)$ are the elements of the $\big[S[\frac{1}{f}]\big]_0$-module $\big[M[\frac{1}{f}]\big]_0$, where $\big[M[\frac{1}{f}]\big]_0$ denotes the degree zero submodule of the $\mathbb{Z}$-graded module $M \otimes_S S[\frac{1}{f}]$. Furthermore, every coherent sheaf $\mathscr{M}$ on $X = \operatorname{Proj} S$ is determined in this way by some finitely generated $\mathbb{Z}$-graded S-module M. A natural question arises: *what graded $\mathbb{Z}$-module determines $F^e_* \mathcal{O}_X$, or more generally, $F^e_* \mathscr{M}$, for a coherent sheaf $\mathscr{M}$ on* $\operatorname{Proj} S$*?*

To answer this question, fix a graded S-module $M = \oplus_{n \in \mathbb{Z}} M_n$. Consider the graded S-submodule of $F^e_* M$ formed by elements of integral degrees:

$$F^e_* M^{(p^e)} = \bigoplus_{n \in \mathbb{Z}} F^e_* M_{p^e n} \subset F^e_* M, \tag{4.4.1}$$

where $M^{(p^e)}$ denotes the "Veronese submodule" $M^{(p^e)} = \bigoplus_{n\in\mathbb{Z}} M_{p^e n}$ of M. The homogeneous elements of $F^e_* M^{(p^e)}$ are precisely those elements of $F^e_* M$ that happen to have integral degrees:

$$\{F^e_* m \mid \deg F^e_* m \in \mathbb{Z}\} = \{F^e_* m \mid p^e \text{ divides } \deg m\}.$$

Since the elements of S have integer degree, the usual action of S on elements of $F^e_* M$ induces a natural S-module action on $F^e_* M^{(p^e)}$.

Lemma 4.5. *Let S be an $\mathbb{N}$-graded ring of characteristic p and let M be a $\mathbb{Z}$-graded S-module. Let $F^e_*M^{(p^e)}$ be the graded submodule of F^e_*M defined in (4.4.1). Then, as quasicoherent sheaves on* $\operatorname{Proj} S$*,*

$$\widetilde{F^e_*M^{(p^e)}} = F^e_*(\widetilde{M}).$$

In particular, the sheaf $F^e_*\mathcal{O}_X$ on $X = \operatorname{Proj} S$ is determined by the graded S-module $F^e_*S^{(p^e)}$.

Proof of Lemma. We compute the sections of $\widetilde{F^e_*M^{(p^e)}}$ over a basic open affine $D_+(f) \subset \operatorname{Proj} S$:

$$\begin{aligned}
\widetilde{F^e_*M^{(p^e)}}(D_+(f)) &= \left[F^e_*M^{(p^e)}[\tfrac{1}{f}]\right]_0 \\
&= \left[F^e_*(M[\tfrac{1}{f^{p^e}}])\right]_0 \\
&= \left[F^e_*(M[\tfrac{1}{f}])\right]_0 \\
&= F^e_*\left(\left[M[\tfrac{1}{f}]\right]_0\right),
\end{aligned}$$

which is precisely $F^e_*\widetilde{M}(D_+(f))$. $\square$

Lemma 4.6. *With notation as in Lemma 4.5, the module inclusion*

$$F^e_*M^{(p^e)} \subset F^e_*M$$

splits *as a map of $F^e_*S^{(p^e)}$-modules, and hence as a map of S-modules.*

Proof. The inclusion $M^{(p^e)} \hookrightarrow M$ splits as a map of $S^{(p^e)}$-modules (by degree considerations). Thus $F^e_*M^{(p^e)} \hookrightarrow F^e_*M$ splits as a map of $F^e_*S^{(p^e)}-$modules, and as hence a map of S-modules via map $S \to F^e_*S^{(p^e)}$ sending $s \mapsto F^e_*s^{p^e}$. $\square$

Proof of Theorem 4.1. Let S be an F-finite Frobenius split graded ring, and let $\phi \in \operatorname{Hom}_S(F_*S, S)$ be a splitting of Frobenius. Because F_*S is a finitely generated $\frac{1}{p}\mathbb{N}$-graded S-module, the module $\operatorname{Hom}_S(F_*S, S)$ is finitely generated over S and $\frac{1}{p}\mathbb{Z}$-graded. So taking $\phi \in \operatorname{Hom}_S(F_*S, S)$ such that $\phi(F_*1) = 1$, and writing ϕ as a finite sum of homogenous components $\sum \phi_i$, we see that we may replace ϕ by its degree zero component ϕ_0 so as to assume with out loss of generality that $\phi(F_*1) = 1$ where ϕ is homogeneous and *degree preserving.*

Restrict ϕ to the $\mathbb{Z}$- graded submodule $F^e_*S^{(p^e)}$ whose homogeneous elements have integer degrees. Now, the composition of degree-preserving maps of $\mathbb{N}$-graded S-modules

$$S \xrightarrow{F} F^e_* S^{(p^e)} \xrightarrow{\phi} S$$

is the identity map, and so the corresponding map of sheaves on $\operatorname{Proj} S$

$$\widetilde{S} \xrightarrow{F} \widetilde{F^e_* S^{(p^e)}} \xrightarrow{\widetilde{\phi}} \widetilde{S}$$

is the identity map as well. Invoking Lemma 4.5, we thus have a splitting of Frobenius on $\operatorname{Proj} S$:

$$\mathcal{O}_X \xrightarrow{F} F_* \mathcal{O}_X \xrightarrow{\widetilde{\phi}} \mathcal{O}_X.$$

Thus $\operatorname{Proj} S$ is globally Frobenius split.

For (ii), assume the $\mathbb{N}$-graded ring S is strongly F-regular. In particular, S is a normal ring, and so $X = \operatorname{Proj} S$ is a normal scheme (Exercise 4.1).

Take any homogeneous nonzerodivisor $c \in S$ of positive degree, and let B be the divisor[23] on X determined by c. Specifically, the prime divisors of B are the codimension one integral subschemes $B_1, \dots, B_t$ defined by the minimal primes $P_1, \dots, P_t$ of c, and the coefficient on each is the order of c in the discrete valuation ring S_{P_i}. Put differently, if $(c) = P_1^{(n_1)} \cap P_2^{(n_2)} \cap \cdots \cap P_t^{(n_t)}$ is an irreducible primary decomposition[24] of the principal ideal c in S, then $B = \sum_{i=1}^t n_i B_i$, where B_i is the closed integral subscheme of codimension one $\mathbb{V}(P_i)$ in $\operatorname{Proj} S$.

We claim that scheme $X \setminus \operatorname{Supp} B$ is globally F-regular. Indeed, $X \setminus \operatorname{Supp} B$ is the affine scheme

$$X \setminus \mathbb{V}(c) = D_+(c) = \operatorname{Spec} \big[S[\frac{1}{c}]\big]_0,$$

which is globally F-regular if and only if the ring $\big[S[\frac{1}{c}]\big]_0$ is strongly F-regular (Proposition 2.8). But $\big[S[\frac{1}{c}]\big]_0$ is strongly F-regular because it is a direct summand of the strongly F-regular ring $S[\frac{1}{c}]$.

Now to show that X is globally F-regular, it suffices to show that X is eventually globally Frobenius split along B (Theorem 2.14). Because S is strongly F-regular, there exists $e \in \mathbb{N}$ and $\phi \in \operatorname{Hom}_S(F^e_*(\frac{1}{c}S), S)$ such that the composition

$$S \hookrightarrow F^e_*(\frac{1}{c}S) \xrightarrow{\phi} S \qquad\qquad 1 \mapsto F^e_* 1 \mapsto 1$$

is the identity map on S; see Exercise 2.1. Again, without loss of generality, we can assume that the splitting $\phi \in \operatorname{Hom}_S(F^e_*(\frac{1}{c}S), S)$ is homogeneous and

[23]Caution! B need not be Cartier. See Exercise 4.5.

[24]Here, $P^{(n)}$ denotes the P-primary ideal $P^n S_P \cap S$, the n-th symbolic power of P.

degree preserving. So the map ϕ restricts to a splitting of the submodule $F^e_*(\frac{1}{c}S)^{(p^e)}$ of integer-degree elements to give a composition

$$S \hookrightarrow F^e_*(\frac{1}{c}S)^{(p^e)} \xrightarrow{\phi} S \tag{4.6.1}$$

which is the identity map on S. Applying the Serre functor $\widetilde{(-)}$ to (4.6.1) induces the identity composition of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X(B) \xrightarrow{\tilde{\phi}} \mathcal{O}_X \tag{4.6.2}$$

since $\widetilde{\frac{1}{c}S} = \mathcal{O}_X(B)$ (by Lemma 4.5 and Exercise 4.3). This shows that X is eventually globally Frobenius split along B, completing the proof of Theorem 4.1. $\square$

4.3. Section rings. We next prove a converse for Theorem 4.1. As we have seen in Example 4.4, it is necessary to restrict the rings S.

Definition 4.7. Let X be a projective variety and let $\mathscr{L}$ be any (typically ample) invertible sheaf on X. The **section ring** of X with respect to $\mathscr{L}$ is the $\mathbb{N}$-graded ring

$$S(X, \mathscr{L}) := \bigoplus_{n \geq 0} H^0(X, \mathscr{L}^n).$$

Equivalently, the section ring is the global sections of the *sheaf of algebras* $\oplus_{n\in\mathbb{N}}\mathscr{L}^n$ on X.

Proposition 4.8. *Let $S = S(X, \mathscr{L})$ be the section ring on a projective variety X as defined in Definition 4.7. Then*

(i) *The section ring S is a finitely generated algebra over its degree zero subring, which is the field $k = H^0(X, \mathcal{L})$;*
(ii) *The projective scheme* Proj S *recovers X and $\widetilde{S(1)} \cong \mathscr{L}$;*
(iii) *The variety X is normal if and only if $S(X, \mathscr{L})$ is normal;*
(iv) *The natural map*

$$\operatorname{Spec} S \setminus \{\mathfrak{m}\} \xrightarrow{\pi} X$$

is a $k^\times$-bundle over X: every point $x \in X$ has a neighborhood U such that $\pi^{-1}(U) \cong U \times_{\operatorname{Spec} k} \operatorname{Spec} k[t, t^{-1}]$.

Example 4.9. In the simplest example, the variety $X \hookrightarrow \mathbb{P}^n_k$ is a projectively normal closed subscheme of $\mathbb{P}^n$, and we can take $\mathscr{L}$ to be $\mathcal{O}_X(1)$, the standard *hyperplane bundle* on X. In this case, the section ring

$$S = S(X, \mathcal{O}_X(1)) := \bigoplus_{n\in\mathbb{N}} H^0(X, \mathcal{O}_X(n)),$$

is precisely the homogenous coordinate ring $R = k[x_0, \ldots, x_n]/I_X$ where I_X is the homogeneous ideal of polynomials vanishing on X.

Caution 4.10. If $X \hookrightarrow \mathbb{P}^n_k$ is not normally embedded, the section ring S built from $\mathcal{O}_X(1)$ is not the homogeneous coordinate ring R: it is some finite integral extension S of R such that $S_n = R_n$ for all $n \gg 0$ [**Har77**, II Ex 5.14]. If X is normal, the section ring with respect to $\mathcal{O}_X(1)$ is the normalization of the homogeneous coordinate ring R.

Caution 4.11. Section rings can be built from non-ample $\mathscr{L}$, but they are much worse behaved and we will not do this in this book. We use the term *section ring* exclusively for section rings of ample invertible sheaves on X.

Theorem 4.12. *Let X be a projective variety of prime characteristic, and let $\mathscr{L}$ be any ample invertible sheaf on X. Then X is globally Frobenius split (respectively, globally F-regular) if and only if the section ring $S(X, \mathscr{L})$ is locally Frobenius split (respectively, strongly F-regular).*

Theorem 4.12 says that the *global versions* of Frobenius splitting and F-regularity for a projective variety $X = \operatorname{Proj} S$ amount to the corresponding *local* properties at the vertex $\mathfrak{m}$ of the affine cone $\operatorname{Spec} S$ over X. A much weaker fact is that the corresponding *local* properties of X amount to the same *local* properties on the punctured cone over X:

Proposition 4.13. *Let X be a projective variety and $S = S(X, \mathscr{L})$ a section ring for some ample $\mathscr{L}$ on X. The variety $X = \operatorname{Proj} S$ has property $\mathcal{P}$ if and only if the punctured cone $\operatorname{Spec} S \setminus \{\mathfrak{m}\}$ has property $\mathcal{P}$, where $\mathcal{P}$ can be any of the following local properties: reduced, normal, Cohen-Macaulay, locally F-regular, locally Frobenius split, non-singular.*

Proof of Proposition 4.13. Because property $\mathcal{P}$ can be checked locally, Proposition 4.8(iv) tells us that Proposition 4.13 amounts to saying that a Noetherian ring R has property $\mathcal{P}$ if and only if $R[t, t^{-1}]$ has property $\mathcal{P}$. We leave as an exercise to check this directly for each property. $\square$

4.4. Saturated graded modules of coherent sheaves on Proj. If $S = S(X, \mathscr{L})$ is a section ring for some ample $\mathscr{L}$ on a projective variety X, then for any coherent sheaf $\mathscr{M}$ on X, we can construct a finitely generated $\mathbb{Z}$-graded S-module

$$M = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathscr{M} \otimes \mathscr{L}^n)$$

such that $\widetilde{M}$ recovers $\mathscr{M}$ on X. This graded S-module M is the "largest" (or unique saturated) S-module which determines $\mathscr{M}$. Any other graded S-module which eventually agrees with M in large degree will also determine $\mathscr{M}$. For example, $\widetilde{S(d)}$ recovers[25] $\mathscr{L}^d$.

[25]Caution: If S is not a section ring, then the sheaves $\widetilde{S(d)}$ need not be invertible!

Proof of Theorem 4.12. One direction has already been proved in Theorem 4.1. It remains to prove that if a projective variety X is globally Frobenius split (or globally F-regular), then any section ring S is locally Frobenius split (respectively strongly F-regular).

First assume that X is globally Frobenius split. So there exists ϕ such that the composition

$$\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$$

is the identity. Now tensor with the sheaf of algebras $\oplus_{n\in\mathbb{N}}\mathscr{L}^n$ and use the projection formula to get

$$\bigoplus_{n\in\mathbb{N}}\mathscr{L}^n \xrightarrow{F} F_*(\bigoplus_{n\in\mathbb{N}}\mathscr{L}^{pn}) \xrightarrow{\phi} \bigoplus_{n\in\mathbb{N}}\mathscr{L}^n.$$

Taking global sections, we have the identity composition of S-modules

$$S \xrightarrow{F} F_*(S^{(p)}) \xrightarrow{\phi} S,$$

where $S^{(p)}$ denotes the p^{th} Veronese subring of S. Since the Veronese subrings split off S, let $\pi \in \operatorname{Hom}_{S^{(p)}}(S, S^{(p)})$ be a splitting of $S^{(p)} \hookrightarrow S$. Now define ψ to be the composition

$$F_*S \xrightarrow{F_*\pi} F_*S^{(p)} \xrightarrow{\phi} S.$$

It is easy to check that ψ is a splitting of the Frobenius map $S \to F_*S$. This completes the proof that if a projective variety X is globally Frobenius split, so are all its section rings.

Now assume that X is globally F-regular. In particular, because X is reduced, there exists a basic open affine $D_+(c) \subset X$ which is regular, where c is some homogeneous non-zerodivisor of S of positive degree. This means that $\left[S[\frac{1}{c}]\right]_0$ is regular; thus the ring $S[\frac{1}{c}]$ is regular as well by Proposition 4.13. Thus c can be used to test F-regularity for S by Theorem 5.1 in Chapter 1.

Let D be the effective divisor on X defined by the homogeneous ideal (c) of S. In this case, the graded module corresponding to the coherent sheaf $\mathcal{O}_X(D)$ is precisely $\frac{1}{c}S$ (which is isomorphic as a graded S-module to the shifted module $S(\deg c)$). By Lemma 4.5, the graded module corresponding to $F^e_*\mathcal{O}_X(D)$ is $F^e_*(\frac{1}{c}S)^{p^e}$.

Since X is globally F-regular, there is an e such that the natural map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X(D) \qquad\qquad 1 \mapsto F^e_*1$$

splits. Tensoring with the sheaf of $\mathcal{O}_X$-algebras $\oplus\mathscr{L}^n$, we have that

$$\bigoplus_{n\in\mathbb{N}}\mathscr{L}^n \longrightarrow \bigoplus_{n\in\mathbb{N}}\mathscr{L}^n \otimes_{\mathcal{O}_X} F^e_*\mathcal{O}_X(D) \qquad\qquad 1 \mapsto F^e_*1,$$

splits as well. Taking global sections, we have that

$$S \longrightarrow F^e_*(\frac{1}{c}S)^{(p^e)} \qquad\qquad 1 \mapsto F^e_* 1$$

splits, as well, where we've used Serre's correspondence and Lemma 4.5 to identify the graded S-module module corresponding to $F^e_*\mathcal{O}_X(D)$. Let $\phi \in \operatorname{Hom}_S(F^e_*(\frac{1}{c}S)^{(p^e)}, S)$ be a splitting. The natural inclusion $F^e_*(\frac{1}{c}S)^{(p^e)} \hookrightarrow F^e_*\frac{1}{c}S$ also splits (Lemma 4.6); let $\psi \in \operatorname{Hom}_S(F^e_*\frac{1}{c}S, F^e_*(\frac{1}{c}S)^{(p^e)})$ be a splitting. Then the map $S \longrightarrow F^e_*(\frac{1}{c}S)$ is split by the composition

$$F^e_*\frac{1}{c}S \overset{\psi}{\twoheadrightarrow} F^e_*(\frac{1}{c}S)^{p^e} \overset{\phi}{\twoheadrightarrow} S \qquad\qquad F^e_*1 \mapsto F^e_*1 \mapsto 1.$$

Restricting to the submodule F^e_*S, we see that the S-module map

$$S \xrightarrow{F^e} F^e_*\frac{1}{c}S \qquad\qquad 1 \mapsto F^e_*1$$

splits as well. This proves that S is strongly F-regular, by Theorem 5.1 in Chapter 1. $\square$

Remark 4.14. We remind the reader that Hochster and Huneke introduced three flavors of F-regularity, all conjectured to be equivalent: strong F-regularity Definition 2.7, weak F-regularity (all ideals are tightly closed) and F-regularity (all ideals tightly closed in all localizations). For finitely generated graded rings over a field, these are known to be equivalent [**LS99**], so we can drop the "strongly" in referring to strongly F-regular graded rings.

4.5. Exercises.

Exercise 4.1. Let S be an $\mathbb{N}$-graded ring. Prove that S is normal (respectively, Frobenius split; respectively strongly F-regular), then $\operatorname{Proj} S$ is normal (respectively, locally Frobenius split; respectively, locally F-regular)

Hint: On $D_+(f)$, the sections $\left[S[\frac{1}{f}]\right]_0$ are a direct sumnand of the $\mathbb{Z}$-graded ring $S[\frac{1}{f}]$.

Exercise 4.2. Let S be a finitely generated $\mathbb{N}$-graded algebra over a field k. Let $\mathcal{P}$ be one of the following local properties: normal, Cohen-Macaulay, reduced, Frobenius split (in the case k is F-finite), or strongly F-regular (in the case k is F-finite). Prove that S has property $\mathcal{P}$ if and only if $S_{\mathfrak{m}}$, where $\mathfrak{m}$ is the unique homogeneous maximal ideal of S.

Hint: Reduce to the case where k is infinite field; then show the locus of points in $\operatorname{Spec} S$ that do not have property $\mathcal{P}$ is defined by a homogeneous ideal of S.

Exercise 4.3. Let S be a normal $\mathbb{N}$-graded ring finitely generated over a field S_0, and let $X = \operatorname{Proj} S$. Let $c \in S$ be a homogeneous element of positive degree. Show that the subscheme of X defined by the homogeneous ideal cS determines a Weil divisor B, and that $\mathcal{O}_X(B) = \widetilde{\frac{1}{c}S}$.

Hint: Caution: B need not be Cartier!

Exercise 4.4. Prove Proposition 4.13.

Hint: Observe that R is a direct summand of $R[t, t^{-1}]$ while $R[t, t^{-1}]$ is a faithfully flat extension of R.

Exercise 4.5. Let $S = k[x, y, z]$ be the polynomial ring with *non-standard* grading given by $\deg(x) = 3, \deg(y) = 2, \deg(z) = 1$. Prove that $\operatorname{Proj} S$ is singular. Show also that the coherent module on $\operatorname{Proj} S$ given by the graded module $\frac{1}{z}S$ is not invertible. In particular, show that the divisor on $\operatorname{Proj} S$ given by the vanishing of z is a prime Weil (non-Cartier) divisor.

Hint: Compute on the chart $D_+(y)$.

Exercise 4.6. Suppose that X is a globally Frobenius split projective variety with splitting $\phi : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ and $Z \subseteq X$ is a subscheme compatibly split by ϕ. Fix an ample line bundle $\mathscr{L}$ on X. Suppose that S is the section ring of X and with respect to $\mathscr{L}$ and

$$I_Z = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{I}_Z \otimes \mathscr{L}^n) \subseteq S,$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z. Show that the ideal I_Z is compatible with the map $\phi_S : F_*S \longrightarrow S$ induced as in the proof of Theorem 4.12.

Exercise 4.7. Prove the converse to Exercise 4.6. In particular, suppose that I_Z is compatible with the induced $\phi_S : F_*S \longrightarrow S$. Prove that Z is compatible with ϕ.

Exercise 4.8. With notation as in the previous problem, let S_Z denote the section ring with respect to $\mathscr{L}|_Z$. Now assume that the base field k is algebraically closed and Z is a variety, so in particular Z is connected. Show that the map $S \longrightarrow S_Z$ is surjective. In particular, any normal projective variety that is globally compatibly Frobenius split in projective space is projectively normally embedded.

Hint: It suffices to show that $H^0(X, \mathcal{O}_X \otimes \mathscr{L}^n) \longrightarrow H^0(Z, \mathcal{O}_Z \otimes \mathscr{L}^n)$ surjects for all $n > 0$. See Theorem 3.10.

Exercise 4.9. Let S be an $\mathbb{N}$-graded domain finitely generated by its degree one elements over its degree zero piece $S_0 = k$, an F-finite field. Show that

$\operatorname{Proj} S$ is globally Frobenius split (respectively, globally F-regular) if and only if the normalization S^N is locally Frobenius split (respectively, strongly F-regular).

Hint: Show that the normalization S^{N} is the section ring $S(X, \mathcal{O}_X(1))$; see [**Har77**, Exercise 5.14].

Exercise 4.10. Suppose that X is a projective variety of dimension > 0. Let $(S, \mathfrak{m})$ denote the section ring with irrelevant ideal $\mathfrak{m}$. Prove that $H^{\dim X+1}_{\mathfrak{m}}(S)$ is zero in all non-negative degrees if X is globally F-regular and zero in all positive degrees if X is globally Frobenius split.

Exercise 4.11. Suppose X is a projective variety and $\mathscr{L}$ is an ample line bundle. Suppose that the section ring of $\mathscr{L}$ is Cohen-Macaulay. Prove that $H^i(X, \mathscr{L}^n) = 0$ for every $0 < i < \dim X$ and every $n \in \mathbb{Z}$.

5. Local cohomology and section rings

Warning, this section is mostly not yet written. Proceed with caution. It should also eventually contain a number of example.

We have seen that the global geometry of a projective variety X is closely related to the local properties at the vertex of an affine cone over X. This can be understood more deeply by connecting the cohomology groups of coherent sheaves on X to the local cohomology modules of the corresponding graded modules at the vertex of the cone. Using this idea, we reprove Theorem 2.16 for projective varieties, and develop a useful criterion for global F-rationality for projective varieties.

Proposition 5.1. *Let X be a projective variety with section ring $S = S(X, \mathscr{L})$ with respect to a fixed ample invertible sheaf $\mathscr{L}$, and let $\mathfrak{m}$ denote its unique homogeneous maximal ideal. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let*

$$M = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F} \otimes \mathscr{L}^n)$$

be the corresponding graded S-module. Then there is a natural degree-preserving isomorphism of S-modules

$$H^{i+1}_{\mathfrak{m}}(M) \cong \bigoplus_{n \in \mathbb{Z}} H^i(X, \mathcal{F} \otimes \mathscr{L}^n)$$

for each $i \geq 1$.

Proof. The local cohomology modules describe the obstruction to extending sections of the sheaf $\widetilde{M}$ on the affine scheme $\operatorname{Spec} S$ from the open set $\mathcal{U} = \operatorname{Spec} S \setminus \{\mathfrak{m}\}$ to $\operatorname{Spec} S$. That is, we have an exact sequence where the map labeled ρ is simply restriction:

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow H^0(\operatorname{Spec} S, \widetilde{M}) \xrightarrow{\rho} H^0(\mathcal{U}, \widetilde{M}) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow$$

$$H^1(\operatorname{Spec} S, \widetilde{M}) \longrightarrow H^1(\mathcal{U}, \widetilde{M}) \longrightarrow H^2_{\mathfrak{m}}(M) \longrightarrow H^2(\operatorname{Spec} S, \widetilde{M}) \longrightarrow \dots$$

This long exact sequence shows that there are natural isomorphisms

$$H^i(\operatorname{Spec} S \setminus \{\mathfrak{m}\}, \widetilde{M}) \to H^{i+1}_m(M)$$

for all $i \geq 1$, because the higher cohomologies of quasi-coherent sheaves vanish on an affine scheme.

Now let M be a graded S-module corresponding to the sheaf $\mathcal{F}$. Consider a homogeneous system of parameters $f_0, \dots, f_d$. The open sets $D(f_0), \dots, D(f_d)$ form an open affine cover for $\mathcal{U} = \operatorname{Spec} S \setminus \{\mathfrak{m}\}$, and the open sets $D_+(f_0), \dots, D_+(f_d)$ form an open affine cover for X. Writing down the Čech complex that computes the cohomology $H^i(\operatorname{Spec} S \setminus \{\mathfrak{m}\}, \widetilde{M})$ from the cover $\{D(f_i)\}$ we observe that its zero graded piece is precisely the Čech complex from the cover $\{D_+(f_i)\}$ of $\operatorname{Proj} S$ computing $H^{i-1}(X, \mathcal{F})$. Similarly, its n-graded piece is the Čech complex that computes $H^{i-1}(X, \mathcal{F} \otimes \mathscr{L}^n)$. This complete the proof. $\square$

5.1. Graded Matlis Duality. Let X be a normal projective variety with canonical module ω_X. Fix any ample invertible sheaf $\mathcal{L}$ on X and let S be the corresponding section ring $S(X, \mathcal{L})$.

5.2. F-rationality and the a-invariant.

5.3. Examples of graded rings.

CHAPTER 4

Frobenius Splitting for embedded schemes

We now turn to criteria for identifying Frobenius splitting, as well as ways to measure “how far" a given variety is from being Frobenius split. In this chapter, we are primarily interested in affine schemes embedded as closed schemes in F-finite regular schemes—that is, rings that are a homomorphic image of a regular ring.[1]

We begin with a useful criterion for Frobenius splitting due to Richard Fedder (Theorem 1.1), which allows us to identity many examples of Frobenius split and non-Frobenius split schemes embedded in a non-singular scheme. Similarly, Glassbrenner’s Theorem (Theorem 1.8) provides an analogous criterion for strong F-regularity. The proof of Fedder’s Criterion, undertaken in Section 2, requires developing a careful understanding the R-module $\mathrm{Hom}_R(F_*R, R)$ for rings that are quotients of F-finite regular rings. This naturally leads to several important ideas, including the *F-pure threshold* as well as *F-singularities and test ideals for pairs*.

In Section 3, we introduce the *F-pure threshold* in the special case of a hypersurface defined by some element f in a regular local ring S. This numerical invariant of the singularities of the *pair* (S, f) can be viewed as a measurement of how far the hypersurface $\operatorname{Spec} S/f$ is from being Frobenius split. The F-pure threshold is a prime characteristic analog of the *log canonical threshold* in complex algebraic geometry, as we explain in Chapter 6.

In Section 4, we introduce Frobenius splitting and strong F-regularity for *pairs* $(R, \mathfrak{a}^t)$, where R is a ring of prime characteristic, $\mathfrak{a} \subseteq R$ is an ideal and $t \geq 0$ is a real number. This natural generalization of the material in sections 4 and 3 in Chapter 1 is analogous to the use of pairs in birational algebraic geometry, and reflects not only the singularities of $\operatorname{Spec} R$, but also the singularities of the closed embedded scheme defined by the ideal $\mathfrak{a}$.

In Chapter 5, we develop a theory of test ideals $\tau(R, \mathfrak{a}^t)$ for pairs $(R, \mathfrak{a}^t)$, generalizing the theory of test ideals from Section 5 of Chapter 1 (which can be viewed as the special case where the ideal $\mathfrak{a}$ is the unit ideal). The theory

[1]Recall, every F-finite Noetherian ring is a quotient of an F-finite regular ring by **[Gab04]**.

will pushed further in more geometric setting in Chapter 5. In this chapter, we stick to the affine setting, focusing first on the rich case where the ambient scheme $\operatorname{Spec} R$ is *regular.* This provides a powerful way to study ideals in R and highlights the analogy with the multiplier ideal in complex geometry. For example, we introduce Frobenius jumping numbers in Section 6, a spectrum of numerical invariants of a (prime characteristic) pair $(R, \mathfrak{a}^t)$, which can be interpreted as characteristic p analogs of the jumping coefficients for multiplier ideal sheaves on smooth complex varieties. Finally, in Section 7, we use the test ideal to prove a theorem on the behavior of symbolic powers of ideals in prime characteristic, following the original multiplier ideal proof of Ein-Lazarsfeld-Smith [**ELS01**] (*cf.* [**HH02**]). The formal relationship between the test ideal and multiplier ideal will be discussed later in Chapter 6.

Studying the test ideal in this case highlights the many ways in which the test ideal is analogous to the multiplier ideal in complex geometry. For example, we introduce Frobenius jumping numbers in Section 6, a spectrum of numerical invariants of a (prime characteristic) pair $(R, \mathfrak{a}^t)$, which can be interpreted as characteristic p analogs of the jumping coefficients for multiplier ideal sheaves on smooth complex varieties. And in Section 7, we use the test ideal to prove the Ein-Lazarsfeld-Smith [**ELS01**] theorem on the behavior of symbolic powers of ideals in prime characteristic, following the original multiplier ideal proof. The formal relationship between the test ideal and multiplier ideal will be discussed later in Chapter 6.

1. Fedder's criterion: statement and applications

The most useful test for Frobenius splitting is due to Fedder:

Theorem 1.1 (Fedder's Criterion, [**Fed83**]). *Let $R = S/I$ be a quotient of a regular local F-finite ring $(S, \mathfrak{m})$. Then R is Frobenius split if and only if*

$$(I^{[p^e]} : I) \not\subseteq \mathfrak{m}^{[p^e]}$$

for some (or equivalently, for every) natural number e. Here, the notation $(I^{[p^e]} : I)$ denotes the ideal of elements in S which multiply I into $I^{[p^e]}$.

Fedder's Criterion is especially convenient to apply for hypersurfaces:

Corollary 1.2. *Let $R = S/(f)$, where $(S, \mathfrak{m})$ is an F-finite regular local ring and f is a non-zero element in $\mathfrak{m}$. Then R is Frobenius split if and only if*

$$f^{p^e - 1} \notin \mathfrak{m}^{[p^e]}$$

for some (or equivalently, for every) $e > 0$.

PROOF. If $I = (f) \subseteq S$ is a principal ideal, then $(I^{[p^e]} : I) = (f^{p^e - 1})$, and the corollary follows. □

There is an analogous statement for complete intersections; see Exercise 1.3.

We explore some applications.

Example 1.3. A reduced normal crossing divisor D on a non-singular variety X is locally Frobenius split. To see this, choose local parameters $x_1, \ldots, x_d$ at a closed point P, so that the maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_{X,P}$ is generated by $x_1, \ldots, x_d$ and the divisor D is defined by the principal ideal $(x_1 x_2 \cdots x_n)$ for some $n \leq d$. By Corollary 1.2, the local ring $\mathcal{O}_{D,P} = \mathcal{O}_{X,P}/(x_1 x_2 \cdots x_n)$ is Frobenius split if $(x_1 x_2 \cdots x_n)^{p-1} \notin \mathfrak{m}^{[p]}$. But if $(x_1 x_2 \cdots x_n)^{p-1} \in (x_1^p, x_2^p, \ldots, x_d^p)$, then $1 \in (x_1, \ldots, x_n, x_{n+1}^p, \ldots, x_d^p) \subseteq \mathfrak{m}$ (see Exercise 1.2). This contradiction allows us to conclude that D is Frobenius split at P.

Example 1.4. Consider the complete local ring $R = k[\![x, y, z]\!]/(xy - z^2)$, where k is a F-finite field of characteristic p. In the power series ring $k[\![x, y, z]\!]$,

$$(xy - z^2)^{p-1} \equiv x^{p-1} y^{p-1} \text{ modulo } (x^p, y^p, z^2),$$

so $(xy - z^2)^{p-1} \notin (x^p, y^p, z^p) \subseteq (x^p, y^p, z^2)$ (again, using Exercise 1.2). This implies R is Frobenius split by Corollary 1.2.

Example 1.5. Consider a hypersurface in $\mathbb{P}^n$ defined by a homogeneous polynomial f of degree d over an F-finite field of characteristic p. Fedder's Criterion easily implies that:

(a) The hypersurface is *never* (globally) Frobenius split if $d > n + 1$;
(b) If $d = n + 1$, then the hypersurface is (globally) Frobenius split if and only if the monomial $(x_0 x_1 \cdots x_n)^{p-1}$ appears in f^{p-1} with non-zero coefficient.

To prove this, first note that we can check Frobenius splitting at the unique homogeneous maximal ideal of the homogeneous coordinate ring $k[x_0, \ldots, x_n]/(f)$ (Theorem 4.12 in Chapter 3). So applying Fedder's criterion, we consider whether or not f^{p-1} is in the ideal $\mathfrak{m}^{[p]} = (x_0^p, x_1^p, \ldots, x_n^p)$.

Expanding out f^{p-1} as a k-linear combination of monomials, we observe that each monomial $x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}$ appearing f^{p-1} is in $\mathfrak{m}^{[p]}$ unless *all* $a_i \leq p-1$, which can happen only when $\deg f^{p-1} = \sum_{i=0}^n a_i \leq (n+1)(p-1)$. This implies (a) immediately, by Fedder's criterion. Statement (b) follows as well, since when $\deg f = n+1$, all monomials of f^{p-1} are in $\mathfrak{m}^{[p]}$ *with the possible exception* of the term $(x_0 x_1 \cdots x_n)^{p-1}$, which is in $\mathfrak{m}^{[p]}$ if and only if its coefficient is zero.

See also Example 1.11 for a generalization to global F-regularity.

Example 1.6. Let us consider some cubic surfaces of characteristic two. Let

$$f = x^3 + y^3 + z^3 + w^3 \qquad \text{and} \qquad g = xyz + x^3 + y^3 + z^3 + w^3$$

be two polynomials over an algebraically closed field k of characteristic two. Both define smooth cubic surfaces in $\mathbb{P}^3$, but the surface defined by g is globally Frobenius split while the surface defined by f is not. Indeed, remembering that $p = 2$, we have

$$f^{p-1} = x^3 + y^3 + z^3 + w^3 \in (x^2, y^2, z^2, w^2) = \mathfrak{m}^{[p]}$$

whereas,

$$g^{p-1} = xyz + x^3 + y^3 + z^3 + w^3 \notin (x^2, y^2, z^2, w^2) = \mathfrak{m}^{[p]},$$

so this follows immediately from Corollary 1.2. Non-singular cubic surfaces in $\mathbb{P}^3$ that are not Frobenius split, it turns out, exist only in characteristic two, and they have extremely special geometry. For example, there are no "triangles" on a non-Frobenius split cubic: every triple of coplanar lines meet at a point. Put differently, there are 45 Eckardt points on a non-Frobenius split cubic surface. See [**KKP$^+$21**].

1.1. Elliptic curves. As a special case of Example 1.5, we again see that Frobenius splitting for elliptic curves is the same as ordinarity (compare with Chapter 3 Example 1.29):

Corollary 1.7. *An elliptic curve over an F-finite field is (globally) Frobenius split if and only if it is ordinary.*

Proof. An elliptic curve E can be embedded as a cubic curve in $\mathbb{P}^2$, say, defined by $f \in k[x, y, z]$. But now using Example 1.5, the curve E is Frobenius split if and only if the term $(xyz)^{p-1}$ appears in f^{p-1} with non-zero coefficient. This condition is one of the many characterizations of ordinarity for elliptic curves [**Har77**, IV Prop 4.21]. $\square$

1.1.1. *The Fermat cubic.* Let us examine more closely a particular elliptic curve, say with equation $f = x^3 + y^3 + z^3$ in characteristic $p \neq 3$. Using the trinomial expansion, we compute

$$f^{p-1} = \sum_{i+j+k=p-1} \binom{p-1}{i, j, k} x^{3i} y^{3j} z^{3k}. \tag{1.7.1}$$

All terms of (1.7.1) are in (x^p, y^p, z^p) with the possible exception of the $(xyz)^{p-1}$ term. This term appears if and only if $3i = 3j = 3k = p - 1$, as its coefficient

$$\binom{p-1}{\frac{p-1}{3}\ \frac{p-1}{3}\ \frac{p-1}{3}} = \frac{(p-1)!}{\left(\frac{p-1}{3}\right)!\left(\frac{p-1}{3}\right)!\left(\frac{p-1}{3}\right)!}$$

is clearly non-zero modulo p. So $k[x,y,z]/(x^3+y^3+z^3)$ is Frobenius split if and only if $p \equiv 1 \pmod 3$. Put differently, the elliptic curve in $\mathbb{P}^2$ defined by $x^3+y^3+z^3$ is ordinary when $p \equiv 1$ mod 3 and supersingular[2] when $p \equiv 2$ mod 3. For more on elliptic curves, see Exercise 1.5.

1.2. Glassbrenner's criterion for strong F-regularity. Donna Glassbrenner proved an analogous criterion for strong F-regularity:

Theorem 1.8 (**[Gla96]**). *Let $R = S/I$ be a quotient of an F-finite regular local ring $(S, \mathfrak{m})$ by some non-zero prime ideal I. Then R is strongly F-regular if and only if for every $c \in S \setminus I$, there exists $e > 0$ so that*

$$c\ (I^{[p^e]} : I) \nsubseteq \mathfrak{m}^{[p^e]}. \tag{1.8.1}$$

Remark 1.9. Glassbrenner's theorem can be stated using just one "test c" as we did in Chapter 1 Section 5. That is, to check that R is strongly F-regular, it suffices to check (1.8.1) for just *one* $c \in S \setminus I$ such that the localization S_c/IS_c is strongly F-regular (for example, regular).

Remark 1.10. Theorem 1.8 (and Remark 1.9) adapt easily to the case I is not prime, but in this case, we should take c not in any minimal prime of I.

Glassbrenner's Criterion tells us that hypersurface singularities of high multiplicity are *never* strongly F-regular.

Example 1.11. If $R = S/(f)$ where $(S, \mathfrak{m})$ is an F-finite regular local ring of dimension n and f is a non-zero element in $\mathfrak{m}^n$, then R is *never* strongly F-regular. To check this, observe that the ideal $(I^{[p^e]} : I)$ is generated by f^{p^e-1}, so $(I^{[p^e]} : I) \subseteq \mathfrak{m}^{(p^e-1)n}$ for all e. In particular,

$$c(I^{[p^e]} : I) = cf^{p^e-1}S \subseteq \mathfrak{m}^{(p^e-1)n+1} \subseteq \mathfrak{m}^{[p^e]}$$

for all $c \in \mathfrak{m}$ and every $e \in \mathbb{N}$. Glassbrenner's criterion therefore tells us that the hypersurface defined by f is not strongly F-regular. Interpreted as a statement in the graded case, this example says that the degree of a globally F-regular hypersurface in $\mathbb{P}^n$ is at most n; See also Example 1.5.

Hypersurfaces of low multiplicity can be strongly F-regular or not, even in fixed degree, depending on the specific example; see Exercise 1.6. However, quadrics are always strongly F-regular:

Example 1.12. Let $R = k[x_1, \ldots, x_n]/(f)$ be a quotient of a polynomial ring in at least three variables over an F-finite field by an irreducible homogeneous element f of degree two. Then R is strongly F-regular. To see this, recall

[2]In this context, supersingular means simply "not ordinary."

that changing coordinates, we can assume[3] that $f = x_1x_2 + g$ where g is a homogeneous degree two polynomial in $x_3, \ldots, x_n$. In this case, $R[x_1^{-1}]$ is regular. Applying Glassbrenner's criterion (as modified by Remark 1.9), we examine

$$x_1 f^{p^e-1} = x_1 \sum_{i=0}^{p^e-1} \binom{p^e-1}{i} (x_1x_2)^i g^{p^e-1-i}.$$

Note that each summand $x_1(x_1x_2)^i g^{p^e-1-i}$ has degree $1+i$ in x_1, so that no cancellation can occur among them. Furthermore, each binomial coefficient $\binom{p^e-1}{i}$ is non-zero by Lucas's theorem (see Exercise 1.4). So thinking about degrees, we see the $x_1 f^{p-1} \in \mathfrak{m}^{[p^e]}$ if and only if each term $x_1(x_1x_2)^i g^{p^e-1-i}$ is in $\mathfrak{m}^{[p^e]}$. In particular, when $i = p^e - 2$, this says

$$x_1(x_1x_2)^{p^e-2} g = x_1^{p^e-1} x_2^{p^e-2} g \in (x_1^{p^e}, \ldots, x_n^{p^e}),$$

which forces

$$g \in (x_1^{p^e}, \ldots, x_n^{p^e}) : x_1^{p^e-1} x_2^{p^e-2} = (x_1, x_2^2, x_3^{p^e} \ldots, x_n^{p^e})$$

(see, *e.g.*, Exercise 1.2). As g has degree two in $x_3, \ldots, x_n$, this is clearly impossible for $e \gg 0$.

Fedder's criterion will be proved in the next section after developing a thorough understanding of the module $\operatorname{Hom}_R(F_*^e R, R)$. The proof of Glassbrenner's theorem is similar, and left to the reader in Exercise 2.4.

1.3. Exercises.

Exercise 1.1. Describe all $f \in (x, y, z) \subseteq \mathbb{F}_2[x, y, z]$ such that $\mathbb{F}_2[x, y, z]/(f)$ is Frobenius split in a neighborhood of (x, y, z).

Exercise 1.2. Let $y_1, \ldots, y_n$ be a regular sequence in a commutative ring. For natural numbers $0 \le a_i \le N_i$, prove that

$$(y_1^{N_1}, y_2^{N_2}, \ldots, y_n^{N_n}) : (y_1^{a_1} y_2^{a_2} \ldots y_n^{a_n}) \subseteq (y_1^{N_1-a_1}, y_2^{N_2-a_2}, \ \ldots, \ y_n^{N_n-a_n}).$$

Hint: Induction on the sum of the N_i works well.

Exercise 1.3. Let $(S, \mathfrak{m})$ be an F-finite regular local ring, and suppose that $R = S/(f_1, \ldots, f_r)$ is a complete intersection.[4] Prove that R is Frobenius

[3] This follows from the classification theory for quadratic forms. We can extend scalars by Corollary 1.24 in Chapter 3 so as to assume k is quadratically closed. When $p \neq 2$, this classification is well-known and can be found in any advanced linear algebra textbook. Because the characteristic two case can be hard to find, we refer the reader to [**KPS**$^+$**21**, Prop 3.4].

[4] Complete intersection means that $\dim R = \dim S - r$, or equivalently, that $f_1, \ldots, f_r$ form a regular sequence.

split if and only if
$$(f_1 \cdots f_r)^{p-1} \notin \mathfrak{m}^{[p]}.$$

Hint: Prove that $I^{[p]} : I = I^{[p]} + (f_1 \cdots f_r)^{p-1}$.

Exercise 1.4 (Lucas's Theorem)**.** Let p be a prime number. Given two positive integers, m and n, let $m = m_k p^k + \cdots + m_0$ and $n = n_k p^k + \cdots + n_0$ be their base p expansions. Prove that $\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p}$.

Exercise 1.5 (Elliptic curves in Legendre normal form)**.** Assuming k is algebraically closed and of characteristic $p \neq 2$, a smooth elliptic curve E can be defined by a polynomial of the form $zy^2 - h(x,z)$, where h is a homogeneous cubic. Show that E is Frobenius split if and only if the dehomogenized polynomial $h(x,1)^{\frac{(p-1)}{2}}$ has a non-zero x^{p-1} term. Again, this is equivalent to ordinarity of E by [**Sil09**, Chapter V, Theorem 4.1(a)].

Exercise 1.6. Use Glassbrenner's criterion to show that the smooth cubic surface in $\mathbb{P}^3$ defined by $xyz + x^3 + y^3 + z^3 + w^3$ is globally F-regular in all characteristics $p \neq 3$, and the cubic surface defined by $x^3 + y^3 + z^3 + w^3$ is globally F-regular if and only if the characteristic $p \geq 5$.

Exercise 1.7. Suppose that R is a regular Noetherian ring of characteristic $p > 0$ and that $\mathfrak{q}$ is a prime ideal. Prove that $\mathfrak{q}^{[p^e]}$ is $\mathfrak{q}$-primary.

Hint: Show that if $f \notin \mathfrak{q}$, then the "multiplication by f" map $R/\mathfrak{q}^{[p^e]} \xrightarrow{f} R/\mathfrak{q}^{[p^e]}$ is injective.

Exercise 1.8. Suppose that S is an F-finite regular ring that is *not necessarily local* and $R = S/I$. Fix a prime ideal $\mathfrak{q} \in \operatorname{Spec} R = \mathbb{V}(I) \subseteq \operatorname{Spec} S$. Show that $\operatorname{Spec} R$ is Frobenius split at $\mathfrak{q}$ if and only if for some fixed (equivalently, all) $e \in \mathbb{N}$,
$$(I^{[p^e]} : I) \nsubseteq \mathfrak{q}^{[p^e]}.$$

Hint: Reduce to the case where $\mathfrak{q}$ is the maximal ideal of a regular local ring using Exercise 1.7 to show that $\mathfrak{q}^{[p^e]} R_{\mathfrak{q}} \cap R = \mathfrak{q}^{[p^e]}$.

Exercise 1.9. State and prove an analog of Exercise 1.8 for strong F-regularity.

2. The proof of Fedder's Criterion

To prove Fedder's Criterion, we examine $\operatorname{Hom}_R(F_*R, R)$, where Frobenius splittings, if they exist, must live. We have already looked carefully at the structure of $\operatorname{Hom}_S(F_*S, S)$ as an F_*S-module, in Chapter 2, in the case where S is a polynomial ring, and seen that this module is free of rank one over F_*S. In this section, in the case where R is a *quotient* of an F-finite regular local ring S, Fedder proved a concrete description of the module $\mathrm{Hom}_R(F_*R, R)$ in terms of the F_*S-module $\mathrm{Hom}_S(F_*S, S) \cong F_*S$, from which his criterion follows easily. This description of $\mathrm{Hom}_R(F_*R, R)$ has other applications, including to computing the defining ideal for the non-Frobenius split locus explicitly; see Subsection 2.3.

2.1. Fedder's Lemma. Recall that in general, whenever $A \longrightarrow B$ is a map of commutative rings, there is a natural (right) B-module structure on $\mathrm{Hom}_A(B, A)$, where $b \in B$ acts on $\phi \in \mathrm{Hom}_A(B, A)$ to produce the map $\phi \circ b \in \mathrm{Hom}_A(B, A)$; see Subsection 1.1 of Chapter 2. In this section, we freely use this idea for the Frobenius map $R \longrightarrow F^e_* R$. In particular, for all $e \in \mathbb{N}$, we consider the natural $F^e_* R$-module structure on $\mathrm{Hom}_R(F^e_* R, R)$ given by pre-multiplication by elements of $F^e_* R$. The action of $F^e_* r$ on $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ produces the map[5] $\phi \star r \in \mathrm{Hom}_R(F^e_* R, R)$.

Theorem 2.1 (Fedder's Lemma, [**Fed83**])**.** *Let S be an F-finite regular ring, and let $R = S/I$. Then there is a natural $F^e_* S$-module map*

$$F^e_*\big(I^{[p^e]} : I\big) \cdot \mathrm{Hom}_S(F^e_* S, S) \longrightarrow \mathrm{Hom}_R(F^e_* R, R) \tag{2.1.1}$$

which is surjective with kernel $F^e_* I^{[p^e]} \cdot \mathrm{Hom}_S(F^e_* S, S)$.

The map (2.1.1) is the natural one discussed in Section 6 of Chapter 1—namely, the image of a map $F^e_* S \xrightarrow{\psi} S$ is the map on the quotient defined by

$$F^e_*(S/I) \xrightarrow{\overline{\psi}} S/I \qquad F^e_*(x \bmod I) \longmapsto \psi(x) \bmod I.$$

The theorem asserts, in particular, that all maps ψ in $F^e_*\big(I^{[p^e]} : I\big) \cdot \mathrm{Hom}_S(F^e_* S, S)$ are compatible with I in the sense of Definition 6.1 in Chapter 1 (that is, $\psi(F^e_* I) \subseteq I$).

Remark 2.2. We pause to clarify the notation $F^e_* J \cdot \mathrm{Hom}_S(F^e_* S, S)$ in Fedder's Lemma[6], as it has been known to cause some confusion. In this chapter, for an arbitrary ring R of positive characteristic and ideal $\mathfrak{b} \subseteq R$, we will always use the notation

$$F^e_* \mathfrak{b} \cdot \mathrm{Hom}_R(F^e_* R, R),$$

to denote the $F^e_* R$-submodule of $\mathrm{Hom}_R(F^e_* R, R)$ generated by

$$\{\phi \circ F^e_* b \mid \forall F^e_* b \in F^e_* \mathfrak{b}, \quad \forall \phi \in \mathrm{Hom}_R(F^e_* R, R)\}.$$

The $\cdot$ alerts us that we are working with the $F^e_* R$ module structure (and not the R module structure). This module should not be confused with $\mathfrak{b}\, \mathrm{Hom}_R(F^e_* R, R)$, which refers to the R-module structure on $\mathrm{Hom}_R(F^e_* R, R)$,

[5] $\phi \star r := \phi \circ F^e_* r$, Subsection 4.2 in Chapter 1.

[6] which appears twice, once for $J = (I^{[p^e]} : I)$ and once for $J = I^{[p^e]}$

nor with $F_*(\mathfrak{b}\operatorname{Hom}_R(F^e_*R, R))$, which is its image under the functor F_*. Yet another potential notation for $F^e_*\mathfrak{b}\cdot\operatorname{Hom}_R(F^e_*R, R)$ is $\operatorname{Hom}_R(F^e_*R, R)\star\mathfrak{b}$, using the operation $\star$ in the Cartier algebra; see Subsection 4.2 in Chapter 1.

2.2. The Proofs. When S is an F-finite polynomial ring, we saw in Proposition 1.3 of Chapter 2 that there is some $\Phi^e \in \operatorname{Hom}_S(F^e_*S, S)$ such that the natural map

$$F^e_*S \longrightarrow \operatorname{Hom}_S(F^e_*S, S) \qquad F^e_*s \mapsto \Phi^e \star s$$

is an *isomorphism* of F^e_*S-modules. Recall that such a map Φ^e is called a **generating map** (see Definition 1.5). The existence of such Φ^e holds beyond the polynomial ring case, as we will recall below.

We recall that we already studied this. Indeed for any (quasi-)Gorenstein (for instance regular) F-finite local ring S, we have that we have an isomorphism of F^e_*S-modules:

$$\operatorname{Hom}_S(F^e_*S, S) \cong F^e_*S. \tag{2.2.1}$$

In other words, $\operatorname{Hom}_S(F^e_*S, S)$ is a free of rank-1 as an F^e_*S-module. See Chapter 2 Corollary 3.16. Hence, there exists $\Phi^e \in \operatorname{Hom}_S(F^e_*S, S)$ which generates the Hom-set as an F^e_*S-module, a *generating map.* Note the e is a simple index here (however, in many cases it can be treated as a composition, see Appendix A Proposition 5.3). In what follows, for most cases of interest, it is easy to reduce to the case that $\operatorname{Hom}_S(F^e_*S, S) = \Phi^e \cdot F^e_*S$.

In addition to generating maps, we also need the following lemma:

Lemma 2.3. *Let S be an F-finite regular ring. Let I and J be arbitrary ideals of S. Then*

$$\phi(F^e_*J) \subseteq I \quad \textit{for all } \phi \in \operatorname{Hom}_S(F^e_*S, S) \tag{2.3.1}$$

if and only if $J \subseteq I^{[p^e]}$. In particular, if Φ^e generates $\operatorname{Hom}_S(F^e_*S, S)$ as an F^e_*S-module, then

$$J \subseteq I^{[p^e]} \quad \textit{if and only if} \quad \Phi^e(F^e_*J) \subseteq I. \tag{2.3.2}$$

Proof of Lemma 2.3. First note that if Φ^e generates $\operatorname{Hom}_S(F^e_*S, S)$ as an F^e_*S-module, then regardless of whether or not S is regular or F-finite,

$$\phi(F^e_*J) \subseteq I \;\; \forall \phi \in \operatorname{Hom}_S(F^e_*S, S) \quad \text{if and only if} \quad \Phi^e(F^e_*J) \subseteq I.$$

So the second statement follows from the first.

To prove (2.3.1), first assume $J \subseteq I^{[p^e]}$. So $F^e_*J \subseteq F^e_*I^{[p^e]} = IF^e_*S$. Now applying any $\phi \in \operatorname{Hom}_S(F^e_*S, S)$, we see that

$$\phi(F^e_*J) \subseteq \phi(IF^e_*S) \subseteq I,$$

whether or not S is regular and F-finite.

Conversely, assume that $\phi(F^e_* J) \subseteq I$ for all $\phi \in \operatorname{Hom}_S(F^e_* S, S)$. Then the same holds locally since the formation of the Hom-set commutes with localization, and because an inclusion of ideals can be checked locally, there is no loss of generality in assuming that S is local. So we may assume that $F^e_* S$ is free over S (by Kunz's theorem), and let

$$\{F^e_* e_1,\, F^e_* e_2,\, \ldots,\, F^e_* e_n\}$$

be a finite set of free generators, where each $e_i \in S$.

Take arbitrary $f \in J$. We want to show that $f \in I^{[p^e]}$. We can write

$$F^e_* f = a_1 F^e_* e_1 + \cdots + a_n F^e_* e_n$$

for some (unique) $a_i \in S$. Because $\phi(F^e_* J) \subseteq I$ for *all* $\phi \in \operatorname{Hom}_S(F^e_* S, S)$, we see that $\pi_i(F^e_* J) \subseteq I$ for the projections $\pi_i \in \operatorname{Hom}_S(F^e_* S, S)$ onto the free summands spanned by $F^e_* e_i$. Thus for each i,

$$a_i = a_1 \pi_i(F^e_* e_1) + \cdots + a_n \pi_i(F^e_* e_n) = \pi_i(F^e_* f) \in I.$$

This means that

$$F^e_* f = F^e_* a_1^{p^e} e_1 + \cdots + F^e_* a_n^{p^e} e_n \in F^e_* I^{[p^e]},$$

or in other words, $f \in I^{[p^e]}$. This proves that $J \subseteq I^{[p^e]}$. $\square$

Proof of Theorem 2.1. The existence of the map (2.1.1) is a consequence of the following claim.

Claim 2.4. *Every $\psi \in F^e_*\big(I^{[p^e]} : I\big) \cdot \operatorname{Hom}_S(F^e_* S, S)$ is compatible with the ideal I.*

Proof of claim. To check this claim, we verify that $\psi(F^e_* I) \subseteq I$. For this, we may assume that $\psi = \phi \star s$ for some $\phi \in \operatorname{Hom}_S(F^e_* S, S)$ and $s \in \big(I^{[p^e]} : I\big)$. Now since $sI \subseteq I^{[p^e]}$, it follows that

$$\psi(F^e_* I) = (\phi \circ F^e_* s)(F^e_* I) = \phi(F^e_*(sI)) \subseteq \phi(F^e_* I^{[p^e]}) \subseteq I\phi(F^e_* S) \subseteq I,$$

so that ψ is compatible with I, proving the claim. $\square$

We now have a well-defined map (2.1.1), which is clearly additive and $F^e_* S$-linear. It remains only to check it is surjective and has the desired kernel. Because the formation of the quotient S/I, the modules $\operatorname{Hom}_S(F^e_* S, S)$ and $\operatorname{Hom}_R(F^e_* R, R)$, and the ideal $(I^{[p^e]} : I)$ all commute with localization, there is no loss of generality in assuming that S is local.

To check the map (2.1.1) is surjective, we use the fact that $F^e_* S$ is a *projective* S-module by Kunz's Theorem (Corollary 2.2 in Chapter 1). For an arbitrary R-module map $F^e_*R \xrightarrow{\psi} R$, the projectivity of F^e_*S allows us to fill in the dotted arrow at the top of the commutative diagram of S-modules

$$\begin{tikzcd} F^e_*S \arrow[r, dotted, "\widetilde{\psi}"] \arrow[d, two heads] & S \arrow[d, two heads] \\ F^e_*R \arrow[r, "\psi"'] & R \end{tikzcd} \tag{2.4.1}$$

where the vertical arrows are the natural quotient maps by F^e_*I and I, respectively. The lifted map $\widetilde{\psi}$ is compatible with I by construction; we must show that $\widetilde{\psi}$ lies in the submodule $F^e_*\big(I^{[p^e]} : I\big) \cdot \operatorname{Hom}_S(F^e_*S, S)$. For this, fix an F^e_*S-module generator Φ^e for $\operatorname{Hom}_S(F^e_*S, S)$ and write

$$\widetilde{\psi} = \Phi^e \circ F^e_*y$$

for some $y \in S$. It now suffices to show that $y \in (I^{[p^e]} : I)$. The compatibility of $\widetilde{\psi}$ and I implies that

$$\widetilde{\psi}(F^e_*I) = (\Phi^e \star y)(F^e_*I) = \Phi^e(F^e_*(yI)) \subseteq I.$$

So by Lemma 2.3, we can conclude that $yI \subseteq I^{[p^e]}$, or equivalently, that $y \in \big(I^{[p^e]} : I\big)$, as needed. We have shown that, when S is regular, the map (2.1.1) is surjective.

Finally, we consider the kernel of (2.1.1). Take an arbitrary

$$\phi = \Phi^e \star y \in F^e_*\big(I^{[p^e]} : I\big) \cdot \operatorname{Hom}_S(F^e_*S, S)$$

in the kernel (2.1.1). Since ϕ induces the zero map on F^e_*R, the image of ϕ is

$$\phi(F^e_*S) = (\Phi^e \star y)(F^e_*S) = \Phi^e(F^e - *yS) \subseteq I.$$

In particular, $\Phi^e(F^e_*y) \in I$. Thus $y \in I^{[p^e]}$ by Lemma 2.3, showing the kernel is $(F^e_*I^{[p^e]}) \operatorname{Hom}_S(F^e_*S, S)$. The theorem is proved. $\square$

We are ready to deduce Fedder's Criterion from Theorem 2.1:

Proof of Theorem 1.1. Recall that R is Frobenius split if and only if the iterated Frobenius map F^e splits for some (equivalently, every) $e \in \mathbb{N}$ (Proposition 3.9 in Chapter 1). Thus it suffices to prove, for fixed $e \in \mathbb{N}$, that F^e splits if and only if $(I^{[p^e]} : I) \not\subseteq \mathfrak{m}^{[p^e]}$.

The map $F^e : R \longrightarrow F^e_*R$ splits if and only if there is $\phi \in \operatorname{Hom}_R(F^e_*R, R)$ such that $\phi(F^e_*1_R)$ is a unit in R. Such a splitting ϕ would be induced by some $\Phi^e \circ F^e_*y \in \operatorname{Hom}_S(F^e_*S, S)$, where $y \in (I^{[p^e]} : I)$ by Theorem 2.1. Therefore, F^e splits if and only if there exists some $y \in (I^{[p^e]} : I)$ such that

$$(\Phi^e \circ F^e_*y)(F^e_*1_S) = \Phi^e(F^e_*y) \notin \mathfrak{m},$$

where Φ^e is an F^e_*S-module generator for $\operatorname{Hom}_S(F^e_*S, S)$. By Lemma 2.3, we conclude that R is Frobenius split if and only if $(I^{[p^e]} : I) \not\subseteq \mathfrak{m}^{[p^e]}$. ☐

Remark 2.5. Theorem 2.1 is false in general for non-regular S, even if S has very mild singularities; see Example 2.12 and Chapter 5 Example 4.4. On the other hand, a generalization to the case where S is Gorenstein and I has finite projective dimension can be found in [**GMS20**, Section 3].

Glassbrenner's criterion is proved similarly, so is left as Exercise 2.4.

2.3. Computing the non-Frobenius split locus. Our proof of Fedder's Lemma suggests an explicit defining ideal for the (closed) loci of non-Frobenius split and non-strongly F-regular points. In practice, this ideal can be computed by machine quite easily.

Corollary 2.6. *Let S be an F-finite regular ring and assume that $\operatorname{Hom}_S(F^e_*S, S)$ is generated by Φ^e as an F^e_*S-module.*

Let $R = S/I$, and consider $\operatorname{Spec} R$ as the closed set $\mathbb{V}(I)$ of $\operatorname{Spec} S$. Then

(a) *For any fixed e, the closed locus of non-Frobenius split points of $\operatorname{Spec} R$ is defined by the ideal $\Phi^e(F^e_*(I^{[p^e]} : I)) \subseteq S$.*
(b) *The closed locus of non-strongly F-regular points of $\operatorname{Spec} R$ is defined by the ideal $\Phi^e(F^e_*c(I^{[p^e]} : I)) \subseteq S$ for $e \gg 0$, where c is any element of S not in any minimal prime of I such that S_c/IS_c is strongly F-regular.*

The ideals described in (a) and (b) contain I, so they cut out closed sets of $\mathbb{V}(I) = \operatorname{Spec} R$.

Proof. Left as Exercise 2.7. ☐

Remark 2.7. While we already saw the closedness of these loci in Chapter 1, Corollary 2.6 is important in practice, because computing the image of an ideal F^e_*J under Φ^e is straightforward, for example with a computer, when S is a polynomial ring. See Exercises 2.1 and 2.2.

Remark 2.8. The ideals $\Phi^e(F^e_*(I^{[p^e]} : I))$ and $\Phi^e(F^e_*c(I^{[p^e]} : I))$ in Corollary 2.6 depend on the choice of e (and c), although the loci they define do not, as they coincide with non-Frobenius split locus and non-strongly F-regular locus, respectively. Up to radical, the latter ideal is the test ideal $\tau(R)$ (when R is F-finite and reduced).

Question 2.9 (Open question)**.** The ideals $\Phi^e(F^e_*(I^{[p^e]} : I))$ defined in Corollary 2.6 form a descending chain of ideals in R (see Exercise 2.6). Does this chain stabilize? This is known to be true if R is a hypersurface or more generally $\mathbb{Q}$-Gorenstein (see [**CEMS18**]). Those results are a corollary of a key step in a famous result of Hartshorne and Speiser [**HS77**], generalized in certain ways by Lyubeznik [**Lyu97**] and also independently proven in greater generality by Gabber [**Gab04**]. We prove a version of this result in Theorem 6.18 later in this chapter, see also Exercise 6.11. This result, and its applications, also plays a staring role later in Chapter 8 Section 2.

2.4. The submodule of compatible maps. We isolate an interesting fact proved about compatibility of ideals and maps from our proof of Theorem 2.1:

Corollary 2.10. *Suppose S is an F-finite regular ring and $I \subseteq R$ is an ideal. Then the set of maps in $\operatorname{Hom}_S(F^e_* S, S)$ that are compatible with I is exactly*

$$\Big(F^e_*\big(I^{[p^e]} : I\big)\Big) \cdot \operatorname{Hom}_S(F^e_* S, S).$$

More generally, fixing any ring R of prime characteristic and ideal J of R, we might try to identify the maps in $\operatorname{Hom}_R(F^e_* R, R)$ compatible with J. Such maps form an $F^e_* R$ submodule

$$\mathcal{M}_J = \{\phi \mid \phi(F^e_* J) \subseteq J\} \subseteq \operatorname{Hom}_R(F^e_* R, R),$$

and can be interpreted as the maps which descend to maps in $\operatorname{Hom}_T(F^e_* T, T)$, where $T = R/J$. An interesting problem is to try to understand when the natural homomorphism

$$\mathcal{M}_J \longrightarrow \operatorname{Hom}_T(F^e_* T, T) \qquad \phi \mapsto \overline{\phi} \tag{2.10.1}$$

is *surjective*. For example, the following consequence of surjectivity is immediate:

Proposition 2.11. *Let R be an arbitrary ring of characteristic $p > 0$ and $T = R/J$ an arbitrary quotient. If the natural map (2.10.1) is surjective and T is Frobenius split, then also R is Frobenius split.*

We just showed that Fedder's Lemma guarantees the map (2.10.1) is *always* surjective when R is regular. When $J = (f)$ is principal R is quasi-Gorenstein, and T is S_2 (for instance if R is Gorenstein) the map (2.10.1) is also surjective as you will show in Exercise 2.15 (one can see this by analyzing the diagram (5.20.2) in Chapter 2).

However, (2.10.1) is not surjective in general:

Example 2.12. Let $R = \mathbb{F}_p[x, y, z]/I = S/I$ where $I = (x^2 - yz)$, and consider the ideal $J/I \subseteq R$ where $J = (x, y)$. The map (2.10.1) is *not* surjective in this case. Indeed, working in the polynomial ring $\mathbb{F}_p[x, y, z]$, we compute

$$I^{[p]} : I = ((x^2 - yz)^{p-1}) \qquad \text{and} \qquad J^{[p]} : J = ((xy)^{p-1}, x^p, y^p).$$

So the element $(xy)^{p-1}$ is in $(J^{[p]} : J)$ but not $(I^{[p]} : I)$. This means that the map $\phi \in \mathrm{Hom}_T(F_* T, T)$ (where $T = R/J$) induced by the map[7] $\Phi \star (xy)^{p-1}$ does not lift to any map in $\mathrm{Hom}_R(F_* R, R)$. The example is studied in more detail in Chapter 5 Example 4.4.

For more discussions on quantifying the failure of surjectivity for height one primes, see Chapter 6 Section 6 and [**Das15**].

We can give a nice description of the submodule $\mathcal{M}_J$ of compatible maps for quotients of regular rings using Fedder's Lemma:

Corollary 2.13. *Let $R = S/I$ be a quotient of an F-finite regular ring S. For an arbitrary ideal $\overline{J} = J/I$ of R, the submodule*

$$\mathcal{M}_{\overline{J}} := \{\phi \mid \phi(F_* \overline{J}) \subseteq \overline{J}\} \subseteq \mathrm{Hom}_R(F^e_* R, R)$$

of all ϕ compatible with $\overline{J}$ is the image of the natural map

$$F^e_* \big((I^{[p^e]} : I) \cap (J^{[p^e]} : J) \big) \cdot \mathrm{Hom}_S(F^e_* S, S) \longrightarrow \mathrm{Hom}_R(F^e_* R, R)$$

obtained by sending ϕ in the source to the map $\overline{\phi} \in \mathrm{Hom}_R(F^e_* R, R)$ defined by $\overline{\phi}(F^e_*(x \text{ mod } I)) = \phi(F^e_* x) \text{ mod } I$.

Proof. This follows from Theorem 2.1 by simply unravelling notation. □

Corollary 2.14. *With notation as in Corollary 2.13, $\overline{J}$ is compatible with every map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ if and only if*

$$(I^{[p^e]} : I) \subseteq (J^{[p^e]} : J). \tag{2.14.1}$$

Equivalently, the inclusion (2.14.1) holds if and only if there is a naturally induced map $\mathrm{Hom}_R(F^e_* R, R) \to \mathrm{Hom}_T(F^e_* T, T)$ where $T = R/J$.

Taking J to be the maximal ideal of a regular local ring $(S, \mathfrak{m})$, we can use Corollary 2.14 to show certain rings are *not* strongly F-regular:

Example 2.15. Consider the ideals $I = (xy) \subseteq J = (x, y)$ in the polynomial ring $\mathbb{F}_p[x, y]$. Then

$$I^{[p^e]} : I = ((xy)^{p^e - 1}) \qquad \text{and} \qquad J^{[p^e]} : J = ((xy)^{p^e-1}, x^{p^e}, y^{p^e}),$$

in $\mathbb{F}_p[x, y]$, so that

$$(I^{[p^e]} : I) \subseteq (J^{[p^e]} : J).$$

Setting $R = \mathbb{F}_p[x, y]/(xy)$, then Corollary 2.14 implies that every map $F^e_* R \longrightarrow R$ sends the maximal ideal (x, y) back into itself. In particular, R can not be strongly F-regular: no map in $\mathrm{Hom}_R(F^e_* R, R)$ can send $F^e_* c$ to 1 for $c \in (x, y)$.

[7]Here, Φ denotes a generating map for $\mathrm{Hom}_S(F_* S, S)$ where S is the polynomial ring $\mathbb{F}_p[x, y, z]$.

We conclude with some examples.

Example 2.16 (F-injective but not F-split)**.** Suppose that $S = \mathbb{F}_3[\![y, z, u, v]\!]$, consider the ideal

$$I = (uv, zu, y^4z - zv)$$

and set $R = S/I$. In Chapter 4 Exercise 2.14 we saw that $R/(\overline{y})$ is Frobenius split and hence F-injective. Since $\overline{y}$ is a regular element of R and $R/(\overline{y})$ is Cohen-Macaulay (since it is 1-dimensional and reduced), we see that R is also Cohen-Macaulay. Thus R is F-injective. But now by Exercise 2.14 we see that R is not F-pure (use Fedder's criterion).

Example 2.17 (F-rational but not F-split)**.** If $S = \mathbb{F}_{11}[a, b, c, d, t]$ with

$$I = (a^2t^5 + a^4 - bc,\ b^2t^5 - dt^5 + a^2b^2 - a^2d - cd,\ b^3 - a^2d - bd)$$

and set $R = S/I$. Since $R/(t)$ is F-regular (by Glassbrenner's criterion), we see that R is F-rational by Chapter 1 Exercise 7.19.

2.5. Exercises.

Exercise 2.1. Suppose S is an F-finite regular ring and F^e_*S has an S-basis $F^e_*b_1, \ldots, F^e_*b_m$. Further suppose that $\Phi^e \in \operatorname{Hom}_S(F^e_*S, S)$ is a generating map. Suppose $f \in S$ and write

$$F^e_*f = \sum_{i=1}^m a_i F^e_* b_i = F^e_* \sum_{i=1}^m a_i^{p^e} b_i.$$

Then

$$\Phi^e(F^e_* fS) = (a_1, \ldots, a_m).$$

Exercise 2.2. Suppose $S = \mathbb{F}_3[x, y]$ and let $f = yx^3 + x^2y^4 - y^{12}$. Prove that $\Phi(F_*fS) = (x, y, y^4) = (x, y)$.

Hint: Write $F_*f = xF_*y + yF_*x^2y + y^4F_*1$, and apply Exercise 2.1.

Exercise 2.3. Let S be an F-finite regular ring, and $R = S/I$ some proper quotient. Show that R is e-Frobenius split along $c \in R$ if and only if the "evaluation at c map"

$$\begin{array}{rcl} F^e_*(I^{[p^e]} : I) \cdot \operatorname{Hom}_S(F^e_*S, S) & \longrightarrow & S \\ \phi & \mapsto & \phi(F^e_*c) \end{array} \tag{2.17.1}$$

is surjective.

Exercise 2.4. Prove Glassbrenner's theorem, Theorem 1.8, and the remark following it about testing along just one c.

Hint: Use Exercise 2.3 and Lemma 2.3.

Exercise 2.5. Let S be an F-finite regular ring and let I be an ideal in S. Assume that $\Phi^e \in \mathrm{Hom}_S(F^e_* S, S)$ is a generator as an $F^e_* S$-module. Show that for any non-zerodivisor $c \in S$,

$$I \subseteq \Phi^e\left(F^e_* c(I^{[p^e]} : I)\right) \tag{2.17.2}$$

for all $e \gg 0$. Furthermore, if c is a unit, prove (2.17.2) for all $e > 0$.

Hint: For $e \gg 0$, find $d \in S$ so that $\Phi^e(F^e_* cd) = 1$. Now given $y \in I$, apply $\Phi^e \circ F^e_* y^{p^e} c$ to the element $F^e_* d$.

Exercise 2.6. With notation as in the previous exercise, show that

$$\Phi^e\big(F^e_*(I^{[p^e]} : I)\big) \supseteq \Phi^{e+1}(F^{e+1}_*(I^{[p^{e+1}]} : I)).$$

Hint: Let $R = S/I$ and use the fact that $\Phi^e(F^e_*(I^{[p^e]} : I)) \cdot R$ is the same as the image of the "evaluation at $F^e_* 1$" map $\mathrm{Hom}_R(F^e_* R, R) \longrightarrow R$.

Exercise 2.7. Prove Corollary 2.6.

Hint: Study the proof of Theorem 5.12 in Chapter 1 in light of Theorem 2.1.

Exercise 2.8. Suppose $(R, \mathfrak{m})$ is an F-finite DVR with $\mathfrak{m} = (t)$. Show that a map $\phi : F^e_* R \longrightarrow R$ generates $\mathrm{Hom}_R(F^e_* R, R)$ if and only if $\phi(F^e_*(t^{p^e-1})) = R$.

Exercise 2.9. More generally, suppose $(R, \mathfrak{m})$ is an F-finite d-dimensional regular local ring with regular system of parameters $x_1, \dots, x_d$ (that is, $(x_1, \dots, x_d) = \mathfrak{m}$). Show that $\phi : F^e_* R \longrightarrow R$ generates $\mathrm{Hom}_R(F^e_* R, R)$ as an $F^e_* R$-module if and only if $\phi(F^e_*(x_1^{p^e-1} \cdots x_d^{p^e-1})) = R$.

Hint: Complete at the maximal ideal and use the Cohen-Structure Theorem. A map will generate that Hom-set if and only if it generates after completion.

Exercise 2.10. Consider the ring $R = \mathbb{F}_3[x, y]/(x^2 + y^5)$. Let J_e be the image of "evaluation at $F^e_* 1$" map $\mathrm{Hom}_R(F^e_* R, R) \longrightarrow R$. Compute the ideals J_1 and J_2, and show they are not equal, but do define the same locus.

Exercise 2.11. Suppose that S is an F-finite regular ring and let I and J be arbitrary ideals of S. Show that

$$\mathrm{Image}\left((F^e_* J) \cdot \mathrm{Hom}(F^e_* S, S) \xrightarrow{\text{eval at } F^e_* 1} S\right) \subseteq I \iff J \subseteq I^{[p^e]}.$$

This gives an alternate interpretation of Lemma 2.3.

Exercise 2.12. Prove that the ring $\frac{\mathbb{F}_p[x,y,z]}{(x^3+y^3+z^3)}$ is not Frobenius split along *any* element of $\mathfrak{m} = (x, y, z)$ (a non-unit that vanishes at the origin). Prove that it is Frobenius split along every unit if and only if $p \equiv 1 \pmod 3$.

Exercise 2.13. Let k be an F-finite field, and let f be a homogeneous polynomial of degree d in the local ring $S = k[x_1, \dots, x_n]_{\mathfrak{m}}$ where $\mathfrak{m} = (x_1, x_2, \dots, x_n)$. Show that if $f^{p^e-1} \in (x_1^{p^e}, x_2^{p^e}, \dots, x_n^{p^e}, (x_1 x_2 \cdots x_n)^{p^e-1})$, then $S/(f)$ is not strongly F-regular.

Hint: See Example 2.15.

Exercise 2.14. Suppose that $S = \mathbb{F}_3[\![y, z, u, v]\!]$, consider the ideal

$$I = (uv, zu, y^4 z - zv)$$

Prove that $R = S/I$ is not F-split even though $R/(\overline{y})$ is F-split.

Hint: A program like Macaulay2 can help compute $I^{[3]} : I$.

Exercise 2.15. Suppose that R is an F-finite quasi-Gorenstein local ring and $f \in R$ is a regular element such that $R/(f)$ is also S_2, for instance normal. Show there is a natural map

$$(F^e_* f^{p^e-1}) \cdot \mathrm{Hom}_R(F^e_* R, R) \longrightarrow \mathrm{Hom}_{R/(f)}(F^e_* R/(f), R/(f)) \tag{2.17.3}$$

which is surjective.

Exercise 2.16. Suppose that R is an F-finite quasi-Gorenstein local ring and $f \in R$ is a regular element. Then if the quotient $R/(f)$ is S_2 and Frobenius split (respectively, strongly F-regular), then so is R. In fact (f) is compatibly Frobenius split in R.

Exercise 2.17. Let S be an F-finite regular local ring, and suppose $R = S/I$ is some Gorenstein quotient. Let $f \in S$ be such that its image in R is a regular element. Prove that

$$(I^{[p]} + (f^p)) : (I + (f)) \subseteq f^{p-1}(I^{[p]} : I) + I^{[p]} + (f^p).$$

Hint: Use the ideas behind Corollary 2.13 and Chapter 2 Corollary 3.16, see also Lemma 5.20 in Chapter 2.

Exercise 2.18. Let $R = \mathbb{F}_p[x, y]/(x, y)^n$ for some $n > 0$. Explain why c is a strong test element if and only c is a height test element in the sense of Chapter 1 Exercise 5.17. Show that the ideal of all strong test elements is $(x, y)^{n-1}$.

Exercise 2.19. Let $R = \mathbb{F}_p[x, y]/(xy, x^n)$ for some $n > 1$. Show that the ideal of all strong test elements is (x^{n-1}, y). Show also that the ideal of all height test elements is (x^{n-1}, y) (as defined in Exercise 5.17).

Hint: Fedder's Lemma is useful for understanding $\mathrm{Hom}_R(F^e_* R, R)$.

Exercise 2.20 (An algorithm to compute $\tau(\omega_R)$)**.** Suppose that S is a polynomial ring over an F-finite ring (or a regular local ring) and write $R = S/I$ for some ideal I. Assume R is S_2. Choose $J \supseteq I$ an ideal such that $\overline{J} = J/I \cong \omega_R$. Consider $T^e : F^e_* \omega_R \longrightarrow \omega_R$ and view this as a map $T^e_{\overline{J}} : F^e_* \overline{J} \longrightarrow \overline{J}$, see Chapter 2.

(a) Show that $T^e_{\overline{J}} : F^e_* \overline{J} \longrightarrow \overline{J}$ extends to a $\overline{\phi} : F^e_* R \longrightarrow R$ by applying the functor $\mathrm{Hom}_R(-, \overline{J})$ to $T^e_{\overline{J}}$.

(b) Let $\phi : F^e_* S \longrightarrow S$ be a map inducing $\overline{\phi}$, which exists by Chapter 4 Theorem 2.1. Show that $\phi(F^e_* J) \subseteq J$.

(c) Show that
$$\frac{(J^{[p^e]} : J) \cap (I^{[p^e]} : I)}{I^{[p^e]}} \cong \mathrm{Hom}_R(F^e_* \omega_R, \omega_R)$$
and hence we may represent T^e as an element of the numerator of the left side.

(d) Let c be a strong test element for R and consider the submodules
$$M_m := \sum_{e=0}^{m} T^e(F^e_* c\, \omega_R)$$
whose sum is $\tau(\omega_R)$ by Proposition 5.6. Show that
$$T(F_* M_m) + c\,\omega_R = M_{m+1}$$
and use this to deduce that if $M_m = M_{m+1}$, then $M_m = \tau(\omega_R)$.

(e) Use what you've done in this exercise to describe an algorithm to compute $\tau(\omega_R)$.

This algorithm essentially appears in [**Kat08**].

3. F-pure thresholds of hypersurfaces

The *F-pure threshold* of a hypersurface is a numerical measurement of "how far" a characteristic p singularity is from being Frobenius split at a given point.[8]

The F-pure threshold is an analog of the log canonical threshold, a complex singularity invariant. The F-pure threshold is more subtle, however, as it accounts for some of the ways singularities can be "worse" in prime characteristic. These connections with complex singularity invariants will be discussed in Chapter 6.

[8]The *F-pure threshold* could just as well be called the "Frobenius splitting threshold", but the term *F-pure threshold* is universal throughout the literature. Recall that for F-finite rings, Frobenius splitting is equivalent to F-purity; see Chapter 1 Proposition 7.20.

3.1. The Definition. Let $(S,\mathfrak{m})$ denote an F-finite regular local ring, and fix some non-zero $f \in \mathfrak{m}$. We are interested in measuring how far the hypersurface $S/(f)$ is from being Frobenius split at $\mathfrak{m}$.

Because $f \in \mathfrak{m}$, of course also $f^{p^e} \in \mathfrak{m}^{[p^e]}$ for all $e \in \mathbb{N}$. Thus, in general, the *largest possible* integer ν such that $f^\nu \notin \mathfrak{m}^{[p^e]}$ is $\nu = p^e - 1$; by Fedder's criterion, this happens if and only if $S/(f)$ is Frobenius split. For "more singular" hypersurfaces, we might expect the largest ν with $f^\nu \notin \mathfrak{m}^{[p^e]}$ to be much smaller. This naturally motivates the following:

Definition 3.1 (**[TW04, MTW05]**)**.** Let $(S,\mathfrak{m})$ be a regular local ring of characteristic $p > 0$. For any non-zero $f \in \mathfrak{m}$, the **F-pure threshold** of f (at $\mathfrak{m}$) is the supremum of the set

$$\left\{ \frac{\nu}{p^e} \in \mathbb{Q}_{\geq 0} \mid f^\nu \notin \mathfrak{m}^{[p^e]} \right\}. \tag{3.1.1}$$

We denote the F-pure threshold by $\operatorname{fpt}_{\mathfrak{m}}(f)$.

It is immediate from the definition that $\operatorname{fpt}_{\mathfrak{m}}(f)$ exists and is bounded above by one. When the hypersurface defined by f is Frobenius split, Fedder's Criterion implies that the set (3.1.1) includes all rational numbers of the form $\frac{p^e-1}{p^e}$, so its F-pure threshold is precisely 1. In fact, the F-pure threshold being one characterizes Frobenius splitting:

Theorem 3.2. *Let $(S,\mathfrak{m})$ is an F-finite regular local ring, and let f be any non-zero element of $\mathfrak{m}$. Then the quotient $S/(f)$ is Frobenius split if and only if* $\operatorname{fpt}_{\mathfrak{m}}(f) = 1$.

Theorem 3.2 follows easily from the following technical lemma:

Lemma 3.3. *Let $f \in \mathfrak{m}$ be a non-zero element in a regular local ring $(S,\mathfrak{m})$. Let $\nu(e) \in \mathbb{N}$ be maximal such that $f^{\nu(e)} \notin \mathfrak{m}^{[p^e]}$. Then*

$$\frac{\nu(e)}{p^e} \leq \operatorname{fpt}_{\mathfrak{m}}(f) \leq \frac{\nu(e)+1}{p^e} \tag{3.3.1}$$

for all $e \in \mathbb{N}$. In particular, the limit

$$\lim_{e \longrightarrow \infty} \frac{\nu(e)}{p^e} \tag{3.3.2}$$

exists and is equal to $\operatorname{fpt}_{\mathfrak{m}}(f)$.

Proof. We repeatedly use the fact that, for any ideal $J \subseteq S$ and any $e \in \mathbb{N}$, we have $z^{p^e} \in J^{[p^e]}$ if and only if $z \in J$ (Lemma 3.11 in Chapter 1). Using this fact, we see that whether or not a rational number $\frac{\nu}{p^e}$ is in the set (3.1.1) is independent of what power of p we use to represent the denominator.

First, we claim that the sequence $\left\{\frac{\nu(e)}{p^e}\right\}$ is a non-decreasing sequence, bounded above by $\mathrm{fpt}(f)$. Indeed, given e, we have $f^{\nu(e)} \notin \mathfrak{m}^{[p^e]}$, so raising to the p-th power, also $f^{p\nu(e)} \notin \mathfrak{m}^{[p^{e+1}]}$. By definition of $\nu(e+1)$, this says that $p\nu(e) \leq \nu(e+1)$. Dividing by p^{e+1}, we have $\frac{\nu(e)}{p^e} \leq \frac{\nu(e+1)}{p^{e+1}}$, so that the sequence is non-decreasing, as claimed. Furthermore, because each $\frac{\nu(e)}{p^e}$ is in the set $\left\{\frac{\nu}{p^e} \in \mathbb{Q}_{\geq 0} \mid f^\nu \notin \mathfrak{m}^{[p^e]}\right\}$, clearly the sequence $\{\frac{\nu(e)}{p^e}\}$ is bounded above by the supremum, $\mathrm{fpt}_{\mathfrak{m}}(f)$, and so the sequence $\{\frac{\nu(e)}{p^e}\}$ has a limit.

Likewise, the sequence $\{\frac{\nu(e)+1}{p^e}\}$ is a non-increasing sequence. Indeed, $f^{\nu(e)+1} \in \mathfrak{m}^{[p^e]}$ implies that $f^{p(\nu(e)+1)} \in \mathfrak{m}^{[p^{e+1}]}$, so by definition of $\nu(e+1)$, we see $p(\nu(e)+1) \geq \nu(e+1)+1$. Dividing by p^{e+1}, we have

$$\frac{\nu(e)+1}{p^e} \geq \frac{\nu(e+1)+1}{p^{e+1}},$$

which shows the sequence is $\{\frac{\nu(e)+1}{p^e}\}$ non-increasing.

Since

$$\lim_{e \longrightarrow \infty} \frac{\nu(e)+1}{p^e} = \lim_{e \longrightarrow \infty} \left(\frac{\nu(e)}{p^e} + \frac{1}{p^e}\right) = \lim_{e \longrightarrow \infty} \frac{\nu(e)}{p^e}$$

it immediately follows that

$$\sup\left\{\frac{\nu(e)}{p^e}\right\} = \lim_{e \longrightarrow \infty} \frac{\nu(e)}{p^e} = \lim_{e \longrightarrow \infty} \frac{\nu(e)+1}{p^e} = \inf\left\{\frac{\nu(e)+1}{p^e}\right\}.$$

The result follows. □

Example 3.4. Suppose that $S/(f)$ is a *non-Frobenius split* hypersurface of characteristic p, where S is a regular local ring. In this case, Fedder's Criterion tells us that $f^{p-1} \notin \mathfrak{m}^{[p]}$, so $\nu(1) \leq p-2$ (with $\nu(e)$ as defined in Lemma 3.3). Therefore, Lemma 3.3 produces the bound

$$\mathrm{fpt}_{\mathfrak{m}}(f) \leq \frac{\nu(1)+1}{p} \leq \frac{p-2+1}{p} = 1 - \frac{1}{p}.$$

This bound is *sharp* in general. For example, the F-pure threshold of the homogenous cubic defining a supersingular elliptic plane curve is precisely $1 - \frac{1}{p}$; see Example 3.11 and **[BS15]**. In particular, for such cubics, the non-increasing sequence $\frac{\nu(e)+1}{p^e}$ is constant.

Another useful description of the F-pure threshold follows easily from Lemma 3.3:

Corollary 3.5. *Let $f \in \mathfrak{m}$ be a non-zero element of regular local ring $(S, \mathfrak{m})$. Then the F-pure threshold of f at $\mathfrak{m}$ is*

$$\operatorname{fpt}_{\mathfrak{m}}(f) = \inf \left\{ \frac{a}{p^e} \mid f^a \in \mathfrak{m}^{[p^e]} \right\}.$$

Example 3.6. Suppose that $f = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in $k[x_1, \ldots, x_n]$, where k has characteristic p. Let $\mathfrak{m} = (x_1, \ldots, x_n)$. Then

$$\operatorname{fpt}_{\mathfrak{m}}(f) = \min \left\{ \frac{1}{a_i} \mid i = 1, 2, \ldots, n \right\}.$$

Indeed, the containment

$$f^b = x_1^{ba_1} \cdots x_n^{ba_n} \in \mathfrak{m}^{[p^e]} = (x_1^{p^e}, \ldots, x_n^{p^e})$$

holds exactly when $ba_i \geq p^e$ for some i, or equivalently, when $\frac{b}{p^e} \geq \frac{1}{a_i}$ for some i. By Corollary 3.5, it follows that the F-pure threshold is

$$\inf \left\{ \frac{b}{p^e} \mid \frac{b}{p^e} \geq \frac{1}{a_i} \quad i = 1, 2, \ldots, n \right\}.$$

Finally, since each fraction $\frac{1}{a_i}$ can be approximated arbitrarily closely from above by rational numbers of the form $\frac{b}{p^e}$—for example, by rounding up base p truncations—we conclude that $\operatorname{fpt}_{\mathfrak{m}}(f) = \min \left\{ \frac{1}{a_i} \mid i = 1, 2, \ldots, n \right\}$.

Remark 3.7. While not obvious, the F-pure threshold $\operatorname{fpt}_{\mathfrak{m}}(f)$ of any non-zero element in a regular local ring is *rational*, see Theorem 6.14.

3.2. Computing the F-pure threshold. Even very simple polynomials can have F-pure thresholds that depend on the characteristic in non-obvious ways, as the next example shows:

Example 3.8. Consider the cubic $f = x^2y - xy^2 \in k[x, y]$, defining a configuration of three lines in the plane meeting at the origin. We compute the F-pure threshold of this line arrangement at the point of concurrency $\mathfrak{m} = (x, y)$.

<u>**Characteristic** $p = 2:$</u> In this case, $\operatorname{fpt}_{\mathfrak{m}}(f) = 1/2$. Indeed, notice $f^{p^e/2} = f^{p^{e-1}} = x^{p^e} y^{p^{e-1}} + x^{p^{e-1}} y^{p^e} \in \mathfrak{m}^{[p^e]}$. So the F-pure threshold is at most $\frac{p^{e-1}}{p^e} = \frac{1}{2}$. On the other hand, $f^{\frac{p^e}{2} - 1}$ has a term

$$(x^2)^{\frac{p^e}{2} - 1} y^{\frac{p^e}{2} - 1} = x^{p^e - 2} y^{\frac{p^e}{2} - 1} \notin \mathfrak{m}^{[p^e]}.$$

So the F-pure threshold is bounded below by the supremum of $\frac{\frac{p^e}{2} - 1}{p^e} = \frac{1}{2} - \frac{1}{p^e}$, which is $\frac{1}{2}$. So indeed, $\operatorname{fpt}(f) = 1/2$.

<u>**Characteristic** $p = 3:$</u> In this case, $\operatorname{fpt} = 2/3$. Notice that

$$f^{2p^e/3} = f^{2p^{e-1}} = (x^2y - xy^2)^{2p^{e-1}} = (x^4y^2 - 2x^3y^3 + x^2y^4)^{p^{e-1}},$$

which is contained in (x^{p^e}, y^{p^e}) since $p = 3$. Thus $\mathrm{fpt}_{\mathfrak{m}}(f) \le 2/3$. On the other hand,

$$f^{2(p^{e-1}-1)} = (x^4y^2 - 2x^3y^3 + x^2y^4)^{p^{e-1}-1},$$

which we leave to the reader to prove is not contained in (x^{p^e}, y^{p^e}); see Exercise 3.5. Thus the F-pure threshold is bounded below by the supremum of the rational numbers $\frac{2p^{e-1}-1}{p^e} = \frac{2}{3} - \frac{1}{p^e}$, which means that $\mathrm{fpt}_{\mathfrak{m}}(f) \ge 2/3$.

Characteristic $p \equiv 2 \pmod 3$: In this case, $\mathrm{fpt}(f) = \frac{2}{3} - \frac{1}{3p} = \frac{2p-1}{3p}$.

Before computing, note that $2p-1$ is divisible by 3. Furthermore, $2p-1$ is odd and hence so is $\frac{2p-1}{3}$.

Expanding out $(x^2y - xy^2)^{\frac{2p-1}{3}}$ using the binomial theorem, we see that all terms involve either x^p or y^p with the possible exception of the two "middle terms"

$$Nx^{2l_1+l_2}y^{l_1+2l_2} \;+\; Nx^{l_1+2l_2}y^{2l_1+l_2}$$

where $l_1 = \lfloor(2p-1)/6\rfloor$ and $l_2 = \frac{2p-1}{3} - l_1 = \lceil(2p-1)/6\rceil$, and N is the appropriate binomial coefficient. However, since

$$l_1 + 2l_2 = \lfloor(2p-1)/6\rfloor + 2\lceil(2p-1)/6\rceil = p$$

both these terms are in fact in (x^p, y^p). So $(x^2y - xy^2)^{\frac{2p-1}{3}} \in \mathfrak{m}^{[p]}$, which implies that $\mathrm{fpt}(f) \le \frac{2p-1}{3p}$.

For the other inequality, it suffices to show that

$$(x^2y - xy^2)^{\frac{(2p-1)p^{e-1}}{3}-1} \notin (x^{p^e}, y^{p^e}) \tag{3.8.1}$$

for $e \ge 2$. Set $m = \frac{2p-1}{3}$, and observe that $2 < m < p$. Then

$$mp^{e-1} - 1 = (m-1)p^{e-1} + (p-1)p^{e-2} + \cdots + (p-1)p^0,$$

via base-p arithmetic. It follows from Lucas' theorem (see Exercise 1.4) that every binomial coefficient of $f^{mp^{e-1}-1}$ is nonzero. Hence it is easy to see that (3.8.1) holds.

Characteristic $p \equiv 1 \pmod 3$: In this case, $\mathrm{fpt}_{\mathfrak{m}}(f) = 2/3$. We leave this final case as Exercise 3.6.

Example 3.9. Consider the plane cusp singularity defined by $f = y^2 - x^3 \in \mathbb{F}_p[x, y]$. Computing similarly to the Example 3.8, one computes:

$$\mathrm{fpt}_{\mathfrak{m}}(f) = \begin{cases} 1/2 & p = 2 \\ 2/3 & p = 3 \\ 5/6 - 1/(6p) & p \equiv 5 \pmod 6 \\ 5/6 & p \equiv 1 \pmod 6 \end{cases}$$

Adopting the philosophy that *smaller* F-pure thresholds indicate *worse* singularities, we see that the cusp is "most singular" in characteristic two, a statement we can confirm in other ways, such as by considering the Jacobian ideal. We also see that the cusp is "more singular" in characteristics congruent to -1 modulo 6 than those congruent to 1 modulo 6, but as p gets very large, this difference is less significant. Finally, we see that the F-pure thresholds approach $5/6$ as p gets large, which is precisely the log canonical threshold of this cusp over $\mathbb{C}$. We will return to this idea in Chapter 6.

Remark 3.10. A general formula for the F-pure threshold of $y^m + \lambda x^n \in k[x, y]$ (where λ is any scalar in the field k of characteristic $p > 0$) can be found in [**Her15**, Thm 3.4].

Example 3.11. The F-pure threshold need not depend on congruence classes of p in general, as it does in the previous examples. For example, given a homogeneous cubic $f \in \mathbb{Z}[x, y, z]$ defining a smooth elliptic curve E in $\mathbb{P}^2_{\mathbb{C}}$,

$$\operatorname{fpt}(f) = \begin{cases} 1 & \text{if } E \text{ is ordinary} \\ 1 - \frac{1}{p} & \text{if } E \text{ is supersingular.} \end{cases}$$

See [**BS15, Pag18**]. Indeed an elliptic curve in characteristic zero reduces to an ordinary elliptic curve for infinitely many $p > 0$ [**Ser81**] [**Sil09**, Exercise 5.11] but also reduces to a super singular elliptic curve for infinitely many $p > 0$, but these need not always form an arithmetic progression as they can have density zero among all primes [**Ser81, Elk87**], [**Sil09**, Theorem 4.7].

Remark 3.12. There are numerous papers that compute the F-pure threshold in special cases. We mention a small set of these here.

(a) For homogeneous polynomials, see [**HNnBWZ16, MÏ8**].
(b) In $k[x, y]$, see [**Har06, Pag22**].
(c) For binomials, see [**ST09, Her14**].
(d) For sums of polynomials in distinct variables, including diagonal hypersurfaces, see [**Her15, GVJVNnB22**].

For lower bounds on the F-pure threshold, see [**KKP$^+$22**].

Remark 3.13. Note the set of all possible F-pure thresholds in a fixed ring (but varying f) satisfies the ascending chain condition (there is no infinite ascending chain) by work of Sato [**Sat19, Sat21**].

Remark 3.14. In the next section, we will define the F-pure threshold more generally for $f \in \mathfrak{m} \subseteq R$, where the ambient ring R is *non-regular*, and even more generally for *pairs* (R, f^t) where $t \in \mathbb{R}_{>0}$ is some formal real exponent and even replacing f with an ideal and so defining it for $(R, \mathfrak{a}^t)$. Later we will extend these ideas naturally to pairs (X, Δ) where X is a normal Noetherian F-finite scheme and Δ is a $\mathbb{Q}$-divisor, and still more general situations.

3.3. Approximating F-pure threshold. Lemma 3.3 can be used to produce a good approximation of the F-pure threshold. Specifically, if we can find $\nu(e)$ as defined in Lemma 3.3 for some e, then

$$\frac{\nu(e)}{p^e} \le \mathrm{fpt}_{\mathfrak{m}}(f) \le \frac{\nu(e)}{p^e} + \frac{1}{p^e}, \tag{3.14.1}$$

and so we know the F-pure threshold within a margin of error of $\frac{1}{p^e}$.

Example 3.15. Consider the plane curve defined by $f = y^2 - x^5$ in the polynomial ring $S = \mathbb{F}_p[x, y]$. The following tables, computed using the function "`frobeniusNu`" in the `FrobeniusThresholds` package for Macaulay2 [**HSTW21**], show $\nu(e)$ at the maximal ideal (x, y) in two different characteristics:

$p = 11$

e	$\nu(e)$	$\nu(e)/p^e$, estimate of fpt	error
1	7	.636364	$\le 1/11$
2	84	.694215	$\le 1/11^2$
3	931	.699474	$\le 1/11^3$
4	10248	.699952	$\le 1/11^4$
5	112735	.699996	$\le 1/11^5$
6	1240092	$\sim .7$	$< .0000006$

$p = 7$

e	$\nu(e)$	$\nu(e)/p^e$, estimate of fpt	error
1	4	.571429	$\le 1/7$
2	33	.673469	$\le 1/7^2$
3	237	.690962	$\le 1/7^3$
4	1665	.693461	$\le 1/7^4$
5	11661	.693818	$\le 1/7^5$
6	81633	.693869	$\le 1/7^6$
7	571437	.693876	$\le 1/7^7$
8	4000065	.693877	$\le 1/7^8$
9	28000461	.693878	$\le 1/7^9$
10	196003233	.693878	$< .000000004$

TABLE 1. Estimates of FPT of $y^2 - x^5$ in distinct characteristics

In characteristic 11, the ratios $\nu(e)/p^e$ appear to converge to $0.7 = 7/10$, and indeed the F-pure threshold is $7/10$ in this case. In characteristic 7 however, the table is slightly different; in this case, the ratios $\nu(e)/p^e$ converge to $34/49$ (which is just shy of $7/10$).

3.4. Exercises.

Exercise 3.1. Let k be a prime characteristic field, and S the localization of $k[x_1, \ldots, x_n]$ at the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. For any field extension k' of k, let $S' = S \otimes_k k'$, and let $\mathfrak{m}'$ be its maximal ideal. For $f \in \mathfrak{m}$, we can consider f as an element of $\mathfrak{m}' \subseteq S'$ under the natural inclusion $S \subseteq S'$. Prove that F-pure threshold of f at $\mathfrak{m}$ is the same as the F-pure threshold of f at $\mathfrak{m}'$ in S'. Do the same for a regular local ring S essentially of finite type over a field k isomorphic to its residue field.

Exercise 3.2. Prove Theorem 3.2.

Hint: Use Lemma 3.3 and Fedder's criterion.

Exercise 3.3. Let $S = \mathbb{F}_2[x, y, z]$. Show that $\operatorname{fpt}(x^3 + y^3 + z^3) = \frac{1}{2}$.

Hint: Use Corollary 3.5 to show $\operatorname{fpt} f \leq \frac{1}{2}$. For the reverse inequality, show $\operatorname{fpt} f \geq \frac{1}{2} - \frac{1}{2^e}$ by considering which monomial terms in $f^{2^{e-1}-1}$ are in $(x^{2^e}, y^{2^e}, z^{2^e})$.

Exercise 3.4. Verify the computations asserted in Example 3.9.

Exercise 3.5. Verify that the $f^{2(p^{e-1}-1)}$ in Example 3.8 is not in (x^{p^e}, y^{p^e}) in the case that $p = 3$. Conclude that $\operatorname{fpt}(f) = 2/3$ in this case.

Exercise 3.6. Show that $\operatorname{fpt}(x^2y - xy^2) = 2/3$ if $p \equiv 1 \pmod 3$, completing Example 3.8.

Exercise 3.7. Let $(S, \mathfrak{m})$ be an F-finite regular local ring $(S, \mathfrak{m})$. For $f \in \mathfrak{m}$, let $\nu(e)$ be the largest integer a such that $f^a \notin \mathfrak{m}^{[p^e]}$. Show that

$$\nu(e)\ (p^{(n-1)e} + \cdots + p^e + 1) \leq \nu(ne)$$

for every $n > 0$. Conclude that

$$\frac{\nu(e)}{p^e - 1} \leq \operatorname{fpt}_{\mathfrak{m}}(f) \leq \frac{\nu(e) + 1}{p^e}.$$

This improves the bounds on the F-pure threshold given by (3.14.1).

Hint: Observe S is e-Frobenius split along $f^{\nu(e)}$. Now use Corollary 4.11 from Chapter 1.

Exercise 3.8. Suppose S is a regular domain and J is any non-zero proper ideal of S. Given a non-zero f in the radical of J, let $\nu_f^J(e)$ denote the largest integer such that $f^{\nu(e)} \notin J^{[p^e]}$. Prove that the limit

$$\lim_{e \longrightarrow \infty} \frac{\nu_f^J(e)}{p^e}$$

exists. This limit is called the *F-threshold of f with respect to J,* see [**MTW05, HMTW08**].

Hint: The proof is essentially the same as the proof of Lemma 3.3.

Exercise 3.9. Suppose R is a regular ring and $J, \mathfrak{a} \subseteq R$ are ideals with $J \subseteq \sqrt{\mathfrak{a}}$. Set $\nu = \nu_{\mathfrak{a}}^J(p^e)$ to be the largest integer such that

$$\mathfrak{a}^\nu \not\subseteq J^{[p^e]}.$$

Show that the limit

$$\lim_{e \longrightarrow \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

exists. We call the *F-threshold of $\mathfrak{a}$ with respect to J*. The case where J is the maximal ideal in a local ring can be called the F-pure threshold of the ideal $\mathfrak{a}$; see Exercise 4.19 and also [**HMTW08**].

Hint: Revisit the proof of Lemma 3.3.

4. Frobenius splitting for pairs

Consider a *pair* (R, f^t) consisting of an local ring $(R, \mathfrak{m})$, a non-zero $f \in \mathfrak{m}$, and a real number $t \geq 0$, considered as formal exponent. Our goal in this section is to define a meaningful notion of *Frobenius splitting* for the pair (R, f^t) so that the following hold:

(a) For regular R, the pair (R, f^1) is Frobenius split if and only if the ring $R/(f)$ is Frobenius split. More generally, Frobenius splitting for the pair (R, f^t) should imply that R is Frobenius split as well as some restriction on the hypersurface cut out by f;

(b) If (R, f^t) is Frobenius split, then also (R, f^s) is Frobenius split for all non-negative $s \leq t$.

(c) The pair $(R, f^{\frac{1}{p}})$ is Frobenius split if and only if R is Frobenius split along f;

(d) The F-pure threshold[9] of an element f in a regular local ring R is the "threshold" at which the pair (R, f^t) becomes Frobenius split:

$$\mathrm{fpt}_{\mathfrak{m}}(f) = \sup\{t \in \mathbb{R} \mid (R, f^t) \text{ is Frobenius split}\}.$$

(e) Frobenius splitting for pairs behaves like Frobenius splitting for rings in other ways[10] as well as like log canonicity for pairs in complex geometry.

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[9] as defined in Section 3

[10] We will get more precise about this soon; see, for example, Remark 4.18

As we will see, there are several different ways to define Frobenius splitting for the pair (R, f^t) that will make some of these items hold, each depending on slightly different ways to approximate the real number t by rational numbers whose denominator is a power of p. But among these, one—which we call *sharp Frobenius splitting*—stands out as the "best behaved" way to define Frobenius splitting of pairs, because it makes the properties above hold, and otherwise has nice properties that mirror the properties of log canonical pairs in complex algebraic geometry. We will also define a natural extension of *strong F-regularity* to pairs which will have properties similar to Kawamata log terminal pairs.

Frobenius splitting can also be defined for pairs $(R, \mathfrak{a}^t)$ where the ideal $\mathfrak{a}$ is not necessarily principal; see Subsection 4.4. In this case, Frobenius splitting of the pair will imply Frobenius splitting for R as well as imposing a restriction on the subscheme defined by $\mathfrak{a}$. In later chapters, we extend Frobenius splitting to pairs of schemes and divisors (Section 3 of Chapter 5), and to the general setting of Cartier algebras (Section 3 of Chapter 8).

4.1. Formal real exponents. Consider an element f in a ring R of prime characteristic p. In Definition 4.1 of Chapter 1, we defined R to be *e-Frobenius split along f* when the R-linear map

$$R \longrightarrow F_*^e R \qquad 1 \mapsto F_*^e f \tag{4.0.1}$$

splits in the category of R-modules. Interpreting the map (4.0.1) as the map $R \to R^{1/p^e}$ sending 1 to f^{1/p^e} (see Subsection 1.5 in Chapter 1), we can think of this as saying that R is Frobenius split along the "fractional power" f^{1/p^e}, or that the *pair* $(R, f^{\frac{1}{p^e}})$ is Frobenius split.

Likewise, for rational numbers t of the form $\frac{n}{p^e}$ where n is a positive integer, we can say that R is Frobenius split along $f^{\frac{n}{p^e}}$ if the map

$$R \longrightarrow F_*^e R \cong R^{1/p^e} \qquad 1 \mapsto F_*^e f^n \cong f^{n/p^e} \tag{4.0.2}$$

splits. Because (4.0.2) splits if and only if

$$R \longrightarrow F_*^{e+1} R \cong R^{1/p^{e+1}} \qquad 1 \mapsto F_*^{e+1} f^{pn} \cong f^{np/p^{e+1}} \tag{4.0.3}$$

splits (Exercise 4.2 in Chapter 1), no ambiguity arises from writing the exponent t as $\frac{np}{p^{e+1}}$ or some other fraction whose denominator is a power of p. When these maps split, we say the *pair* $(R, f^{\frac{n}{p^e}})$ is Frobenius split.

Now, given any real $t > 0$, we can approximate t in various ways by rational numbers of the form $\frac{n_e}{p^e}$. One simple way is to think of t as a limit of the sequence of successive truncations of its base p expansion, $\left\{ \frac{\lfloor p^e t \rfloor}{p^e} \right\}_{e \in \mathbb{N}}$.

Other possibilities include sequences of the form

$$\left\{\frac{\lceil p^e t\rceil}{p^e}\right\}_{e\in\mathbb{N}}, \qquad \left\{\frac{\lfloor (p^e-1)t\rfloor}{p^e}\right\}_{e\in\mathbb{N}}, \qquad \text{or} \qquad \left\{\frac{\lceil (p^e-1)t\rceil}{p^e}\right\}_{e\in\mathbb{N}},$$

all of which converge to t as e goes to infinity. The last of these options turns out to work well for defining a notion of *Frobenius splitting for pairs*:

Definition 4.1. Let R be a ring of characteristic $p > 0$, $f \in R$, and t a non-negative real number. The pair (R, f^t) is said to be **sharply Frobenius split** if there exists some $e > 0$, such that the R-module map

$$R \longrightarrow F_*^e R \qquad \text{sending} \qquad 1 \mapsto F_*^e f^{\lceil t(p^e-1)\rceil} \tag{4.1.1}$$

splits in the category of R-modules. Put differently, (R, f^t) is sharply Frobenius split if there exists an e such that R is e-Frobenius split along $f^{\lceil t(p^e-1)\rceil}$.

Remark 4.2. Sharp Frobenius splitting of the pair (R, f^t) immediately forces restrictions on R and f. Namely, the ring R must be Frobenius split, and if $t > 0$, then f must be a non-zerodivisor. See Propositions 4.5 and 4.6 in Chapter 1, respectively.

In studying Frobenius splitting of pairs, we will repeatedly use the fact that if R is e-Frobenius split along some c, then R is e-Frobenius split along any c' dividing c (Proposition 4.6 in Chapter 1). For example, this fact immediately implies:

Proposition 4.3. *If (R, f^t) is sharply Frobenius split and $0 \le s \le t$, then (R, f^s) is also sharply Frobenius split.*

The next lemma highlights an important feature of our choice of approximating sequence for t in Definition 4.1:

Lemma 4.4. *Fix a pair (R, f^t) where R is a prime characteristic ring and $f \in R$. If the R-module map (4.1.1) splits for one value of $e > 0$, then it splits for infinitely many values of e.*

Proof of Lemma 4.4. The map (4.1.1) splits if and only if R is e-Frobenius split along $f^{\lceil t(p^e-1)\rceil}$. By Corollary 4.11 in Chapter 1, this implies that R is ne-Frobenius split along $(f^{\lceil t(p^e-1)\rceil})^{\frac{p^{ne}-1}{p^e-1}}$. Now because

$$(\lceil t(p^e-1)\rceil)(\frac{p^{ne}-1}{p^e-1}) \ge \lceil t(p^{ne}-1)\rceil, \tag{4.4.1}$$

we conclude that R is ne-Frobenius split along $f^{\lceil t(p^{ne}-1)\rceil}$ as well (Proposition 4.6 in Chapter 1). That is, the map (4.1.1) splits when e is replaced by any ne with $n \in \mathbb{N}$. The lemma is proved. $\square$

Remembering that "F-purity" is essentially another word for Frobenius splitting,[11] we can now explain the sense in which the F-pure threshold really is a "threshold" for Frobenius splitting:

Theorem 4.5. *Let $(S, \mathfrak{m})$ be an F-finite regular local ring, and let $f \in \mathfrak{m}$ be a non-zero element. Then the F-pure threshold of f is the supremum of the set*

$$\{t \mid (S, f^t) \textit{ is sharply Frobenius split}\}.$$

Theorem 4.5 suggests a definition of F-pure threshold for an element in a ring even if that ring is not necessarily regular; this is discussed in Subsection 4.5.

The proof of Theorem 4.5 uses the following straightforward lemma:

Lemma 4.6. *Let $(S, \mathfrak{m})$ be an F-finite regular local ring. For any non-zero $f \in \mathfrak{m}$, and any natural number $e > 0$, the ring S is e-Frobenius split along f^a if and only if $f^a \notin \mathfrak{m}^{[p^e]}$. In particular, for $t > 0$, (S, f^t) is sharply Frobenius split if and only if*

$$f^{\lceil t(p^e-1)\rceil} \notin \mathfrak{m}^{[p^e]}$$

for some (equivalently, infinitely many) $e > 0$.

Proof. Because F^e_*S is a finitely generated free S-module, any element that is part of a minimal S-module generating set can be assumed part of a free S-module basis for F^e_*S, and hence will split off F^e_*S. By Nakayama's lemma, an element F^e_*g is part of a minimal generating set if and only if $F^e_*g \notin \mathfrak{m}F^e_*S$ or equivalently, $g \notin \mathfrak{m}^{[p^e]}$. Applying this to f^a, the first statement of the lemma follows. Taking $a = \lceil t(p^e - 1)\rceil$, the second does too, (invoking Lemma 4.4 to get infinitely many e). $\square$

Proof of Theorem 4.5. Lemma 4.6 says that the pair (S, f^t) is sharply Frobenius split if and only if

$$\frac{\lceil t(p^e-1)\rceil}{p^e} \in \left\{ \frac{a}{p^E} \mid f^a \notin \mathfrak{m}^{[p^E]} \right\} \tag{4.6.1}$$

for infinitely many e. So, if (S, f^t) is sharply Frobenius split, then because the sequence $\{\frac{\lceil t(p^e-1)\rceil}{p^e} \mid e \in \mathbb{N}\}$ converges to t, t is bounded above by the supremum of $\{\frac{a}{p^E} \mid f^a \notin \mathfrak{m}^{[p^E]}\}$—that is, $t \le \operatorname{fpt}(f)$. This shows that

$$\sup\{t \mid (S, f^t) \text{ is sharply Frobenius split}\} \le \operatorname{fpt}(f).$$

For the reverse inequality, observe that if $t > \operatorname{fpt}(f)$, then the convergence of $\{\frac{\lceil t(p^e-1)\rceil}{p^e}\}_e$ to t implies that, for sufficiently large e, $\frac{\lceil t(p^e-1)\rceil}{p^e} >$

[11] the two notions coincide for F-finite rings; see Chapter 1 Subsection 7.6.

$\operatorname{fpt}(f) = \sup\{\frac{a}{p^E} \mid f^a \notin \mathfrak{m}^{[p^E]}\}$. By Lemma 4.6, this means that S fails to be e-Frobenius split along $f^{\lceil t(p^e-1)\rceil}$ for all $e \gg 0$, so that (S, f^t) can be not sharply Frobenius split. This proves Theorem 4.5. $\square$

Remark 4.7. When S is regular, local and F-finite, the pair (S, f^1) is sharply Frobenius split if and only if the quotient ring $S/(f)$ is Frobenius split. Indeed, taking $t = 1$ in Lemma 4.6, we have (S, f^1) is sharply Frobenius split if and only if $f^{(p^e-1)} \notin \mathfrak{m}^{[p^e]}$ for some e. By Fedder's Criterion, this is equivalent to $S/(f)$ being Frobenius split.

Caution 4.8. Confusion can arise because of an ambiguity in our notation: it is not the case that the pair (R, f^{nt}) is sharply Frobenius split if and only if (R, g^t) is sharply Frobenius split for $g = f^n$. See Exercise 4.7. However, this issue is easily fixed by working with $\mathbb{Q}$-divisors, which formally identifies the pairs $(\operatorname{Spec} R, t \operatorname{div}(f^n))$ and $(\operatorname{Spec} R, nt \operatorname{div}(f))$; we will adopt this point of view in Chapter 5.

Remark 4.9. Alternative ways to define Frobenius splitting of pairs will be discussed in Subsection 4.3.

4.2. Strong F-regularity of a pair. There is a natural generalization to strong F-regularity:

Definition 4.10 (**[HW02]**)**.** Let R be a ring of characteristic $p > 0$, $f \in R$, and t a non-negative real number. The pair (R, f^t) is **strongly F-regular** if for all non-zerodivisors $c \in R$, there exists $e \in \mathbb{N}$ such that the map

$$R \longrightarrow F^e_* R \qquad 1 \mapsto F^e_* c f^{\lceil (p^e-1)t \rceil} \tag{4.10.1}$$

splits as a map of R-modules.

Example 4.11. Clearly the pair $(R, 1^t)$ (for any t), as well as the pair (R, f^0) (for any f), is strongly F-regular if and only if R is strongly F-regular (as defined in Chapter 1).

The following basic facts follow straightforwardly from the definitions (and Proposition 4.6 in Chapter 1):

Proposition 4.12. *Let R be a ring of positive characteristic, $f \in R$, and t a non-negative real number.*

(a) *If the pair (R, f^t) is strongly F-regular, then it is sharply Frobenius split;*
(b) *If the pair (R, f^t) is strongly F-regular, then so is (R, f^s) for all non-negative $s \leq t$;*
(c) *If (R, f^t) is a strongly F-regular pair with $t > 0$, then R is strongly F-regular and f is a non-zerodivisor.*

The next lemma (and especially its proof) will be useful:

Lemma 4.13. *Let R be a ring of characteristic $p > 0$, $f \in R$, and t a non-negative real number.*

(i) If the map (4.10.1) splits for one value of e, then it splits for infinitely many values of e;
(ii) If the pair (R, f^t) is strongly F-regular, then for any non-zerodivisor c, the map (4.10.1) splits for all $e \gg 0$.

Proof of Lemma 4.13. Note that (i) follows as did Lemma 4.4: if R is e-Frobenius split along $cf^{\lceil (p^e-1)t \rceil}$, then R is ne-Frobenius split along $(cf^{(\lceil (p^e-1)t \rceil})^{\frac{p^{en}-1}{p^e-1}}$, hence also ne-Frobenius split along $cf^{\lceil (p^{ne}-1)t \rceil}$ (by Proposition 4.6 in Chapter 1).

For (ii), fix a non-zerodivisor c. Consider the non-zerodivisor $c' = cf^{\lceil t \rceil}$. Because (R, f^t) is strongly F-regular, there exists e such that R is e-Frobenius split along $c' f^{\lceil t(p^e-1) \rceil}$. We claim that R is e'-Frobenius split along $cf^{\lceil t(p^{e'}-1) \rceil}$ for all $e' \geq e$. To see this, note that

$$\lceil t(p^e - 1) \rceil + \lceil t \rceil \geq \lceil tp^e \rceil,$$

so, using Proposition 4.6 in Chapter 1, we have that R is e-Frobenius split along $c' f^{\lceil tp^e \rceil - \lceil t \rceil} = cf^{\lceil tp^e \rceil}$. In particular, R is e-Frobenius split along the factor $cf^{\lceil t(p^e-1) \rceil}$. But also, R is $(e+1)$-Frobenius split along $(cf^{\lceil tp^e \rceil})^p$ (by Exercise 4.2 in Chapter 1); so because

$$p\lceil tp^e \rceil \geq \lceil (t(p^{e+1} - 1) \rceil,$$

we can conclude also that R is $(e+1)$-Frobenius split along the factor $cf^{\lceil t(p^{e+1}-1) \rceil}$. In other words, the map (4.10.1) splits with $e+1$ in place of e, and by induction, for all $e \gg 0$. ☐

While the definition of strongly F-regularity for the pair (R, f^t) demands that we check infinitely many splitting conditions—one for each non-zerodivisor c— analogously to Theorem 5.1 in Chapter 1, it is enough to test for splitting only for one "test element" c:

Theorem 4.14. *Let R be a Noetherian F-finite ring, $f \in R$, and $t \geq 0$ a real number. Suppose that $d \in R$ is such that the pair $(R[\frac{1}{d}], (\frac{f}{1})^t)$ is strongly F-regular. Then the pair (R, f^t) is strongly F-regular if and only if the R-module map $R \to F^e_* R$ sending $1 \mapsto F^e_* df^{\lceil (p^e-1)t \rceil}$ splits for some e.*

To prove Theorem 4.14, we recall the following special case of a result from Chapter 1.

Lemma 4.15 (Chapter 1 Corollary 4.11)**.** *Let d and f be arbitrary elements of a ring R of characteristic $p > 0$. Fix a non-negative real number t, and suppose that R is e-Frobenius split along $df^{\lceil (p^e-1)t\rceil}$. Then for all $N \in \mathbb{N}$, there exists $E \in \mathbb{N}$ such that R is E-Frobenius split along $d^N f^{\lceil (p^E-1)t\rceil}$.*

Proof of Lemma 4.15. Fix $N \in \mathbb{N}$. Choose n so that $p^{(n-1)e} > N$. By Corollary 4.11 in Chapter 1, R is ne-Frobenius split along $(df^{\lceil (p^e-1)t\rceil})^{\frac{p^{ne}-1}{p^e-1}}$. Because

$$\frac{p^{en}-1}{p^e-1}\lceil (p^e-1)t\rceil \geq \lceil (p^{ne}-1)t\rceil,$$

setting $E = ne$, the result follows from Proposition 4.6 in Chapter 1. $\square$

Proof of Theorem 4.14. Fix a non-zerodivisor c. We need to show that there exists b such that the R-module map

$$R \to F^b_* R \qquad 1 \mapsto F^b_* c f^{\lceil (p^b-1)t\rceil} \tag{4.15.1}$$

splits. Localizing at d, we know that there exists e such that $R[d^{-1}]$ is e-Frobenius split along $\frac{cf^{\lceil (p^e-1)t\rceil}}{1}$. So there exists $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that

$$\phi(F^e_* c f^{\lceil (p^e-1)t\rceil}) = d^N \tag{4.15.2}$$

for some $N \in \mathbb{N}$ (this follows exactly as in the proof of Theorem 5.1 in Chapter 1, and uses the finiteness conditions on R).

Now Lemma 4.15 implies that there exists $E > 0$ and $\psi \in \operatorname{Hom}_R(F^E_* R, R)$ such that

$$\psi(F^E_* d^N f^{\lceil (p^E-1)t\rceil}) = 1. \tag{4.15.3}$$

We define $\Psi \in \operatorname{Hom}_R(F^{E+e}_* R, R)$ to be the composition $\Psi = \psi \star \phi$ using the Cartier algebra notation. Is is easy to verify that

$$\Psi(F^{E+e}_* c f^{p^e\lceil (p^E-1)t\rceil + \lceil (p^e-1)t\rceil}) = 1.$$

This shows that R is $(E+e)$-Frobenius split along

$$c f^{p^e\lceil (p^E-1)t\rceil + \lceil (p^e-1)t\rceil}, \tag{4.15.4}$$

and hence along $cf^{\lceil (p^{E+e}-1)t\rceil}$ since $\lceil (p^{E+e}-1)t\rceil \leq p^e\lceil (p^E-1)t\rceil + \lceil (p^e-1)t\rceil$ (again, Proposition 4.6 in Chapter 1). This establishes (4.15.1) with $b = E+e$, completing the proof of Theorem 4.14. $\square$

4.3. Variants of sharp Frobenius splitting. Our definition of (sharp) Frobenius splitting for pairs (R, f^t) depended on a particular choice of sequence of rational numbers whose denominators are powers of p converging to t. Other choices can and have been made:

Definition 4.16 (**[HW02, Tak04a]**)**.** Let R be a ring of characteristic $p > 0$, $f \in R$ and $t \geq 0$.

(a) The pair (R, f^t) is **weakly Frobenius split** if the R-module map

$$R \longrightarrow F^e_* R \qquad 1 \mapsto F^e_* f^{\lfloor t(p^e-1)\rfloor} \tag{4.16.1}$$

splits for all $e \gg 0$.

(b) The pair (R, f^t) is **strongly Frobenius split** if the R-module map

$$R \longrightarrow F^e_* R \qquad 1 \mapsto F^e_* f^{\lceil tp^e\rceil} \tag{4.16.2}$$

splits for all $e \gg 0$.

The sequences approximating t used to define weak, sharp and strong Frobenius splitting are related as follows: for each e,

$$\frac{\lfloor t(p^e-1)\rfloor}{p^e} \leq \frac{\lceil t(p^e-1)\rceil}{p^e} \leq \frac{\lceil tp^e\rceil}{p^e}.$$

This leads to the following comparison result:

Proposition 4.17. *Let R be a ring of characteristic $p > 0$, $f \in R$ and $t \geq 0$.*

(a) If the pair (R, f^t) is strongly Frobenius split, then it is sharply Frobenius split.
(b) If the pair (R, f^t) is sharply F-split, then it is weakly Frobenius split.
(c) If the pair (R, f^t) is weakly Frobenius split with $t > 0$, then for every $\epsilon \in (0, t]$, the pair $(R, f^{t-\epsilon})$ is strongly (and hence sharply) Frobenius split.

See Exercise 4.13 for a fourth variant of Frobenius splitting that arises by approximating t by the sequence of its base p truncations $\{\frac{\lfloor p^e t\rfloor}{p^e}\}$.

Proof. We leave (a) and (b) as exercises, and focus on (c). Given that (R, f^t) is weakly Frobenius split, we have some $e_0 > 0$ such that R is e-Frobenius split along $f^{\lfloor t(p^e-1)\rfloor}$ for all $e \geq e_0$. Now for any $\epsilon \in (0, t]$, we have

$$\lfloor t(p^e-1)\rfloor \geq \lceil (t-\epsilon)p^e\rceil$$

for all $e \gg 0$. So R is e-Frobenius split along $f^{\lceil (t-\epsilon)p^e\rceil}$ for all sufficiently large e (Proposition 4.6 in Chapter 1). That is, $(R, f^{t-\epsilon})$ is strongly Frobenius split. $\square$

Remark 4.18. Despite their similarities, the different variants of Frobenius splitting for pairs are *not* the same (see Exercises 4.8 and 4.9). For example, it is not true that if the map (4.16.1) in the definition of weak F-splitting splits for some e, then it splits for infinitely many e. This is unlike the situation for sharp Frobenius splitting (see Lemma 4.4). Later, we'll see examples of weakly Frobenius split pairs that do not have radical test ideals [**MY09**], as we would expect from an analog of log-canonicity in complex geometry.

Remark 4.19. One could attempt to similarly define variants for strong F-regularity by choosing a different way to approximate t, but they all lead to the same notion. See Remark 4.26.

Remark 4.20. Our terminology differs slightly from the literature. Hara and Watanabe first defined weak Frobenius splitting for pairs in the local case (using a different approach), but called it *F-purity* of the pair. Likewise, strong Frobenius splitting for pairs was first defined by Takagi and Watanabe, who called it *strong F-purity*. Similarly, sharp Frobenius splitting was first called sharp F-purity when introduced by Schwede[12] [**Sch10b**]. The terms strong, weak and sharp F-splitting can also be found in the literature.

4.4. Frobenius splitting for higher codimension pairs. We now generalize Frobenius splitting to pairs $(R, \mathfrak{a}^t)$ where $\mathfrak{a}$ is an arbitrary ideal of R. When $\mathfrak{a}$ is principal— say generated by f— Frobenius splitting of the pair $(R, \mathfrak{a}^t)$ is the same as Frobenius splitting of (R, f^t) (Exercise 4.10).

To take *all* the elements of $\mathfrak{a}$ into account, we use the module

$$F^e_*\mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \operatorname{Hom}_R(F^e_*R, R)$$

encountered in Section 2. As an R-submodule of $\operatorname{Hom}_R(F^e_*R, R)$, this is generated by all compositions $\phi\star g$, where $g \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$ and $\phi \in \operatorname{Hom}_R(F^e_*R, R)$. There is a natural R-module map

$$F^e_*\mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \operatorname{Hom}_R(F^e_*R, R) \xrightarrow{\text{eval at } F^e_*1} R$$

given by evaluation at F^e_*1. Its image is the ideal of R generated by *all* elements of the form $\phi(F^e_*g)$ where $\phi \in \operatorname{Hom}_R(F^e_*R, R)$ and $g \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$.

Definition 4.21 ([**HW02, Tak04a**])**.** Let R be a Noetherian F-finite ring of characteristic $p > 0$, $\mathfrak{a}$ an ideal of R, and $t \geq 0$ a real number.

(a) The pair $(R, \mathfrak{a}^t)$ is **sharply Frobenius split** if there exists $e > 0$ such that the map

$$F^e_*\mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \operatorname{Hom}_R(F^e_*R, R) \longrightarrow R \qquad \phi \mapsto \phi(F^e_*1)$$

[12] although his definition there is slightly different from the one here for non-principal ideals in non-local case, which we will study shortly

is surjective;

(b) The pair $(R, \mathfrak{a}^t)$ is **strongly F-regular** if for every non-zerodivisor $c \in R$, there exists $e > 0$ such that the map

$$F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil} \cdot \operatorname{Hom}_R(F^e_* R, R) \longrightarrow R \qquad\qquad \phi \mapsto \phi(F^e_* c)$$

is surjective.

Remark 4.22. If there exists $e > 0$ such that the map in (a) (or (b)) is surjective, then infinitely many such e exist. We leave the proof as Exercise 4.16, as it is similar to Lemma 4.4.

Remark 4.23. Clearly a strongly F-regular pair $(R, \mathfrak{a}^t)$ is sharply Frobenius split. Also, if the pair $(R, \mathfrak{a}^t)$ is strongly F-regular (or sharply Frobenius split), then so is the pair $(R, \mathfrak{a}^s)$ for all non-negative $s \leq t$.

Theorem 4.24. *Let R be a Noetherian, F-finite ring, $\mathfrak{a}$ an ideal of R, and $t \geq 0$ a real number. Then the locus of points of $\operatorname{Spec} R$ where the pair $(R, \mathfrak{a}^t)$ is sharply Frobenius split (respectively, strongly F-regular) is open.*

Proof. We prove the statement for strong F-regularity; the statement for sharp Frobenius splitting is proved similarly and left to the reader. We assume $t > 0$, since when $t = 0$, this is simply Theorem 5.12 in Chapter 1.

Suppose that $Q \in \operatorname{Spec} R$ is such that $(R_Q, \mathfrak{a}_Q^t)$ is strongly F-regular. We need to find an open neighborhood $\mathcal{U}$ of Q such that for each $P \in \mathcal{U}$, the pair $(R_P, \mathfrak{a}_P^t)$ is strongly F-regular. Because $(R_Q, \mathfrak{a}_Q^t)$ is strongly F-regular, the ambient ring R_Q is strongly F-regular, and hence a normal domain (Chapter 1 Theorem 4.30), and the ideal $\mathfrak{a}R_Q$ is not the zero ideal. So we can choose $c \in \mathfrak{a}$ such that the pair $(R_c, \mathfrak{a}_c^t)$ (which is the same as the pair $(R_c, 1^t)$) is strongly F-regular because $R_c = R[\frac{1}{c}]$ is *regular.* Now, we can test for strong F-regularity of the pair $(R, \mathfrak{a}^t)$ by checking whether there is some $e \in \mathbb{N}$ such that the map

$$\begin{array}{ccc} F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil} \cdot \operatorname{Hom}_R(F^e_* R, R) & \xrightarrow{\text{eval at } F^e_* c} & R \\ \phi & \mapsto & \phi(F^e_* c) \end{array} \tag{4.24.1}$$

is surjective (Exercise 4.17). Since $(R_Q, \mathfrak{a}R_Q^t)$ is strongly F-regular, there is some e such that (4.24.1) becomes surjective when we localize at Q. This means there is an open neighborhood $\mathcal{U} \subseteq \operatorname{Spec} R$ of Q for which the map (4.24.1) becomes surjective when localizing at each point $P \in \mathcal{U}$. Each of these points P, therefore, is such that $(R_P, \mathfrak{a}R_P^t)$ is strongly F-regular, and we have found the needed open neighborhood of Q contained in the strongly F-regular locus of the pair $(R, \mathfrak{a}^t)$. $\square$

Remark 4.25. Weak and strong Frobenius splitting of the pair $(R, \mathfrak{a}^t)$ can be defined analogously, and many nice properties hold. For instance, we can define a pair $(R, \mathfrak{a}^t)$ to be **strongly F-split** if for some $e > 0$ so that the map

$$F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil} \cdot \mathrm{Hom}_R(F^e_* R, R) \longrightarrow R \qquad\qquad \phi \mapsto \phi(F^e_* 1)$$

is surjective. For instance, it is not difficult to see that if $(R, \mathfrak{a}^t)$ is sharply F-split, then $(R, \mathfrak{a}^{t-\epsilon})$ is strongly F-split for each $t \geq \epsilon > 0$ (the argument is essentially identical to that in Proposition 4.17 (c)).

Remark 4.26. One might define variants of strong F-regularity by choosing different roundings to approximate t, analogously to the variants of Frobenius splitting above. However, it turns out that they are all equivalent. Indeed, assuming $\mathfrak{a}$ contains a non-zerodivisor, we can find a non-zerodivisor d such that $d\mathfrak{a}^{\lfloor t(p^e-1) \rfloor} \subseteq \mathfrak{a}^{\lceil tp^e \rceil}$ for every e. Then replacing any c that appears in the definition of a strongly F-regular pair with $c' = cd$ is enough to absorb any difference in roundings that might occur; see Exercise 4.11. We saw a similar trick, which might be loosely called *absorbing differences in rounding into the test element*, in the proof of Lemma 4.13 (ii).

On the other hand, if $\mathfrak{a}$ contains no non-zerodivisor, the pair $(R, \mathfrak{a}^t)$ for $t > 0$ does not satisfy any of the variants of Frobenius splitting.

Remark 4.27. One might wonder[13] whether in general the pair $(R, \mathfrak{a}^t)$ is sharply Frobenius split if and only if there exists $e > 0$ and $g \in \mathfrak{a}^{\lceil (p^e-1)t \rceil}$ such that R is e-Frobenius split along g. This is not known in general, unless $\mathfrak{a}$ is principal, R is local, or $\mathrm{Hom}_R(F^e_* R, R)$ is a cyclic $F^e_* R$-module. See Exercises 4.10, 4.14, and 4.15, respectively.

4.5. F-pure threshold.

Our definition of Frobenius splitting for pairs suggests the following definition of the F-pure threshold:

Definition 4.28 (**[TW04, MTW05]**)**.** Let R be a Frobenius split ring, $\mathfrak{a}$ a proper ideal of R, and $t \geq 0$ a real number. The **F-pure threshold** of the pair $(R, \mathfrak{a}^t)$ is the supremum of the set

$$\{s \geq 0 \mid (R, \mathfrak{a}^t) \text{ is sharply Frobenius split}\}. \tag{4.28.1}$$

By *F-pure threshold of the ideal* $\mathfrak{a}$ in R, we mean the F-pure threshold of the pair $(R, \mathfrak{a}^1)$.

When $\mathfrak{a}$ contains a non-zerodivisor and is contained within a maximal ideal generated by m elements, the set (4.28.1) is bounded above by $\lceil \frac{m}{t} \rceil$ (Exercise 4.18), so the F-pure threshold is a finite real number for all proper ideals in a Noetherian Frobenius split ring, provided the set (4.28.1) is non-empty.

[13] indeed, sharp F-purity was first defined this way in [**Sch10b**].

Remark 4.29. The set (4.28.1) can be empty—for example, when $\mathfrak{a}$ consists only of zerodivisors (or if R is *not* Frobenius split). In this case, we can make the convention that the F-pure threshold is $-\infty$.

Remark 4.30. The F-pure threshold of a pair $(R, \mathfrak{a}^t)$ can be computed using any of the three variants of Frobenius splitting for pairs. That is, replacing the word "sharply" with "weakly" or with "strongly" in Definition 4.28 produces the same number. This follows from Proposition 4.17.

The F-pure threshold is a local invariant:

Proposition 4.31. *Let R be a Noetherian F-finite ring, $\mathfrak{a}$ an ideal of R and $t \geq 0$ a real number. Then the F-pure threshold of the pair $(R, \mathfrak{a}^t)$ is equal to*

$$\inf \left\{ \mathrm{fpt}(R_Q, \mathfrak{a}_Q^t) \;\middle|\; Q \in \operatorname{Spec} R \right\} = \inf \left\{ \mathrm{fpt}(R_{\mathfrak{m}}, \mathfrak{a}_{\mathfrak{m}}^t) \;\middle|\; \mathfrak{m} \in \text{m-}\operatorname{Spec} R \right\}.$$

Proof. This follows as Frobenius splitting of a pair $(R, \mathfrak{a}^t)$ can be checked locally, as the formation of

$$(F_*^e \mathfrak{a}^{\lceil tp^e \rceil}) \operatorname{Hom}_R(F_*^e R, R) \xrightarrow{\text{eval@1}} R$$

commutes with localization. □

Lemma 4.32. *Suppose $(S, \mathfrak{m})$ is a Noetherian F-finite regular local ring and $\mathfrak{a} \subseteq S$ is an ideal. For each $e > 0$, define $\nu(e)$ to be the largest integer such that $\mathfrak{a}^{\nu(e)} \not\subseteq \mathfrak{m}^{[p^e]}$. Then*

$$\mathrm{fpt}_{\mathfrak{m}}(\mathfrak{a}) = \lim_{e \longrightarrow \infty} \frac{\nu(e)}{p^e}.$$

Proof. Left to the reader in Exercise 4.20, *cf.* Exercise 4.19. □

Remark 4.33. It is natural to form other similar ratios. Indeed, for any ideals $\mathfrak{a}, J$ in an arbitrary Noetherian ring R of prime characteristic, set $\nu_{\mathfrak{a}}^J$ to be the largest natural number n with $\mathfrak{a}^n \not\subseteq J^{[p^e]}$. The the $\limsup$ and $\liminf$ of $\frac{\nu_{\mathfrak{a}}^J}{p^e}$ were considered in [**HMTW08**] where they were called the ***F*-threshold of $\mathfrak{a}$ with respect to** J if they coincided. Indeed, they do coincide and so the limit

$$\mathrm{fpt}_{\mathfrak{m}}(\mathfrak{a}) = \lim_{e \longrightarrow \infty} \frac{\nu_{\mathfrak{a}}^J}{p^e}.$$

exists thanks to [**NnBS20**]. Also see [**?**] in the case of a regular ring.

4.6. Singularities at the F-pure threshold. Let R be a Frobenius split ring, and let $f \in R$. Then the pair (R, f^t) is always sharply Frobenius split for $t < \operatorname{fpt}(f)$, and never sharply Frobenius split for $t > \operatorname{fpt}(f)$. This leaves open, however, what happens for $t = \operatorname{fpt}(f)$.

If t is a rational number whose denominator is not divisible by p, then one can check that (R, f^t) is sharply Frobenius split when $t = \operatorname{fpt}(f)$. However, there are examples where $(R, f^{\operatorname{fpt}(f)})$ is not sharply Frobenius split; see Exercise 4.8.

A result of Hernández clarifies the situation:

Theorem 4.34 ([**Her12**, Theorem 4.9])**.** *Let $(R, \mathfrak{m})$ be a Noetherian F-finite Frobenius split ring, $f \in R$ a non-zerodivisor, and $\lambda = \operatorname{fpt}_{\mathfrak{m}}(f)$. Then:*

(a) (R, f^λ) is sharply Frobenius split if and only if $\lambda = a/b$ where p does not divide b.
(b) (R, f^λ) is weakly Frobenius split.
(c) (R, f^λ) is not *strongly Frobenius split.*

Proof. We sketch one direction of (a) in 4.4. The others are left to the reader. $\square$

Caution 4.35. For non-principal $\mathfrak{a}$, it is not true that if $\lambda = \operatorname{fpt}_{\mathfrak{m}}(\mathfrak{a})$ and $t = a/b$ is a rational number without p in its denominator, then $(R, \mathfrak{a}^\lambda)$ is sharply Frobenius split, in contrast to the principal case, Theorem 4.34.

For instance, if $R = \mathbb{F}_2[x, y]$ and $\mathfrak{a} = (x^2, y^2)$ it is not difficult to see that $\operatorname{fpt}_{\mathfrak{m}}(\mathfrak{a}) = 1$ which has no $p = 2$ in its denominator. On the other hand since $x^{2^e-1}y^{2^e-1} \notin \mathfrak{a}^{2^e-1}$, and because is the only monomial of degree $2(2^e-1)$ not in (x^{2e}, y^{2e}), we see that $\mathfrak{a}^{2^e-1} \subseteq (x^{2^e}, y^{2^e})$ for all e. This shows that $(R, \mathfrak{a}^1)$ is not sharply F-split.

4.7. Exercises.

Exercise 4.1. Let $(R, \mathfrak{m})$ be a local ring, and $f \in \mathfrak{m}$. Show that the pair (R, f^t) is not sharply Frobenius split if $t > 1$.

Exercise 4.2. Let R be an F-finite Noetherian ring. Prove that R is strongly F-regular if and only if for all non-zerodivisors $c \in R$, there exists an $e > 0$ such that the pair $(R, c^{\frac{1}{p^e-1}})$ is sharply Frobenius split.

Exercise 4.3. Let R be an F-finite Noetherian ring, $f \in R$. Let $t = \frac{a}{p^e-1}$, where $a \in \mathbb{N}$. Show that (R, f^t) is sharply Frobenius split if and only if $R \xrightarrow{1 \mapsto F^e_* f^a} F^e_* R$ splits.

Hint: Use that $t(p^e - 1) = a$ is an integer.

Exercise 4.4. Let $(R, \mathfrak{m})$ be a Frobenius split local ring of characteristic $p > 0$. Let $f \in \mathfrak{m}$ be such that $\lambda = \operatorname{fpt}_{\mathfrak{m}}(f)$ is a rational number whose denominator is not divisible by p. Show that (R, f^{λ}) is sharply Frobenius split.

Conversely, if $\lambda = \operatorname{fpt}_{\mathfrak{m}}(f)$ and (R, f^{λ}) is sharply F-split, show that we may write λ as a rational number without p in the denominator.

Hint: Suppose that $a = \lambda(p^e - 1)$ is an integer. To show that R is e-Frobenius split along f^a, it suffices to show it is $e+d$-Frobenius split along f^{ap^d}. Compare ap^d with $(\lambda - \epsilon)(p^{e+d} - 1)$. For the converse, suppose we cannot write $\lambda = \frac{a}{p^e - 1}$. Deduce that (R, f^t) is also sharply F-split for some $t > \lambda$ contradicting the maximality of the F-pure threshold.

Exercise 4.5. Fix a non-zerodivisor f in a Frobenius split local ring R. Show that there exists a $t > 0$ such that (R, f^t) is sharply Frobenius split if and only if R is eventually Frobenius split along f. Use this to find an example where the set

$$\{t \mid (R, f^t) \text{ is sharply Frobenius split}\} \tag{4.35.1}$$

has supremum zero.

Exercise 4.6. Prove parts (a) and (b) of Proposition 4.17.

Hint: Both statements follow from Proposition 4.6 in Chapter 1. Note that if $e > d$, then $p^{e-d}\lfloor t(p^d - 1)\rfloor \leq \lceil (p^e - 1)t \rceil$.

Exercise 4.7. Consider $R = \mathbb{F}_2[x]$ and set $f = x^2$. Show that $(R, f^{1/2})$ is not sharply Frobenius split even though (R, x^1) is sharply Frobenius split. This emphasizes that we must be cautious interpreting the formal exponents in discussing strongly Frobenius split pairs.

Exercise 4.8. Let S be the localization of the polynomial ring $\mathbb{F}_2[x, y, z]$ at the maximal ideal (x, y, z), and let f be the polynomial $x^3 + y^3 + z^3$. Show that the pair $(S, f^{\frac{1}{2}})$ is weakly but not sharply Frobenius split. [Note also that $\operatorname{fpt}(f) = \frac{1}{2}$ by Exercise 3.3.]

Exercise 4.9. Let $(S, \mathfrak{m})$ be an F-finite regular local ring, and suppose that the quotient ring $S/(f)$ is Frobenius split. Show that the pair (S, f^1) is not strongly Frobenius split. Use this to find an example of a pair that is sharply but not strongly Frobenius split.

Exercise 4.10. Suppose R is a ring and $(f) = \mathfrak{a} \subseteq R$ is a principal ideal. Show that $(R, \mathfrak{a}^t)$ is sharply Frobenius split (Definition 4.21) if and only if (R, f^t) is sharply Frobenius split (Definition 4.1). In other words, show that Definition 4.1 and Definition 4.21 agree.

Exercise 4.11. Let R be an F-finite Noetherian ring, $\mathfrak{a} \subseteq R$ an ideal, and $t \leq 0$ a real number. Show that the following are equivalent:

(a) The pair $(R, \mathfrak{a}^t)$ is strongly F-regular;
(b) For every non-zerodivisor $c \in R$, the evaluation-at-c map $F^e_* \mathfrak{a}^{\lfloor t(p^e-1) \rfloor} \cdot \mathrm{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* c)} R$ is surjective;
(c) For every non-zerodivisor $c \in R$, the evaluation-at-c map $F^e_* \mathfrak{a}^{\lfloor tp^e \rfloor} \cdot \mathrm{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* c)} R$ is surjective;
(d) For every non-zerodivisor $c \in R$, the evaluation-at-c map $F^e_* \mathfrak{a}^{\lceil tp^e \rceil} \cdot \mathrm{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* c)} R$ is surjective.

Hint: Use the trick that test elements can absorb differences in rounding as in the proof of Lemma 4.13 (ii).

Exercise 4.12. Suppose that the R-module map

$$R \longrightarrow F^e_* R, \qquad 1 \mapsto F^e_* f^{\lfloor tp^e \rfloor} \tag{4.35.2}$$

splits for some value e_0 of e. Show that (4.35.2) splits for all positive $e < e_0$.

Exercise 4.13 (Keen Frobenius Splitting)**.** Let R be an F-finite Noetherian ring, $\mathfrak{a} \subseteq R$ an ideal, and $t \geq 0$ a real number. Define the pair (R, f^t) to be *keenly Frobenius split*[14] if the R-module map

$$R \longrightarrow F^e_* R \qquad 1 \mapsto F^e_* f^{\lfloor tp^e \rfloor}$$

splits for all $e > 0$ (equivalently, for infinitely many $e > 0$ by Exercise 4.12). Prove the following.

(a) For $t < 1$, a sharply F-split pair (R, f^t) is keenly F-split.
(b) The pair $(\mathbb{F}_p[x], x^1)$ is sharply F-split but not keenly F-split.
(c) Propositions 4.3 and 4.17, and 4.30 all hold with the word *keenly* in place of *sharply*.
(d) Keen and sharp Frobenius splitting are equivalent for rational $t < 1$ whose denominator is not divisible by p.

Exercise 4.14. Suppose $(R, \mathfrak{m})$ is a *local* ring and $\mathfrak{a} \subseteq R$. Show that $(R, \mathfrak{a}^t)$ is sharply Frobenius split if and only if for some $e > 0$ there exists $g \in \mathfrak{a}^{\lceil t(p^e-1) \rceil}$ and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* g) = 1$.

Exercise 4.15. Let $\mathfrak{a} \subseteq R$ be an ideal in a ring R of characteristic $p > 0$. Suppose that $\mathrm{Hom}_R(F^e_* R, R)$ is a cyclic $F^e_* R$-module. Show that for any $t \geq 0$, the pair $(R, \mathfrak{a}^t)$ is sharply Frobenius split if and only if for some $e > 0$ there exists $g \in \mathfrak{a}^{\lceil t(p^e-1) \rceil}$ and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* g) = 1$.

[14]Keenly Frobenius split pairs first appeared implicitly as early as **[Wat91]**; this rounding choice has been used in several other papers but does not appear to have been given a name.

Exercise 4.16. Let R be an F-finite Noetherian ring, $\mathfrak{a}$ an ideal in R, and $t \geq 0$ a real number. Fix any $c \in R$. Show that if

$$F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \mathrm{Hom}_R(F^e_* R, R) \xrightarrow{\text{eval at } F^e_* c} R \quad \text{sending} \quad \phi \mapsto \phi(F^e_* c) \tag{4.35.3}$$

is surjective for some $e \in \mathbb{N}$, then also

$$F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \mathrm{Hom}_R(F^e_* R, R) \xrightarrow{\text{eval at } F^{ne}_* c} R \quad \text{sending} \quad \phi \mapsto \phi(F^{ne}_* c) \tag{4.35.4}$$

is surjective for all integers $n > 0$.

Hint: Look at the proof of Lemma 4.4.

Exercise 4.17. Let R be a Noetherian F-finite ring, $\mathfrak{a}$ an ideal in R, and $t \geq 0$ a real number. Suppose there exists $d \in R$ such that the pair $(R[\frac{1}{d}], (\mathfrak{a}R[\frac{1}{d}])^t)$ is strongly F-regular. Prove that the pair $(R, \mathfrak{a}^t)$ is strongly F-regular if and only if there exists $e > 0$ such that the map

$$F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil} \cdot \mathrm{Hom}_R(F^e_* R, R) \longrightarrow R \quad \text{sending} \quad \phi \mapsto \phi(F^e_* d)$$

is surjective for some e.

Exercise 4.18. Suppose that R is a ring of characteristic $p > 0$ and $\mathfrak{a} \subseteq R$ is an ideal which can be generated by m elements. Show that

(a) $\mathfrak{a}^{p^e(m-1)}\, \mathfrak{a}^{[p^e]} = \mathfrak{a}^{p^e m}$;
(b) The F-pure threshold of $\mathfrak{a}$ is bounded above by m.

Exercise 4.19. Suppose $\mathfrak{a}$ is an ideal in an F-finite regular local ring $(S, \mathfrak{m})$. Prove that

$$\mathrm{fpt}_{\mathfrak{m}}(\mathfrak{a}) = \sup\{\frac{\nu}{p^e} \mid \mathfrak{a}^\nu \not\subseteq \mathfrak{m}^{[p^e]}\}.$$

Hint: The proof of the first part is similar to Theorem 4.5.

Exercise 4.20. More generally, if $\mathfrak{a}$ is an ideal in an F-finite regular local ring $(S, \mathfrak{m})$, show that

$$\mathrm{fpt}_{\mathfrak{m}}(\mathfrak{a}) = \lim_{e \longrightarrow \infty} \frac{\nu_e}{p^e}$$

where ν_e is the largest integer such that $\mathfrak{a}^{\nu_e} \not\subseteq \mathfrak{m}^{[p^e]}$.

5. Test ideals for pairs: a first look

We begin this first look at test ideals for *pairs* $(S, \mathfrak{a}^t)$ by studying the case where the ambient ring S is an F-finite *regular* ring. We will move beyond this case shortly, but we start by emphasizing this important and classical case where many of the keys ideas are more transparent.

Like the F-pure threshold, the *test ideal* of the pair (S, f^t) measures the failure of Frobenius splitting. For rational numbers t of the form $\frac{\nu}{p^e}$, we saw that the pair $(S, f^{\frac{\nu}{p^e}})$ is Frobenius split if and only if the "evaluation at $f^{\frac{\nu}{p^e}}$" map

$$\text{(5.0.1)} \qquad \operatorname{Hom}_S(F^e_* S, S) \longrightarrow S \quad \text{sending} \quad \phi \mapsto \phi(F^e_* f^\nu)$$

is *surjective*, see Subsection 4.1. This suggests that the *image* of the "evaluation at f^t" map —which is an ideal of S determined by f^t— can be considered a natural obstruction to Frobenius splitting for the pair (S, f^t). This is the *test ideal* of the pair (S, f^t) in this special case where $t \in \mathbb{Q}$ has denominator[15] a power of p and the ambient ring S is regular.

For arbitrary positive real numbers t, we approximate t by rational numbers $\{t_e\}_{e\in\mathbb{N}}$ whose denominators are powers of p, although, as before, there are several different ways to do so. For defining the test ideal, it turns out that the sequence $\{\frac{\lceil tp^e \rceil}{p^e}\}_{e\in\mathbb{N}}$ works well because it *descends* to t and so gives rise to an *ascending chain* of image ideals in (5.0.1), which stabilizes by Noetherianity:

Definition 5.1 (**[HY03, Tak06]**)**.** Let S be an F-finite regular domain, $\mathfrak{a}$ an ideal of S, and $t \geq 0$ a real number. The **test ideal** $\tau(S, \mathfrak{a}^t)$ is the image of the map

$$\text{(5.1.1)} \quad F^e_* \mathfrak{a}^{\lceil p^e t \rceil} \cdot \operatorname{Hom}_S(F^e_* S, S) \xrightarrow{\text{eval at } F^e_* 1} S \quad \text{sending} \quad \psi \mapsto \psi(F^e_* 1)$$

for sufficiently large e.

In order to be sure Definition 5.1 makes sense, we must check that the images of the maps (5.1.1) form an ascending chain of ideals. In fact, this is true more generally:

Lemma 5.2. *Let R be a Frobenius split ring, $\mathfrak{a} \subseteq R$ an ideal of R and $t \geq 0$ a real number. For each $e \in \mathbb{N}$, let $J_e(R, \mathfrak{a}^t)$ be the image of the R-module map*

$$\text{(5.2.1)} \qquad \begin{aligned} F^e_* \mathfrak{a}^{\lceil p^e t \rceil} \cdot \operatorname{Hom}_R(F^e_* R, R) &\xrightarrow{\text{eval at } 1} R \\ \psi &\mapsto \psi(F^e_* 1). \end{aligned}$$

Then the $J_e(R, \mathfrak{a}^t)$ form an ascending chain of ideals

$$J_1(R, \mathfrak{a}^t) \subseteq J_2(R, \mathfrak{a}^t) \subseteq J_3(R, \mathfrak{a}^t) \subseteq \cdots .$$

Proof. The ideal $J_e(R, \mathfrak{a}^t)$ is generated by all $\phi(F^e_* g)$ where $g \in \mathfrak{a}^{\lceil p^e t \rceil}$ and $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. It suffices to show such $\phi(F^e_* g)$ are in $J_{e+1}(R, \mathfrak{a}^t)$.

[15]There is no ambiguity if we represent $t = \frac{\nu}{p^e}$ by a different fraction with denominator a power of p, such as $\frac{p\nu}{p^{e+1}}$. See Exercise 5.1.

Let $\psi = \phi \star \pi = \phi \circ F^e_* \pi \in \mathrm{Hom}_R(F^{e+1}_* R, R)$, where $\pi \in \mathrm{Hom}_R(F_* R, R)$ is a splitting of Frobenius. Then since $\pi \circ F$ is the identity on R, we have

$$\phi(F^e_* g) = \phi(F^e_* \pi(F(g))) = \psi(F^{e+1}_* g^p).$$

But $\mathfrak{a}^{p\lceil p^e t\rceil} \subseteq \mathfrak{a}^{\lceil p^{e+1} t\rceil}$ for all e, so $g^p \in \mathfrak{a}^{\lceil p^{e+1} t\rceil}$. This means $\psi(F^{e+1}_* g^p) \in J_{e+1}(R, \mathfrak{a}^t)$ as needed. $\square$

Because the images of (5.1.1) play such an important role, it is helpful to name them:

Definition 5.3 ([**BMS08**], *cf.* [**KLZ09**])**.** Let S be an F-finite regular ring. For any ideal $\mathfrak{b} \subseteq S$, the image of the map

$$F^e_* \mathfrak{b} \cdot \mathrm{Hom}_S(F^e_* S, S) \xrightarrow{\psi \mapsto \psi(F^e_* 1)} S$$

is denoted by $\mathfrak{b}^{[1/p^e]} = J_e(R, \mathfrak{b})$ and called the **e-th Frobenius root of $\mathfrak{b}$**. In particular, the **test ideal** of a pair $(S, \mathfrak{a}^t)$ is $(\mathfrak{a}^{\lceil tp^e\rceil})^{[1/p^e]}$ for $e \gg 0$.

Remark 5.4. The Frobenius root $\mathfrak{b}^{[1/p^e]}$ of an ideal $\mathfrak{b}$ in an F-finite regular ring S can also be described as the *unique smallest ideal* J such that $\mathfrak{b} \subseteq J^{[p^e]}$. See Exercise 5.6. This idea of Frobenius roots and using it as an approach to the test ideal was first described in [**BMS08**].

Caution 5.5. Lemma 5.2 produces a stable ideal $J(R, \mathfrak{a}^t)$ associated to any pair $(R, \mathfrak{a}^t)$ where R a Noetherian Frobenius split ring—this ideal, unfortunately, is *not* the test ideal in general. However, in the special case that R is strongly F-regular (instead of being split), $J_e(R, \mathfrak{a}^t)$ is the test ideal for $e \gg 0$. See Subsection 5.2 below for how to correct for this issue.

5.1. Computation of test ideals in regular rings. The following useful tool lets us compute many test ideals.

Proposition 5.6 (Key computational tool)**.** *Suppose that S is an F-finite regular ring such that $F^e_* S$ has free S-basis $F^e_* b_1, \ldots, F^e_* b_m$. For $f \in S$, write*

$$F^e_* f = \sum_{i=1}^m a_i F^e_* b_i = F^e_* \sum_{i=1}^m a_i^{p^e} b_i.$$

Then

$$(f)^{[1/p^e]} = (a_1, \ldots, a_m).$$

Furthermore, if an ideal J is generated by $f_1, \ldots, f_t$, then for all $e \in \mathbb{N}$

$$J^{[1/p^e]} = (f_1)^{[1/p^e]} + \cdots + (f_t)^{[1/p^e]}.$$

Proof. The first statement was proven in Exercise 2.1, for the second statement, observe that by linearity,

$$\phi(F^e_*(f_1, \ldots, f_t)) = \phi(F^e_* f_1 S) + \cdots + \phi(F^e_* f_t S)$$

for all $\phi \in \mathrm{Hom}_S(F^e_* S, S)$. So $(f_1, \ldots, f_t)^{[1/p^e]} = (f_1)^{[1/p^e]} + \cdots + (f_t)^{[1/p^e]}$. $\square$

Corollary 5.7. *Let S be an F-finite regular ring, and $\mathfrak{a} = (f) \subseteq S$ a principal ideal of S. Then then $\tau(S, \mathfrak{a}^{\frac{1}{p^e}}) = (\mathfrak{a})^{[1/p^e]}$.*

Proof. This follows immediately from Exercise 5.1 and Proposition 5.6. $\square$

Example 5.8. Suppose $S = \mathbb{F}_3[x,y]$ and let $f = yx^3 + x^2y^4 - y^{12}$. Then

$$(f)^{[1/3]} = (x, y, y^4) = (x, y).$$

This follows immediately from Proposition 5.6, after writing

$$F_*f = xF_*y + yF_*x^2y - y^4F_*1.$$

Finally, by Corollary 5.7, the test ideal is

$$\tau(S, f^{1/3}) = (x, y).$$

Caution 5.9. Corollary 5.7 is not true if $\mathfrak{a}$ is not (at least locally) principal. For instance, in $R = \mathbb{F}_2[x,y]$ with $\mathfrak{a} = (x^2, y^2)$, we easily see that $\mathfrak{a}^{[1/2]} = (x, y)$, but

$$(\mathfrak{a}^2)^{[1/4]} = (x^4, x^2y^2, y^4)^{[1/4]} = R.$$

It follows that $\tau(R, \mathfrak{a}^1) = R \neq \mathfrak{a}^{[1/2]}$.

Remark 5.10. If $\mathfrak{a} = S$ with S still regular, then $\tau(S, \mathfrak{a}^t) = S$ for all $t \geq 0$. If $\mathfrak{a} = (0)$, then $\tau(S, \mathfrak{a}^t) = 0$ for all $t > 0$. We make the convention that $(0)^0 = S$ so that $\tau(S, (0)^0) = S$.

5.2. Test ideals of pairs in reduced rings.

If R is F-finite and *regular*, it turns out that for any fixed non-zerodivisor c, the test ideal $\tau(S, \mathfrak{a}^t)$ is the image of the "evaluation at $F^e_* c$" map

(5.10.1) $$F^e_*\mathfrak{a}^{\lceil p^e t\rceil} \cdot \mathrm{Hom}_S(F^e_*S, S) \longrightarrow S \quad \text{sending} \quad \phi \mapsto \phi(F^e_*c)$$

for all sufficiently large e. We will show this immediately below in Lemma 5.12. However, this perspective also leads us to an alternate definition of the test ideal of pairs even when the ring is non-regular.

Definition 5.11 ([HY03, Tak04b]). Let R be a Noetherian F-finite reduced ring, $\mathfrak{a} \subseteq R$ an ideal, and $t \geq 0$ a real number.

(5.11.1)
$$\tau(R, \mathfrak{a}^t) = \sum_{e\in\mathbb{N}} \mathrm{image}\left[\left(F^e_*(\mathfrak{a}^{\lceil tp^e\rceil}\tau(R))\right) \cdot \mathrm{Hom}_R(F^e_*R, R) \xrightarrow{\phi\mapsto\phi(F^e_*1)} R\right].$$

In particular, since in a regular ring $\tau(R) = R$, we see our definition of a test ideal agrees with the one we gave before in Definition 5.1.

If instead of computing $\tau(R)$, you know a single strong test element c that is a non-zerodivisor, we can still compute $\tau(R, \mathfrak{a}^t)$.

Lemma 5.12. *With notation as in Definition 5.11,*

(●) *suppose $c \in R$ is a strong test element and a non-zerodivisor.*

Then for any integer $e_0 > 0$

$$\begin{aligned}
&\textstyle\sum_{e \geq e_0} \text{image}\left[\left(F^e_* \mathfrak{a}^{\lceil tp^e \rceil}\right) \cdot \operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* c)} R\right] \\
(5.12.1) \qquad &= \textstyle\sum_{e \geq e_0} \text{image}\left[\left(F^e_*(c\, \mathfrak{a}^{\lceil tp^e \rceil})\right) \cdot \operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* 1)} R\right] \\
&= \tau(R, \mathfrak{a}^t).
\end{aligned}$$

In particular, the formula for the test ideal in (5.10.1) holds in a regular ring for any non-zerodivisor c.

Remark 5.13. Assuming $t > 0$ and $\mathfrak{a}$ has height 0, we can weaken the hypothesis that $c \in \tau(R)$ is a *non-zerodivisor* in Lemma 5.12. Those primarily interested in the domain setting (or the setting where $\mathfrak{a}$ has positive height) are invited to skip what follows straight to the proof of Lemma 5.12.

Suppose R is reduced with minimal primes $Q_1, \dots, Q_t, \dots, Q_m$ with $0 \neq \mathfrak{a}_{Q_i} \subseteq R_{Q_i}$ for $i = 1, \dots, t$ and with $\mathfrak{a}_{Q_i} = 0$ for $i = t+1, \dots, m$. In the statement of Lemma 5.12, it then suffices to instead assume instead that:

(●′) *c is a strong test element which is not contained in any of the minimals primes $Q_1, \dots, Q_t$.*

We make some preliminary remarks. Recall by Exercise 6.25 in Chapter 1 that $\tau(R) = \tau_{\not\subseteq Q_1,\dots,Q_t}(R) + \tau_{\not\subseteq Q_{t+1},\dots Q_m}(R)$[16]. Therefore, since $\mathfrak{a}\, \tau_{\not\subseteq Q_{t+1},\dots Q_m} = 0$, we see that

$$\tau(R, \mathfrak{a}^t) = \sum_{e \in \mathbb{N}} \text{image}\left[\left(F^e_*(\mathfrak{a}^{\lceil tp^e \rceil} \tau_{\not\subseteq Q_1,\dots,Q_t}(R))\right) \cdot \operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* 1)} R\right].$$

Since $\tau_{\not\subseteq Q_1,\dots,Q_t}(R) \cap \tau_{\not\subseteq Q_{t+1},\dots Q_m}(R) = 0$ any $c \in \tau(R)$ can be written uniquely as $c = c_1 + c_2$ with $c_1 \in \tau_{\not\subseteq Q_1,\dots,Q_t}(R) \subseteq Q_{t+1} \cap \dots \cap Q_m$ and

[16] Here $\tau_{\not\subseteq Q_1,\dots,Q_m}(R)$ is the smallest uniformly compatible ideal not contained in $Q_1, \dots, Q_t$; it is contained in $Q_{t+1} \cap \dots \cap Q_m$ however. Indeed, it is the set of strong test elements contained in $Q_{t+1} \cap \dots \cap Q_m$. For details, see the exercises in Subsection 5.6.1 and Subsection 6.5.1 in Chapter 1

$c_2 \in \tau_{\not\subseteq Q_{t+1},\dots Q_m}(R)$. It immediately follows that

$$\begin{aligned}&\textstyle\sum_{e\geq e_0} \operatorname{image}\left[\left(F^e_*\mathfrak{a}^{\lceil tp^e\rceil}\right)\cdot \operatorname{Hom}_R(F^e_*R,R) \xrightarrow{\phi\mapsto\phi(F^e_*c)} R\right]\\ =&\textstyle\sum_{e\geq e_0} \operatorname{image}\left[\left(F^e_*\mathfrak{a}^{\lceil tp^e\rceil}\right)\cdot \operatorname{Hom}_R(F^e_*R,R) \xrightarrow{\phi\mapsto\phi(F^e_*c_1)} R\right].\end{aligned}$$

Hence, instead of assuming c is a non-zerodivisor, we may assume that

($\bullet''$) *$c \in Q_{t+1}\cap\cdots\cap Q_m$ is a strong test element which is not contained in any of the minimals primes $Q_1,\dots,Q_t$.*

Under this latter assumption, we will add footnotes to the proof below to explain where modification is necessary to obtain our desired generalization.

Proof of Lemma 5.12. Since $c\in\tau(R)$, the containment $\ldots\subseteq\tau(R,\mathfrak{a}^t)$ is clear from the definition.

Next, fix another non-zerodivisor[17] $d\in\tau(R)$ and corresponding integer f_0. The sum in (5.12.1) is generated by elements $\phi(F^e_*cg)$ as we range over all $g\in\mathfrak{a}^{\lceil tp^e\rceil}$ and all $\phi\in\operatorname{Hom}_R(F^e_*R,R)$ (for all $e\geq e_0$). We claim that $\phi(F^e_*cg)$ can be written as $\phi'(F^{e'}_*dh)$ where $h\in\mathfrak{a}^{\lceil tp^{e'}\rceil}$ and $\phi'\in\operatorname{Hom}_R(F^{e'}_*R,R)$ for $e'\geq e'_0$. Indeed, because d is a non-zerodivisor[18] and c is a strong test element, there exists[19] $\psi\in\operatorname{Hom}_R(F^f_*R,R)$ such that $\psi(F^f_*d)=c$ for some $f\gg 0$ (in particular, we may assume $f\geq f_0$). Now let $e'=e+f\geq f_0$, $\phi'=\phi\star\psi\in\operatorname{Hom}_R(F^{e'}_*R,R)$, and $h=g^{p^f}\in(\mathfrak{a}^{\lceil tp^e\rceil})^{[p^f]}\subset\mathfrak{a}^{\lceil tp^{e+f}\rceil}$. One easily checks that

$$\phi(F^e_*cg)=\phi'(F^{e'}_*dh).$$

This implies that

$$\begin{aligned}&\sum_{e\geq e_0} \operatorname{image}\left[\left(F^e_*\mathfrak{a}^{\lceil tp^e\rceil}\right)\cdot \operatorname{Hom}_R(F^e_*R,R) \xrightarrow{\phi\mapsto\phi(F^e_*c)} R\right]\\ \subseteq&\sum_{e\geq f_0} \operatorname{image}\left[\left(F^e_*\mathfrak{a}^{\lceil tp^e\rceil}\right)\cdot \operatorname{Hom}_R(F^e_*R,R) \xrightarrow{\phi\mapsto\phi(F^e_*d)} R\right]\end{aligned} \tag{5.13.1}$$

Taking $d=c$ shows that these sums are independent of the choice of e_0 (and so we may assume $e_0=f_0=1$ and our sums are over $e\in\mathbb{N}$).

[17] Respectively, $d\in Q_{t+1},\dots,Q_m$, $d\in\tau(R)$ is not contained in any $Q_1,\dots,Q_t$ from Remark 5.13 if doing the more general statement.

[18] Respectively, not contained in any of $Q_1,\dots,Q_t$.

[19] by Exercise 6.27 from Chapter 1

On the other hand, reversing the roles of d and c shows the sums in (5.13.1) are equal. Finally, notice that the double sum

$$\sum_{c\in\tau(R)}\sum_{e\in\mathbb{N}} \text{image}\left[\left(F^e_*\mathfrak{a}^{\lceil tp^e\rceil}\right)\cdot \operatorname{Hom}_R(F^e_*R,R) \xrightarrow{\phi\mapsto\phi(F^e_*c)} R\right]$$

is equal[20] to $\tau(R,\mathfrak{a}^t)$. Furthermore, we may restrict to non-zerodivisor c's in that sum since $\tau(R)$ is generated by non-zerodivisors by Corollary 5.23[21] in Chapter 1. Thus by our work immediately above, the outer sum $\sum_{c\in\tau(R)}$ is superfluous and the result follows. $\square$

Note, our proof also shows that for any $e_0 \geq 0$,

(5.13.2)
$$\tau(R,\mathfrak{a}^t) = \sum_{e\geq e_0} \text{image}\left[\left(F^e_*(\mathfrak{a}^{\lceil tp^e\rceil}\tau(R))\right)\cdot \operatorname{Hom}_R(F^e_*R,R) \xrightarrow{\phi\mapsto\phi(F^e_*1)} R\right].$$

When $\mathfrak{a}$ is the unit ideal (or when $t=0$), the test ideal $\tau(R,\mathfrak{a}^t)$ recovers the test ideal $\tau(R)$ defined in Chapter 1; this follows from the characterization in Corollary 6.16 in Chapter 1.

Another application of this formula is that it lets us address the ambiguity of notation problem mentioned earlier.

Proposition 5.14. *Suppose R is a Noetherian F-finite reduced ring, $\mathfrak{a}\subseteq R$ is an ideal, $t\in\mathbb{R}_{\geq 0}$ and $n\in\mathbb{N}$. Then*

$$\tau(R,\mathfrak{a}^{nt}) = \tau(R,(\mathfrak{a}^n)^t).$$

PROOF. Since $n\lceil tp^e\rceil \geq \lceil tnp^e\rceil$ we have the containment $\tau(R,(\mathfrak{a}^n)^t)\subseteq \tau(R,\mathfrak{a}^{nt})$.

On the other hand, $\lceil tnp^e\rceil + n \geq n\lceil tp^e\rceil$ and so if we choose $c\in\mathfrak{a}^n$ a non-zerodivisor (or, if $\mathfrak{a}$ does not have positive height, choose c not contained in $Q_1,\dots,Q_t$ using the notation of Remark 5.13), then

$$c\,\mathfrak{a}^{\lceil tnp^e\rceil}\subseteq \mathfrak{a}^{n\lceil tp^e\rceil}.$$

[20] respectively, in the double sum take $c\in\tau_{\not\subseteq Q_1,\dots,Q_t}(R)=\tau(R)\cap Q_{t+1}\cap\cdots\cap Q_m$, see Exercise 6.26 in Chapter 1.

[21] respectively generated by elements not in $Q_1,\dots,Q_t$ by Chapter 1 Exercise 5.16

Now, if $d \in \tau(R)$ is a non-zerodivisor (or not in $Q_1, \dots, Q_t$), so is $cd \in \tau(R)$, and so we have by Lemma 5.12 (or Remark 5.13) that

$$\begin{aligned} \tau(R, \mathfrak{a}^{nt}) = &\ \textstyle\sum_{e \in \mathbb{N}} \operatorname{image}\left[\left(F^e_*(dc\, \mathfrak{a}^{\lceil ntp^e \rceil})\right) \cdot \operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* 1)} R \right] \\ \subseteq &\ \textstyle\sum_{e \in \mathbb{N}} \operatorname{image}\left[\left(F^e_*(d\, \mathfrak{a}^{n\lceil tp^e \rceil})\right) \cdot \operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* 1)} R \right] \\ = &\ \tau(R, (\mathfrak{a}^n)^t), \end{aligned}$$

as desired. $\square$

Via a similar argument, we can prove another basic property of test ideals:

Proposition 5.15. *Suppose S is a Noetherian F-finite domain and $\mathfrak{a} \subseteq S$ is an ideal. Then for all $t \geq 0$,*

$$\tau(S, \mathfrak{a}^t) = \tau(S, \overline{\mathfrak{a}}^t)$$

where $\overline{a}$ denotes the integral closure of $\mathfrak{a}$.

PROOF. We may assume that $\mathfrak{a} \neq 0$. Since $\mathfrak{a} \subseteq \overline{\mathfrak{a}}$, the inclusion $\tau(S, \mathfrak{a}^t) \subseteq \tau(S, \overline{\mathfrak{a}}^t)$ follows from Proposition 5.17. For the other inclusion, we use the following fact[22] about integral closure: there exists $\ell \in \mathbb{N}$ such that for all n, $\overline{\mathfrak{a}}^{n+\ell} \subseteq \mathfrak{a}^n$. So taking a non-zerodivisor $c \in \mathfrak{a}^\ell$, we have

$$c\overline{a}^{\lceil tp^e \rceil} \subseteq a^{\lceil tp^e \rceil},$$

for all $e \geq 0$. The result then follows from Lemma 5.12 via an argument quite similar to the one in Proposition 5.14 above. $\square$

5.3. Basic properties of test ideals and Skoda-type theorems. Several basic properties of test ideals follow straightforwardly from the definition:

Proposition 5.16. *Let $\mathfrak{a}$ be an ideal in a Noetherian F-finite reduced ring S, and $t \geq 0$ a real number. For any multiplicative set $W \subseteq S$,*

$$\tau(S, \mathfrak{a}^t) W^{-1} S = \tau(W^{-1} S,\, (\mathfrak{a} W^{-1} S)^t).$$

Furthermore, if S is local and $\widehat{S}$ denotes completion at any ideal, then

$$\tau(S, \mathfrak{a}^t)\widehat{S} = \tau(\widehat{S}, \widehat{\mathfrak{a}}^t).$$

PROOF. The proof follows easily from the fact that all relevant objects commute with localization and completion, so we leave it as Exercise 5.2. $\square$

Proposition 5.17. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in a Noetherian F-finite reduced ring.*

[22] See, *e.g.* [**SH06**, Proposition 1.1.7], *cf.* Chapter 7 Theorem 1.15.

(a) If $\mathfrak{b} \subseteq \mathfrak{a}$, then $\tau(S, \mathfrak{b}^t) \subseteq \tau(S, \mathfrak{a}^t)$ for all $t \geq 0$.
(b) If $s \geq t$, then $\tau(S, \mathfrak{b}^s) \subseteq \tau(S, \mathfrak{b}^t)$ for all $t \geq 0$.

PROOF. See Exercise 5.4. □

Proposition 5.18. *Suppose R is a Noetherian F-finite reduced ring and $\mathfrak{a} \subseteq R$ is an ideal. Then for all $1 \gg \epsilon > 0$ we have that*

$$\tau(R, \mathfrak{a}^t) = \tau(R, \mathfrak{a}^{t+\epsilon}).$$

PROOF. Note we always have the containment $\supseteq$ by Proposition 5.17, so we we prove the reverse. Choose $c \in \tau(R)$ and $x \in \mathfrak{a}$ non-zerodivisors (or not in $Q_1, \dots, Q_t$ if $\mathfrak{a}$, as in Remark 5.13, does not have positive height). By Noetherianity and Lemma 5.12, we know that the test ideal is the finite sum:

$$\tau(R, \mathfrak{a}^t) = \sum_{e=1}^{N} \text{image}\left[\left(F^e_* cx\, \mathfrak{a}^{\lceil tp^e \rceil}\right) \cdot \text{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* 1)} R\right]$$

for some integer N. Observe that $x\, \mathfrak{a}^{\lceil tp^e \rceil} \subseteq \mathfrak{a}^{\lceil tp^e + 1 \rceil} \subseteq \mathfrak{a}^{\lceil (t+1/N)p^e \rceil}$ for all $1 \leq e \leq N$. Thus,

$$\begin{aligned} \tau(R, \mathfrak{a}^t) \subseteq &\ \textstyle\sum_{e=1}^{N} \text{image}\left[\left(F^e_* c\, \mathfrak{a}^{\lceil (t+1/N)p^e \rceil}\right) \cdot \text{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* 1)} R\right] \\ \subseteq &\ \textstyle\sum_{e=1}^{\infty} \text{image}\left[\left(F^e_* c\, \mathfrak{a}^{\lceil (t+1/N)p^e \rceil}\right) \cdot \text{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* 1)} R\right] \\ = &\ \tau(R, \mathfrak{a}^{t+1/N}) \end{aligned}$$

which completes the proof. □

The skew commutativity[23] of the Cartier Algebra can sometimes be helpful in computing test ideals:

Example 5.19. Let $S = \mathbb{F}_2[x, y]$ and $f = xy(x-y) = x^2y - xy^2 \in S$, and let $\Phi \in \text{Hom}_S(F_* S, S)$ be the standard monomial generating map (defined in Proposition 1.3 in Chapter 2). Then

$$\tau(S, f^{1/2}) = \Phi(F_* f S) = (x, y)$$

using Corollary 5.7 and Proposition 5.6. Letting $\phi = \Phi \star f = \Phi \circ F_* f$, recall that

$$\phi^{\star n} = \Phi^{\star n} \circ F^n_* f^{1+2+\cdots+2^{n-1}}$$

for all n (Proposition 4.13 in Chapter 1). This means that

$$\begin{array}{rcccl} \tau(S, f^{1/2}) & = & \Phi(F_* f S) & = & \phi(F_* S) \\ \tau(S, f^{3/4}) & = & \Phi^{\star 2}(F^2_* f^{1+2} S) & = & \phi^{\star 2}(F^2_* S) \\ \tau(S, f^{7/8}) & = & \Phi^{\star 3}(F^3_* f^{1+2+4} S) & = & \phi^{\star 3}(F^3_* S) \end{array}$$

[23]Proposition 4.13 in Chapter 1

and more generally $\tau(S, f^{\frac{2^n-1}{2^n}}) = \phi^n(F^n_* S)$ for all $n > 0$. So the test ideals $\tau(S, f^{\frac{2^n-1}{2^n}})$ form a decreasing sequence of ideals, which stabilizes immediately because $\phi(F_* S) = \phi^{\star 2}(F_* S) = (x, y)$ (Exercise 5.8). It follows that

$$\tau(S, f^{\frac{2^n-1}{2^n}}) = (x, y)$$

for all n. So $\tau(S, f^t) = (x, y)$ for all $t \in [1/2, 1)$ by Proposition 5.17.

Theorem 5.20 (Skoda's theorem for test ideals, [**HY03**, Theorem 2.1], [**HT04**, Theorem 4.2])**.** *Suppose R is a Noetherian F-finite reduced ring and $\mathfrak{a} \subseteq S$ is an ideal generated by m elements. For any $t \geq m$,*

$$\tau(S, \mathfrak{a}^t) = \mathfrak{a}\, \tau(S, \mathfrak{a}^{t-1}).$$

Proof. The trick is the following simple fact (Exercise 4.18 (a)), which holds for any ideal $\mathfrak{a}$ generated by m elements in any ring:

$$\mathfrak{a}^{p^e m} = \mathfrak{a}^{[p^e]} \mathfrak{a}^{p^e(m-1)}. \tag{5.20.1}$$

It follows that for any real $t \geq m$ and any $e \in \mathbb{N}$

$$\mathfrak{a}^{\lceil p^e t \rceil} = \mathfrak{a}^{\lceil p^e(t-m) + p^e m \rceil} = \mathfrak{a}^{p^e m} \mathfrak{a}^{\lceil p^e(t-m) \rceil} = \mathfrak{a}^{[p^e]} \mathfrak{a}^{p^e(m-1)} \mathfrak{a}^{\lceil p^e(t-m) \rceil} = \mathfrak{a}^{[p^e]} \mathfrak{a}^{\lceil p^e(t-1) \rceil}.$$

So also

$$F^e_* \mathfrak{a}^{\lceil p^e t \rceil} = F^e_* (\mathfrak{a}^{[p^e]} \mathfrak{a}^{\lceil p^e(t-1) \rceil}) = \mathfrak{a} F^e_* \mathfrak{a}^{\lceil p^e(t-1) \rceil}.$$

Now applying any $\phi \in \operatorname{Hom}_R(F^e_* R, R)$, we see that

$$\phi(F^e_* \mathfrak{a}^{\lceil p^e t \rceil} \tau(R)) = \mathfrak{a} \phi(F^e_* \mathfrak{a}^{\lceil p^e(t-1) \rceil} \tau(R)).$$

We then see that

$$\begin{aligned}
\tau(R, \mathfrak{a}^t) &= \sum_{e \in \mathbb{N}} \operatorname{image} \left[\big(F^e_* \mathfrak{a}^{\lceil t p^e \rceil} \tau(R)\big) \cdot \operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* 1)} R \right] \\
&= \sum_{e \in \mathbb{N}} \sum_{\substack{\phi \in \\ \operatorname{Hom}_R(F^e_* R, R)}} \phi\big(F^e_* \mathfrak{a}^{\lceil t p^e \rceil} \tau(R)\big) \\
&= \sum_{e \in \mathbb{N}} \sum_{\substack{\phi \in \\ \operatorname{Hom}_R(F^e_* R, R)}} \mathfrak{a} \phi\big(F^e_* \mathfrak{a}^{\lceil p^e(t-1) \rceil} \tau(R)\big) \\
&= \mathfrak{a} \tau(R, \mathfrak{a}^{t-1})
\end{aligned}$$

which is what we wanted to show. $\square$

5.4. The test ideal via generating maps. Any time there exists a generating map $\Phi \in \mathrm{Hom}_R(F_* R, R)$, we may define the test ideal by taking images of under $\Phi^e := \Phi^{\star e}$. This relatively straightforward observations has several important consequences.

Indeed, the following general fact, whose proof is a straightforward exercise in unravelling notation, holds for any ring for which $\mathrm{Hom}_R(F^e_* R, R)$ is a cyclic $F^e_* R$-module:

Lemma 5.21. *Let R be an arbitrary ring of characteristic $p > 0$ such that $\mathrm{Hom}_R(F^e_* R, R)$ is generated by some $\Phi^e \in \mathrm{Hom}_R(F^e_* R, R)$ as an $F^e_* R$-module. Then for any ideal $\mathfrak{b} \subseteq R$, image of the map*

$$F^e_* \mathfrak{b} \cdot \mathrm{Hom}_R(F^e_* R, R) \xrightarrow{\textit{eval at } F^e_* 1} R \tag{5.21.1}$$

is precisely the image $\Phi^e(F^e_* \mathfrak{b})$ of the R-module $F^e_* \mathfrak{b} \subseteq F^e_* R$ under Φ^e.

Remark 5.22. We note some facts related to generating maps.

(a) If $\Phi^d \in \mathrm{Hom}_R(F^d_* R, R)$ is a generating map, so is $\Phi^{de} := (\Phi^d)^{\star e} \in \mathrm{Hom}_S(F^{de}_* S, S)$ by Proposition 5.3 in Appendix A (justifying the notation Φ^{de}).
(b) Note that if Ψ^e and Φ^e are both generators for $\mathrm{Hom}_R(F^e_* R, R)$, then Ψ^e and Φ^e agree up to pre-multiplication by a unit. Such multiplication doesn't change the image of any submodule—in particular, $\Psi^e(F^e_* \mathfrak{b}) = \Phi^e(F^e_* \mathfrak{b})$ for all ideals $\mathfrak{b}$ in R.
(c) Although, in general, an F-finite regular (or quasi-Gorenstein) ring S need not admit a generating map $\Phi^e \in \mathrm{Hom}_S(F^e_* S, S)$, it always does *locally* by (2.2.1).

Generating maps even makes working with $\tau(R)$ easier, so we record a fact we will need shortly.

Lemma 5.23. *Suppose R is a Noetherian F-finite reduced ring and $\Phi \in \mathrm{Hom}_R(F^e_* R, R)$ is a generating map. Then*

$$\Phi(F^e_* \tau(R)) = \tau(R).$$

Proof. Note the containment $\subseteq$ follows from the fact that $\tau(R)$ is compatible with all maps, by Theorem 6.15 in Chapter 1. Since $\Phi(F^e_* \tau(R))$ is also easily seen to be compatible with all maps, the result follows since $\tau(R)$ is the smallest such ideal nonzero at each minimal prime. $\square$

In the regular case, we defined the test ideal via an ascending chain of ideals from Lemma 5.2. The same result holds in the presence of a generating map (so for instance in the quasi-Gorenstein and sufficiently local case).

Corollary 5.24. *Suppose R is a Noetherian F-finite reduced ring and $\Phi \in \mathrm{Hom}_R(F_*R, R)$ is a generating map. Then the ideals*

$$\Phi(F_*\mathfrak{a}^{\lceil tp \rceil}\tau(R)) \subseteq \Phi^2(F_*^2\mathfrak{a}^{\lceil tp^2 \rceil}\tau(R)) \subseteq \cdots \subseteq \Phi^e(F_*^e\mathfrak{a}^{\lceil tp^e \rceil}\tau(R)) \subseteq \dots$$

form an ascending chain which stabilizes to the test ideal $\tau(R, \mathfrak{a}^t)$.

Proof. The proof is in essence the same as that of Lemma 5.2. Using Lemma 5.23 we see that

$$\begin{aligned} \Phi^e(F_*^e\mathfrak{a}^{\lceil tp^e \rceil}\tau(R)) = &\quad \Phi^e(F_*^e\mathfrak{a}^{\lceil tp^e \rceil}\Phi(F_*\tau(R))) \\ = &\quad \Phi^{e+1}(F_*^{e+1}(\mathfrak{a}^{\lceil tp^e \rceil})^{[p]}\tau(R))) \\ \subseteq &\quad \Phi^{e+1}(F_*^{e+1}\mathfrak{a}^{\lceil tp^{e+1} \rceil}\tau(R)) \end{aligned}$$

as desired. That these stabilize to the test ideal follows from the fact that

$$\tau(R, \mathfrak{a}^t) = \sum_{e \in \mathbb{N}} \Phi^e(F_*^e\mathfrak{a}^{\lceil tp^e \rceil}\tau(R)).$$

by Lemma 5.21. $\square$

We immediately obtain the following corollary describing a transformation rule for test ideals under a generating map Φ.

Corollary 5.25. *Suppose R is a Noetherian F-finite reduced ring and $\Phi \in \mathrm{Hom}_R(F_*R, R)$ is a generating map. Then for any ideal $\mathfrak{a} \subseteq R$ and any $e > 0$, we have that*

$$\Phi^e(F_*^e\tau(R, \mathfrak{a}^t)) = \tau(R, \mathfrak{a}^{t/p^e}).$$

Remark 5.26. We will see later in Chapter 5 Section 7 that Corollary 5.29 is a special case of a formula for test ideals under finite ring extensions.

Frobenius roots can also be described using a generating map.

Corollary 5.27. *If S is an F-finite regular ring and $\Phi^e \in \mathrm{Hom}_S(F_*^eS, S)$ is a generating map, then for any ideal $\mathfrak{b} \subseteq S$,*

$$\mathfrak{b}^{[1/p^e]} = \Phi^e(F_*^e\mathfrak{b}).$$

Using Corollary 5.27, we see that Frobenius roots behave very much like Frobenius powers:

Proposition 5.28. *If J is an ideal in an F-finite regular ring S, then $(J^{[1/p^e]})^{[1/p^d]} = J^{[1/p^{e+d}]}$ for all $d, e \in \mathbb{N}$.*

Proof. Equality of ideals can be checked locally, and the formation of $J^{[1/p^e]}$ commutes with localization, so by Remark 5.22 we may assume there is a generating map $\Phi \in \operatorname{Hom}_S(F_*S, S)$. In this case Φ^f is a generating map for $\operatorname{Hom}_S(F^f_*S, S)$, so that

$$J^{[1/p^f]} = \Phi^f(F^f_* J)$$

for all $f > 0$ (Corollary 5.27). Since $\Phi^d \star \Phi^e = \Phi^d \circ F^d_*\Phi^e$ is a generating map for $\operatorname{Hom}_S(F^{d+e}_*S, S)$, Corollary 5.27 again gives

$$J^{[1/p^{d+e}]} = \Phi^d \star \Phi^e(F^{d+e}_* J) = \Phi^d(F^d_*\Phi^e(F^e_* J)) = (J^{[1/p^e]})^{[1/p^d]}.$$

□

In the regular case, Corollary 5.25 becomes the following:

Corollary 5.29. *Let S be an F-finite regular ring, $\mathfrak{a} \subseteq S$ an ideal and $t \geq 0$ a real number. Then*

$$\tau(S, \mathfrak{a}^t)^{[1/p^e]} = \tau(S, \mathfrak{a}^{t/p^e})$$

for all $e > 0$.

5.5. The defining property of the test ideal. In Theorem 6.15 from Chapter 1, we saw that the test ideal was characterized as the unique smallest ideal, containing a nonzero divisor. An analogous result is true here.

For each integer $e \geq 0$ set

$$(\mathscr{C}_R^{\mathfrak{a}^t})_e := (F^e_*\mathfrak{a}^{\lceil t(p^e-1)\rceil}) \cdot \operatorname{Hom}_R(F^e_*R, R) \subseteq \operatorname{Hom}_R(F^e_*R, R) =: (\mathscr{C}_R)_e$$

and form the subring of the Cartier algebra:

$$\mathscr{C}_R^{\mathfrak{a}^t} := \bigoplus_{e\geq 0}(\mathscr{C}_R^{\mathfrak{a}^t})_e \subseteq \bigoplus_{e\geq 0}(\mathscr{C}_R)_e =: \mathscr{C}_R.$$

The fact that this forms a ring (under composition), is essentially the argument in Lemma 4.4, with the key inequality (4.4.1). This is done in detail later in Chapter 8 Example 3.2.

Proposition 5.30. *Let R be a Noetherian F-finite reduced ring. Suppose $\mathfrak{a} \subseteq R$ is an ideal containing a non-zerodivisor. Then the test ideal $\tau(R, \mathfrak{a}^t)$ is the* unique smallest ideal *in R*

(a) compatible with every element of $(\mathscr{C}_R^{\mathfrak{a}^t})_e$ (for each $e \geq 0$), and
(b) containing a non-zerodivisor (i.e. not contained in any minimal prime).

Proof. We first must show that $\tau(R, \mathfrak{a}^t)$ is compatible with each element of $(\mathscr{C}_R^{\mathfrak{a}^t})_e$. It suffices to check this on the generators of $\mathscr{C}(R, \mathfrak{a}^t)$:

$\psi \star cg = \psi \circ F^e_* g$, with $\psi \in \operatorname{Hom}_R(F^e_* R, R)$ and $g \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$. Consider $J_f = \phi(F^f_* \mathfrak{a}^{\lceil tp^f \rceil} \tau(R)) \subseteq \tau(R, \mathfrak{a}^t)$ where $\phi \in \operatorname{Hom}_R(F^f_* R, R)$. Observe that

$$\begin{array}{rl} (\psi \star g) J_f = & (\psi \star g)\phi(F^f_* \mathfrak{a}^{\lceil tp^f \rceil} \tau(R)) \\ = & (\psi \star \phi)\big(F^{e+f}_* (g^{p^f} \mathfrak{a}^{\lceil tp^f \rceil} \tau(R))\big) \\ \subseteq & (\psi \star \phi)\big(F^{e+f}_* (\mathfrak{a}^{p^f \lceil t(p^e-1)\rceil} \mathfrak{a}^{\lceil tp^f \rceil} \tau(R))\big) \\ \subseteq & (\psi \star \phi)\big(F^{e+f}_* (\mathfrak{a}^{\lceil t(p^{e+f})} \tau(R))\big) \\ \subseteq & \tau(R, \mathfrak{a}^t). \end{array}$$

Since $\tau(R, \mathfrak{a}^t)$ is the sum of such J_f, we have proven that $\tau(R, \mathfrak{a}^t)$ is compatible with all maps in $(\mathscr{C}^{\mathfrak{a}^t}_R)_e$.

Now suppose that J is an ideal containing a non-zerodivisor and compatible with all $\phi \in (\mathscr{C}^{\mathfrak{a}^t}_R)_e$ for all $e > 0$. We need to show that $\tau(R, \mathfrak{a}^t) \subseteq J$. Since both are ideals containing non-zerodivisor, there must exist a non-zerodivisor $x \in \tau(R, \mathfrak{a}^t) \cap J \subseteq \tau(R)$. Picking $y \in \mathfrak{a}^{\lceil t \rceil}$ a non-zerodivisor, and since $\mathfrak{a}^{\lceil tp^e \rceil} \subseteq \mathfrak{a}^{\lceil t(p^e-1)\rceil}$, we see from Lemma 5.12 that:

$$\begin{array}{rll} \tau(R, \mathfrak{a}^t) = & \sum_e \sum_{\phi \in (\mathscr{C}_R)_e} \phi(F^e_*(xy\, \mathfrak{a}^{\lceil tp^e \rceil})) & \\ \subseteq & \sum_e \sum_{\phi \in (\mathscr{C}_R)_e} \phi(F^e_*(xy\, \mathfrak{a}^{\lceil t(p^e-1)\rceil})) & \\ = & \sum_e \sum_{\psi \in (\mathscr{C}^{\mathfrak{a}^t}_R)_e} \psi(F^e_*(xyR)) & \text{(this is } \subseteq J \text{ since } x \in J) \\ \subseteq & \sum_e \sum_{\phi \in (\mathscr{C}_R)_e} \phi(F^e_*(x\, \mathfrak{a}^{\lceil t \rceil}\, \mathfrak{a}^{\lceil t(p^e-1)\rceil})) & \\ \subseteq & \sum_e \sum_{\phi \in (\mathscr{C}_R)_e} \phi(F^e_*(x \mathfrak{a}^{\lceil tp^e \rceil})) & \\ = & \tau(R, \mathfrak{a}^t). & \end{array}$$

Since the third line is a subset of J, our result is proven. ☐

Remark 5.31. By a similar argument, one can also show that $\tau(R, \mathfrak{a}^t)$ is the smallest ideal containing a non-zerodivisor compatible with all elements of $(F^e_* \mathfrak{a}^t) \cdot \operatorname{Hom}_R(F^e_* R, R)$ (for all $e > 0$). One can also show that $\tau(R, \mathfrak{a}^t)$ is set made up of all strong test elements for the Cartier algebra $\mathscr{C}^{\mathfrak{a}^t}_R$, see Chapter 8 Example 3.2 and **??**.

Remark 5.32. If $\mathfrak{a}$ does not have positive height but R has minimal primes $Q_1, \dots, Q_t, \dots, Q_m$ and $\mathfrak{a}_{Q_1}, \dots, \mathfrak{a}_{Q_t} \neq 0$ but $\mathfrak{a}_{Q_{t+1}}, \dots, \mathfrak{a}_{Q_m} = 0$, then $\tau(R, \mathfrak{a}^t)$ is the smallest

(a) compatible with every element of $\mathcal{C}_R(\mathfrak{a}^t)_e$ (for each $e \geq 0$), and
(b') not contained in $Q_1, \dots, Q_t$.

The proof is unchanged in view of Remark 5.13.

Corollary 5.33. *Suppose R is a Noetherian F-finite reduced ring and $\mathfrak{a} \subseteq R$ is an ideal containing a non-zerodivisor. Then for any $c \in \tau(R, \mathfrak{a}^t)$ a non-zerodivisor, we have that*

$$\tau(R, \mathfrak{a}^t) = \sum_e \sum_{\phi \in (\mathscr{C}^{\mathfrak{a}^t}_R)_e} \phi(F^e_* cR).$$

PROOF. The sum is by definition the smallest compatible ideal containing c. It thus must equal $\tau(R, \mathfrak{a}^t)$ by Proposition 5.30. $\square$

Corollary 5.34 (*cf.* [**Vas98, Sch08**]). *Suppose R is a Noetherian F-finite ring and $(R, \mathfrak{a}^t)$ is a sharply F-split pair. Then $R/\tau(R, \mathfrak{a}^t)$ is F-pure.*

PROOF. Note since the pair is sharply F-split, we see that $\mathfrak{a}$ contains a non-zerodivisor and that R is reduced. We see that $\tau(R, \mathfrak{a}^t)$ is compatible with some Frobenius splitting by Proposition 5.30. The result follows. $\square$

5.6. Triviality of the test ideal and strong F-regularity. As in the non-pair setting, the test ideal defines the locus of non-strongly F-regular points:

Theorem 5.35. *Let R be an F-finite Noetherian reduced ring, $\mathfrak{a} \subseteq R$ an ideal, and $t \geq 0$. The test ideal $\tau(R, \mathfrak{a}^t)$ defines the closed locus of* $\operatorname{Spec} R$ *where the pair $(R, \mathfrak{a}^t)$ fails to be strongly F-regular. In particular, the pair $(R, \mathfrak{a}^t)$ is strongly F-regular if and only if $\tau(R, \mathfrak{a}^t) = R$.*

PROOF OF THEOREM 5.35. Without loss of generality, we may assume $t > 0$, since the case where $t = 0$ follows from Corollary 5.25 in Chapter 1. Note also that the first statement follows immediately from the second, because both strong F-regularity and the test ideal behave well under localization (Theorem 4.24 and Exercise 5.13). To prove the second statement, we may assume that R is local and that $\mathfrak{a}$ contains a non-zerodivisor, for otherwise $(R, \mathfrak{a}^t)$ is never strongly F-regular and $\tau(R, \mathfrak{a}^t) \neq R$.

Assume $(R, \mathfrak{a}^t)$ is strongly F-regular and pick $c \in \tau(R, \mathfrak{a}^t)$ a non-zerodivisor. We know that the map

$$\mathscr{C}_R^{\mathfrak{a}^t} = (F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil}) \cdot \operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* c)} R$$

is surjective for some e. This implies that $1 = \phi(F^e_* c) \subseteq \phi(F^e_* \tau(R, \mathfrak{a}^t)) \subseteq \tau(R, \mathfrak{a}^t)$ for some $\phi \in (\mathscr{C}_R^{\mathfrak{a}^t})_e$ proving that $\tau(R, \mathfrak{a}^t) = R$.

Conversely, suppose $\tau(R, \mathfrak{a}^t) = R$ and pick $c \in R$ a non-zerodivisor. We want to show that $\phi(F^e_* c) = 1$ for some $\phi \in \mathscr{C}_R^{\mathfrak{a}^t}$. It suffices to show that $\psi(F^e_* cd) = 1$ for some $d \in R$ (since we can set $\phi = \psi \star d$). Thus we may assume that $cd \in \tau(R, \mathfrak{a}^t)$. By Corollary 5.33, we have that $1 \in \tau(R, \mathfrak{a}^t) = \sum_e \sum_{\phi \in (\mathscr{C}_R^{\mathfrak{a}^t})_e} \phi(F^e_* cdR)$. But R is local so this means one of the terms in the sum contains 1. The result follows. $\square$

5.7. Exercises.

Exercise 5.1. Let S be a Frobenius split ring, $f \in S$, and $t = \frac{\nu}{p^e}$ where $\nu \in \mathbb{N}$. Show that the image of the map

$$\mathrm{Hom}_S(F^e_* S, S) \xrightarrow{\text{eval at } f^\nu} S \quad \text{sending} \quad \phi \mapsto \phi(F^e_* f^\nu) \tag{5.35.1}$$

is the same as the image of

$$\mathrm{Hom}_S(F^{e+1}_* S, S) \xrightarrow{\text{eval at } f^{p\nu}} S \quad \text{sending} \quad \phi \mapsto \phi(F^{e+1}_* f^{p\nu}). \tag{5.35.2}$$

In particular, conclude that $\tau(S, f^{\frac{\nu}{p^e}})$ is the image of the map (5.35.1).

Hint: Examine the proof of Lemma 5.2.

Exercise 5.2. Prove Proposition 5.16.

Exercise 5.3. Let $S = \mathbb{F}_p[x_1, \dots, x_n]$ and suppose that $\Phi : F^e_* S \longrightarrow S$ is a generating map for $\mathrm{Hom}_S(F^e_* S, S)$. Let $\phi = \Phi \circ F^e_* x_1^{p^e-1} \cdots x_n^{p^e-1} = \Phi \star x_1^{p^e-1} \cdots x_n^{p^e-1}$. Show that ϕ is compatible to every ideal of the form $(x_{i_1}, \dots, x_{i_t})$. Conclude that every ideal generated by square free monomial ideals[24] is compatible with ϕ.

Hint: Use Exercise 2.1 and Chapter 1 Proposition 6.4. For the statement that these are the only compatible ideals, see Chapter 5 Proposition 2.9 and Corollary 2.10.

Exercise 5.4. Prove Proposition 5.17.

Exercise 5.5. Let $f \in \mathfrak{m}$ be an element in a regular local F-finite ring $(S, \mathfrak{m})$. Suppose that $F^e_* S$ is freely generated by $F^e_* e_1, F^e_* e_2, \dots, F^e_* e_n$ for some elements $e_i \in S$. Expressing $F^e_* f$ uniquely as $a_1 F^e_* e_1 + a_2 F^e_* e_2 + \dots + a_n F^e_* e_n$, show that $(a_1, a_2, \dots, a_n)$ is the smallest ideal J of S such that $f \in J^{[p^e]}$.

Hint: Show that $f \in J^{[p^e]}$ if and only if $F^e_* f \in J F^e_* S$.

Exercise 5.6. Let S be an F-finite regular ring.

(a) For an arbitrary collection $\{J_\lambda\}$ of ideals in S, show that
$$\bigcap_\lambda J_\lambda^{[p^e]} = (\bigcap_\lambda J_\lambda)^{[p^e]}.$$
(b) For any ideal $\mathfrak{b}$ in S, show that there exists a *smallest* ideal J such that $\mathfrak{b} \subseteq J^{[p^e]}$.
(c) Prove that $\mathfrak{b}^{[1/p^e]}$ is the smallest ideal $J \subseteq S$ such that $\mathfrak{b} \subseteq J^{[p^e]}$.

[24]Those more geometrically minded might observe that these ideals are the strata of $\mathrm{div}(x_1 \dots x_n)$, see Appendix B Definition 7.3.

Hint: For (a), reduce to the local case and use the fact that F_*S is free over S. For (b), consider the intersection of all such J.

Exercise 5.7. Show that if $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2$ is an ideal in a regular F-finite product ring $S = S_1 \times S_2$, then for all $t \geq 0$, $\tau(S, \mathfrak{a}^t) = \tau(S_1, \mathfrak{a}_1^t) \times \tau(S_2, \mathfrak{a}_2^t)$.

Exercise 5.8. Let $S = \mathbb{F}_p[x, y]$ and $\mathfrak{a} = (xy(x - y))$. Compute $\tau(S, \mathfrak{a}^t)$ for $t = 3/4$ and $p = 2$.

Exercise 5.9. Let R be a strongly F-regular ring. For any ideal $\mathfrak{a} \subseteq$ and $t \geq 0$, show that $(R, \mathfrak{a}^t)$ is strongly F-regular if and only if $(R, \mathfrak{a}^t)$ is strongly Frobenius split.

Hint: Use the test ideal.

Exercise 5.10. Let R be a Frobenius split ring, $\mathfrak{a} \subseteq R$ an ideal in R, and $t \geq 0$ a real number. Fix any $c \in R$, let J_e denote the image of the map $\mathfrak{a}^{\lceil tp^e \rceil} \cdot \operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\phi \mapsto \phi(F^e_* c)} R$. Prove that the J_e are an increasing sequence of ideals.

Hint: Adapt the proof of Lemma 5.2.

Exercise 5.11. Suppose S is an F-finite regular ring. Show that there exists $c \in S$ so that the test ideal $\tau(S, \mathfrak{a}^t)$ is equal to each of the following for some $e \gg 0$:

$$(c\mathfrak{a}^{\lceil tp^e \rceil})^{[1/p^e]},\ (c\mathfrak{a}^{\lfloor tp^e \rfloor})^{[1/p^e]},\ (c\mathfrak{a}^{\lceil t(p^e-1) \rceil})^{[1/p^e]},\ (c\mathfrak{a}^{\lfloor t(p^e-1) \rfloor})^{[1/p^e]}.$$

Exercise 5.12. Suppose that R is an F-finite ring and that $g^n = f \in R$. Show that (R, f^t) is strongly F-regular if and only if (R, g^{nt}) is strongly F-regular. Thus we avoid the ambiguity issues that plagued sharp Frobenius splitting; see Exercise 4.7.

Exercise 5.13. Let R be an F-finite Noetherian reduced ring, $\mathfrak{a} \subseteq R$ an ideal, and $t \geq 0$. Prove that for any multiplicative set $W \subseteq R$,

$$\tau(W^{-1}R, (\mathfrak{a}W^{-1}R)^t) = \tau(R, \mathfrak{a}^t)W^{-1}R.$$

Exercise 5.14. Consider the ring $R = \mathbb{F}_p[u, x, y, z]/(x^2 - y^2 z)$ where $p \neq 2$. Verify that R is Frobenius split but not strongly F-regular. Use this fact to show that $\tau(R, \mathfrak{a}^t)$ is proper for all non-negative $t \in \mathbb{R}$. Compute the F-pure threshold of the pair (R, u) to find an example showing that the strong F-regularity hypothesis in Theorem 6.1 can not be weakened to Frobenius split.

Hint: For the first statement, use Fedder's criterion and the fact that strongly F-regular rings are normal.

Exercise 5.15 (Mixed Test Ideals)**.** Suppose that S is an F-finite regular ring and $\mathfrak{a}_1, \ldots, \mathfrak{a}_m \subseteq R$ are ideals. For any $t_1, \ldots, t_m \in \mathbb{R}_{\geq 0}$ Define

$$\tau(S, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m}) = (\mathfrak{a}_1^{\lceil t_1 p^e \rceil} \cdots \mathfrak{a}_m^{\lceil t_m p^e \rceil})^{[1/p^e]}$$

for $e \gg 0$. Show that this definition is indeed independent of $e \gg 0$. This is sometimes called a *mixed test ideal.*

Exercise 5.16 (Skoda's theorem for mixed test ideals)**.** Suppose S is an F-finite regular ring, and $\mathfrak{a} \subseteq S$ is an ideal generated by m elements and $\mathfrak{b} \subseteq S$ an arbitrary ideal. Show that for all $t \geq m$ and $s \geq 0$,

$$\tau(S, \mathfrak{a}^t \mathfrak{b}^s) = \mathfrak{a} \cdot \tau(S, \mathfrak{a}^{t-1} \mathfrak{b}^s).$$

Exercise 5.17. With the notion of mixed test ideals as defined in Exercise 5.15, show that

$$\tau(R, \mathfrak{a}^s \mathfrak{a}^t) = \tau(R, \mathfrak{a}^{s+t}).$$

This avoids any ambiguity when $\mathfrak{a} = \mathfrak{b}$.

Exercise 5.18. The **analytic spread** of an ideal $\mathfrak{a}$ in a Noetherian ring R is the smallest integer ℓ such that $\mathfrak{a}$ is integral over an ideal generated by ℓ elements. Prove the following version of the Skoda theorem: for a Noetherian F-finite reduced ring R and ideal $\mathfrak{a}$ with analytic spread ℓ, $\tau(R, \mathfrak{a}^t) = \mathfrak{a}\tau(R, \mathfrak{a}^{t-1})$ for all $t \geq \ell$. .

6. Frobenius jumping numbers

We now discuss *Frobenius jumping numbers,* or *F-jumping numbers* for short, a whole spectrum of numbers generalizing the F-pure threshold. We take a moment to connect the F-pure threshold to the behavior of the test ideal.

Let R be an F-finite ring, and $\mathfrak{a} \subseteq R$ an ideal of R. We know that

$$\tau(R) = \tau(R, \mathfrak{a}^{\epsilon})$$

for $1 \gg \epsilon > 0$ by Proposition 5.18. If R is strongly F-regular, this also equals R.

Theorem 6.1. *Let R be a strongly F-regular ring, and $\mathfrak{a} \subseteq R$ a proper ideal of positive height. Then for each $t \geq 0$, $\tau(R, \mathfrak{a}^t) = R$ if and only if $t < \operatorname{fpt}(\mathfrak{a})$. That is,* $\operatorname{fpt}(\mathfrak{a})$ *is the minimum value of t such that $\tau(R, \mathfrak{a}^t)$ is proper.*

Remark 6.2. If $\mathfrak{a}$ does not have positive height, then it consists of zerodivisors and $\operatorname{fpt}(\mathfrak{a}) = -\infty$; in this case, the first statement in Theorem 6.1 is vacuous and the second meaningless.

Remark 6.3. The hypothesis that R is strongly F-regular in Theorem 6.1 is necessary: if R is not strongly F-regular, then $\tau(R, \mathfrak{a}^t) \subseteq \tau(R)$ is a *proper* ideal of R for *all* $t \geq 0$. It is not hard to find examples of Frobenius split rings that are not strongly F-regular and admit ideals of positive F-pure threshold; such pairs never satisfy the conclusion of Theorem 6.1. See Exercise 5.14.

To prove Theorem 6.1 we use the following lemma.

Lemma 6.4. *Suppose R is strongly F-regular, $\mathfrak{a} \subseteq R$ is an ideal and $(R, \mathfrak{a}^t)$ is sharply F-split for some $t > 0$. Then $(R, \mathfrak{a}^s)$ is strongly F-regular for all $0 \leq s < t$.*

Proof. Since $\mathfrak{a}^{\lceil t(p^e-1)\rceil} \subseteq \mathfrak{a}^{\lceil sp^e \rceil}$ (*cf.* Proposition 4.17 (c)) we notice that

$$(F^e_* \mathfrak{a}^{\lceil sp^e \rceil}) \cdot \operatorname{Hom}_R(F^e_* R, R) \xrightarrow{\text{eval at } F^e_* 1} R$$

surjects for some $e > 0$. In other words, we just showed that $J_e(R, \mathfrak{a}^t) = R$ for some $e > 0$ (and hence for all $e \gg 0$). But since R is strongly F-regular, $\tau(R) = R$ and so $\tau(R, \mathfrak{a}^t) = J_e(R, \mathfrak{a}^t)$ for $e \gg 0$. The result follows. ☐

Proof of Theorem 6.1. If $\tau(R, \mathfrak{a}^t) = R$, then $(R, \mathfrak{a}^t)$ is strongly F-regular. It follows that $(R, \mathfrak{a}^{t+\epsilon})$ is strongly F-regular by Proposition 5.18 and hence also sharply F-split, and so $t < t + \epsilon \leq \operatorname{fpt}(\mathfrak{a})$.

Conversely, if $t < t' < \operatorname{fpt}(\mathfrak{a})$, then we know that $(R, \mathfrak{a}^{t'})$ is sharply F-split by definition, and so by Lemma 6.4 we see that $(R, \mathfrak{a}^t)$ is strongly F-regular, the result follows. ☐

Corollary 6.5. *Let $\mathfrak{a}$ be an ideal in a strongly F-regular ring R. Then*

$$\begin{aligned}\operatorname{fpt}(\mathfrak{a}) &= \sup\left\{t \mid (R, \mathfrak{a}^t) \ \text{ is strongly } F\text{-regular}\right\} \\ &= \min\left\{t \mid (R, \mathfrak{a}^t) \ \text{ is not strongly } F\text{-regular}\right\}.\end{aligned}$$

6.1. The definition of F-jumping numbers. We have seen that $\tau(R, \mathfrak{a}^t) = R$ for very small (positive) t, and that the "first" value of t where $\tau(R, \mathfrak{a}^t)$ "jumps" to a proper ideal occurs when $t = \operatorname{fpt}(\mathfrak{a})$ (Theorem 6.1). More generally, we know that

$$\tau(R, \mathfrak{a}^s) \supseteq \tau(R, \mathfrak{a}^t) \qquad \text{whenever } s < t,$$

so we can consider other values of t for which the test ideal "jumps":

Definition 6.6. Let $\mathfrak{a}$ be an ideal in an F-finite Noetherian ring R. A positive real number t is an **F-jumping number for the pair** $(R, \mathfrak{a})$ if

$$\tau(R, \mathfrak{a}^{t-\epsilon}) \supsetneq \tau(R, \mathfrak{a}^t)$$

for all $0 < \epsilon \leq t$. In the case that $\mathfrak{a} = (f)$ is principal, we say that t is an **F-jumping number for** f.

Example 6.7. (Compare with Example 3.6) Let $S = \mathbb{F}_p[x, y]$ and $\mathfrak{a} = (x^2y)$. The jumping numbers of $\mathfrak{a}$ are the positive half-integers:

- For $0 \leq t < 1/2$, we have $\tau(S, \mathfrak{a}^t) = S$. To see this, observe that for t in this range, $\lceil p^e t \rceil \leq \lfloor \frac{p^e-1}{2} \rfloor$ for all $e \gg 0$. This implies that $F^e_*(x^2y)^{\lceil tp^e \rceil}$ is part of a free basis for F^e_*S over S, so by Proposition 5.6,
$$1 \in ((x^2y)^{\lceil tp^e \rceil})^{[1/p^e]} \subseteq \tau(S, \mathfrak{a}^t).$$

- For $1/2 \leq t < 1$, we have $\tau(S, \mathfrak{a}^t) = (x)$. Indeed, in this case,
$$p^e > \lceil tp^e \rceil \geq \frac{p^e}{2},$$
so that the monomial $(x^2y)^{\lceil tp^e \rceil}$ factors as $x^{p^e} x^{\mu_1} y^{\mu_2}$ where $0 \leq \mu_1, \mu_2 \leq p^e - 1$. In particular, $F^e_*(x^2y)^{\lceil tp^e \rceil} = x^{p^e} F^e_* x^{\mu_1} y^{\mu_2}$ where $F^e_* x^{\mu_1} y^{\mu_2}$ is part of a free basis for F^e_*S over S. By Proposition 5.6, for $e \gg 0$,
$$\tau(S, \mathfrak{a}^t) = ((x^2y)^{\lceil tp^e \rceil})^{[1/p^e]} = (x).$$

Similar computations give:

- For $t \in [n, n + \frac{1}{2})$ with $n \in \mathbb{N}$, $\tau(S, \mathfrak{a}^t) = (x^{2n}y^n)$.

- For $t \in [n + \frac{1}{2}, n + 1)$, $\tau(S, \mathfrak{a}^t) = (x^{2n+1}y^n)$.

Example 6.8. Example 6.7 can be generalized to show that for a monomial $x_1^{a_1} x_2^{a_2} \cdots\ x_n^{a_n}$ in a polynomial ring S,
$$\tau(S, (x_1^{a_1} x_2^{a_2} \cdots\ x_n^{a_n})^t) = (x_1^{\lfloor ta_1 \rfloor} x_2^{\lfloor ta_2 \rfloor} \cdots\ x_n^{\lfloor ta_n \rfloor}).$$
Thus the test ideal in this case is constant on half-open intervals with "jumps" as we pass from $t = \frac{n}{a_i}$ (where $n \in \mathbb{N}$) to a slightly smaller real number $t - \epsilon$. The jumping numbers for the principal ideal $(x_1^{a_1} x_2^{a_2} \cdots\ x_n^{a_n})$ are all rational numbers of the form $\frac{a_i}{n}$ for some $n \in \mathbb{N}$.

Example 6.9. If S is an F-finite regular ring and $f \in S$ is a non-zero element such that $S/(f)$ is also regular, then the jumping numbers of (S, f) are precisely the positive integers. See Exercise 6.3.

Example 6.10. Let $S = \mathbb{F}_5[x, y]$, and consider the element $f = x^2y - xy^2 \in S$. We saw in Example 3.8 that $\operatorname{fpt}(f) = \frac{2p-1}{3p} = \frac{3}{5}$. Thus $\tau(R, f^t) = R$ for all $t < 3/5$. For $t = 3/5$, test ideal can be computed explicitly using Corollary 5.7 and Proposition 5.14:
$$\tau(S, f^{3/5}) = \tau(S, (f^3)^{1/5}) = (f^3)^{[1/5]} = (x^6y^3 - 3x^5y^4 + 3x^4y^5 - x^3y^6)^{[1/5]}.$$
By Proposition 5.6, therefore, writing
$$F_*(x^6y^3 - 3x^5y^4 + 3x^4y^5 - x^3y^6) = xF_*xy^3 - 3xF_*y^4 + 3yF_*x^4 - yF_*x^3y,$$

we see that
$$\tau(S, f^{3/5}) = (f^3)^{[1/5]} = (x, y).$$
In summary, we have
$$\tau(S, f^t) = \begin{cases} S & 0 \le t < 3/5 \\ (x, y) & t = 3/5. \end{cases}$$
In fact, $\tau(S, f^t) = (x, y)$ for all $3/5 \le t < 1$; see Example 6.21.

6.2. First properties of jumping numbers. In example Example 6.7, larger jumping numbers were all integer shifts of smaller ones. This is a general property of jumping numbers following from Skoda's theorem, Theorem 5.20.

Corollary 6.11. *Let R be an F-finite Noetherian reduced ring, and let $\mathfrak{a} \subseteq R$ be an ideal generated by m elements. For any $t \ge m$, let t' be the unique real number in the interval $[m-1, m)$ such that $t - t' \in \mathbb{Z}$. If t is an F-jumping number for $\mathfrak{a}$, then also t' is an F-jumping number of $\mathfrak{a}$.*

Proof. This is an immediate consequence of Theorem 5.20. Indeed, suppose that $t \ge m$, and write $t = t' + n$ where $n \in \mathbb{Z}$ and $t' \in [m-1, m)$. Since
$$\mathfrak{a}^n \, \tau(R, \mathfrak{a}^{t-n}) = \tau(R, \mathfrak{a}^t)$$
we see that if t is a F-jumping number, so is t'. □

Another important property of jumping numbers follows immediately from the nice behavior of the test ideal.

Corollary 6.12. *Suppose S is an F-finite ring such that* $\operatorname{Hom}_R(F^e_* R, R)$ *has a generating map Φ (or more generally if the* Hom*-set is locally principal, for instance if R is regular or quasi-Gorenstein), that $\mathfrak{a} \subseteq S$ is an ideal and $t \ge 0$ is a real number. Then if t is a jumping number of $\mathfrak{a}$, then so is $p^e t$ for every $e \in \mathbb{N}$.*

Proof. This follows immediately from the fact that
$$\tau(S, \mathfrak{a}^{t/p^e}) = \Phi(\tau(S, \mathfrak{a}^t))$$
for all $e > 0$ (Corollary 5.25). The more general statement follows since the test ideal can be computed locally. □

Example 6.13. Consider the polynomial $y^2 - x^3 \in \mathbb{F}_p[x, y]$, where $p \equiv 5 \pmod 6$. We claim that $\frac{5}{6}$ can not be its F-pure threshold (compare with Example 3.9). Indeed, if $\frac{5}{6}$ were its F-pure threshold, then $\frac{5}{6}$ would be a jumping number, and so $\frac{5p}{6}$ would be a jumping number as well (by Corollary 6.12). But $p \equiv 5 \pmod 6$ implies that $\frac{5p}{6} = \frac{1}{6} + k$ for some positive integer k, and so $\frac{1}{6}$ would have to be an F-jumping number (by Corollary 6.11).

This contradicts the fact that the F-pure threshold is the *smallest* F-jumping number (Theorem 6.1).

6.3. Discreteness and rationality of F-jumping numbers. We now discuss a deeper property of Frobenius jumping numbers:

Theorem 6.14. *Let S be an F-finite quasi-Gorenstein (for instance regular) ring, and $\mathfrak{a}$ an ideal of S. The set of all F-jumping numbers of $(S, \mathfrak{a})$ is a discrete set of rational numbers.*

Remark 6.15. Even without the principal hypothesis, Theorem 6.14 was first proved in [**BMS08**] in the case where S is a regular ring essentially of finite type over a field (see [**KLZ09**] where the F-finite hypothesis was replaced with excellent local). Later, this was done when R is $\mathbb{Q}$-Gorenstein in [**ST14**] (with various intermediate and related results, see [**BSTZ10, STZ12, GS18, CEMS18**], including results in the non-F-finite case).

We will prove Theorem 6.14 in this section *ONLY for principal ideals* in quasi-Gorenstein rings, although the $\mathbb{Q}$-Gorenstein generalization is not much more difficult. We will obtain the non-principal case as a corollary in Chapter 10 (essentially following the argument of [**GS18**]). It is an open question if the set of F-jumping numbers form a discrete set in general. Based on the multiplier ideal case, we expect that rationality fails [**Urb12**].

We already know from Proposition 5.18 that $\tau(R, \mathfrak{a}) = \tau(R, \mathfrak{a}^{t+\epsilon})$ for all $1 \gg \epsilon > 0$. Since F-jumping numbers are bounded below by zero, we immediately obtain the following.

Corollary 6.16. *Let $\mathfrak{a}$ be an ideal in an F-finite Noetherian reduced ring R. There is no infinite strictly decreasing sequence of F-jumping numbers for the pair $(R, \mathfrak{a})$.*

In other words, we have no accumulations of jumping numbers from above.

We can also see in general that the set of jumping numbers is *closed:*

Proposition 6.17. *Let $\mathfrak{a}$ be an ideal in an F-finite Noetherian reduced ring R. Then the set of F-jumping numbers of $(R, \mathfrak{a})$ is a closed subset of $\mathbb{R}$. In particular, a limit of F-jumping numbers is an F-jumping number.*

Proof. Let C denote the set all F-jumping numbers of $(R, \mathfrak{a})$. Suppose that $t \in \mathbb{R}_{>0}$ is the limit of some sequence $t_n \in C$. We must show that $t \in C$. By Proposition 5.18, we may assume that $\{t_n\}$ is strictly increasing to t. In this case, for any $\epsilon > 0$, there exists N such that $t - \epsilon < t_n < t$ for all $n \geq N$. But then

$$\tau(R, \mathfrak{a}^{t-\epsilon}) \supsetneq \tau(R, \mathfrak{a}^{t_N}) \supseteq \tau(R, \mathfrak{a}^t),$$

showing that t is an F-jumping number as well. $\square$

In order to prove Theorem 6.14, we also need to prove that no F-jumping number is a limit of F-jumping numbers from *below.* We will do this here only for principal ideals in a regular ring, making use of the following theorem due to Hartshorne-Speiser, Lyubeznik, and Gabber:

Theorem 6.18 ([**HS77, Lyu97, Gab04**])**.** *Suppose that R is a Noetherian ring, M is a finite generated R-module (for us, typically $M = R$) and $\phi \in \operatorname{Hom}_R(F^e_* M, M)$ (we do not assume that R is F-finite). Then the descending chain of ideals of submodules*

$$M \supseteq \phi(F^e_* M) \supseteq \phi^2(F^{2e}_* M) \supseteq \phi^3(F^{3e}_* M) \supseteq \dots \tag{6.18.1}$$

eventually stabilizes (again, $\phi^n = \phi^{\star n}$).

Remark 6.19. Theorem 6.18 is a variant of an important result due to Hartshorne-Speiser, Lyubeznik, and Gabber, in different guises [**HS77, Lyu97, Gab04**]. We will revisit a more general version later in Chapter 8 Theorem 2.1.

Before proving Theorem 6.18, we use it to prove a critical step in the proof of the discreteness of the set of jumping numbers:

Corollary 6.20. *Let f be an element in an F-finite quasi-Gorenstein ring R. Then no* rational *number is an accumulation point of F-jumping numbers for (R, f).*

Proof. Suppose that $t \in \mathbb{Q}$ is an accumulation point of F-jumping numbers. Then there is a strictly increasing sequence of jumping numbers converging to t (Proposition 5.18). In this case, there is also a strictly increasing sequence of jumping number converging to $p^d t$ for every d by Corollary 6.12. So replacing t by tp^d for suitable d, we may assume the rational number t has denominator not divisible by p.

Now, we can write $t = a/(p^e - 1)$ for some $a, e \in \mathbb{N}$ (see Exercise 6.7). Consider the sequence $\{t_n \mid n \in \mathbb{N}\}$ where

$$t_n = t\,\frac{p^{ne}-1}{p^{ne}} = \frac{a}{p^e-1}\,\frac{p^{ne}-1}{p^{ne}} = \frac{a(1 + p^e + p^{2e} + \cdots + p^{(n-1)e})}{p^{ne}}.$$

Clearly $\{t_n\}$ is an increasing sequence converging to t. We will show that the ideals $\tau(R, f^{t_n})$ are constant for $n \gg 0$. This will contradict the assumption that t is an accumulation point of jumping numbers from below.

Indeed, consider the descending chain of (test) ideals $\tau(R, f^{t_n})$

$$\tau(R, f^{t_1}) \supseteq \tau(R, f^{t_2}) \supseteq \tau(R, f^{t_3}) \supseteq \dots$$

It suffices to show that this chain eventually stabilizes on each open set of a finite affine cover of $\operatorname{Spec} R$. In particular, without loss of generality, we assume that there exists $\Phi \in \operatorname{Hom}_R(F_* R, R)$ which is a generating map. Recall that Φ^e is also a generating map by Appendix A Proposition 5.3, and so by Corollary 5.24, our chain becomes

$$\Phi^e(F^e_* f^a \tau(R)) \supseteq \Phi^{2e}(F^{2e}_* f^{a(1+p^e)}\tau(R)) \supseteq \Phi^{3e}(F^{3e}_* f^{a(1+p^e+p^{2e})}\tau(R)) \supseteq \dots$$

Since $\operatorname{Spec} R$ has a finite cover by open affine sets with generating maps, if we can show the stabilization in the presence of a generating map, the general case also follows.

Let $\phi = \Phi^e \star f^a$. Then it follows from Chapter 1 Proposition 4.13 that $\phi^n(F^{ne}_* \tau(R)) = \Phi^{ne}(F^{ne}_* f^{a(1+p^e+\dots+p^{(n-1)e})}\tau(R))$ and so our descending chain of test ideals can be rewritten as

$$\phi(F^e_* S) \supseteq \phi^2(F^{2e}_* S) \supseteq \phi^e(F^{3e}_* S) \supseteq \dots$$

But now Theorem 6.18 precisely says that these ideals are constant for $n \gg 0$. This completes the proof of Corollary 6.20. $\square$

Example 6.21. Building on Example 6.10, we let $S = \mathbb{F}_5[x, y]$ and $f = x^2y - xy^2$. We have seen that $3/5$ is a jumping number for f (the smallest jumping number, or F-pure threshold). We claim the next smallest jumping number is 1. For this, we should show that $\tau(S, f^t)$ is constant for $3/5 \leq t < 1$, with $\tau(S, f^1)$ is strictly smaller.

Using the idea in the proof of Corollary 6.20, it suffices to show that $\tau(S, f^{\frac{5^n-1}{5^n}}) = (x, y)$ for all n, but that $\tau(S, f^1) \subsetneq (x, y)$. We have

$$\tau(S, f^{\frac{5^n-1}{5^n}}) = \Phi^n(F^n_* f^{5^n-1} S) = \Phi^n(F^n_*(f^{5-1})^{1+5+5^2+\dots+5^{n-1}} S).$$

Using the relation $\Phi^n \star (f^4)^{1+5+5^2+\dots+5^{n-1}} = (\Phi^n \star f^4)^{\star n}$, in the Cartier algebra, this becomes

$$\tau(S, f^{\frac{5^n-1}{5^n}}) = (\Phi^n \star f^4)^{\star n}(F^n_* S),$$

so that setting $\phi = \Phi \star f^4$, we have $\tau(S, f^{\frac{5^n-1}{5^n}}) = \phi^n(F^n_* S)$.

When $n = 1$, we compute $\tau(S, f^{\frac{4}{5}}) = \Phi(F_*(f^4)) = (x, y)$, using Proposition 5.6. When $n = 2$, we compute

$$\tau(S, f^{24/25}) = \phi^2(F^2_* S) = \phi(F_*(x, y)) = \big((f^4)(x, y)\big)^{[1/5]},$$

which again

$$\tau(S, f^{24/25}) = (x, y)$$

by Proposition 5.6. So the descending chain

$$\phi^1(F^1_* S) \supseteq \phi^2(F^2_* S) \supseteq \phi^3(F^3_* S) \supseteq \dots$$

stabilizes immediately at the first step, and all

$$\tau(S, f^{\frac{p^n-1}{p^n}}) = \phi^n(F^n_* S) = (x, y).$$

This proves that

$$\tau(S, f^t) = \begin{cases} S & 0 \leq t < 3/5 \\ (x, y) & 3/5 \leq t < 1, \end{cases}$$

Finally, when $t = 1$, $\tau(S, f^1) \subseteq (f)$, which strictly smaller that (x, y). So 1 is a jumping number,

We complete the story by proving the Hartshorne-Speiser-Lyubeznik-Gabber result:

Proof of Theorem 6.18. First note that if $\phi^n(F^{ne}_* M) = \phi^{n+1}(F^{(n+1)e}_* M)$, then applying $\phi(F^e_* -)$ yields $\phi^{n+1}(F^{(n+1)e}_* M) = \phi^{n+2}(F^{(n+2)e}_* M)$. It follows that if there is an open set $U \subseteq \operatorname{Spec} R$ such that $\phi^n(M)$ and $\phi^{n+1}(M)$ agree (after localizing) at all points of U, then $\phi^n(F^{ne}_* M)$ agrees with $\phi^{n+i}(F^{(n+i)e}_* M)$ for all $i \geq 0$ again at all points for U.

For each integer n let $Z_n = \operatorname{Supp}\left(\phi^n(F^{ne}_* M)/\phi^{n+1}(F^{(n+1)e}_* M)\right)$. By our initial observation, we see that $Z_0 \supseteq Z_1 \supseteq Z_2 \dots$ is a descending chain of closed subsets of $\operatorname{Spec} R$. Hence $Z_i = Z_{i+1} = \dots$ for $i \gg 0$ since $\operatorname{Spec} R$ is Noetherian. By replacing M by $\phi^i(F^{ne}_* M)$, we can assume that $Z_0 = Z_1 = Z_2 = \dots$. We assume for a contradiction that $Z_i \neq \emptyset$ for any $i > 0$. We let η be a generic point of the now constant Z_i (in other words, a minimal prime of the ideal defining Z_i) and replace R by R_η, now a local ring with maximal ideal $\mathfrak{m}$. With our current assumptions, we now see that $\operatorname{Supp}\left(\phi^n(F^{ne}_* M)/\phi^{n+1}(F^{(n+1)e}_* M)\right) = \{\mathfrak{m}\}$ for all n.

Write $\mathfrak{m} = (x_1, \dots, x_t)$. Choose N such that $x_i^N M \subseteq \phi(M)$ for each $i = 1, \dots, t$. We claim for all $n \geq 0$ that

$$x_i^{2N} M \subseteq \phi^n(F^{ne}_* M).$$

We argue by induction on n, the base case is clear so suppose it is true for n. Now observe that

$$\begin{array}{rcl} x_i^{2N} M & \subseteq & x_i^N \phi(F^e_* M) \\ & = & \phi(F^e_* x_i^{p^e N} M) \\ & \subseteq & \phi(F^e_* x_i^{2N} M) \\ & \subseteq & \phi(F^e_* \phi^n(F^{ne}_* M)) \\ & = & \phi^{n+1}(F^{(n+1)e}_* M) \end{array}$$

as desired.

It follows that $(x_1^{2N}, \dots, x_t^{2N})M \subseteq \phi^n(F^{ne}_* M)$ for all n. But observe that $M/(x_1^{2N}, \dots, x_t^{2N})M$ is a module of finite length and that we have a sequence of properly descending submodules $\{\phi^n(F^{ne}_* M)/(x_1^{2N}, \dots, x_t^{2N})M\}_n$. This is a contradiction. $\square$

We conclude the section by completing the proof of Theorem 6.14 in the case where $\mathfrak{a}$ is principal:

Proof of Theorem 6.14 assuming Theorem 6.18. We assume $\mathfrak{a} = (f)$ is principal. In light of Corollary 6.20 and Proposition 6.17, in order to show discreteness of F-jumping numbers, it suffices to show that there are no irrational F-jumping numbers.

In view of Corollary 6.11, we may restrict ourselves to F-jumping numbers in $[0, 1]$. The set X of *irrational* jumping numbers in $[0, 1]$ is closed, since any point in its closure would be an accumulation point of F-jumping numbers, hence an F-jumping number (by Proposition 6.17), but not rational (by Corollary 6.20). In addition, X has no accumulation points from above by Proposition 5.18. Furthermore, for each $e > 0$, the fractional parts $p^e t - \lfloor tp^e \rfloor$ of $p^e t$ for all $e > 0$ are all in X, by Corollary 6.12 and Corollary 6.11. Such a closed set X of irrational numbers can be shown to be empty with an argument in real analysis; see Exercise 6.8. $\square$

6.4. Exercises.

Exercise 6.1. Complete the computation in Example 6.7.

Exercise 6.2. Suppose that $S = \mathbb{F}_p[x_1, \dots, x_n]$ and $\mathfrak{a} = (x_1^{a_1} \dots x_n^{a_n})$. Show that
$$\tau(S, \mathfrak{a}^t) = (x_1^{\lfloor ta_1 \rfloor} \dots x_n^{\lfloor ta_n \rfloor}).$$

Exercise 6.3. If S is an F-finite regular ring and $f \in S$ is a non-zero element such that $S/(f)$ is also regular, show that the jumping numbers of (S, f) are precisely the positive integers. More generally, suppose that S is quasi-Gorenstein and $S/(f)$ is quasi-Gorenstein and strong F-regular. Prove the same result.

Exercise 6.4. Let R be an F-finite regular local ring of characteristic $p > 0$. Suppose some $f \in R$ has $\mathrm{fpt}(f) = (b-1)/b$. Use the method of Example 6.13 to deduce what you can about the relationship between b and p.

Exercise 6.5. Suppose that R is regular. Show that $\tau(R, f^{a/p^d}) = (f^a)^{[1/p^d]}$. Generalize this statement and proof to when R is strongly F-regular quasi-Gorenstein.

Exercise 6.6. Prove the following result of Mustaţă-Yoshida [**MY09**]. Suppose that k is an F-finite field and $S = k[x_1, \ldots, x_n]$. For every ideal $J \subseteq S$ show that there exists $f \in S$ such that $\tau(S, f^{1/p}) = J$. That is, show that every ideal is a test ideal.

Hint: Use generators of J to construct f. Then use Exercise 2.1.

Exercise 6.7. Suppose p is prime. Prove that every rational number $t \geq 0$ can be written as $t = \frac{a}{p^d(p^e-1)}$ for some integers $a, d \geq 0$ and $e > 0$.

Hint: If n is relatively prime to p, then the class of p is in the group of units $(\mathbb{Z}/n\mathbb{Z})^\times$.

Exercise 6.8. Suppose p is a prime number and let $D \subseteq [0, 1]$ be a closed set of *irrational numbers* such that the following two properties are satisfied.

(1) If $t \in D$, then there exists an $\epsilon > 0$ such that $D \cap (t, t + \epsilon) = \emptyset$.
(2) The fractional part $\{p^e t\}$ of $p^e t$ is in D for all integers $e > 0$.

Then $D = \emptyset$.

Hint: If you are stuck, you can find a full solution in [**KLZ09**].

Exercise 6.9. Let R be an F-finite regular ring, $(f) = \mathfrak{a} \subseteq R$ is a principal ideal and $t = \frac{a}{p^e-1}$. Suppose that for ome n:

$$\left(\mathfrak{a}^{\frac{a(p^{ne}-1)}{p^e-1}}\right)^{[1/p^{ne}]} = \left(\mathfrak{a}^{\frac{a(p^{(n+1)e}-1)}{p^e-1}}\right)^{[1/p^{(n+1)e}]} \tag{6.21.1}$$

Prove that the ideal (6.21.1) is equal to $\tau(R, \mathfrak{a}^{t-\epsilon})$ for $1 \gg \epsilon > 0$.

Hint: Use $\phi = \Phi^e \circ F^e_* f^a$ where f is a generator for $\mathfrak{a}$.

Exercise 6.10. Let $R = \mathbb{F}_p[x, y]$ and $\mathfrak{a} = (xy(x - y))$. Compute $\tau(R, \mathfrak{a}^t)$ for all t and $p = 2, 3, 4, 5$.

Hint: This work was started in Example 5.19 for $p = 2$.

Exercise 6.11. Suppose that S is an F-finite regular ring, and $R = S/I$ is quasi-Gorenstein, then show that the ideals of R

$$J_e = (I^{[p^e]} : I)^{[1/p^e]} \Big/ I$$

are descending as e increases, stabilizes for $e \gg 0$, and cuts out the locus where R/I is not F-split.

Hint: After localizing if necessary, show that these ideals are images of some appropriate ϕ^e as in Theorem 6.18. See also Corollary 2.6.

6.4.1. *Exercises on parameter test modules.* We begin with a definition.

Definition 6.22. Suppose R is an F-finite locally equidimensional ring with test module ω_R. For any ideal $\mathfrak{a} \subseteq R$ and $t \geq 0$, we define the **parameter test module of the pair** $(R, \mathfrak{a}^t)$ to be $T^e(F^e_* \mathfrak{a}^{\lceil tp^e \rceil} \tau(\omega_R))$ for $e \gg 0$ where $T^e : F^e_* \omega_R \longrightarrow \omega_R$ is the dual to Frobenius from Chapter 2. The resulting object is denoted by $\tau(\omega_R, \mathfrak{a}^t)$. Note, when $\mathfrak{a} = (f)$ is principal, we denote $\tau(\omega_R, \mathfrak{a}^t)$ by $\tau(\omega_R, g^t)$.

Exercise 6.12. Show that $T^e(F^e_* \mathfrak{a}^{\lceil tp^e \rceil}) \subseteq T^{e+1}(F^{e+1}_* \mathfrak{a}^{\lceil tp^{e+1} \rceil})$ to conclude that $\tau(\omega_R, \mathfrak{a}^t)$ is well defined.

Hint: We know that $T(F_* \tau(\omega_R)) = \omega_R$ by Exercise 5.1 in Chapter 2.

Exercise 6.13. With notation as in Definition 6.22, show that $T(F_* \tau(\omega_R, \mathfrak{a}^t)) = \tau(\omega_R, \mathfrak{a}^{t/p})$.

Exercise 6.14. Assume R is a Noetherian F-finite domain (the domain assumption is for simplicity). Suppose $\mathfrak{a} \subseteq R$ is an ideal. Prove that

$$\tau(\omega_R, \mathfrak{a}^t) = \tau(\omega_R, \mathfrak{a}^{t+\epsilon})$$

for all $1 \gg \epsilon > 0$.

Hint: Mimic the proof of Proposition 5.18.

Definition 6.23. With notation as in Definition 6.22, we say that $t > 0$ is a **test module F-jumping number** if $\tau(\omega_R, \mathfrak{a}^t) \neq \tau(\omega_R, \mathfrak{a}^{t-\epsilon})$ for all $1 \gg \epsilon > 0$.

Exercise 6.15. Show that an accumulation point of test module F-jumping numbers is a test module F-jumping number.

Exercise 6.16. Suppose that $\mathfrak{a} = (f)$ is principal. Show that no rational number of the form $a/(p^e - 1)$ can be a test module F-jumping number.

Hint: Use Theorem 6.18 with $M = \tau(\omega_R)$ and $\phi = T \star f^a$.

Exercise 6.17. Use the preceding exercises and the strategy of this section to show that, in the case that $\mathfrak{a} = (f)$ is principal, the test module F-jumping numbers are rational numbers with no accumulations points. In the case that R is quasi-Gorenstein, this coincides with what we already showed in this section since $R \cong \omega_R$ and so $\tau(\omega_R, \mathfrak{a}^t) \cong \tau(R, \mathfrak{a}^t)$.

7. Restriction, subadditivity, and symbolic powers

Test ideals can be viewed as a "prime characteristic analog" of the *multiplier ideals* in complex algebraic geometry. Variants of multiplier ideals appeared in several different guises in the work Grauert-Riemenschneider, Kohn, Esnault-Viehweg, Kollár, Lipman, Nadel, also see for instance work of Siu and Demailly, [**GR70, Koh79, EV83, Kol86, Lip94, Nad90, Siu01, DEL00**]. For an algebraic perspective, we recommend [**Laz04b**, Chapter 9]. The connection has inspired a great deal of work on test ideals. In particular, test ideals enjoy many of the same features and applications, such as the Skoda theorem, as we saw in Theorem 5.20. In this section, we discuss two properties of test ideals—subadditivity and the restriction theorem—and one application—the Ein-Lazarsfeld-Smith theorem on symbolic powers of ideals, all of which were first proven in the complex setting with multiplier ideals. We will return to how and why the test ideal is related to the multiplier ideal later in Chapter 6.

7.1. The restriction theorem. The original *restriction theorem* concerned the behavior of multiplier ideals on a smooth ambient variety under restriction to a smooth divisor, or by induction, to any smooth subvariety. Here we do the same for test ideals in regular F-finite rings. We'll later study the behavior of test ideals under restriction in much greater generality when we get to F-adjunction Chapter 5 Section 4. Note we proved a variant of this in the non-pair setting in Theorem 5.21 in Chapter 2. For now we have:

Theorem 7.1 (The restriction theorem for test ideals)**.** *Consider a surjective homomorphism $S \longrightarrow \overline{S}$ of F-finite regular rings. Then for every $e > 0$, we have that*

$$(\mathfrak{b}\overline{S})^{[1/p^e]} \subseteq (\mathfrak{b}^{[1/p^e]})\overline{S}. \tag{7.1.1}$$

In particular, for any ideal $\mathfrak{a} \subseteq S$ and any $t \geq 0$,

$$\tau(\overline{S}, \overline{\mathfrak{a}}^t) \subseteq \tau(S, \mathfrak{a}^t)\overline{S}. \tag{7.1.2}$$

Equation (7.1.2) also holds as long as $S \longrightarrow \overline{S}$ is a map of quasi-Gorenstein rings which has a kernel generated by a regular sequence (we leave this as an exercise in Exercise 7.2).

Proof. Both statements may be checked locally, so we assume the regular ring $(S, \mathfrak{m})$ is local, in which case the surjection $S \longrightarrow \overline{S}$ factors as a succession of quotients by a single regular element. So by induction, we can assume $\overline{S} = S/(f)$, where $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.

Fix $e > 0$, and let $\Phi^e_S \in \operatorname{Hom}_S(F^e_* S, S)$ be an $F^e_* S$-module generator. By Theorem 2.1, the map

$$\Phi^e_S \star f^{p^e-1} = \Phi^e_S \circ F^e_* f^{p^e-1} \in \operatorname{Hom}_S(F^e_* S, S)$$

descends to a map $\Phi^e_{\overline{S}} : F^e_* \overline{S} \longrightarrow \overline{S}$, which is a $F^e_* \overline{S}$-module generator for $\mathrm{Hom}_{\overline{S}}(F^e_* \overline{S}, \overline{S})$ (this also follows from Chapter 2 Lemma 5.20). Thus

$$\Phi^e_{\overline{S}}(F^e_* \overline{\mathfrak{b}}) = \Phi^e_S(F^e_* f^{p^e-1}\mathfrak{b})\overline{S}.$$

So by Remark 5.4 and the fact that $f^{p^e-1}\mathfrak{b} \subseteq \mathfrak{b}$, we can conclude that

$$\overline{\mathfrak{b}}^{[1/p^e]} = (f^{p^e-1}\mathfrak{b})^{[1/p^e]}\overline{S} \subseteq \mathfrak{b}^{[1/p^e]}\overline{S},$$

proving (7.1.1).

To deduce the result for test ideals, recall that $\tau(S, \mathfrak{a}^t) = \Phi^e(F^e_*(\mathfrak{a}^{\lceil tp^e \rceil})$ for $e \gg 0$ (Corollary 5.24). So the desired statement follows by setting $\mathfrak{b} = \mathfrak{a}^{\lceil tp^e \rceil}$ and by observing that $\mathfrak{a}^n \overline{S} = (\mathfrak{a}\overline{S})^n$ for all $n > 0$. $\square$

Remark 7.2. We can view Theorem 7.1 as an instance of a familiar intuition about singularities: they can only get worse, never better, when restricted to (or intersected with) a divisor. Thinking of the test ideals $\tau(S, \mathfrak{a}^t)$ as a measurement of the singularities of the subscheme $\mathbb{V}(\mathfrak{a}) \subseteq \operatorname{Spec} S$, we have argued that $\tau(S, \mathfrak{a}^t)$ is a deeper (in other words smaller) ideal when the scheme $\mathbb{V}(\mathfrak{a})$ is more singular. The restriction theorem says that, as measured by test ideals, the singularities of the intersection $\mathbb{V}(\mathfrak{a}) \cap \operatorname{Spec} \overline{S}$ can only be *as bad or worse, never better* than those of $\mathbb{V}(\mathfrak{a})$ at points of $\operatorname{Spec} S$, as the corresponding test ideals are deeper.

7.2. Subadditivity.

Subadditivity is a property of *mixed test ideals*:

Definition 7.3. Let $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_m$ be ideals in an F-finite regular ring S. For any non-negative real numbers $t_1, t_2 \dots, t_m$, the **mixed test ideal** can be defined as

$$\tau(S, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m}) = (\mathfrak{a}_1^{\lceil t_1 p^e \rceil} \cdots \mathfrak{a}_m^{\lceil t_m p^e \rceil})^{[1/p^e]}$$

for $e \gg 0$.

We can now state and prove the subadditivity theorem.

Theorem 7.4 (Subadditivity, [**HY03**, Theorem 4.5], [**Tak06**])**.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in an F-finite regular ring S. For all real numbers $s, t \geq 0$,*

$$\tau(S, \mathfrak{a}^s \mathfrak{b}^t) \subseteq \tau(S, \mathfrak{a}^s)\tau(S, \mathfrak{b}^t).$$

An important special case is where all ideals $\mathfrak{a}_i$ are the same and the t_i are all 1:

Corollary 7.5. *Let $\mathfrak{a}$ be an ideal in an F-finite regular ring S. Then*

$$\tau(S, \mathfrak{a}^n) \subseteq \tau(S, \mathfrak{a})^n$$

for all $n \in \mathbb{N}$.

PROOF OF THEOREM 7.4. Fix $e \gg 0$. Then

$$\tau(S, \mathfrak{a}^s \mathfrak{b}^t) = \left(\mathfrak{a}^{\lceil sp^e \rceil} \mathfrak{b}^{\lceil tp^e \rceil}\right)^{[1/p^e]} \subseteq \left((\tau(S, \mathfrak{a}^s))^{[p^e]} \mathfrak{b}^{\lceil tp^e \rceil}\right)^{[1/p^e]} \tag{7.5.1}$$

with the inclusion following from the fact that if $I^{[\frac{1}{p^e}]} \subseteq J$ for any ideals I and J, then $I \subseteq J^{[p^e]}$ (applied with $I = \mathfrak{a}^{\lceil sp^e \rceil}$ and $J = \tau(S, \mathfrak{a}^t)$). Now, the ideals in (7.5.1) are contained in

$$\tau(S, \mathfrak{a}^s)(\mathfrak{b}^{\lceil tp^e \rceil})^{[1/p^e]} = \tau(S, \mathfrak{a}^s)\tau(S, \mathfrak{b}^t)$$

using the fact that for any ideals I and J, and any $e > 0$, $(I^{[p^e]}J)^{[\frac{1}{p^e}]} = IJ^{[\frac{1}{p^e}]}$. The result follows. $\square$

Remark 7.6. The proof of subadditivity provided above is from [**BMS08**]. It is essentially (Matlis) dual to a proof of Takagi [**Tak06**], who extended the idea to prove variants of subadditivity for non-regular rings.

Remark 7.7. There is another proof of subadditivity via the restriction theorem combined restricting to the diagonal. This strategy was first deployed by Demailly, Ein and Lazarsfeld to prove the subadditivity property for multiplier ideals in the complex setting. A Matlis dual version for test ideals was first proved in [**HY03**]. A explanation of that proof can also be found in [**SZ15**]. We sketch that proof in Exercise 7.5 below.

7.3. Application to symbolic powers. The symbolic power $Q^{(n)}$ of a prime ideal Q in an arbitrary commutative ring R consists of the elements that vanish to order n along Q. More precisely,

$$Q^{(n)} = Q^n R_Q \cap R,$$

where Q^n is the ordinary power and R_Q is the localization at Q. For radical ideals, the symbolic power $I^{(n)}$ can be described as the intersection of the *minimal* primary components in a primary decomposition of I^n. See the excellent survey [**DDSG$^+$18**] or Appendix A for background on symbolic powers.

While is it clear that $Q^n \subseteq Q^{(n)}$, the reverse inclusion is false in general (unless Q is maximal). In fact, it is not obvious whether or not, given n, we can find *any* m such that $Q^{(m)} \subseteq Q^n$. Irena Swanson proved that quite generally, there is a *linear bound* k such that $Q^{(kn)} \subseteq Q^n$ for all n, but her k depends on Q and may be very large [**Swa00**].

Remarkably, for regular rings, there is a very nice uniform bound depending only on the dimension of the ring that works for all prime ideals:

Theorem 7.8 ([**HH02**], *cf.* [**ELS01**])**.** *Suppose that S is an F-finite regular ring of dimension d. Then for all radical ideals I,*

$$I^{((d-1)n)} \subseteq I^n$$

for every $n > 0$. More precisely, if h is the maximal height (codimension) of the minimal primes of I, then $I^{(hn)} \subseteq I^n$ for every $n > 0$.

Example 7.9. Let $S = k[x, y, z]$ and $I = (x, y) \cap (x, z) \cap (y, z) = (xy, xz, yz)$. Theorem 7.8 says in this case that

$$I^{(2n)} \subseteq I^n$$

for all n.

Remark 7.10. Theorem 7.8 was originally proven by Ein, Lazarsfeld and Smith, for finite type $\mathbb{C}$-algebras, using multiplier ideals [**ELS01**]. At the time it was proven, Theorem 7.8 was quite surprising, and hadn't been suspected even in simple cases, like Example 7.9 above. Soon after, Hochster and Huneke found a tight closure proof for the characteristic p case [**HH02**]. Theorem 7.8 has now been generalized to mixed characteristic, with proofs mimicking the ideas of Ein-Lazarsfeld-Smith presented here; see [**MS18**] for excellent regular rings and [**Mur22b**] for the generalization to the non-excellent case. Many authors have investigated generalizations, for example, to non-regular rings [**HKV09, HK19, Wal18, CRS20, BJNnB19, GMS22**].

Our proof of Theorem 7.8 closely follows the Ein-Lazarsfeld-Smith proof, using test ideals in place of multiplier ideals.

Proof of Theorem 7.8. Since $(I^{(hn)})^m \subseteq I^{(hmn)}$ (Exercise 7.9) we have that

$$\tau(R, I^{(hn)}) \subseteq \tau(R, I^{(hnm)^{\frac{1}{m}}}) \subseteq \tau(R, (I^{(hmm'n)})^{\frac{1}{mm'}})$$

for all $n, m, m' \in \mathbb{N}$. In particular, by the Noetherian property, for m sufficiently large and divisible, $\tau(R, (I^{(hmn)})^{\frac{1}{m}})$ stabilizes. We denote this stable value by

$$\tau_\infty(R, I^{(hn)}).$$

Now observe that $\mathfrak{a} \subseteq \tau(R, \mathfrak{a})$ for any ideal $\mathfrak{a}$ (see Exercise 7.1), so

$$I^{(hn)} \subseteq \tau(R, I^{(hn)}) \subseteq \tau_\infty(R, I^{(hn)}).$$

Now, for m sufficiently divisible,

$$\tau_\infty(R, I^{(hn)}) = \tau(R, (I^{(hnm)^{\frac{n}{nm}}})) = \tau(R, \underbrace{I^{(hnm)^{\frac{1}{nm}}} \cdots I^{(hnm)^{\frac{1}{nm}}}}_{n\text{-times}})$$

where the second equality is Exercise 5.17. But by repeated applications of subadditivity, Theorem 7.4, this is contained in

$$\underbrace{\tau(R, (I^{(hnm)})^{\frac{1}{nm}}) \cdots \tau(R, (I^{(hnm)})^{\frac{1}{nm}})}_{n \text{ times}}.$$

Therefore, to complete the proof, it suffices to show that

$$\tau(R, (I^{(hnm)})^{\frac{1}{nm}}) \subseteq I. \tag{7.10.1}$$

Because I is radical, it is the intersection of its minimal primes, and the inclusion (7.10.1) may be checked locally at each of its minimal primes (Exercise 7.6). Furthermore, since the formation of the test ideal commutes with localization, we may localize at a minimal prime of I to assume that R is local and $I = \mathfrak{m}$ is maximal.

Suppose $\mathfrak{m} = (x_1, \dots, x_d)$ where $d = \dim R \leq h$. Since $\mathfrak{m}$ is maximal, its symbolic and ordinary powers are the same. But now notice that $\tau(R, (\mathfrak{m}^{hnm})^{\frac{1}{nm}}) = \tau(R, \mathfrak{m}^h) \subseteq \tau(R, \mathfrak{m}^d)$ by Proposition 5.14. For $e \gg 0$, $\tau(R, \mathfrak{m}^d) = (\mathfrak{m}^{dp^e})^{[1/p^e]}$ and we must show that this is contained in $\mathfrak{m}$.

Note that $\mathfrak{m}^{dp^e}$ is generated by monomials $x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}$ in the x_i of degree dp^e and so must have $a_i \geq p^e$ for some i. Therefore $\Phi^e(F^e_* x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d})$ is divisible by x_i. So

$$(\mathfrak{m}^{dp^e})^{[1/p^e]} = \Phi^e(F^e_* \mathfrak{m}^{dp^e}) \subseteq \mathfrak{m}$$

as desired. This completes the proof. $\square$

7.4. Exercises.

Exercise 7.1. For any ideal $\mathfrak{a}$ in a regular F-finite ring R, prove that $\mathfrak{a} \subseteq \tau(R, \mathfrak{a})$.

Hint: Use the fact that $\mathfrak{a}^{[p^e]} \subseteq \mathfrak{a}^{p^e}$ for all $e > 0$.

Exercise 7.2. Consider a surjective homomorphism $\pi : S \longrightarrow R$ of F-finite quasi-Gorenstein rings where $\ker(\pi)$ is generated by a regular sequence. For any ideal $\mathfrak{a} \subseteq S$ and any $t \geq 0$, show that

$$\tau(R, (\mathfrak{a}R)^t) \subseteq \tau(S, \mathfrak{a}^t)R.$$

Hint: Suppose $\ker \pi = (f_1, \dots, f_t)$ with the f_i's form a regular sequence. Deduce that each $R/(f_1, \dots, f_m)$ is S_2 and hence quasi-Gorenstein, see Appendix C Lemma 6.10.

Exercise 7.3. Formulate and prove the Restriction Theorem, Theorem 7.1, for mixed test ideals in regular rings defined as in Exercise 5.15.

Exercise 7.4. Suppose k is a perfect field, A and B are finitely generated k-algebras and $\Phi_A \in \operatorname{Hom}_A(F^e_* A, A)$ and $\Phi_B \in \operatorname{Hom}_B(F^e_* B, B)$ generate their respective Hom-sets. Let $S = A \otimes_k B$ and consider the map Φ_S

$$\begin{aligned} F^e_* S &\xrightarrow{\quad \Phi_S \quad} S \\ F^e_*(a \otimes b) &\longmapsto \Phi_A(a) \otimes \Phi_B(b). \end{aligned}$$

Prove that Φ_S generates $\mathrm{Hom}_S(F^e_* S, S)$ as an $F^e_* S$-module.

Hint: If you are stuck, you can find a proof in [**Smo20**, Corollary 3.10].

Exercise 7.5. This exercise gives another proof of subadditivity Theorem 7.4 under a certain hypothesis. Suppose that S is Noetherian F-finite and regular is a localization of a finite type k-algebra where k is a perfect field of characteristic $p > 0$. We let $T = S \otimes_k S$.

(a) To prove subadditivity, show that we may assume S is a smooth k-algebra (in particular, that S is finite type over k).

(b) Consider the diagonal surjection

$$S \otimes_k S \xrightarrow{\Delta} S$$

$$a \otimes b \longmapsto ab.$$

Let $\mathfrak{a}' = \mathfrak{a} \otimes_k S \subseteq T$ and $\mathfrak{b}' = S \otimes_k \mathfrak{b} \subseteq T$. Show that $\Delta(\mathfrak{a}') = \mathfrak{a}' S = \mathfrak{a}$ and $\Delta(\mathfrak{b}') = \mathfrak{b}' S = \mathfrak{b}$, and more generally that $\Delta(\mathfrak{a}'\mathfrak{b}') = \mathfrak{a}\mathfrak{b}$.

(c) Complete the proof by showing that $\tau(T, \mathfrak{a}'^s \mathfrak{b}'^t) = \tau(T, \mathfrak{a}'^s)\tau(T, \mathfrak{b}'^t)$ and then using the restriction theorem Theorem 7.1.

Exercise 7.6. Suppose Q is a P-primary ideal in a Noetherian domain. Show that $I \subseteq Q$ if and only if $IR_P \subseteq QR_P$.

More generally, if $J = Q_1 \cap \cdots \cap Q_m$ is an ideal where all of its associated primes $P_i = \sqrt{Q_i}$ are minimal, then $I \subseteq J$ if and only if $IR_{P_i} \subseteq JR_{P_i}$ for all i.

Exercise 7.7. Suppose R is a Noetherian ring, $I \subseteq R$ is a radical ideal and $n > 0$ is an integer. Suppose that $I^n = Q_1 \cap \cdots \cap Q_m \cap Q_{m+1} \cap \dots Q_l$ is a primary decomposition of I^n where $Q_1, \dots, Q_m$ are the primary ideals associated to the minimal primes. Show that

$$I^{(n)} = Q_1 \cap \cdots \cap Q_m.$$

Exercise 7.8. Suppose R is a normal Noetherian domain and $Q \subseteq R$ is a height one prime ideal. Prove that $Q^{(n)} = (Q^n)^{\S 2}$ where here $(-)^{\S 2}$ means reflexification (applying $\mathrm{Hom}_R(\bullet, R)$ twice) or S_2-ification.

Exercise 7.9. Suppose R is a Noetherian ring and I is a radical ideal. Show that

$$(I^{(a)})^b \subseteq I^{(ab)}.$$

CHAPTER 5

Anti-canonical Divisors and Maps in $\mathrm{Hom}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$

For a normal variety X of prime characteristic p, there is a correspondence between maps of $\mathcal{O}_X$-modules $F^e_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ and certain effective anti-canonical divisors on X. This chapter develops this point of view, deepening our understanding of Frobenius splitting and test ideal for pairs (X, Δ) where Δ is a $\mathbb{Q}$-divisor on X.

At the heart of our story is the natural isomorphism of sheaves of $F^e_*\mathcal{O}_X$-modules

$$\mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X, \mathcal{O}_X) \cong F^e_*\omega_X^{(1-p^e)} \cong F^e_*\mathcal{O}_X((1-p^e)K_X). \tag{0.0.1}$$

proved in Theorem 2.26 in Chapter 3. Using this isomorphism we will see that each global section $\phi \in \mathrm{Hom}_X(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ determines an effective Weil divisor D_ϕ on X in the linear system $|(1-p^e)K_X|$, where K_X is a canonical divisor. Normalizing D_ϕ appropriately, we associate to each such ϕ some effective anti-canonical $\mathbb{Q}$-divisor Δ_ϕ, which turns out to be—in the case where ϕ is a splitting of Frobenius— the maximal effective divisor Δ such that the pair (X, Δ) is Frobenius split by ϕ. More generally, we'll show that the divisor Δ_ϕ, an in particular the singularities of that divisor, govern the subschemes of X compatible with ϕ.

We also revisit, in Section 3, the subject of vanishing theorems for projective varieties, using our new theory of F-singularities for pairs to prove a version of the Kawamata-Viehweg vanishing theorem for Frobenius split varieties. In Section 3, we'll show that the divisors associated to maps enjoy an adjunction-type property we call "F-adjunction," and use this to develop new tools to establish Frobenius splitting. In Section 5, we develop the test ideal for pairs, with applications in...

Setting 0.1. Throughout this chapter, the reader is invited to assume that X is a normal variety over an algebraically closed field k of characteristic $p > 0$, so that the canonical sheaf can be taken to be the unique reflexive sheaf $\omega_X = (\wedge^{\dim X}\Omega_{X/k})^{\mathrm{S}_2}$ that agrees with the top exterior power of the sheaf of Kähler differentials $\wedge^{\dim X}\Omega_{U/k}$ on the non-singular locus U of X; see Chapter 2. However, the natural generality for this chapter is the setting of an arbitrary normal integral Noetherian F-finite scheme X, so we will work in this setting. In this case, we always take ω_X to be the *canonical* canonical module as defined in Chapter 2 Section 4. In particular,

$$\mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X, \omega_X) \cong F^e_*\omega_X$$

for all $e \in \mathbb{N}$.

1. The Divisor of a map in $\mathrm{Hom}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$

Let X be a normal Noetherian F-finite integral scheme. The sheaf

$$\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$$

is a reflexive $\mathcal{O}_X$-module (Lemma 4.3 in Appendix B) and hence it is torsion free and satisfies Serre's S_2 condition (Lemma 4.4 in Appendix B). In particular, it also satisfies Serre's S_2 condition when considered as an $F^e_*\mathcal{O}_X$-module (Corollary 4.6 in Appendix B), and so it is a rank one reflexive $F^e_*\mathcal{O}_X$-module on the normal scheme $\operatorname{Spec} F^e_*\mathcal{O}_X \cong X$. In particular, every non-zero global section

$$\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X) = \Gamma\big(X, \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)\big)$$

determines[1] a unique divisor D_ϕ on X such that

$$F^e_*\mathcal{O}_X(D_\phi) \cong \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X). \tag{1.0.1}$$

Our goal in this section is to reinterpret the right side of (1.0.1), and understand what D_ϕ tells us about Frobenius splitting when ϕ is a splitting of Frobenius.

Recall that by Theorem 2.26 in Chapter 3, there is a natural isomorphism of sheaves of $F^e_*\mathcal{O}_X$-modules:

$$\mathscr{H}\mathrm{om}(F^e_*\mathcal{O}_X, \mathcal{O}_X) \cong F^e_*\omega_X^{(1-p^e)}, \tag{1.0.2}$$

where the notation $\omega_X^{(1-p^e)} = \mathcal{O}_X((1-p^e)K_X)$ denotes the unique reflexive sheaf on X which agrees with $\omega_X^{\otimes 1-p^e}$ on the non-singular locus[2] of X. In particular, each non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$ is identified with a non-zero global section of the sheaf $F^e_*\omega_X^{(1-p^e)}$, and so— because F^e is affine— ϕ is identified with a non-zero section of the rank one reflexive sheaf $\omega_X^{1-p^e}$ on X. Its corresponding divisor of zeros is the divisor D_ϕ from (1.0.1). We package this idea into the following:

Definition 1.1 ([**MR85**])**.** Fix a non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$, where X is a normal integral Noetherian F-finite scheme. The **divisor associated to** ϕ is the divisor D_ϕ of zeros of the global section of the sheaf $F^e_*\omega_X^{(1-p^e)}$ obtained from ϕ via the isomorphism (1.0.2).

[1] by Appendix B Proposition 4.15

[2] Alternatively, $\omega_X^{(1-p^e)}$ can be defined as the reflexive hull $(\omega_X^{\otimes 1-p^e})^{\mathrm{S}_2}$ of $\omega_X^{\otimes 1-p^e}$

The following properties of D_ϕ are immediate from the definition:

Proposition 1.2. *Let X be a normal integral Noetherian F-finite scheme, and fix a canonical divisor*[3] *K_X for X. For any non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$ of coherent $\mathcal{O}_X$-modules, the associated divisor D_ϕ is an effective Weil divisor on X linearly equivalent to $(1-p^e)K_X$, that is,*

$$D_\phi \geq 0 \quad \text{and} \quad D_\phi \sim (1-p^e)K_X.$$

Furthermore, two maps $\phi, \phi' \in \mathrm{Hom}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ determine the same divisor if and only if they differ by a unit—that is, $D_\phi = D_{\phi'}$ if and only if $\phi = \phi' \circ F^e_* s$ for some $s \in \Gamma(X, \mathcal{O}_X)^\times$.

The divisor of a map commutes with localization; this follows from general properties of the divisor of zeros of a non-zero section of any rank-one reflexive sheaf on a normal integral scheme X:

Lemma 1.3. *Let D_ϕ be the divisor of a non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$, with notation as in Definition 1.1. For any open set $U \subset X$, the restriction $\phi_{|U} : F^e_*\mathcal{O}_U \longrightarrow \mathcal{O}_U$ determines the divisor $(D_\phi)_{|U}$. That is,*

$$(D_\phi)_{|U} = D_{\phi_{|U}}.$$

Example 1.4. Let R be a normal Noetherian F-finite domain, and suppose that $\Phi \in \mathrm{Hom}_R(F^e_*R, R)$ generates $\mathrm{Hom}_R(F^e_*R, R)$ as an F^e_*R-module (for example, R could be a regular local ring). Then each non-zero $\phi \in \mathrm{Hom}_R(F^e_*R, R)$ can be written as $\phi = \Phi \circ F^e_* g$ for some non-zero $g \in R$. In this case, letting X denote $\mathrm{Spec}\, R$, the divisor D_ϕ on X is the divisor $\mathrm{div}_X(g)$, whose support is precisely $\mathbb{V}(g) \subset X$.

Example 1.5. As a special case of Example 1.4, let V be an F-finite discrete valuation ring, and suppose t is a generator for its maximal ideal $\mathfrak{m}$. Then every $\phi \in \mathrm{Hom}_V(F^e_*V, V)$ is, up to precomposition with a unit, of the form $\phi = \Phi^e \star ut^n$ for some non-negative integer n, some unit u, and where $\Phi \in \mathrm{Hom}_V(F_*V, V)$ is a generating map. The corresponding divisor D_ϕ is the divisor $n[\mathfrak{m}] \in \mathrm{Div}(V) \cong \mathbb{Z}$.

A global consequence of the previous example follows.

Corollary 1.6. *Let X be a normal integral Noetherian F-finite scheme. Fix a non-zero map $\phi : F^e_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ and let D_ϕ be the divisor of ϕ. Then $D_\phi = 0$ if and only if ϕ globally generates $\mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ as an $F^e_*\mathcal{O}_X$-module.*

Proof of Corollary 1.6. Since $\mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ and its $F^e_*\mathcal{O}_X$-module generated by ϕ are both reflexive, we can check that they agree by

[3] meaning, any Weil divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$

checking on the non-singular locus, which has codimension at least two. So we assume that $\mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ is an invertible $F^e_*\mathcal{O}_X$-module. In general, an invertible sheaf is generated by some global section s at precisely the points where s is not zero. In particular, $\phi \in \mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ generates as an $F^e_*\mathcal{O}_X$-module at any point outside the divisor D_ϕ. This means $\mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ is globally generated by ϕ if and only if $D_\phi = 0$. $\square$

Example 1.7. Let $\mathbb{P}^n$ be the projective space over a perfect field k of prime characteristic. Consider the canonical toric Frobenius splitting π of $\mathbb{P}^n$ constructed in Example 1.4 of Chapter 3. We claim that the associated divisor is

$$D_\pi = (p^e - 1)\sum_{i=0}^{n} H_i,$$

where the H_i are the torus invariant hyperplanes $\mathbb{V}(x_i)$ in $\operatorname{Proj} k[x_0, x_1, \ldots, x_n]$. Note that D_π is an effective divisor in the linear system $|(1-p^e)K_{\mathbb{P}^n}|$, as expected.

To check this, recall that π can be described on each torus invariant chart $D_+(x_i) = \operatorname{Spec} k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ as the map which send the free basis elements

$$\left\{ F^e_* \left(\frac{x_0}{x_i}\right)^{a_0} \cdots \left(\frac{x_n}{x_i}\right)^{a_n} \right\}_{0 \leq a_j \leq p^e - 1}$$

to zero except for F^e_*1, which it sends to 1. One readily checks that

$$\pi = \Phi^e \circ F^e_* \left(\frac{x_0}{x_i}\right)^{p^e-1} \cdots \left(\frac{x_n}{x_i}\right)^{p^e-1} \tag{1.7.1}$$

as maps in

$$\mathrm{Hom}_{k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]}(F^e_* k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}], k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]),$$

where Φ^e is the standard monomial F^e_*R-module generator for $\mathrm{Hom}_R(F^e_*R, R)$ (see Proposition 1.3 in Chapter 2). So in the affine chart $D_+(x_i) \subseteq \mathbb{P}^n$, the divisor

$$D_{\pi|D_+(x_i)} = (p^e - 1)(H_0 + H_1 + \cdots + \widehat{H_i} + \cdots H_n)_{|D+(x_i)},$$

where each H_i is coordinate plane $\mathbb{V}(x_i)$ and the notation $\widehat{H_i}$ means that the ith hyperplane is omitted. Putting these together, we see that

$$D_\pi = (p^e - 1)\sum_{i=0}^{n} H_i,$$

as claimed.

1.1. Divisors and factorization of maps. The divisor D_ϕ can be characterized as the largest effective divisor D such that the map ϕ extends to $F^e_*\mathcal{O}_X(D)$:

Theorem 1.8. *Let X be a normal integral Noetherian F-finite scheme, and let D be an effective divisor on X. For any non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$, the map ϕ extends to a map $F^e_*\mathcal{O}_X(D) \xrightarrow{\tilde{\phi}} \mathcal{O}_X$ if and only if $D \le D_\phi$. In other words, the diagram*

$$
\begin{tikzcd}
F^e_*\mathcal{O}_X \arrow[rr, "\phi"] \arrow[dr, hook] & & \mathcal{O}_X \\
& F^e_*\mathcal{O}_X(D), \arrow[ur, dotted, "\widetilde{\phi}"'] &
\end{tikzcd}
\tag{1.8.1}
$$

can be completed to a commutative diagram of $\mathcal{O}_X$-module maps with a map $\widetilde{\phi}$ if and only if $D \le D_\phi$.

Theorem 1.8 implies that if $\phi \in \mathrm{Hom}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ is a *splitting of Frobenius,* then X is Frobenius e-split along D_ϕ, and D_ϕ is the *largest* effective divisor with this property.

Remark 1.9. The inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$ becomes the identity map at the generic point of X, so the map ϕ in Theorem 1.8 becomes a map $F^e_*\mathcal{K}(X) \to \mathcal{K}(X)$ at the generic point of X. Thus $\widetilde{\phi}$ in diagram (1.8.1), when it exists, is the restriction of the generic point of ϕ to $F^e_*\mathcal{O}_X(D)$, and hence the *unique* extension of ϕ to the sheaf $F^e_*\mathcal{O}_X(D)$. For this reason, we usually use the notation ϕ instead of $\widetilde{\phi}$ to denote this extension (when it exists).

To prove the theorem, we will use the following characterization, in terms of D_ϕ, of the subsheaf of $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ generated by ϕ:

Proposition 1.10. *Fix a non-zero global section ϕ of $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$, where X is as in Theorem 1.8. Then the coherent $F^e_*\mathcal{O}_X$-subsheaf of the sheaf $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ generated by ϕ is precisely $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D_\phi), \mathcal{O}_X)$.*

Proof of Proposition 1.10. Since both $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D_\phi), \mathcal{O}_X)$ and the subsheaf of $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ generated by the global section ϕ are reflexive $F^e_*\mathcal{O}_X$-subsheaves of $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$, we may check that they are the same in codimension 1, that is at the generic point of each prime divisor on X. So without loss of generality, we may assume that $X = \mathrm{Spec}\, R$ where R is an F-finite discrete valuation ring. Now the statement follows from Example 1.5. □

Proof of Theorem 1.8. Assume that $D \le D_\phi$. Since

$$\mathcal{O}_X \subseteq \mathcal{O}_X(D) \subseteq \mathcal{O}_X(D_\phi),$$

it suffices to show that the arrow $\widetilde{\phi}$ can be filled in when $D = D_\phi$. This follows immediately from Proposition 1.10.

Conversely, assume that ϕ extends to a map $F^e_*\mathcal{O}_X(D) \xrightarrow{\phi} \mathcal{O}_X$. We need to show that $D \leq D_\phi$. Since ϕ generates $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D_\phi), \mathcal{O}_X)$, every section $\psi \in \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D_\phi), \mathcal{O}_X)$ (on any open set) can be written as $\psi = \phi \star g$ for some section g of $\mathcal{O}_X$ (over that open set). In particular, $\psi \in \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X)$, since ϕ is in the $F^e_*\mathcal{O}_X$-module $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X)$. In other words,

$$\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D_\phi), \mathcal{O}_X) \subseteq \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X).$$

Applying $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(-, \mathcal{O}_X)$, we obtain an inclusion of reflexive sheaves (Appendix B Section 4)

$$F^e_*\mathcal{O}_X(D) \subseteq F^e_*\mathcal{O}_X(D_\phi),$$

and conclude that $\mathcal{O}_X(D) \subseteq \mathcal{O}_X(D_\phi)$. Since these inclusions are the identity at the generic point, this says that $D \leq D_\phi$. The theorem is proved. $\square$

2. Maps in $\mathrm{Hom}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ and anti-canonical $\mathbb{Q}$-divisors

Fix a non-zero map $\phi \in \mathrm{Hom}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$. Scaling appropriately, we produce an effective anti-canonical $\mathbb{Q}$-divisor on X associated to ϕ:

Definition 2.1. Let X be a normal integral F-finite Noetherian scheme. For any non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$ of $\mathcal{O}_X$-modules, define the **anti-canonical $\mathbb{Q}$-divisor** of ϕ to be the $\mathbb{Q}$-divisor

$$\Delta_\phi = \frac{1}{p^e - 1} D_\phi,$$

where D_ϕ is the divisor of ϕ (as in Definition 1.1).

By construction, the divisor Δ_ϕ is an effective $\mathbb{Q}$-divisor, with the property that $(p^e - 1)(\Delta_\phi + K_X)$ is an Weil divisor linearly equivalent to the trivial divisor. In particular, Δ_ϕ is $\mathbb{Q}$-linearly equivalent[4] to $-K_X$.

We have the following important correspondence between maps and anti-canonical effective $\mathbb{Q}$-divisors:

---

[4]Given two $\mathbb{Q}$-divisors Γ_1, Γ_2, we say that Γ_1 and Γ_2 are $\mathbb{Q}$-linearly equivalent (written $\Gamma_1 \sim_{\mathbb{Q}} \Gamma_2$) if there exists an $n > 0$ such that $n\Gamma_1$ and $n\Gamma_2$ are linearly equivalent Weil divisors.

Theorem 2.2. *Let X be a normal integral Noetherian F-finite scheme, with fixed canonical divisor K_X. There is a bijection*

$$\left\{ \begin{array}{c} \textit{non-zero } \mathcal{O}_X\textit{-linear maps} \\ \phi : F^e_*\mathcal{O}_X \longrightarrow \mathcal{O}_X \end{array} \right\} \Big/ \equiv \quad \to \quad \left\{ \begin{array}{c} \textit{Effective } \mathbb{Q}\textit{-divisors } \Delta \\ \textit{such that for some } e > 0 \\ (p^e - 1)(K_X + \Delta) \sim 0 \end{array} \right\}$$

induced by sending $\phi \mapsto \Delta_\phi$. Here the equivalence relation $\equiv$ on maps is generated by the relations $\phi \equiv \phi^n$ and $\phi \equiv \phi \star u$ where $u \in H^0(X, \mathcal{O}_X)$ is a unit. Again, $\phi^n := \phi^{\star n}$ denotes the n-fold self-composition of ϕ in the Cartier Algebra.[5]

To prove this, we use the following

Lemma 2.3 (*cf.* Chapter 3 Proposition 2.5 (c))**.** *Let X be a normal F-finite Noetherian scheme. Assuming that $\phi \in \mathrm{Hom}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ and $\psi \in \mathrm{Hom}(F^d_*\mathcal{O}_X, \mathcal{O}_X)$ are non-zero maps, then the divisor of the map $\phi \star \psi \in \mathrm{Hom}(F^{e+d}_*\mathcal{O}_X, \mathcal{O}_X)$ is*

$$D_{\phi\star\psi} = p^d D_\phi + D_\psi. \tag{2.3.1}$$

In particular, for any natural number n,

$$D_{\phi^n} = (1 + p^e + \cdots + p^{e(n-1)})D_\phi.$$

Proof. Because divisors are determined in codimension 1, we may assume that $X = \operatorname{Spec} R$ where R is a discrete valuation ring. In this case fix an F_*R-module generator Φ for $\mathrm{Hom}_R(F_*R, R)$. By Proposition 5.3 in Appendix A, we know that the compositions Φ^e, Φ^d, and Φ^{e+d} generate the modules $\mathrm{Hom}_R(F^e_*R, R)$, $\mathrm{Hom}_R(F^d_*R, R)$, and $\mathrm{Hom}_R(F^{e+d}_*R, R)$, respectively. Write

$$\phi = \Phi^e \circ F^e_*c = \Phi^e \star c \quad \text{and} \quad \psi = \Phi^d \circ F^d_*c' = \Phi^d \star c'$$

for some $c, c' \in R$, where $\star$ denotes the product in the Cartier algebra as discussed in Subsection 4.2 of Chapter 1. Then

$$\phi \star \psi = (\Phi^e \star c) \star (\Phi^d \star c')) = (\Phi^e \star \Phi^d \star c^{p^e} c') = \Phi^{e+d} \circ F^{e+d}_* c^{p^e} c'.$$

Here, we are using only the relation $c \star \Phi^d = \Phi^d \star c^{p^e}$ (see (4.12.4) in Chapter 1). Now the desired statements follow immediately from Example 1.4. $\square$

Proof of Theorem 2.2. We have already seen that the association $\phi \mapsto \Delta_\phi$ produces a $\mathbb{Q}$-divisor with the desired properties, and that ϕ and $\phi \star u$, where u is a unit, determine the same Δ. We must also check that for any non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$,

$$\Delta_\phi = \Delta_{\phi^n} \tag{2.3.2}$$

[5] See Subsection 4.2 of Chapter 1.

for all $n \geq 0$. This follows easily from Lemma 2.3:

$$\Delta_{\phi^n} = \frac{1}{p^{en}-1} D_{\phi^n} = \frac{1 + p^e + \cdots + p^{e(n-1)}}{p^{en}-1} D_\phi = \frac{1}{p^e - 1} D_\phi = \Delta_\phi.$$

This proves that $[\phi] \mapsto \Delta_\phi$ is well defined.

Finally, for $\phi, \psi \in \mathrm{Hom}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$, we know that $\Delta_\phi = \Delta_\psi$ if and only if ϕ and ψ differ by pre-multiplication by a unit in $H^0(X, F^e_*\mathcal{O}_X)$ (Proposition 1.2). Thus the assignment $[\phi] \mapsto \Delta_\phi$ is a well-defined injective mapping.

To see this mapping is bijective, take a $\mathbb{Q}$-divisor $\Delta \geq 0$ such that $(p^e - 1)(K_X + \Delta)$ is a Weil divisor linearly equivalent to zero. In particular, $(p^e - 1)\Delta$ is some effective Weil divisor D. Now use the isomorphism of coherent $F^e_*\mathcal{O}_X$-modules

$$F^e_*\mathcal{O}_X((1-p^e)(K_X + \Delta)) \cong \mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X) \tag{2.3.3}$$

(Exercise 2.5) to conclude that the sheaf $\mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X)$ is isomorphic to the trival sheaf $F^e_*\mathcal{O}_X$, and hence globally generated as an $F^e_*\mathcal{O}_X$-module by some non-vanishing global section ($F^e_*\mathcal{O}_X$ is globally generated by F^e_*1); in particular, there is a non-zero map

$$F^e_*\mathcal{O}_X(D) \xrightarrow{\phi} \mathcal{O}_X.$$

Because $D \geq 0$, the map ϕ restricts to a map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$. Finally, we leave it to the reader to verify that the assignment $\Delta \mapsto [\phi]$ is inverse to the assignment $[\phi] \mapsto \Delta_\phi$. $\square$

Another basic property of Δ_ϕ is the following restriction on the *index*[6] of $\Delta_\phi + K_X$ whose proof we omit.

Proposition 2.4. *With notation as in Definition 2.1, the index of the $\mathbb{Q}$-Cartier divisor $K_X + \Delta_\phi$ divides $p^e - 1$. In particular, the index of $K_X + \Delta_\phi$ is not divisible by p.*

As a corollary, we get the following special version of Theorem 2.2 in the *local* case where Cartier divisors all become principal:

Corollary 2.5. *Let R be an F-finite Noetherian normal local ring, and let $X = \mathrm{Spec}\, R$. The assignment $\phi \mapsto \Delta_\phi$ defines a bijection:*

$$\left\{ \begin{array}{c} \textit{non-zero } \mathcal{O}_X\textit{-linear maps} \\ \phi : F^e_*\mathcal{O}_X \longrightarrow \mathcal{O}_X \\ \textit{(ranging over all } e > 0) \end{array} \right\} \Big/ \equiv \quad \longrightarrow \quad \left\{ \begin{array}{c} \mathbb{Q}\textit{-divisors } \Delta \geq 0 \\ \textit{such that} \\ K_X + \Delta \textit{ is } \mathbb{Q}\textit{-Cartier} \\ \textit{with index not divisible by } p \end{array} \right\}$$

[6] Recall that the **index** of a $\mathbb{Q}$-Cartier divisor Γ is the smallest positive integer n such that $n\Gamma$ is a Cartier divisor (with integer coefficients).

where the equivalence relation on the maps is as in Theorem 2.2.

PROOF. A divisor on the spectrum of a local ring is Cartier if and only if it linearly equivalent to zero. On the other hand, the prime number p fails to divide n if and only if n divides $p^e - 1$ for some $e > 0$ (by Fermat's Little Theorem). The corollary follows by putting these two facts together with Theorem 2.2. $\square$

We make one more useful observation.

Proposition 2.6. *Suppose that $\phi_1 : F^{e_1}_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ and $\phi_2 : F^{e_2}_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ are nonzero maps on an F-finite normal scheme. Then*

$$\Delta_{\phi_1 \star \phi_2} = \frac{p^{e_1} - 1}{p^{e_1+e_2} - 1}\Delta_{\phi_1} + \frac{p^{e_1}(p^{e_2} - 1)}{p^{e_1+e_2} - 1}\Delta_{\phi_2}.$$

PROOF. This is a straightforward consequence of Lemma 2.3. $\square$

2.1. The geometry of Δ_ϕ and compatible subschemes. The geometry of Δ_ϕ has interesting implications for ϕ. For example, Δ_ϕ easily determines the codimension one integral subschemes compatible with ϕ:

Proposition 2.7 (*cf.* Chapter 3 Lemma 3.7)**.** *Fix a non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$, where X is a normal Noetherian integral F-finite scheme, and let Δ_ϕ be the associated anti-canonical $\mathbb{Q}$-divisor. Then an effective divisor G on X is ϕ-compatible*[7] *if and only if $G \leq \lfloor \Delta_\phi \rfloor$.*

PROOF. To check that $G = \sum a_i G_i$ is ϕ-compatible, where the G_i are the irreducible components of $\operatorname{supp} G$, it suffices to check this generically along each G_i by Chapter 1 Proposition 6.7 and Chapter 3 Lemma 3.4. So without loss of generality, we may assume that $X = \operatorname{Spec} V$, where V is a discrete valuation ring, and $G = a \operatorname{div}(t)$, where $a \geq 0$ and t is a uniformizing parameter for V In this case, we can write $\phi = \Phi^e \circ F^e_* u t^n$ for some $n > 0$, u is a unit, and where Φ^e is an F^e_*V-module generator for $\mathrm{Hom}_V(F^e_*V, V)$. So ϕ is compatible with G if and only if it sends $F^e_*\mathcal{O}_X(-G)$ into $\mathcal{O}_X(-G)$. In other words, if and only if

$$\phi(F^e_*(t^a)) \subseteq (t^a) \qquad \text{or equivalently,} \qquad \Phi^e(F^e_*(t^{n+a})) \subseteq (t^a).$$

But this happens if and only if $n + a \geq ap^e$ (Lemma 2.3 in Chapter 4), or equivalently, if and only if $\frac{n}{p^e-1} \geq a$. Finally, since $\Delta_\phi = \frac{n}{p^e-1} \operatorname{div}(t)$ (Example 1.4), we conclude that G is ϕ-compatible if and only if $G \leq \lfloor \Delta_\phi \rfloor$. $\square$

[7] meaning that the underlying subscheme of G is compatible with ϕ in the sense of Chapter 3 Definition 3.1

Proposition 2.8. *Let X be a normal Noetherian integral F-finite scheme. Consider a non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$, and let Δ_ϕ be the associated anti-canonical $\mathbb{Q}$-divisor. If $X \setminus \operatorname{Supp}\Delta_\phi$ is locally F-regular*[8]*, then there are no proper ϕ-compatible subschemes contained in $X \setminus \operatorname{Supp}(\Delta_\phi)$.*

Proof. Because compatibility is a local issue and we are only interested in $X \setminus \operatorname{Supp}(\Delta_\phi)$, we can assume that $X = \operatorname{Spec} R$ for some strongly F-regular domain R and $D_\phi = 0$, which implies that ϕ globally generates $\mathrm{Hom}_R(F^e_*R, R)$ (Corollary 1.6). Now consider an arbitrary proper non-zero ideal $I \subseteq R$, and take any non-zero $c \in I$. Because R is strongly F-regular, we can find $\psi \in \mathrm{Hom}_R(F^{en}_*R, R)$ such that $\psi(F^{en}_*c) = 1$ (see Lemma 4.7 in Chapter 1), and because ϕ^n generates $\mathrm{Hom}_R(F^{en}_*R, R)$ (Proposition 5.3 in Appendix A), we can write $\psi = \phi^n \circ F^{en}_*s$ for some $s \in S$. It follows that $1 \in \phi^n(F^{ne}_*I)$. So I is not ϕ^n compatible, and hence neither can I be ϕ-compatible (Exercise 2.6). This completes the proof. $\square$

In some cases, we can completely determine all the compatible subschemes of ϕ in terms of Δ_ϕ:

Proposition 2.9. *Let X be a regular integral F-finite scheme. For a fixed non-zero map $F^e_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$, suppose that the corresponding anti-canonical $\mathbb{Q}$-divisor has the following properties:*

(i). $\lceil \Delta_\phi \rceil$ *is reduced;*
(ii.) $\operatorname{Supp}\Delta_\phi$ *is a simple normal crossing divisor.*

Then the irreducible ϕ-compatible subschemes are precisely the strata[9] *of the Weil divisor $\lfloor \Delta_\phi \rfloor$—that is, the irreducible subschemes of X obtained as arbitrary intersections of those components of Δ_ϕ with coefficient exactly one.*

Before proving Proposition 2.9, we point out the following corollary, whose proof is left as an exercise:

Corollary 2.10. *With hypothesis as in Proposition 2.9, the compatible subschemes of X are precisely the unions of strata of $\lfloor \Delta_\phi \rfloor$*

Proof of Proposition 2.9. Compatibility is a local issue, so localizing along the generic point of an arbitrary integral closed subscheme, we may assume that $X = \operatorname{Spec} R$, where $(R, \mathfrak{m})$ is a d-dimensional regular local

[8]meaning the local ring at each point is strongly F-regular; see Chapter 1 Definition 4.27.

[9]See Appendix B Definition 7.3.

ring (Proposition 6.7). It suffices to show that $\mathfrak{m}$ is compatible if and only if $\mathbb{V}(\mathfrak{m})$ is among the strata of $\lfloor \Delta_\phi \rfloor$.

Indeed, we already showed that the ideals of the strata are compatible in Chapter 4 Exercise 5.3 so suppose conversely that $\mathbb{V}(\mathfrak{m})$ is not among the strata. We can again write

$$\phi = \Phi^e \circ F^e_* u x_1^{a_1} \cdots x_d^{a_d}$$

where $0 \leq a_i \leq p^e - 1$ and the x_i generate $\mathfrak{m}$. The condition that $\mathfrak{m}$ is not a stratum of $\lfloor \Delta_\phi \rfloor$ ensures that at least one of the a_i—say, a_1— is strictly less than $p^e - 1$. Passing to the completion $\widehat{R} \cong k[\![x_1, \dots, x_d]\!]$, we may choose a generator Φ^e for $\mathrm{Hom}_{\widehat{R}}(F^e_* \widehat{R}, \widehat{R})$ sending $x_1^{p^e-1} \dots x_d^{p^e-1}$ to 1 and all other basis monomials to 0. It follows that $\phi(F^e_*(x_1)) = \Phi^e(F^e_*(x_1^{a_1+1} x_2^{a_2} \cdots x_d^{a_d})) = R$ and hence $\mathfrak{m}$ is not compatible (Proposition 6.7 in Chapter 1). $\square$

2.2. Exercises.

Exercise 2.1. Consider $R = \mathbb{F}_p[x, y]$, fix $0 \leq a, b, \leq p-1$ and let $\phi : F_* R \longrightarrow R$ be the map that sends $F_* x^a y^b \mapsto 1$ and the other basis monomials to zero. Compute D_ϕ.

Exercise 2.2. Suppose that $X = \operatorname{Spec} R$ is a normal F-finite domain. Suppose that $\psi, \phi \in \mathrm{Hom}(F^e_* R, R)$ satisfy the relation $\psi = \phi \star c$.Prove that $D_\psi = \mathrm{div}_X(c) + D_\phi$.

Exercise 2.3. Consider $R = \mathbb{F}_p[x, y]$, fix $0 \leq a, a', b, b' \leq p - 1$ and let $\phi : F_* R \longrightarrow R$ be the map that sends $F_* x^a y^b \mapsto 1$ and $F_* x^{a'} y^{b'} \mapsto 1$ and sends the other basis monomials to zero. Compute D_ϕ.

Exercise 2.4. Consider the ring

$$R = \mathbb{F}_p[x, y, z]/(xy - z^{p-1}) \cong \mathbb{F}_p[u^{p-1}, v^{p-1}, uv] \subseteq \mathbb{F}_p[u, v].$$

Let $D = \mathrm{div}(x, z)$. Verify that $(p - 1)D \sim 0$ and find a map $\phi : F_* R \longrightarrow R$ with $D_\phi = D$.

Exercise 2.5. Let X be a normal F-finite integral scheme. Prove that for any Weil divisor D,

$$\mathscr{H}om(F^e_* \mathcal{O}_X(D), \mathcal{O}_X) \cong F^e_* \mathcal{O}_X((1 - p^e)K_X - D). \tag{2.10.1}$$

Exercise 2.6. Suppose that R is a normal Noetherian F-finite ring. Show that an ideal I is ϕ-compatible, then I is $\phi^n := \phi^{\star n}$-compatible for all $n \geq 1$.

Exercise 2.7. Suppose $\phi : F^e_* R \longrightarrow R$ is surjective. Prove that every coefficient of Δ_ϕ is ≤ 1.

3. Local and global Frobenius splitting for pairs (X, Δ)

The anti-canonical $\mathbb{Q}$-divisors associated to maps in $\mathrm{Hom}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ shed new light on the subject of Frobenius splitting. Indeed, if X is a normal Frobenius split variety, we will see that the $\mathbb{Q}$-divisors of the form Δ_ϕ are "extremal" for the property that the pair (X, Δ) is Frobenius split.

We have already defined sharp Frobenius splitting and strong F-regularity for pairs of the form (R, f^t) for $t \geq 0$ in Definition 4.1 within Chapter 4. The definitions easily adapt to arbitrary real divisors $\Delta \geq 0$ on normal schemes; in fact, by working with divisors, we produce a slightly "better" definition without the ambiguity discussed in Caution 4.8 of Chapter 4. See Caution 3.5 below.

3.1. Sharp Frobenius Splitting of pairs (X, Δ). For any effective Weil divisor D on a normal scheme X, there is a natural inclusion of subsheaves of $\mathcal{K}(X)$

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D),$$

and hence in characteristic $p > 0$, an inclusion

$$F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$$

for all $e > 0$. Thus we can define:

Definition 3.1 ([**HW02, SS10**])**.** Let X be a normal integral Noetherian F-finite scheme, and let Δ be an effective $\mathbb{R}$-divisor on X. Consider the composition map

$$\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil), \tag{3.1.1}$$

where the first map is Frobenius and the second is induced by the natural inclusion $\mathcal{O}_X \subseteq \mathcal{O}_X(\lceil (p^e-1)\Delta \rceil)$ of subsheaves of $\mathcal{K}(X)$. The pair (X, Δ) is said to be **globally sharply Frobenius split** (or simply F**-split**[10]) if there exists some $e > 0$ such that (3.1.1) splits in the category of $\mathcal{O}_X$-modules.

Likewise, (X, Δ) is said to be **locally (sharply) Frobenius split** if there exists some $e > 0$ such that the composition map (3.1.1) splits at each point $x \in X$.

Remark 3.2. Note that (X, Δ) is Frobenius split if and only if X is e-Frobenius split along the Weil divisor $\lceil (p^e-1)\Delta \rceil$.

Remark 3.3. Global Frobenius splitting obviously implies local Frobenius splitting for a pair (X, Δ), but the converse is false, even in the case where $\Delta = 0$ (see, e.g., Example 1.7 in Chapter 3). On the other hand, for *affine* X,

[10] We usually drop the adverb "sharply" from the terminology, as we will not consider any other variant in this context, as well as the adverb "globally" which we include only for emphasis when confusion might arise.

local and global Frobenius splitting of (X, Δ) are equivalent, since splitting for a map of coherent sheaves on a Noetherian affine scheme can be checked locally. In particular, for an arbitrary scheme X, the pair (X, Δ) is locally Frobenius split if and only if X has a cover by open affine sets $\mathcal{U}_i$ with $(\mathcal{U}_i, \Delta_{|\mathcal{U}_i})$ Frobenius split. See Exercise 3.1.

Remark 3.4. A pair (X, Δ) is (locally or globally) Frobenius split if and only if the map (3.1.1) splits (locally or globally, respectively) for infinitely many e; see Exercise 3.3.

Caution 3.5. When $X = \operatorname{Spec} R$ and $\Delta = t \operatorname{div} f$, it is worth comparing Definition 3.1 to Definition 4.1 in Chapter 4 for sharp Frobenius splitting of (R, f^t). They are not quite equivalent! Indeed, if the pair (R, f^t) is sharply Frobenius split in the sense of Chapter 4 Definition 4.1, then (X, Δ) satisfies Definition 3.1, but the converse does *not* quite hold. While almost equivalent, Definition 3.1 is slightly more robust: it eliminates the inconvenient ambiguity we encountered earlier in Caution 4.8 of Chapter 4 when comparing (R, f^t) and $(R, (f^n)^{\frac{t}{n}})$. See Exercises 3.6 and 3.7.

We can characterize Frobenius splitting for pairs using the anti-canonical $\mathbb{Q}$-divisors associated to maps as follows:

Proposition 3.6. *Let (X, Δ) be a globally Frobenius split pair, where X is a normal F-finite integral scheme, and $\Delta \geq 0$ is an $\mathbb{R}$-divisor on X. Then there exists a Frobenius splitting $\phi : F^e_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ such that*

(a) $\Delta \leq \Delta_\phi$;
(b) (X, Δ_ϕ) *is globally Frobenius split; and*
(c) There is an integer n, not divisible by p, such that the $\mathbb{Q}$-divisor $n(K_X + \Delta_\phi)$ is linearly equivalent to the trivial divisor (that is, Δ_ϕ is an anticanonical $\mathbb{Q}$-divisor).

In this case, both (X, Δ) and the pair (X, Δ_ϕ) are globally Frobenius split by a map ϕ corresponding to Δ_ϕ under the correspondence in Theorem 2.2.

Proof. We know there exists $\phi : F^e_* \mathcal{O}_X \subseteq F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil) \longrightarrow \mathcal{O}_X$ splitting $\mathcal{O}_X \subseteq F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$ where $\Delta_\phi \geq \Delta$. By Theorem 1.8 we see that $\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(D_\phi) = F^e_* \mathcal{O}_X((p^e - 1)\frac{1}{p^e-1}\Delta_\phi)$ splits and therefore (X, Δ_ϕ) is globally Frobenius split by ϕ as well. $\square$

3.2. Local and global F-regularity for (X, Δ). Eventual Frobenius splitting along effective divisors generalizes naturally to pairs:

Definition 3.7. Let (X, Δ) be a pair, where X is normal integral Noetherian F-finite scheme and $\Delta \geq 0$ is an $\mathbb{R}$-divisor on X. For an effective Weil-divisor

D, we say that (X, Δ) is **(globally sharply) e-Frobenius split along** D if the composition

$$\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta\rceil + D) \tag{3.7.1}$$

splits in the category of $\mathcal{O}_X$-modules. The pair (X, Δ) is said to be **eventually (globally sharply) Frobenius split along** D if it is e-Frobenius split along D for some e.

Likewise, **local** e-Frobenius splitting along D and **local** eventual Frobenius splitting along D are defined analogously as the splitting of (3.7.1) on an affine cover of X, or equivalently, at the stalk of each point of X.

Eventually globally Frobenius split along D can also be rephrased as follows.

Lemma 3.8. *A pair (X, Δ) is eventually globally Frobenius split along D if and only if the pair $(X, \Delta + \epsilon D)$ is globally Frobenius split for some $\epsilon > 0$.*

PROOF. This is a direct consequence of the definition. Indeed, note that if $\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta\rceil + D)$ splits, then the pair $(X, \frac{1}{(p^e-1)}\lceil (p^e-1)\Delta\rceil + \frac{1}{p^e-1}D)$ is also Frobenius split. Hence, so is $(X, \Delta + \epsilon D)$ for $\epsilon = \frac{1}{p^e-1}$. Conversely, if $(X, \Delta+\epsilon D)$ is globally Frobenius split, then

$$\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X(\lceil (p^e-1)(\Delta + \epsilon D)\rceil)$$

splits for $e > 0$ sufficiently divisible. But for large enough $e > 0$, we have that

$$\lceil (p^e-1)(\Delta + \epsilon D)\rceil \geq \lceil (p^e-1)\Delta\rceil + D.$$

This completes the proof. $\square$

Frobenius splitting along D is easily characterized using anti-canonical $\mathbb{Q}$-divisors of maps:

Proposition 3.9. *Let X be normal F-finite integral scheme, with effective $\mathbb{R}$-divisor Δ. Then the pair (X, Δ) is* (globally) e-Frobenius split along D *if and only if there exists a splitting of Frobenius $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ such that $\Delta_\phi \geq \Delta + \frac{1}{p^e-1}D$.*

PROOF. This is an easy consequence of Theorem 1.8. $\square$

Definition 3.10 (**[HW02, SS10]**)**.** Let (X, Δ) be a pair, where X is normal integral Noetherian F-finite scheme and $\Delta \geq 0$ is an $\mathbb{R}$-divisor on X.

The pair (X, Δ) is **globally F-regular**[11] if (X, Δ) is eventually (globally) Frobenius split along *every* effective Weil divisor D on X.

Remark 3.11. When X is a quasi-projective (over an affine base) normal integral Noetherian F-finite scheme, then (X, Δ) is globally F-regular if and only if the pair (X, Δ) is eventually (globally) Frobenius split along every effective *Cartier* divisor D, see Remark 2.10 in Chapter 3. However, the reader is cautioned that an arbitrary variety may not have *any* non-zero Cartier divisors, so this is not an appropriate definition in general.

Fortunately, to prove a pair (X, Δ) is globally F-regular, we do not have to check eventual splitting along *all* divisors, only *one well-chosen* divisor:

Theorem 3.12. *Let (X, Δ) be a pair, where X is a normal F-finite integral scheme and Δ is an effective $\mathbb{R}$-divisor. Suppose that there exists an effective Weil divisor $B \subseteq X$ such that:*

(i). The pair (X, Δ) is eventually globally Frobenius split along B (equivalently, $(X, \Delta + \epsilon B)$ is globally Frobenius split for some $\epsilon > 0$); and
(ii). The pair $(\mathcal{U}, \Delta|_{\mathcal{U}})$ is globally F-regular, where $\mathcal{U} := X \setminus B$.

Then (X, Δ) is globally F-regular.

Proof. We have seen versions of this theorem already in Theorem 5.1 in Chapter 1, Theorem 2.14 in Chapter 3, and Theorem 4.14 in Chapter 4, so we leave the proof as Exercise 3.13. $\square$

3.3. Perturbing real divisors Δ. A natural question is the extent to which the singularities of a pair (X, Δ) remain nice under small perturbations of Δ. First note that shrinking Δ preserves Frobenius splitting:

Proposition 3.13. *Let X be normal integral Noetherian F-finite scheme. Suppose Δ and Δ' are effective $\mathbb{R}$-divisors such that $\Delta' \leq \Delta$. Then*

(a) If (X, Δ) is (locally or globally) Frobenius split, then so is (X, Δ');
(b) If (X, Δ) is (locally or globally) F-regular, then so is (X, Δ').

We can also slightly increase Δ in the direction of any effective divisor without affecting strong F-regularity:

[11] We drop the adverb "strongly" from the terminology, although we may sometimes include it for emphasis. Likewise, we always mean *globally* F-regular if we do not explicitly modify with the word "locally".

Proposition 3.14. *Suppose that (X, Δ) is (locally or globally) F-regular and $D > 0$ is an effective divisor. Then for all sufficiently small positive ϵ, the pair $(X, \Delta + \epsilon D)$ is (locally or globally, respectively) F-regular as well.*

PROOF. Because (X, Δ) is F-regular, there exists $e > 0$ such that the Frobenius map

$$\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + 2D) \tag{3.14.1}$$

splits. We claim that the pair $(X, \Delta + \frac{1}{p^e-1}D)$ is globally F-regular, whence it will follow from Proposition 3.13 that $(X, \Delta + \epsilon D)$ is F-regular for all $\epsilon \leq \frac{1}{p^e-1}$, completing the proof.

To prove the claim, we apply Theorem 3.12 to the pair $(X, \Delta + \frac{1}{p^e-1}D)$, using D as the "test divisor" B. To check that condition (a) in Theorem 3.12 holds, observe that

$$\lceil (p^e - 1)\Delta \rceil + 2D = \lceil (p^e - 1)(\Delta + \frac{1}{p^e - 1}D) \rceil + D.$$

So the splitting of (3.14.1) implies that the pair $(X, \Delta + \frac{1}{p^e-1}D)$ is eventually globally Frobenius split along D, and (a) holds. To check condition (b), note that $(X, \Delta + \frac{1}{p^e-1}D)$ restricts to (X, Δ) on $\mathcal{U} = X \setminus D$, so (b) holds by hypothesis. This completes the proof. $\square$

Next we prove a convexity-type result for the space of $\mathbb{R}$-divisors Δ such that the pair (X, Δ) has good properties with respect to Frobenius splitting. The first part of what follows is simply a rescaling of Chapter 3 Proposition 2.5 (c) which itself is a global restatement of Chapter 1 Proposition 4.9

Proposition 3.15. *Let Δ_1 and Δ_2 be effective $\mathbb{R}$-divisors on a normal F-finite integral scheme X. Assume that both (X, Δ_1) and (X, Δ_2) are globally Frobenius split. Then there exists arbitrarily small rational $\epsilon > 0$ such that the pair*

$$(X, (1 - \epsilon)\Delta_1 + \epsilon\Delta_2)$$

is globally Frobenius split. Additionally, if either (X, Δ_1) or (X, Δ_2) is globally F-regular, then so is the pair

$$(X, (1 - \epsilon)\Delta_1 + \epsilon\Delta_2).$$

Finally, writing ϵ as a fraction of integers, we may assume its denominator is not divisible by p.

PROOF. The first statement follows either from Proposition 2.6 (replacing Δ_i by $\Delta_{\phi_i} \geq \Delta_i$) or from Chapter 3 Proposition 2.5 (c) since (X, Δ) is F-split if and only if X is e-Frobenius split along $\lceil (p^e - 1)\Delta \rceil$ for some $e > 0$.

Next suppose that either (X, Δ_2) or (X, Δ_1) is globally F-regular. We can choose an effective Weil divisor C whose support contains both Δ_1 and Δ_2 and such that, if we set $\mathcal{U} = X \setminus C$, then the pairs $(\mathcal{U}, \Delta_i|_{\mathcal{U}}) = (\mathcal{U}, 0)$ are globally F-regular. For some $1 \gg \delta > 0$, either the pair $(X, \Delta_1 + \delta C)$ or $(X, \Delta_2 + \delta C)$ is globally F-regular (Proposition 3.14) and hence globally Frobenius split. Now by the globally Frobenius split case, there exists $\epsilon > 0$ such that either

$$\text{(3.15.1)} \quad \Big(X, (1-\epsilon)(\Delta_1 + \delta C) + \epsilon \Delta_2\Big) \text{ or } \Big(X, (1-\epsilon)\Delta_1 + \epsilon(\Delta_2 + \delta C)\Big)$$

is globally Frobenius split, either one of which implies that $\big(X, (1-\epsilon)\Delta_1 + \epsilon\Delta_2\big)$ is eventually globally Frobenius split along C by Lemma 3.8. Notice that the pair (3.15.1) restricts to $(\mathcal{U}, 0)$ which is globally F-regular. Thus $\big(X, (1-\epsilon)\Delta_1 + \epsilon\Delta_2\big)$ is globally F-regular (by Theorem 3.12), as desired. $\square$

Remark 3.16. It is natural to ask that if (X, Δ_1) and (X, Δ_1) are F-split, whether $(X, s\Delta_1 + t\Delta_2)$ is F-split for every $s, t \geq 0$ with $s + t = 1$. Unfortunately this is false, see for instance **[PÍ3]**.

Corollary 3.17. *Suppose (X, Δ) is globally F-regular and $(X, \Delta + tD)$ is globally Frobenius split for some $t > 0$ and effective Weil divisor D. Then $(X, \Delta + t'D)$ is globally F-regular for every $t' \in [0, t)$.*

PROOF. This is left to the reader in Exercise 3.16. $\square$

3.4. More vanishing theorems. We can now generalize the vanishing theorems of Chapter 3 to globally Frobenius split and F-regular pairs.

We know from Theorem 2.32 in Chapter 3 that if $L - K_X$ is ample and X is F-split, then $H^i(X, \mathcal{O}_X(L)) = 0$. The value of pairs in this context is that yields the same vanishing even though $L - K_X$ is only "close to ample" (as measured by a pair).

Theorem 3.18. *Let L be a Weil divisor on a normal integral projective scheme X over an F-finite field. Suppose there exists some effective Weil divisor $\Delta \geq 0$ on X such that*

(i). The pair (X, Δ) is globally Frobenius split; and
(ii). The $\mathbb{Q}$-divisor $L - K_X - \Delta$ is ample.

Then

$$H^i(X, \mathcal{O}_X(L)) = 0$$

for all $i > 0$.

PROOF. Since (X, Δ) is globally Frobenius split, there exists $e > 0$ (and hence, there are infinitely many e) such that the map

$$\mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil), \mathcal{O}_X) \xrightarrow{\text{eval at } F^e_* 1} \mathcal{O}_X \tag{3.18.1}$$

is a surjection on global sections. Since $\mathcal{O}_X$ is free, this is a *split* surjection. Making use of the isomorphism from Exercise 2.5, we obtain a split surjection of sheaves

$$F^e_*\mathcal{O}_X(\lfloor (1-p^e)(K_X+\Delta) \rfloor) \twoheadrightarrow \mathcal{O}_X. \tag{3.18.2}$$

Twisting by L (and reflexifying if L is not Cartier), we have a split surjective map

$$F^e_*\mathcal{O}_X(\lfloor (1-p^e)(K_X+\Delta) \rfloor + p^e L) \twoheadrightarrow \mathcal{O}_X(L).$$

Now, let A denote the ample $\mathbb{Q}$-divisor $L - K_X - \Delta$, and fix n such that nA is an ample Cartier divisor. For each e, use the division algorithm to write $p^e - 1 = nq_e + r_e$ with $q_e \in \mathbb{Z}$ and $0 \leq r_e < n$. Next observe that

$$\lfloor (1-p^e)(K_X+\Delta) \rfloor + p^e L = nq_e A + (r_e+1)L + \lfloor -r_e(K_X+\Delta) \rfloor.$$

As we range over infinitely many e for which (3.18.2) splits, there are only finitely many possible modules $\mathscr{M}_{r_e} = \mathcal{O}_X((r_e+1)L + \lfloor -r_e(K_X+\Delta) \rfloor)$, since there are only finitely many possible remainders r_e when dividing by n. Hence, if e is large enough,

$$H^i\big(X, F^e_*\mathcal{O}_X(\lfloor (1-p^e)(K_X+\Delta) \rfloor + p^e L)\big) \cong H^i\big(X, F^e_*(\mathscr{M}_{r_e} \otimes \mathcal{O}_X(nA)^{q_e})\big)$$

vanishes for all $i > 0$ by Serre vanishing. Finally, the split surjectivity of (3.18.2) implies that $H^i(X, \mathcal{O}_X(L)) = 0$ as well. $\square$

Under the stronger hypothesis that (X, Δ) is globally F-regular, we can weaken the ample hypothesis to big and nef:

Corollary 3.19. *Let L be a Weil divisor on a normal integral projective scheme X over an F-finite field. Suppose there exists some effective $\mathbb{Q}$-divisor $\Delta \geq 0$ on X such that*

(i). The pair (X, Δ) is globally F-regular; and
(ii). The $\mathbb{Q}$-divisor $L - K_X - \Delta$ is nef and big.

Then

$$H^i(X, \mathcal{O}_X(L)) = 0$$

for all $i > 0$.

Proof. Since $L - K_X - \Delta$ is big and nef, there exists an effective divisor D so that for sufficiently small $\epsilon > 0$, $L - K_X - \Delta - \epsilon D$ is ample.[12] Furthermore, we may assume that $(X, \Delta + \epsilon D)$ is globally F-regular and hence globally F-split. But now we can directly apply Theorem 3.18. □

3.5. Global F-regularity and log Fano varieties. Global Frobenius splitting has significant consequences for the geometry of a normal projective variety of characteristic p. For example, we have essentially alreasdy proved that globally Frobenius split pairs (X, Δ) are log Calabi-Yau in Proposition 3.6. If the pair (X, Δ) is globally F-regular, we get even stronger consequences on the global geometry of X:

Theorem 3.20 (Globally F-regular is log Fano, **[SS10]**). *Let (X, Δ') be a globally F-regular pair where X is projective and finite type over an F-finite affine scheme. Then there exists an effective $\mathbb{Q}$-divisor $\Delta \geq \Delta'$ such that*

(a) The pair (X, Δ) is globally F-regular.
(b) The $\mathbb{Q}$-divisor $-K_X - \Delta$ is ample and has $\mathbb{Q}$-Cartier index not divisible by p.

In fact, for every ample Cartier divisor A, there exists Δ satisfying (a) and such that $-K_X - \Delta$ is $\mathbb{Q}$-linearly equivalent to a multiple of A with index not divisible by $p > 0$.

Proof of Theorem 3.20. By Proposition 3.6, there exists $\Delta_1 = \Delta' + \Gamma_1 \geq \Delta'$ such that (X, Δ_1) is globally Frobenius split and such that $K_X + \Delta_1 \sim_{\mathbb{Q}} 0$.

Next choose $H > 0$ an ample effective Cartier divisor such that $H \sim nA$ for some integer $n > 0$ and where $\operatorname{Supp} \Delta_1 \subseteq \operatorname{Supp} H$. By Proposition 3.14 $(X, \Delta' + \epsilon H)$ is globally F-regular and so by Proposition 3.6 we can find a $\Delta_2 + \epsilon H = \Delta' + +\Gamma_2 + \epsilon H$ with $(X, \Delta_2 + \epsilon H)$ Frobenius split and $(p^e - 1)(K_X + \Delta_2 + \epsilon H) \sim 0$ for infinitely many e.

At this point the solution seems at hand, $K_X + \Delta_2 \sim_{\mathbb{Q}} -\epsilon H$ is anti-ample and (X, Δ_2) is Frobenius split, and we will perturb it to make it globally F-regular. We do this by "averaging" with (X, Δ_1). Let us recall what we know so far.

(a) (X, Δ_1) is globally Frobenius split and $K_X + \Delta_1 \sim_{\mathbb{Q}} 0$.
(b) (X, Δ_2) is globally Frobenius split and $K_X + \Delta_2 \sim_{\mathbb{Q}} -\epsilon H$ is anti-ample.
(c) $(X, \Delta_2 + \delta H)$ is globally Frobenius split for $\epsilon \geq \delta > 0$.

[12] Add reference

Applying Proposition 3.15 we have $s, t \geq 0$ with $s + t = 1$ and $1 \gg t > 0$ such that

$$(X, s\Delta_1 + t(\Delta_2 + \delta H)) = (X, \Delta' + s\Gamma_1 + t\Gamma_2 + t\delta H)$$

is globally Frobenius split. Now, for sufficiently small r, $r\Gamma_1 < t\delta H$ (since H contains the support of Δ_1 which contains the support of Γ_1) and so

$$(X, \Delta' + (s + r)\Gamma_1 + t\Gamma_2)$$

is also globally Frobenius split. Furthermore, since t is sufficiently small, $(X, \Delta' + t\Gamma_2)$ is also globally F-regular either by Proposition 3.14 or Corollary 3.17.

Now, apply Corollary 3.17 to the globally F-regular pair $(X, \Delta' + t\Gamma_2)$ and the globally Frobenius split pair $(X, \Delta' + (s + r)\Gamma_1 + t\Gamma_2)$ and we see that

$$(X, \Delta' + s\Gamma_1 + t\Gamma_2)$$

is globally F-regular.

Finally, observe that

$$\begin{array}{rl} & K_X + \Delta' + s\Gamma_1 + t\Gamma_2 \\ = & s(K_X + \Delta' + \Gamma_1) + t(K_X + \Delta' + \Gamma_2) \\ = & s(K_X + \Delta_1) + t(K_X + \Delta_2) \\ \sim_{\mathbb{Q}} & -t\epsilon H \end{array}$$

We set $\Delta = \Delta' + s\Gamma_1 + t\Gamma_2$.

Finally, we address the $\mathbb{Q}$-Cartier index assertion. By construction, we may assume that $(p^e - 1)(K_X + \Delta_1) \sim 0$ for e sufficiently divisible. Likewise $(p^e - 1)(K_X + \Delta_2 + \epsilon H) \sim 0$ for such divisible e. Furthermore, ϵ, s and t maybe chosen without p in their denominator. Hence for sufficiently divisible e, $(p^e - 1)(K_X + \Delta) \sim -(p^e - 1)t\epsilon H$ is linearly equivalent to an anti-ample Cartier divisor (a negative multiple of H). This completes the proof. $\square$

Even in the case that X is local, the previous result says something interesting.

Corollary 3.21. *Suppose that R is an F-finite normal domain, $X = \operatorname{Spec} R$ and $\Delta' \geq 0$ is an $\mathbb{R}$-divisor such that (X, Δ') is strongly F-regular. Then there exists $\Delta_\phi \geq \Delta'$ for some $\phi : F^e_* R \longrightarrow R$ such that (X, Δ_ϕ) is strongly F-regular.*

Proof. Choose $A \sim 0$ which is ample since X is affine, then there exists $\Delta \geq \Delta'$ where (X, Δ) is strongly F-regular and such that $(p^e - 1)(K_X + \Delta)$ is linearly equivalent to a multiple of A, and hence linearly equivalent to zero. Now apply Theorem 2.2. $\square$

3.6. Exercises.

Exercise 3.1. Prove *local* and *global* Frobenius splitting (respectively strong F-regularity) of (X, Δ) are equivalent when $X = \operatorname{Spec} R$ and R is a normal Noetherian F-finite domain.

In particular, show for that an arbitrary normal Noetherian F-finite scheme X, the pair (X, Δ) is locally Frobenius split (respectively strongly F-regular) if and only if X has a cover by open sets $\mathcal{U}_i$ with $(\mathcal{U}_i, \Delta_{\mathcal{U}_i})$ globally Frobenius split (respectively globally F-regular).

Exercise 3.2. Suppose (X, Δ) is globally e-Frobenius split along $D > 0$. Prove it is also globally ne-Frobenius split along D for each integer $n \geq 1$.

Exercise 3.3. Prove the statements in Remark 3.4.

Hint: See the proof of Lemma 4.4 in Chapter 4.

Exercise 3.4. Show that the evaluation-at-$F^e_* 1$ map $\operatorname{Hom}_R(F^e_* R(D), R) \longrightarrow R$ is surjective if and only if there exists $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ surjective such that $D_\phi \geq D$. Here $R(D) := \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}(D))$.

Exercise 3.5. Find an example of a ring R, and a map $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that the extension of ϕ to $F^e_* R(D_\phi)$ is surjective but ϕ is not.

Exercise 3.6. Consider $R = \mathbb{F}_2[x]$, set $X = \operatorname{Spec} R$ and set $f = x^2$. Recall from Chapter 4 Exercise 4.7 that $(R, f^{1/2})$ is not sharply Frobenius split even though (R, x^1) is sharply Frobenius split. Furthermore, show that $(X, \frac{1}{2} \operatorname{div}(f))$ is sharply Frobenius split. Because of this, the divisorial version of sharp F-purity is preferred as it does not have these ambiguities; also see the next exercise.

Exercise 3.7. Let $R = \mathbb{F}_2[x, y, z]/(xy - z^2)$ and let $X = \operatorname{Spec} R$. Notice that $\operatorname{div}(x) = 2D$ for the prime divisor D corresponding to the height-one prime ideal (x, z). Prove that $(X, D) = (X, \frac{1}{2} \operatorname{div}(x))$ is sharply Frobenius split even though $(R, x^{1/2})$ is not.

Exercise 3.8. Let R be a normal F-finite Noetherian domain, and let Δ_ϕ be an effective $\mathbb{R}$-divisor on $X = \operatorname{Spec} R$ associated to some $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. Prove that

(a) The pair (X, Δ_ϕ) is sharply Frobenius split if and only if there exists $d > 0$ and $r \in R$ such that $\phi^d(F^{de}_* r) = 1$.
(b) The pair (X, Δ_ϕ) is strongly F-regular if and only if for every non-zero $c \in R$, there exists a $d > 0$ and $r \in R$ such that $\phi^d(F^{de}_* cr) = 1$.

Exercise 3.9. Let R be a normal Noetherian F-finite domain. Let $X = \operatorname{Spec} R$. Then (X, Δ) is *strongly F-regular* if and only if for every non-zero $c \in R$ there exists $e > 0$ and $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that

(a) $\Delta_\phi \geq \Delta$ and,
(b) $\phi(F^e_*(c)) = R$.

Exercise 3.10. Let R be a normal Noetherian F-finite domain, and let $X = \operatorname{Spec} R$, with Δ some effective $\mathbb{R}$-divisor. We say that (X, Δ) is **weakly Frobenius split** if for all $e \gg 0$, we there exists $\phi \in \mathrm{Hom}_R(F^e_* R(\lfloor (p^e - 1)\Delta \rfloor), R)$ such that $\phi(F^e_* R) = R$. Prove that any sharply Frobenius split pair is weakly Frobenius split.

Exercise 3.11. Let R be a normal Noetherian F-finite domain, and let $X = \operatorname{Spec} R$, with Δ some effective $\mathbb{R}$-divisor. Suppose that there is some $\phi \in \mathrm{Hom}_R(F^e_* R(\lfloor p^e \Delta \rfloor), R)$ with $\phi(F^e_* R) = R$. Show that for every $0 \leq a \leq e$, there exists $\phi_a \in \mathrm{Hom}_R(F^a_* R(\lfloor p^a \Delta \rfloor), R)$ with $\phi_a(F^a_* R) = R$.

Exercise 3.12. Suppose that (X, Δ) is globally Frobenius split and projective over an F-finite field. Prove that $-K_X - \Delta$ is pseudo-effective. In fact, show the stronger statement that multiple of it is effective. In particular, the Kodaira dimension of (X, Δ) is non-positive.

Exercise 3.13. Prove Theorem 3.12.

Hint: Use the same idea as Chapter 3 Theorem 2.14.

Exercise 3.14. Prove Proposition 3.13.

Hint: The issue for the equivalence of global F-regularity and strong F-regularity in the affine case $X = \operatorname{Spec} R$ is the distinction between a Weil divisor $D \geq 0$ and a non-zero $c \in R$. See Chapter 3 Proposition 2.8 for the proof in the non-pair setting.

Exercise 3.15. Suppose that (X, Δ) is globally F-regular where $\Delta \geq 0$ is an $\mathbb{R}$-divisor. Use Theorem 3.12 (and do not use Theorem 3.20) to show that there exists $\Delta' \geq \Delta$ such that (X, Δ') is globally F-regular and such that Δ' is a $\mathbb{Q}$-divisor where p is not a factor of the denominator of any component (in other words, Δ' is a $\mathbb{Z}_{(p)}$-divisor).

Hint: Choose C containing the support of Δ and set $\Delta' = \frac{1}{p^e-1}\lceil (p^e - 1)\Delta \rceil$ for appropriately large e.

Exercise 3.16. Use Proposition 3.15 to directly show Corollary 3.17.

4. Compatible subschemes and F-adjunction

We begin with an important and useful method for checking whether a scheme is globally Frobenius split.

4.1. A global criteria for Frobenius splitting. We recall the following implicit in [**MR85**, Proposition 6].

Proposition 4.1. *Suppose X is normal proper variety over an F-finite field k with $H^0(X, \mathcal{O}_X) = k$, and pick non-zero $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ with associated $\Delta = \Delta_\phi$. Further suppose that $x \in X$ is a closed k-rational point that is compatible with ϕ and such that the induced $\overline{\phi} : F^e_* \mathcal{O}_X/\mathfrak{m}_x \longrightarrow \mathcal{O}_X/\mathfrak{m}_x$ is nonzero. Then (X, Δ) is globally Frobenius split.*

Proof. We have the following diagram:

$$
\begin{tikzcd}
H^0(X, F^e_* \mathcal{O}_X) \arrow[r] \arrow[d, "H^0(X,\phi)"'] & H^0(X, F^e_* \mathcal{O}_X/\mathfrak{m}_x) = F^e_* k \arrow[d] \\
H^0(X, \mathcal{O}_X) \arrow[r] & H^0(X, \mathcal{O}_X/\mathfrak{m}_x) = k
\end{tikzcd}
\tag{4.1.1}
$$

It suffices to prove that ϕ is surjective on global sections (ie, that $H^0(X, \phi)$ surjects), since we can then replace ϕ by a unit-multiple that sends $F^e_* 1$ to 1. That property will descend to every open subset of X.

Back to the diagram, the horizontal arrows are isomorphisms since $k = H^0(X, \mathcal{O}_X) = k(x)$. But now the result follows as our right vertical map is surjective by hypothesis. $\square$

We will see later that x need not be k-rational, we need only that $k \subseteq k(x)$ is separable, see Corollary 7.5.

The value of the previous proposition is that it can show that X is globally Frobenius split by proving a statement about a single closed point.

It can sometimes be straightforward to identify such x. Indeed, the following corollary has been used many times in the literature (again, see [**MR85**]).

Corollary 4.2. *Suppose X is a d-dimensional normal proper variety over an F-finite field $k = H^0(X, \mathcal{O}_X)$. Consider some non-zero $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ with associated $\Delta = \Delta_\phi$. Further suppose that $q \in X$ is a closed k-rational point such that $\Delta = D_1 + \cdots + D_d + \Gamma$ where D_i are distinct prime divisors that are in normal crossings at q and such that $q \notin \Gamma$. Then X is globally Frobenius split.*

Proof. We only need to show that ϕ is surjective at q and compatible with q. But we can check that after passing to the completion R of $\mathcal{O}_{X,q}$. In that case, we may write $R = k[\![y_1, \ldots, y_d]\!]$ where $\mathrm{div}(y_i) = D_i|_{\mathrm{Spec}\, R}$. The result follows since the induced $\phi : F^e_* R \longrightarrow R$ is essentially the canonical toric splitting up to pre-multiplication by a unit. $\square$

Corollary 4.3. *Suppose that X is a d-dimensional proper variety over a field $k = H^0(X, \mathcal{O}_X)$ and there exists an effective Weil divisor $D \sim -K_X$ such that, for some closed k-rational point $q \in X$, D has normal crossings with d distinct directions at q. Then X is globally Frobenius split.*

PROOF. Left to the reader in Exercise 4.4. $\square$

Of course, there are other divisorial arrangements that might occur that also imply the same global F-splitting. Suppose for instance $p \equiv_6 1$, $q \in X$ is a k-rational point, and $R \cong k[\![x, y]\!]$ is the completion of $\mathcal{O}_{X,q}$. If $\Delta|_{\mathrm{Spec}\, R} \cong \frac{5}{6}\operatorname{div}(x^2 - y^3)$, then we also can show that X is globally Frobenius split. Indeed, by Chapter 4 Example 3.9 $\frac{5}{6}$ is the F-purethreshold of the cusp in our situation. Fedder's Lemma can be applied in a straightforward way to prove that $(\mathrm{Spec}\, R, \Delta|_{\mathrm{Spec}\, R})$ is sharply Frobenius split and compatible with the maximal ideal. It the follows from Proposition 4.1 that X is globally F-split.

4.2. A local example. In the above, we used a non-zero map at a *compatible point* to deduce global F-splitting. For the rest of the section we will work in the local case, but we won't think about what happens if we are compatible with a point. Instead we'll study what happens when we are conmpatible with a subscheme of higher dimension.

Indeed, if Δ is a divisor on X, corresponding to $\phi : F^e_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$, and ϕ is compatible with $Z \subseteq X$ and Z is normal, then the induced map $\phi_Z : F^e_*\mathcal{O}_Z \longrightarrow \mathcal{O}_X$ has an associated divisor Δ_Z as well. In particular, Δ induces a $\mathbb{Q}$-divisor Δ_Z on Z. This sort of behavior we call *F-adjunction*. Note, this doesn't occur when Z is a point because points can't support divisors.

We begin with an explicit example.

Example 4.4 (*cf.* Chapter 4 Example 2.12)**.** Suppose $p \neq 2$, $R = \mathbb{F}_p[x, y, z]/(xy - z^2) = S/(xy - z^2)$ and set $\Delta = V(x, z) = \frac{1}{2}\operatorname{div}(x)$. We will first show that $(X = \mathrm{Spec}\, R, \Delta)$ is Frobenius split. We see that $\phi_\Delta(F_*-) = \Phi_R(F_* x^{(p-1)\frac{1}{2}} \cdot -)$ where Φ_R generates $\mathrm{Hom}_R(F_*R, R)$. Now, using Fedder's Lemma Chapter 4 Theorem 2.1, we see that Φ_R is obtained from $\Phi_S(F_*(xy - z^2)^{p-1} \cdot -)$ (in this case $I = (xy - z^2)$). Hence, ϕ_Δ is induced from

$$\Phi_S(F_*(xy - z^2)^{p-1} x^{(p-1)\frac{1}{2}} \cdot -).$$

We next observe that $h = (xy - z^2)^{p-1} x^{(p-1)\frac{1}{2}}$ has a non-zero term

$$\begin{array}{cl} & \binom{p-1}{(p-1)/2}(xyz^2)^{(p-1)/2} x^{(p-1)/2} \\ = & \binom{p-1}{(p-1)/2} x^{p-1} y^{(p-1)/2} z^{(p-1)} \\ \notin & (x^p, y^p, z^p). \end{array}$$

Furthermore all other terms are in (x^p, z^p). Hence $(\operatorname{Spec} R, \Delta)$ is Frobenius split at the origin, and since X and Δ are both non-singular away from the origin, we see that (X, Δ) is Frobenius split.

Now, notice that Δ is compatible with ϕ_Δ by Proposition 2.7 (or since $h \cdot (x, z) \subseteq (x^p, z^p)$). Therefore ϕ_Δ also induces surjective map on $\mathcal{O}_\Delta$ itself, that is a map

$$\overline{\phi_\Delta} : F_* R/(x, z) \longrightarrow R/(x, z).$$

Setting $T = \Gamma(\Delta, \mathcal{O}_\Delta) = R/(x, z) = S/(x, z) \cong \mathbb{F}_p[y]$, we see that the generating map Φ_T is induced from $\Phi_S(F_* x^{p-1} z^{p-1} \cdot -)$. If we expand out $(xy - z^2)^{p-1} x^{(p-1)\frac{1}{2}} = \binom{p}{(p-1)/2} x^{p-1} y^{(p-1)/2} z^{(p-1)} + g$ we see that

$$\phi_\Delta(F_* -) = \Phi_S\left(F_* \binom{p}{(p-1)/2} x^{p-1} y^{(p-1)/2} z^{(p-1)} \cdot -\right) + \Phi_S(F_* g \cdot -).$$

Both maps are compatible with (x, z) but in the second case we are pre-multiplying by $g \in (x^p, z^p)$, hence when inducing the map on T, we need only concern ourselves with the first map by Theorem 2.1. In particular, we now see that

$$\overline{\phi_\Delta} = \Phi_T \star \left(\binom{p}{(p-1)/2} y^{(p-1)/2}\right).$$

Therefore

$$\Delta_{\overline{\phi_\Delta}} = \frac{1}{p-1} \operatorname{div}(y^{(p-1)/2}) = \frac{1}{2} \operatorname{div}(y).$$

A more algebraic way to interpret this example is that we have just shown that not every map $\overline{\psi} : F_* T \longrightarrow T$ comes from a map $\psi : F_* R \longrightarrow R$. However, every pre-multiple of $\overline{\phi_\Delta}$ does, compare with Chapter 4 Remark 2.5 and Example 2.12. The divisorial notation can be viewed as a quick way of telling us how far off $\overline{\phi_\Delta}$ is from generating $\operatorname{Hom}_T(F_* T, T)$.

4.3. The local F-different. We now define the F-different . For other interpretations of the F-different, see [**DS17**].

Definition 4.5. Suppose that R is an F-finite normal domain, $\phi : F^e_* R \longrightarrow R$ is non-zero and $\Delta = \Delta_\phi$ is the induced divisor. Suppose that $V(J) = Z \subseteq \operatorname{Spec} R = X$ is a ϕ-compatible normal[13] subscheme such that ϕ is surjective at the generic points of Z (that is, $\phi : F^e_* R_\eta \longrightarrow R_\eta$ surjects for each minimal prime of J). Then the **F-different of (X, Δ) along Z (or along J)**

[13]Some results on the F-different can be generalized to the non-normal case by normalizing Z, we will mostly avoid this however.

is defined to be Δ_{ϕ_Z}. Frequently the F-different Δ_{ϕ_Z} will be denoted by $F\mathrm{Diff}_Z(\Delta)$.

Remark 4.6. By [**Das15**], our F-different is closely related to Shokurov's different [**Sho92**] from birational algebraic geometry, also see [**Kaw98**].

In the case that $R = \mathbb{F}_p[x, y, z]/(xy - z^2)$, $X = \operatorname{Spec} R$ and $D = V(x, z)$ and $Z = V(x, z) = D$, Example 4.4 says that the F-different of (X, D) along $D = Z$ is $\frac{1}{2} \operatorname{div}_Z(y)$.

Recall that a pair $(X = \operatorname{Spec} R, \Delta_\phi)$ is locally sharply Frobenius split if and only if $\phi : F^e_* R \longrightarrow R$ is surjective. Therefore, arguing just as in the proof of Fedder's Lemma, we see that if ϕ is compatible with a closed subscheme $Z \subseteq X$, then $\phi : F^e_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ is surjective near points of Z if and only if $\overline{\phi} : F^e_* \mathcal{O}_Z \longrightarrow \mathcal{O}_Z$ is surjective. Hence, we have just shown the following.

Theorem 4.7 (Affine F-adjunction I)**.** *Suppose that $(X, \Delta = \Delta_\phi)$ is a pair and $Z \subseteq X$ is a normal subscheme compatible with ϕ. Then (X, Δ) is locally sharply Frobenius split in a neighborhood of Z if and only if $(Z, F\mathrm{Diff}_Z(\Delta))$ is locally sharply Frobenius split.*

One can use this method to prove that (X, Δ) is locally sharply Frobenius split. Indeed, Z is lower dimensional so it may be easier to work with. If Z is a normal curve for instance, then checking F-splitting is simply making sure the coefficients of Δ_Z are ≤ 1.

4.3.1. *What to do if Z is not normal.* In everything we have done so far, we have assumed that the compatible scheme Z is normal. This hypothesis can be weakened substantially in two ways we now describe.

There is a theory of divisors on non-normal schemes that so far we have left untouched. Instead of requiring that an integral scheme is normal (in other words R_1+S_2), we can require that integral scheme to be G_1+S_2 (where G_1 means Gorenstein in codimension 1). In this case, one good way to replace divisors is with reflexive subsheaves $\mathscr{F}$ of $\mathcal{K}(X)$ (here $\mathcal{K}(X)$ is the sheaf of fractions of X) such that $\mathscr{F}$ is a line bundle when localized at all height one points of X. See [**Har94**], [**Kc92**, Chapter 16] or [**Kol13**, Chapter 1] where such sheaves are called Weil-divisorial sheaves or almost Cartier divisors, one can also work with $\mathbb{Q}$-divisorial version of them although extreme care must be taken when rounding as it can happen that two distinct "divisors" D_1 and D_2 satisfy $nD_1 = nD_2$. Regardless, if X or Z is G_1+S_2, then there is still an associated D_{ϕ_Z} and Δ_{ϕ_Z} for any map $\phi : F^e_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$. It still follows that (X, Δ_ϕ) is locally sharply Frobenius split near Z if and only if (Z, Δ_{ϕ_Z}) is locally sharply Frobenius split.

Alternately, and much more simply, if Z is not normal then we may normalize it using the following.

Theorem 4.8 (*cf.* [**BK05**, Exercise 1.2.E(4)])**.** *Suppose that R is a reduced Noetherian ring, $\phi : F^e_* R \longrightarrow R$ is an R-linear map. Let $\nu : R \longrightarrow R^{\mathrm{N}}$ denote the normalization. Then there exists $\phi^{\mathrm{N}} : F^e_* R^{\mathrm{N}} \longrightarrow R^{\mathrm{N}}$ such that the following diagram commutes.*

$$\begin{tikzcd} F^e_* R \arrow[r, "\phi"] \arrow[d, hook] & R \arrow[d, hook] \\ F^e_* R^{\mathrm{N}} \arrow[r, "\phi^{\mathrm{N}}"'] & R^{\mathrm{N}} \end{tikzcd}$$

PROOF. This is left to a series of exercises, see Exercise 6.28, Exercise 6.30 in Chapter 1 and Exercise 4.2 in this chapter. $\square$

It follows immediately that if R is Frobenius split then so is R^{N}. However, the converse is false. In fact, there are examples of rings R where $\phi : F^e_* R \longrightarrow R$ is not surjective but where $\phi^{\mathrm{N}} : F^e_* R^{\mathrm{N}} \longrightarrow R^{\mathrm{N}}$ is surjective (and so $(R^{\mathrm{N}}, \Delta_{\phi^{\mathrm{N}}})$) is locally sharply Frobenius split, see Exercise 4.3. Therefore, we obtain the following.

Theorem 4.9 (Affine F-adjunction II)**.** *Suppose that $(X, \Delta = \Delta_\phi)$ is a pair and $Z \subseteq X$ is a reduced subscheme compatible with ϕ. Then (X, Δ) is locally sharply Frobenius split in a neighborhood of Z then $(Z^{\mathrm{N}}, F\mathrm{Diff}_{Z^{\mathrm{N}}}(\Delta))$ is locally sharply Frobenius split.*

4.4. F-adjunction along a Cartier divisor. We now discuss what happens when $Z = V(f)$ is a Cartier divisor. In fact, this is a case we secretly studied before in Chapter 4 Exercise 2.15.

Lemma 4.10. *Suppose $(R, \mathfrak{m})$ is an F-finite normal quasi-Gorenstein local ring, $f \in R$ is a regular element, $J = (f)$, and $T = R/J$ is also normal. Let $D = \operatorname{div}(f)$, then the F-different of $(\operatorname{Spec} R, D)$ along J is 0.*

PROOF. Since R is quasi-Gorenstein and local, there exists $\Phi_R \in \operatorname{Hom}_R(F^e_* R, R)$ generating it as Hom-set. Set

$$\phi = \Phi \star f^{p^e - 1}$$

and observe that $\Delta_\Phi = \frac{1}{p^e-1}(p^e - 1)\operatorname{div}(f) = D$. We have seen in Proposition 2.7 that $V(f)$ is compatible with ϕ. By Chapter 4 Exercise 2.15 we see that ϕ induces a generating map $\overline{\phi} = \Phi_T \in \operatorname{Hom}_T(F^e_* T, T)$, and so $\Delta_{\overline{\phi}} = 0$ as claimed. $\square$

Indeed, without a quasi-Gorenstein hypothesis, or more generally a $\mathbb{Q}$-Gorenstein hypothesis with index not divisible by $p > 0$ (see [**Sch09**]), this result is false. You are asked to use a computer algebra system compute this in an example in Exercise 4.6, which is really a special case of an example of Anurag K. Singh [**Sin99b**]. Of course, we re-obtain the following result.

Corollary 4.11 (*cf.* Chapter 1 Corollary 7.24)**.** *Suppose* $(R, \mathfrak{m})$ *is an* F*-finite normal quasi-Gorenstein local ring,* $f \in R$ *is a regular element, and* $R/(f)$ *is normal. If* $R/(f)$ *is* F*-split, then so is* R*.*

See [**PS22**] for a generalization to the $\mathbb{Q}$-Gorenstein case.

4.5. Exercises.

Exercise 4.1. With notation as in Chapter 1 Definition 6.24 and Exercise 6.28, find an example $R \subseteq R'$ that shows that $\mathfrak{c}_{R'/R}$ is not necessarily compatible with every $\psi \in \mathrm{Hom}_{R'}(F^e_* R', R')$.

Exercise 4.2. Prove Theorem 4.8 by showing that the induced $\phi : F^e_*\mathcal{K}(R) \longrightarrow \mathcal{K}(R)$ sends $F^e_* R^{\mathrm{N}}$ into R^{N}. We thus obtain a map $\phi^{\mathrm{N}} : F^e_* R^{\mathrm{N}} \longrightarrow R^{\mathrm{N}}$.

Hint: Use the following characterization of R^{N}. An element $x \in \mathcal{K}(R)$ is in R^{N} if and only if there exists $c \in R$ (which we may assume is in $\mathfrak{c}$), not in any minimal prime of R, such that $cx^n \in R$ for all $n \geq 0$, see [**SH06**, Exercise 2.26].

Exercise 4.3. Consider $R = \mathbb{F}_2[x, y, z]/(xy^2 - z^2) \cong \mathbb{F}_2[a^2, b, ab] \subseteq \mathbb{F}_2[a, b] \cong R^{\mathrm{N}}$ and let $\phi = \Phi_R$ be a map generating $\mathrm{Hom}_R(F_* R, R)$ as an $F_* R$-module. Use Fedder's lemma to prove that ϕ is not surjective but show that $\phi^{\mathrm{N}} : F_* R^{\mathrm{N}} \longrightarrow R^{\mathrm{N}}$ is surjective.

Exercise 4.4. Prove Corollary 4.3.

Exercise 4.5. Use Proposition 4.1 or its corollaries to give a different proof that projective space, or more generally any toric variety, is globally Frobenius split.

Exercise 4.6. Consider the following ring $S = \mathbb{F}_{11}[a, b, c, d, t]$ with

$$I = (a^2t^5 + a^4 - bc,\ b^2t^5 - dt^5 + a^2b^2 - a^2d - cd,\ b^3 - a^2d - bd)$$

and set $J = (t) + I$ with $Z = V(J)$. Set $R = S/I$. Use a computer algebra system such as Macaulay2 [**GS**] to show that we have a strict containment

$$t^{p-1} \cdot (I^{[p]} : I) + J^{[p]} \subsetneq J^{[p]} : J.$$

and then explain why this shows that for any $\Delta_\phi \geq \mathrm{div}(t)$ on $\operatorname{Spec} R$, that $F\mathrm{Diff}_Z(\Delta_\phi) \neq 0$. Also verify that $R = S/I$ is not Frobenius split even though $R/(t)$ is Frobenius split.

5. Test ideals for pairs (R, ϕ) and $(\operatorname{Spec} R, \Delta)$

In an F-finite regular ring S, we introduced the notion of a test ideal $\tau(S, \mathfrak{a}^t)$ in Chapter 4 §4. In that case, we defined

$$\tau(S, \mathfrak{a}^t) = \operatorname{Image}\left(F^e_* \mathfrak{a}^{\lceil tp^e \rceil} \cdot \operatorname{Hom}_R(F^e_* S, S) \xrightarrow{\text{eval at } F^e_* 1} S\right)$$

for $e \gg 0$. We saw, for example, that this ideal is the same as

$$\operatorname{Image}\left(F^e_* \mathfrak{a}^{\lceil tp^e \rceil} \cdot \operatorname{Hom}_S(F^e_* S, S) \xrightarrow{\text{eval at } F^e_* d} S\right)$$

where d is any non-zerodivisor and again $e \gg 0$. Instead of viewing this as an operation on the ideal $\mathfrak{a}$, we could instead consider this as a failure of Frobenius splitting when we restrict which maps in $\operatorname{Hom}_S(F^e_* S, S)$ we utilize. In this section, we will introduce the test ideal of pairs (R, ϕ) where $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ is nondegenerate in a certain sense. Analogous to the case of the full test ideal $\tau(R)$, it turns out that, assuming a nondegeneracy condition on ϕ, the test ideal $\tau(R, \phi)$ can be viewed as the smallest ideal of positive height compatible with just ϕ. Eventually, we will define test ideals for pairs (R, Δ) where Δ is a $\mathbb{R}$-divisor on $\operatorname{Spec} R$, and later in Chapter 8 we will define test ideals associated to more general Cartier algebras.

In the case that R is normal, the test ideal $\tau(R, \phi)$ measures the singularities of both R and Δ_ϕ. In particular, the more singular R and Δ_ϕ are, the *smaller* or *deeper* the associated test ideal is. On the other hand, when ϕ generates $\operatorname{Hom}_R(F^e_* R, R)$ as an $F^e_* R$-module, then $\Delta_\phi = 0$, and $\tau(R, \phi)$ only measures the singularities of R.

5.1. Test ideals $\tau(R, \phi)$ of a map ϕ.

We'll start by studying test ideals associated with a single fixed ϕ, a case that is in some sense easier than what we did in Chapter 1.

Definition 5.1. Let R be a Noetherian F-finite reduced ring. For some $e > 0$, fix some $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. A **strong test element** for (R, ϕ) is an element c with the property that for all non-zerodivisors d, there exists $n_0 > 0$ such that for all $n \geq n_0$

$$c \in \phi^n(F^{ne}_*(dR)),$$

the image under ϕ^n of the ideal of $F^{ne}_* R$ generated by $F^{ne}_* d$.

Lemma 5.2. *The set of all strong test elements for a pair (R, ϕ), as in Definition 5.1, is an* ideal *of R.*

Proof. The set of all strong test elements for (R, ϕ) is closed under multiplication by elements of R, because if $c = \phi^n(F^{ne}_* rd)$, then for any $g \in R$,

$$gc = g\phi^n(F^{ne}_* rd) = \phi^n(F^{ne}_* g^{ne} rd) \in \phi^n(F^{ne}_*(dR))$$

by (4.12.4) in Chapter 1. To see the set of all strong test elements for (R, ϕ) is closed under addition, let c_1 and c_2 be strong test elements for (R, ϕ). Fix a non-zerodivisor d. By definition of strong test element, there exist n_1 and n_2 such for all $n \geq \max(n_1, n_2)$,

$$\phi^n(F^n_*(r_1 d)) = c_1 \quad \text{and} \quad \phi^n(F^n_*(r_2 d)) = c_2$$

for some $r_1, r_2 \in R$ (depending on n). Thus

$$c_1 + c_2 = \phi^n(F^n_*(r_1 + r_2)d) \in \phi^n(F^n_*(dR))$$

for all $n \gg 0$, showing that $c_1 + c_2$ is a strong test element as well. □

Definition 5.3. Let (R, ϕ) be a pair, where R is a Noetherian F-finite reduced ring and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ is some fixed map. The **test ideal** of the pair (R, ϕ), denoted $\tau(R, \phi)$, is the set of all strong test elements for (R, ϕ).

The proof of the following analog of Chapter 1 Theorem 6.15 is left as an exercise:

Proposition 5.4. *Let (R, ϕ) be a pair, where R is a Noetherian F-finite reduced ring and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ is some fixed map.*

(i). The test ideal $\tau(R, \phi)$ is ϕ-compatible.
(ii). The test ideal $\tau(R, \phi)$ is contained in every ϕ-compatible ideal J of positive height.

Definition 5.5. A map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ is called **non-degenerate** if for each minimal prime $Q \subseteq R$ we have that $\phi_Q : F^e_* R_Q \longrightarrow R_Q$ is *not* the zero map. A ϕ which is not non-degenerate is called **degenerate**.

The test ideal of a map $\tau(R, \phi)$ *can* be contained in a minimal prime of R if ϕ is degenerate, unlike the situation for the absolute test ideal $\tau(R)$. See Exercise 5.3.

On the other hand, if all stalks of ϕ at the generic points of $\operatorname{Spec} R$ are *non-zero*, then the test ideal $\tau(R, \phi)$ has positive height:

Theorem 5.6. . *Let R be a Noetherian F-finite reduced ring (R, ϕ) be a pair, let $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ be nondegenerate. Then $\tau(R, \phi)$ contains a non-zerodivisor.*

This follows from the following analog of Chapter 1 Theorem 5.21

Lemma 5.7. *Suppose R is a Noetherian F-finite reduced ring and that $b \in R$ is such that:*

(a) $R_b = R[b^{-1}]$ is strongly F-regular (for example regular) and

(b) $D_\phi|_{\operatorname{Spec} R \setminus V(b)} = 0$, *in other words if*

$$\frac{\phi}{1} \in R[b^{-1}] \otimes_R \operatorname{Hom}_R(F^e_* R, R) \cong \operatorname{Hom}_{R[b^{-1}]}(F^e_* R[b^{-1}], R[b^{-1}])$$

generates as an $F^e_* R[b^{-1}]$*-module.*

Then b *has a power that is a strong test element.*

Proof. First observe that we have $1 \in \phi^n_b(F^{ne}_* R_b)$ for all $n \geq 0$ since R_b is F-split and ϕ_b is a generating map (a statement which does not involve any d). It follows that $b^m \in \phi(F^e_* R)$ for some $m \geq 0$. We then see that

$$b^{2m} \in \phi^n(F^{ne}_* b^{mp^{ne}} R)$$

for all n. We now pick $c := b^{2m+1}$.

Fix an arbitrary d a non-zerodivisor. Since R_b is strongly F-regular, there exists $\psi_{n'} \in \operatorname{Hom}_R(F^{n'e}_* R_b, R_b)$ so that $1 \in \psi_{n'}(F^{n'e}_* dR_b)$ for all $n \gg 0$. But since $\phi^{n'}_b$ generates that Hom-set, we also have that $1 \in \phi^{n'}_b(F^{n'e}_* dR_b)$ for all $n' \gg 0$. It follows that for some fixed $n' \gg 0$, there exists $l \geq 0$ such that $b^l \in \phi^{n'}(F^{n'e}_* dR)$. The idea is to *decrease* this l. Choose $n_1 > 0$ so that $p^{n_1 e} > l$. We then obtain that

$$c = b^{2m+1} \in \phi^{n_1}(F^{n_1 e}_* b^{p^{n_1 e}} \cdot R) \subseteq \phi^{n_1}(F^{n_1 e}_* b^l \cdot R) \subseteq \phi^{n_1+n'}(F^{(n_1+n')e}_*(d)).$$

Taking $n_0 = n_1 + n'$ completes the proof. □

Corollary 5.8. *Let* R *be a Noetherian* F*-finite reduced ring, and let* $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ *be nondegenerate. Then* $\tau(R, \phi)$ *is the unique smallest* ϕ*-compatible ideal containing a non-zerodivisor.*

As a consequence, we see that the test ideal $\tau(R, \phi)$ is generated, in a certain sense, by *any* non-zerodivisor strong test element for (R, ϕ):

Corollary 5.9. *With notation as above and assuming that* ϕ *is nondegenerate, we have that*

$$\tau(R, \phi) = \sum_{n \geq 0} \phi^n(F^{ne}_*(cR))$$

for any non-zerodivisor c *that is a strong test element for the pair* (R, ϕ)*.*

Proof. Choose a non-zerodivisor $c \in \tau(R, \phi)$. The ideal

$$J = \sum_{n \geq 0} \phi^n(F^{ne}_*(c))$$

is clearly the smallest ϕ-compatible ideal containing c. Thus J is the test ideal. □

We don't even need the summation to define the test ideal, we just need a ϕ^n for some $n \gg 0$.

Corollary 5.10. *For any c a non-zerodivisor strong test element for (R, ϕ), we have that*

$$\tau(R, \phi) = \phi^n(F^{ne}_*(cR))$$

for all $n \gg 0$.

Proof. The right side is contained in $\tau(R, \phi)$ by Corollary 5.9. On the other hand, picking $f_1, \ldots, f_t$ generators for $\tau(R, \phi)$, we see that for each f_i, there exists n_i so that $f_i \in \phi^n(F^{ne}_*(c))$ for $n \geq n_i$. Thus if we choose $n \geq \max\{n_1, \ldots, n_t\}$ we are done. □

5.2. Properties of test ideal. We now list some basic properties of test ideals $\tau(R, \phi)$.

Proposition 5.11. *Suppose that R is F-finite, Noetherian, and reduced and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ is nondegenerate. Then the following hold:*

(a) For any multiplicative set $W \subseteq R$, we have that $\tau(W^{-1}R, W^{-1}\phi) = W^{-1}\tau(R, \phi)$.
(b) If R is local with maximal ideal $\mathfrak{m}$ and $\widehat{-}$ denotes the $\mathfrak{m}$-adic completion, then $\widehat{\tau(R, \phi)} = \tau(\widehat{R}, \widehat{\phi})$.
(c) If $a \in R$, and we define $\psi = \phi \star a$, then $\tau(R, \psi) \subseteq \tau(R, \phi)$. Furthermore, if a is a unit, then $\tau(R, \psi) = \tau(R, \phi)$
(d) $\tau(R, \phi) = \tau(R, \phi^m)$ for all integers $m > 0$.

Proof. We leave parts (a), (b) and (c) to the exercises, and prove (d).

By Lemma 5.7 we may choose a non-zerodivisor $c \in R$ that is a strong test element for both (R, ϕ) and (R, ϕ^m). But then for $n \gg 0$, we see that

$$\tau(R, \phi) = \phi^{nm}(F^{nme}_*(cR)) = \tau(R, \phi^m)$$

using Corollary 5.10. □

At this point we lack examples, however the following provides plenty as it connects this test ideal with the one we introduced back in Chapter 4 Section 5.

Theorem 5.12. *Suppose that R is an F-finite regular domain with $\Phi^e \in \mathrm{Hom}_R(F^e_* R, R)$ a $F^e_* R$-module generator. Suppose that $0 \neq x \in R$, and $t = \frac{a}{p^e - 1}$ for some $a, e \in \mathbb{Z}_{\geq 0}$. Set $\phi = \Phi^e \star x^a$. Then*

$$\tau(R, (x)^t) = \tau(R, \phi).$$

Proof. Observe that $\operatorname{Supp} \operatorname{div}(x) = \operatorname{Supp} \Delta_\phi$. Hence by Lemma 5.7, for $n \gg \lceil t \rceil$, we know that $x^{n - \lceil t \rceil}$ of x is a strong test element for (R, ϕ). On the other hand, by Chapter 4 Lemma 5.12,

$$\tau(R, (x)^t) = \Phi^{me}(F_*^{me}(x^{n-\lceil t \rceil} x^{\lceil tp^{me} \rceil}))$$

for all $m \gg 0$.We see that

$$\lceil tp^{me} \rceil = \left\lceil \frac{a}{p^e - 1}(p^{me} - 1 + 1) \right\rceil = \left\lceil a\frac{p^{me} - 1}{p^e - 1} + t \right\rceil = a\frac{p^{me} - 1}{p^e - 1} + \lceil t \rceil.$$

Therefore, using that $\phi^m(F_*^{me} -) = \Phi^{me}(F_*^{me} x^{a\frac{p^{me}-1}{p^e-1}} \cdot -)$, we obtain that

$$\tau(R, (x)^t) = \Phi^{me}(F_*^{me}(x^{n-\lceil t \rceil} \cdot x^{\lceil t \rceil} \cdot x^{\lceil a\frac{p^{me}-1}{p^e-1} \rceil})) = \phi^m(F_*^{me}(x^n)).$$

Applying Corollary 5.10 completes the proof. □

Likewise we have the following connection with the test ideal we introduced in Chapter 1.

Theorem 5.13. *Suppose that R is an F-finite reduced Noetherian ring and $\Phi \in \operatorname{Hom}_R(F_* R, R)$ is a generating map. Then*

$$\tau(R, \Phi) = \tau(R).$$

Proof. This follows from the fact that an ideal J is uniformly compatible if and only if it is compatible with Φ. □

We conclude with one more variant of strong F-regularity.

Definition 5.14. Suppose R is F-finite, Noetherian and reduced and $\phi \in \operatorname{Hom}_R(F_*^e R, R)$ is nondegenerate. We say that (R, ϕ) is *strongly F-regular* if $\tau(R, \phi) = R$.

It easily follows that if (R, ϕ) is strongly F-regular, then R is normal by Chapter 1 Theorem 4.30. We then obtain that:

Lemma 5.15. *(R, ϕ) is strongly F-regular if and only if $(\operatorname{Spec} R, \Delta_\phi)$ is strongly F-regular.*

Proof. Left to the reader in Exercise 5.7, or see Subsection 5.3 below. □

5.3. Test ideals of pairs (R, Δ). We consider test ideals associated to $\mathbb{R}$-divisors $\Delta \geq 0$. Note that since we are working with divisors, we assume that R is normal. Once we assume R is normal, it is harmless to assume that R is a domain as well. We also note that we will frequently use the following notation.

Notation 5.16. Suppose R is a normal domain and D is a Weil divisor on R. By $R(D)$ we mean the fractional ideal

$$R(D) := \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}(D)).$$

Setting 5.17. Suppose R is an F-finite normal Noetherian domain. Fix $\Delta \geq 0$ an $\mathbb{R}$-divisor. We set

$$\mathscr{C}^{\Delta,e} := \mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R) \subseteq \mathrm{Hom}_R(F^e_* R, R).$$

In other words, $\mathscr{C}^{\Delta,e} \subseteq \mathrm{Hom}_R(F^e_* R, R)$ is the set of ϕ with $\Delta_\phi \geq \Delta$.

The idea is to use the maps $\mathscr{C}^{\Delta,e}$ to define the test ideal, instead of all the maps. Note we also could have defined the test ideal of the pair $(R, \mathfrak{a}^t)$ via the maps $(F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil}) \cdot \mathrm{Hom}_R(F^e_* R, R)$, see Chapter 4 Proposition 5.30.

Definition 5.18. Suppose that (R, Δ) is as in Setting 5.17. A **strong test element for** (R, Δ) is a $c \in R$ that satisfies for the following condition. For every non-zerodivisor $d \in R$ there exists an $e_0 > 0$ so that for every $e \geq e_0$ there exists some $\phi_e \in \mathscr{C}^{\Delta,e}$ such that $c \in \phi_e(F^e_*(dR))$.

Before we start working with these pairs, we need the following result.

Lemma 5.19. *With notation as in Setting 5.17, suppose that $\phi \in \mathscr{C}^{\Delta,e}$ and $\psi \in \mathscr{C}^{\Delta,f}$. Then $\phi \star \psi \in \mathscr{C}^{e+f}(\Delta)$.*

PROOF. By Proposition 2.6

$$\Delta_{\phi\star\psi} = \frac{p^e - 1}{p^{e+f} - 1}\Delta_\phi + \frac{p^e(p^f - 1)}{p^{e+f} - 1}\Delta_\psi.$$

In this case, since $\Delta_\phi, \Delta_\psi \geq \Delta$ and

$$\frac{p^e - 1}{p^{e+f} - 1} + \frac{p^e(p^f - 1)}{p^{e+f} - 1} = 1.$$

we see that $\Delta_{\phi\star\psi} \geq \Delta$ as well. The result follows. □

Again, we have the following result on the existence of test elements.

Lemma 5.20 (Strong test elements exist for divisor pairs)**.** *Suppose (R, Δ) is as in Setting 5.17. Then there exist a non-zerodivisor $c \in R$ that is a strong test element for (R, Δ). Furthermore, if $b \in R$ is such that R_b is strongly F-regular and such that* $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(\mathrm{div}(b))$, *then we may take c to be a power of b.*

PROOF. This is left to the reader in Exercise 5.14. $\square$

Definition 5.21 ([**Tak04b**]). Suppose (R, Δ) is as in Setting 5.17. The **test ideal of** (R, Δ), denoted $\tau(R, \Delta)$, is the set of strong test elements for (R, Δ).

Theorem 5.22 (*cf.* [**Tak04b**] [**HT04**, Lemma 2.1]). *Suppose that (R, Δ) is as in Setting 5.17. Furthermore, if a non-zerodivisor $c \in R$ is a strong test element for (R, Δ), then*

$$\tau(R, \Delta) = \sum_{e \geq 0} \sum_{\phi} \phi(F^e_*(c)).$$

Here ϕ runs over all elements of $\mathscr{C}^e(\Delta)$. As a consequence, $\tau(R, \Delta)$ is the smallest ideal, nonzero at every minimal prime, and which is compatible with all $\phi \in \mathscr{C}^e(\Delta)$.

It is straightforward to see that the right side of the displayed equation above is also equal to

$$\sum_{e \geq 0} \operatorname{Image}\Big(\mathscr{C}^e(\Delta) \xrightarrow{\phi \mapsto \phi(F^e_* c)} R\Big).$$

PROOF. Let J denote the sum in the theorem. Using Lemma 5.19, we see that $\phi(F^e_* J) \subseteq J$ for any $\phi \in \mathscr{C}^{\Delta, e}$ for any e. Thus, any ideal compatible with all $\phi \in \mathscr{C}^{\Delta, e}$ for all e, which contains c, must also contain J. But by Lemma 5.20, any non-zero ideal compatible with all elements of $\mathscr{C}^{\Delta, e}$ contains c. The result follows. $\square$

We note the following, whose proof is left as an exercise.

Lemma 5.23. *Suppose that (R, Δ) is defined as in Setting 5.17.*

(a) *For any multiplicative set $W \subseteq R$, $W^{-1}\tau(R, \Delta) = (W^{-1}R, \Delta|_{\operatorname{Spec} W^{-1}R})$.*
(b) *For any ideal $J \subseteq R$ with $\widehat{R}$ the J-adic completion, $\tau(R, \Delta)\widehat{R} = (\widehat{R}, \Delta|_{\operatorname{Spec} \widehat{R}})$.*

PROOF. Left to the reader in Exercise 5.15. $\square$

However, we point out the following very useful result.

Theorem 5.24. *Suppose (R, Δ) is as in Setting 5.17. Suppose further that $\Delta = \Delta_\phi$ for some $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. Then*

$$\tau(R, \Delta) = \tau(R, \phi).$$

PROOF. We can pick c a strong test element for both (R, ϕ) and for (R, Δ). Since $\phi^n \in \mathscr{C}^{ne}(\Delta)$ for all n, we note that

$$\tau(R, \phi) = \sum_n \phi^n(F^{ne}_*(c)) \subseteq \sum_{f \geq 0} \sum_{\psi \in \mathscr{C}^{\Delta, f}} \psi(F^f_*(c)) = \tau(R, \Delta).$$

In other words, the right side sums over those ψ with $\Delta_\psi \geq \Delta$. But observe that $\Delta_{\phi^n} \geq \Delta$ (since it is equality), and so the left sum is a subset of the right.

We need to prove the reverse containment ($\supseteq$). Suppose first that $f = ne$ for some integer n. In that case if $\Delta_\psi \geq \Delta_{\phi^n}$ then $\psi = \phi^n \star r$. We thus have that

$$\phi^n(F^{ne}_*(c)) = \sum_{\psi \in \mathscr{C}^{\Delta, f}} \psi(F^f_*(c)).$$

We must deal with the f that are not divisible by e. Now, set $c' = c^{p^{e-1}}$ and note it is also a strong test element for (R, ϕ) and (R, Δ).

Fix an $f > 0$, and $\psi \in \mathscr{C}^{\Delta, f}$ and write $f = ne + r$ where $0 \leq r < e$.

Claim 5.25. *The map*

$$\begin{aligned} \mathscr{C}^{\Delta, r} &\longrightarrow \mathscr{C}^{\Delta, f} \\ \gamma &\longmapsto \phi^n \star \gamma \end{aligned}$$

is surjective.

PROOF OF CLAIM. The idea of the proof is the same as that of Appendix A Lemma 5.1. Indeed, our surjectivity is equivalent to the assertion that

$$\begin{aligned} \mathscr{C}^{ne}(\Delta) \otimes_{F^{ne}_* R} F^{ne}_* \mathscr{C}^{\Delta, r} &\longrightarrow \mathscr{C}^{\Delta, f} \\ \theta \otimes F^{ne}_* \gamma &\longmapsto \theta \star F^{ne}_* \gamma. \end{aligned}$$

is an isomorphism. Note $\mathscr{C}^{ne}(\Delta)$ is generated by ϕ^n as an $F^{ne}_* R$-module, thus the tensor product on the left side of the above is simply the $F^f_* R$-module $F^{ne}_* \mathscr{C}^{\Delta, r}$. But now we have a map between two reflexive $F^f_* R$-modules of rank 1, and so it suffices to show it is an isomorphism in codimension 1. Hence from here out, we may assume that $(R, (xR))$ is a DVR with uniformizer x.

In this case, we may write $\Delta = \frac{a}{p^e - 1} \operatorname{div}(x)$. Suppose we have $\psi \in \mathscr{C}^{\Delta, f}$, in other words $\psi = \Phi^f \star u x^b$ for some unit u and integer $b \geq 0$ such that $\frac{b}{p^f - 1} \geq \frac{a}{p^e - 1}$. Thus

$$b \geq (p^{ne+r} - 1) \frac{a}{p^e - 1} = \frac{a p^r (p^{ne} - 1)}{p^e - 1} + \frac{a(p^r - 1)}{p^e - 1}.$$

Set $c = b - \frac{ap^r(p^{ne}-1)}{p^e-1}$ and fix $\gamma = \Phi^r \star ux^c$. Notice that $\Delta_\gamma = \frac{c}{p^r-1}\operatorname{div}(x) \geq \Delta$. We then have that

$$\psi = \Phi^{ne} \star x^{a\frac{p^{ne}-1}{p^e-1}} \star \Phi^r \star ux^c = \phi^n \star \gamma$$

which completes the proof of the claim. □

With the claim in hand, for a fixed $f = ne + r$, we have that

$$\sum_{\psi \in \mathscr{C}^{\Delta,f}} \psi(F^f_*(c')) = \sum_{\gamma \in \mathscr{C}^{\Delta,r}} \phi^n(F^{ne}_*\gamma(F^r_*(c^{p^{e-1}}))) \subseteq \phi^n(F^{ne}_*(c)).$$

We then obtain that

$$\sum_{f \geq 0} \sum_{\psi \in \mathscr{C}^{\Delta,f}} \psi(F^f_*(c)) \subseteq \tau(R, \phi),$$

which completes the proof. □

We record the following basic properties, compare with various properties from Chapter 4, such as Proposition 5.18.

Proposition 5.26. *Suppose R is a Noetherian F-finite normal domain and $\Gamma \geq 0$ is a $\mathbb{Q}$-divisor. Then:*

(a) If $\Delta' \geq \Gamma$ then $\tau(R, \Delta') \subseteq \tau(R, \Delta)$.
(b) $\tau(R, \Delta + \operatorname{div}(f)) = f\tau(R, \Delta)$ for any $0 \neq f \in R$.
(c) If $D \geq 0$ is any other divisor, then $\tau(\omega_R, \Gamma) = \tau(\omega, \Gamma + \epsilon D)$ for all $1 \gg \epsilon > 0$.

Proof. The first part is immediate from the definition. Part (b) is left to the reader in Exercise 5.11.

For (c), the containment $\supseteq$ is immediate and so we must prove $\subseteq$. Choose a non-zerodivisor $c \in R$ which is a strong test element for (R, Δ). Let $d \in R$ be a non-zerodivisor such that $\operatorname{div}(d) \geq H$. Using Theorem 5.22 and the fact that R is Noetherian, we can choose e_0 so that

$$\tau(R, \Delta) = \sum_{e=0}^{e_0} \sum_{\phi} \phi(F^e_*(dc)).$$

where ϕ runs over all elements of $\mathscr{C}^e(\Delta)$. For any $e \leq e_0$, notice that if $\phi \in \mathscr{C}^e(\Delta)$, then if we define $\psi = \phi \star d$, we have that

$$\psi \in \mathscr{C}^e(\Delta + \frac{1}{p^e - 1}\operatorname{div}(d)) \subseteq \mathscr{C}^e(\Delta + \frac{1}{p^{e_0} - 1}\operatorname{div}(d)).$$

In particular, we see that

$$\sum_{e=0}^{e_0} \sum_{\phi} \phi(F^e_*(dc)) \subseteq \tau(R, \Delta + \frac{1}{p^{e_0}} \operatorname{div}(d)) \subseteq \tau(R, \Delta + \frac{1}{p^{e_0}} H)$$

$\square$

5.4. Exercises.

Exercise 5.1. Suppose R is an F-finite reduced ring. Show that there exists some $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ that is non-zero when localized at any minimal prime of R.

Exercise 5.2. Suppose that $c \in R$ is a strong test element of (R, ϕ). Show for any multiplicative set W, that $c/1 \in W^{-1}R$ is a strong test element of $(W^{-1}R, W^{-1}\phi)$.

Exercise 5.3. Consider the ring $R = \mathbb{F}_p[x, y]/(xy)$.

(a) Show that $F_* R$ is generated as an R-module by the elements:

$$F_* 1, F_* x, \ldots, F_* x^{p-1}, F_* y, \ldots, F_* y^{p-1}.$$

(b) Show that there is a well-defined R-linear map that sends $F_* x \mapsto x$ and the other generators from (a) to 0. (One way to do this is to use Fedder's Lemma).

(c) Compute the test ideal of (R, ϕ) and prove that it is contained in (x).

Exercise 5.4. Suppose that $(R, \mathfrak{m})$ is an F-finite local ring and that $c \in R$ is a strong test element of (R, ϕ). Show that the image of $c \in \widehat{R}$, the $\mathfrak{m}$-adic completion of R is a strong test element of $(\widehat{R}, \widehat{\phi})$.

Hint: Recall since F-finite rings are excellent (and so have geometrically regular fibers), that if R is F-finite and $b \in R$ is such that R_b is regular, then we also have that $\widehat{R}_b$ is regular.

Exercise 5.5. Prove Proposition 5.11 parts (a) and (b) by using the previous exercises and the description of the test ideal in Corollary 5.9.

Exercise 5.6. Prove Proposition 5.11 (c).

Exercise 5.7. Prove Lemma 5.15.

Exercise 5.8. Suppose that R is an F-finite reduced ring with normalization R^{N}. Suppose $\phi : F^e_* R \longrightarrow R$ extends to $\phi^{\mathrm{N}} : F^e_* R^{\mathrm{N}} \longrightarrow R$ via in Chapter 5 Theorem 4.8. Prove that

$$\tau(R, \phi) = \tau(R^{\mathrm{N}}, \phi^{\mathrm{N}}).$$

Hint: We can pick $c \in \mathfrak{c} \cap R$ a strong test element and non-zerodivisor for both (R, ϕ) and for $(R^{\mathrm{N}}, \phi^{\mathrm{N}})$. Notice that c^2 is also such a strong test element.

Exercise 5.9. Suppose that (R, ϕ) with ϕ non-degenerate. Further suppose that $Q \subseteq R$ is a compatible prime ideal with ϕ and that the induced $\phi_Q : F^e_* R_Q \longrightarrow R_Q$ is surjective. Prove that there exists a smallest ideal J, not contained in Q, such that $\phi(F^e_* J) \subseteq J$. It is sometimes denoted by $\tau_{\not\subseteq Q}(R, \phi)$.

Exercise 5.10. With notation as in Exercise 5.9, let $\overline{\phi} : F^e_* R/Q \longrightarrow R/Q$ denote the induced map. Prove that

$$\tau_{\not\subseteq Q}(R, \phi) \cdot (R/Q) = \tau(R/Q, \overline{\phi}).$$

Exercise 5.11. Suppose that R is an F-finite normal domain $\Delta \geq 0$ is a $\mathbb{Q}$-divisor and $D = \operatorname{div}(f)$ is a principal divisor. Show that

$$\tau(R, \Delta + D) = f \cdot \tau(R, \Delta).$$

This should be viewed as a variant of the Skoda-type theorem we saw in Chapter 4 Theorem 5.20.

Hint: For an idea of the argument, see the proof of Proposition 6.15 in the next section.

Exercise 5.12. Suppose that (R, Δ) is as in Setting 5.17. Prove that $\tau(R, \Delta) \subseteq R(-\lfloor \Delta \rfloor) = I_{\lfloor \Delta \rfloor}$.

Hint: If $\phi \in \mathscr{C}^{\Delta, e}$, prove that ϕ is compatible with the ideal $I_{\lfloor \Delta \rfloor}$.

Exercise 5.13. Suppose that (R, Δ) is as in Setting 5.17 and that $\Delta = D + \Gamma$ where D is a prime divisor and the coefficient of D in Γ is zero. Show that there exists a smallest ideal J, not contained in $I_D = R(-D)$, that is compatible with all $\phi \in \mathscr{C}^{\Delta, e}$ for all $e \geq 0$.

Exercise 5.14. Prove Lemma 5.20.

Exercise 5.15. Suppose that (R, Δ) is defined as in Setting 5.17.

(a) For any multiplicative set $W \subseteq R$, show that $W^{-1}\tau(R, \Delta)$ is the test ideal for $(W^{-1}R, \Delta|_{\operatorname{Spec} W^{-1}R})$.
(b) For any ideal $J \subseteq R$ with $\widehat{R}$ the J-adic completion, show that $\tau(R, \Delta)\widehat{R}$ is the test ideal of $(\widehat{R}, \Delta|_{\operatorname{Spec} \widehat{R}})$.

Exercise 5.16. Suppose that (R, Δ) is as in Setting 5.17, that $\mathfrak{a} \subseteq R$ is an ideal, and $t \geq 0$ is a real number. Define the *test ideal of the triple* $(R, \Delta, \mathfrak{a}^t)$, denoted $\tau(R, \Delta, \mathfrak{a}^t)$ to be the smallest non-zero ideal $J \subseteq R$ compatible with every element of $(F^e_* \mathfrak{a}^{\lceil t(p^e - 1) \rceil}) \cdot \mathscr{C}^{\Delta, e}$. Show that this ideal exists.

Exercise 5.17. With notation as in Exercise 5.16, suppose that $\mathfrak{a} = (f)$ is a principal ideal. Prove that

$$\tau(R, \Delta, \mathfrak{a}^t) = \tau(R, \Delta + t\,\mathrm{div}(f)).$$

6. The splitting prime and other compatible ideals

We've defined the test ideal as the *smallest* non-zero compatible ideal. We have also constructed some other ideals that are compatible, see Exercise 5.9. It is natural then to ask what all the other compatible ideals are? Is there a biggest one (besides the ring itself)?

Before continuing, we need the following simple observation.

Lemma 6.1. *Suppose R is a ring of characteristic p and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$. Suppose that J is ϕ-compatible and $\overline{\phi} : F^e_* R/J \longrightarrow R/J$ is the induced map. Then the set of ϕ-compatible ideals that contain J are in bijection with the $\overline{\phi}$-compatible ideals of R/J.*

PROOF. The bijection is given by $I \mapsto I/J$. $\square$

6.1. The splitting prime, the biggest compatible ideal.

We begin our work in the following setting.

Setting 6.2 (Setting for the splitting prime of single map)**.** Suppose that $(R, \mathfrak{m})$ is a an F-finite *local* ring. Further suppose that $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ is surjective.

We now define the splitting prime of [**AE05**].

Definition 6.3 (Splitting prime of a single map)**.** With notation as in Setting 6.2, we define the *splitting prime of* (R, ϕ) as

$$\mathscr{P}(R, \phi) = \{x \in R \mid \phi^n(F^{ne}_*(x)) \subseteq \mathfrak{m} \text{ for all } n.\}.$$

It satisfies the following properties.

Proposition 6.4. *Suppose (R, ϕ) is as in Setting 6.2 and set $Q = \mathscr{P}(R, \phi)$.*

(a) Q is the unique largest proper ϕ-compatible ideal of R.
(b) Q is prime.
(c) $(R/Q, \overline{\phi})$ is strongly F-regular and hence normal and Cohen-Macaulay.

PROOF. First we must show that $Q = \mathscr{P}(R, \phi)$ is an ideal. If $x_1, x_2 \in Q$, then since $\phi^n(F^{ne}_*(x_i R)) \subseteq \mathfrak{m}$, we have that $\phi^n(F^{ne}_*(x_1 + x_2)R) \subseteq \mathfrak{m}$. If $a \in R$, then $\phi^n(F^{ne}_*(axR)) \subseteq \phi^n(F^{ne}_*(xR)) \subseteq \mathfrak{m}$. Hence Q is an ideal.

We claim that Q is ϕ-compatible. We see that $x \in Q$ if and only if $\phi^n(F^{ne}_* rx) \in \mathfrak{m}$ for every n and $r \in R$. But now if $y = \phi(F^e_* x)$ then

$$\phi^n(F^{ne}_* ry) = \phi^n(F^{ne}_*(r\phi(F^e_* x))) = \phi^{n+1}(F^{(n+1)e}_* r^{p^e} x) \in \mathfrak{m}.$$

Thus $y \in Q$ as well and Q is ϕ-compatible.

If J is ϕ-compatible and proper (and so contained in $\mathfrak{m}$), we see that $\phi(F^e_* J) \subseteq J \subseteq \mathfrak{m}$. Thus $J \subseteq Q = \mathscr{P}(R, \phi)$. This shows (a).

At this point we see that Q is radical by Chapter 3 Exercise 3.1, but we will show it is prime by showing R/Q is normal, since a normal local ring is automatically an integral domain, hence (c) $\Rightarrow$ (b).

Choose $\overline{c} \in R/Q$ not in any minimal prime (a nonzero divisor on R/Q). By construction,

$$(\overline{c}) + \sum_{n>0} \overline{\phi}^n(F^{ne}_*(\overline{c}))$$

is $\overline{\phi}$-compatible and contains $\overline{c}$. But by Lemma 6.1 and part (a) there are no ϕ-compatible ideals of R/Q except for 0 and R/Q itself. □

Setting 6.5 (Setting for the splitting prime of a local ring)**.** Suppose that $(R, \mathfrak{m})$ is a an F-finite Frobenius split *local* ring.

Definition 6.6 ([**AE05**])**.** With notation as in Setting 6.2, we define the *splitting prime of* R as

$$\mathscr{P}(R) = \{x \in R \mid \phi(F^e_*(x)) \in \mathfrak{m} \text{ for all } e > 0 \text{ and all } \phi \in \operatorname{Hom}_R(F^e_* R, R)\}.$$

Proposition 6.7 ([**AE05**])**.** *Suppose R is as in Setting 6.5 and set $Q = \mathscr{P}(R)$.*

(a) Q is the unique largest proper ideal compatible with all $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ for all $e > 0$.
(b) Q is prime.
(c) R/Q is strongly F-regular, and in particular R/Q is a normal and Cohen-Macaulay domain.

PROOF. Left to the reader in Exercise 6.5. □

6.2. Finitely many compatible ideals. We conclude the section by proving that there are finitely many ideals compatible with a surjective map $\phi : F^e_* R \longrightarrow R$. In the case that R is local, this was proven independently in [**Sha07, EH08**]. The general case was then shown independently in [**KM09, Sch09**] (and our proof essentially follows that strategy).

Theorem 6.8 ([**KM09, Sch09**])**.** *Suppose R is an F-finite ring and $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ is surjective. Then there are finitely many ϕ-compatible ideals.*

PROOF. We proceed by induction on the dimension of R. Since ϕ is surjective, R is reduced. Each compatible ideal is likewise radical. Hence by Chapter 1 Proposition 6.7 (b) it suffices to show that there are only finite many compatible *prime* ideals since a radical ideal is the intersection of its minimal primes and an intersection of compatible ideals is compatible.

Since any compatible prime contains some minimal prime (which is also compatible again Chapter 1 Proposition 6.7 (b)), using Lemma 6.1 it suffices to work modulo a minimal prime and hence assume R is a domain.

Since $J = \tau(R, \phi)$ is the unique smallest non-zero ϕ-compatible ideal, again by Lemma 6.1 it suffices to show that R/J has only finitely many $\overline{\phi}$-compatible ideals. But $\dim R > \dim(R/\tau(R, \phi))$, and the result follows. $\square$

In Chapter 8 Theorem 2.23 we will substantially generalize this removing the surjectivity hypothesis on ϕ ([**BB11**]). However, instead of considering all compatible ideals we will consider ideals J such that $\phi(F^e_* J) = J$. The proof is more involved though.

Remark 6.9. In the case that $(R, \mathfrak{m}, k)$ is local, $\phi : F^e_* R \longrightarrow R$ is surjective, and $n = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ is the embedding dimension, there at most $\binom{n}{d}$ ϕ-compatible prime ideals Q such that $\dim R/Q = d$ [**ST10b**]. In fact, work of Huneke-Watanabe [**HW15**] , even shows that the multiplicity can be bounded above by $\binom{n}{\dim R}$. Outside of the local case however, there is no bound on the number of compatible ideals, see Exercise 6.6.

6.3. Test (sub)modules of pairs. If R is Gorenstein, then we may take $\omega_R = R$ and then it is not difficult to see that $\tau(R) = \tau(\omega_R)$ where $\tau(\omega_R)$ is as defined in Chapter 2 Section 5. However, there is another way to relate $\tau(R)$ and $\tau(\omega_R)$, and that is by building a pair into $\tau(\omega_R)$. This perspective will be very useful for us in future chapters when comparing test ideals with multiplier ideals from birational geometry.

We work in the following setting.

Setting 6.10. Suppose R is an F-finite normal domain with canonical canonical module ω_R. Fix an embedding $R \subseteq \omega_R$ which itself fixes a choice of a canonical divisor K_R. Finally suppose that $\Gamma \geq 0$ is a $\mathbb{R}$-divisor on $\mathrm{Spec}\, R$.

Note, since we are working with divisors, we assume R is normal. On the other hand if R is normal, it is harmless to assume R is a domain.

Definition 6.11. With R, ω_R and Γ as in Setting 6.10, we define the **test module of the pair** (ω_R, Γ), denoted $\tau(\omega_R, \Gamma)$ to be the smallest non-zero submodule $J \subseteq \omega_R$ such that

$$\phi(F^e_* J) \subseteq J$$

for every

$$\phi \in \operatorname{Hom}_R(F^e_* \omega_R(\lceil (p^e - 1)\Gamma \rceil), \omega_R) \subseteq \operatorname{Hom}_R(F^e_* \omega_R, \omega_R).$$

Here $\omega_R(\lceil (p^e - 1)\Gamma \rceil) = R(K_R + \lceil (p^e - 1)\Gamma \rceil)$.

The techniques we have seen so far also imply that the test module of $\tau(\omega_R, \Gamma)$ exists.

Theorem 6.12. *With notation as in Setting 6.10, $\tau(\omega_R, \Gamma)$ exists and its formation commutes with localization. Furthermore, if c is any non-zerodivisor which is a strong test element for (R, Γ), then*

$$\tau(\omega_R, \Gamma) = \sum_{e \geq 0} \sum_{\phi} \phi(c\, \omega_R)$$

where ϕ ranges over $\operatorname{Hom}_R(F^e_* \omega_R(\lceil (p^e - 1)\Gamma \rceil), \omega_R)$.

PROOF. Left to the reader in Exercise 6.8 and Exercise 6.9. □

The following follows easily, via the same argument as Lemma 5.23 (left to the reader in Exercise 6.9 and Exercise 5.15 respectively).

Lemma 6.13. *With notation as in Definition 6.11,*

(a) For any multiplicative set $W \subseteq R$, $W^{-1}\tau(\omega, \Gamma) = (W^{-1}\omega_R, \Gamma|_{\operatorname{Spec} W^{-1}R})$.
(b) For any ideal $J \subseteq R$ with $\widehat{R}$ the J-adic completion, $\tau(\omega, \Gamma) \otimes_R \widehat{R} \cong (\omega_{\widehat{R}}, \Gamma|_{\operatorname{Spec} \widehat{R}})$.

We now relate this object to the test ideal.

Theorem 6.14. *Suppose R is an F-finite normal domain, $\omega_R \supseteq R$ is a canonical module (note $K_R \geq 0$) and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor. Then*

$$\tau(R, \Delta) = \tau(\omega_R, K_R + \Delta)$$

viewed as submodules of the fraction field $\mathcal{K}(R)$ (or as submodules of ω_R). In particular, $\tau(R) = \tau(\omega_R, K_R)$.

PROOF. We begin our argument in dimension 1 when R is a DVR. In that case $\omega_R = \frac{1}{f} \cdot R \cong R$, with $\mathrm{div}(f) = K_R$. The dual-to-Frobenius map[14] T^e is identified with Φ^e via the isomorphism $\omega_R \cong R$ above (since both generate their respective Hom-sets):

$$\begin{tikzcd} F^e_* \omega_R \arrow[r, "T^e"] \arrow[d, "\sim"'] & \omega_R \arrow[d, "\sim"] \\ F^e_* R \arrow[r, "\Phi^e"'] & R. \end{tikzcd}$$

In this case, because we are computing $\tau(\omega_R, K_R + \Delta)$, and $K_R + \Delta \geq K_R$, the associated $\tau(\omega_R, K_R + \Delta) \subseteq f \cdot \omega_R = R$ by Exercise 5.12.

Now, since R is S_2, if $\tau(\omega_R, K_R + \Delta) \subseteq R$ in dimension 1, the containment holds generally. We now return to R of arbitrary dimension.

Notice that

$$\begin{aligned} & \mathrm{Hom}_R \big(F^e_* \omega_R(\lceil (p^e - 1)(K_R + \Delta)\rceil), \omega_R\big) \\ = \; & \mathrm{Hom}_R \big(F^e_* R(K_R + \lceil (p^e - 1)(K_R + \Delta)\rceil), R(K_R)\big) \\ \cong \; & \mathrm{Hom}_R \big(F^e_* R(K_R + \lceil (p^e - 1)(K_R + \Delta)\rceil - p^e K_R), R\big) \\ = \; & \mathrm{Hom}_R \big(F^e_* R(\lceil (p^e - 1)\Delta\rceil), R\big) \\ = \; & \mathscr{C}^{\Delta, e}. \end{aligned}$$

Thus we may view any $\phi \in \mathrm{Hom}_R(F^e_* \omega_R(\lceil (p^e - 1)(K_R + \Delta)\rceil), \omega_R)$ as an element of $\mathscr{C}^{\Delta, e}$. Because $\tau(\omega_R, K_R + \Delta) \subseteq R$, it is the smallest non-zero submodule of R compatible with elements of $\mathscr{C}^{\Delta, e}$, in other words, it equals the test ideal as desired. $\square$

We point out some other basic properties of test modules of pairs.

Proposition 6.15. *Suppose R is a Noetherian F-finite normal domain and $\Gamma \geq 0$ is a $\mathbb{Q}$-divisor. Then*

(a) If $\Gamma' \geq \Gamma$ then $\tau(\omega_R, \Gamma') \subseteq \tau(\omega_R, \Gamma)$.
(b) $\tau(\omega_R, \Gamma + \mathrm{div}(f)) = f \tau(\omega_R, \Gamma)$ for any $0 \neq f \in R$.
(c) If $D \geq 0$ is any other divisor, then $\tau(\omega_R, \Gamma) = \tau(\omega, \Gamma + \epsilon D)$ for all $1 \gg \epsilon > 0$.

PROOF. The first property is immediate from the definition. Property (b) follows immediately from Theorem 6.12 since

$$\mathrm{Hom}_R(F^e_* \omega_R(\lceil (p^e - 1)(\Gamma + \mathrm{div}(f))\rceil), \omega_R) = (F^e_* f^{p^e - 1}) \mathrm{Hom}_R(F^e_* \omega_R(\lceil (p^e - 1)\Gamma\rceil), \omega_R)$$

and we may replace c by cf without harm.

[14] see Chapter 2

For (c), by utilizing Theorem 6.12 to increase Γ by a principal divisor, we may assume that $\Gamma \geq K_R$. Setting $\Delta = \Gamma + K_R$ we see that

$$\tau(\omega_R, \Gamma) = \tau(R, \Delta) = \tau(R, \Delta + \epsilon D) = \tau(R, \Gamma + \epsilon D)$$

by Proposition 5.26 (c). $\square$

6.4. Exercises.

Exercise 6.1. Suppose R is an F-finite not necessarily local ring and that $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ is nonzero. Use Chapter 1 Exercise 4.12 to show that any proper ideal that is maximal with respect to being ϕ-compatible is prime. This gives an alternate proof of Proposition 6.4 (b).

Exercise 6.2. Suppose that R is an F-finite ring and suppose that $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. Let $W \subseteq R$ be a multiplicative set and let $\phi' : F^e_* W^{-1}R \longrightarrow W^{-1}R$ be the induced map. Let $Q \in \operatorname{Spec} R$ be a prime such that $Q \cap W = \emptyset$ so that $Q' := W^{-1}Q$ is a prime in $\operatorname{Spec} W^{-1}R$. Further suppose that Q' is ϕ'-compatible. Prove that Q is ϕ-compatible.

Exercise 6.3. Suppose that R is an F-finite ring and that $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. If J is ϕ-compatible, show that $\sqrt{J}$ is ϕ-compatible.

Hint: Localize at each minimal prime of J and consider separately the cases when ϕ is surjective or not. Recall (or prove) that an ideal compatible with a surjective map is always radical. Then use the previous exercise to undo the localization.

Exercise 6.4. Suppose that R is an F-finite ring and that $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. If $Q \in \operatorname{Spec} R$ is ϕ-compatible but $\phi_Q : F^e_* R_Q \longrightarrow R_Q$ is not surjective, show that every prime $P \supseteq Q$ is also ϕ-compatible.

Exercise 6.5. Prove Proposition 6.4.

Exercise 6.6. Let $R = \mathbb{F}_p[x]$. Show that for every integer n, there exists a surjective $\phi : F^e_* R \longrightarrow R$ that is compatible with exactly n non-zero prime ideals.

Exercise 6.7. Let $R = \mathbb{F}_p[x_1, \ldots, x_n]$ and let $\phi : F^e R \longrightarrow R$ be the canonical toric splitting implicitly defined in Chapter 3 Example 1.4. Show that ϕ is compatible with exactly $\binom{n}{d}$ prime ideals Q such that $\dim R/Q = d$.[15]

Exercise 6.8. Prove that $\tau(\omega_R, \Gamma)$, as defined in Definition 6.11, exists by showing that

$$\tau(\omega_R, \Gamma) = \sum_{e \geq 0} \sum_{\phi} \phi(c\, \omega_R)$$

where ϕ ranges over $\operatorname{Hom}_R(F^e_* \omega_R(\lceil (p^e - 1)\Gamma \rceil), \omega_R)$.

[15] In fact, if R is an F-finite local ring with embedding dimension n and ϕ is surjective, then there are at most $\binom{n}{d}$ compatible ideals by **[ST10b]**.

Exercise 6.9. Prove that the formation of $\tau(\omega_R, \Gamma)$ commutes with localization and completion.

Exercise 6.10. For a Noetherian F-finite normal domain, prove that $\tau(\omega_R, g^t) = \tau(\omega_R, t\,\mathrm{div}(g))$ where $\tau(\omega_R, g^t)$ is defined in Chapter 4 Definition 6.22.

Hint: Use Proposition 6.15 (c) and Chapter 4 Exercise 6.14 to handle any difference in rounding that may occur.

7. Test ideals and finite ring maps

We begin with the following question:

If $R \subseteq S$ is a finite extension of rings, and $\phi : F^e_* R \longrightarrow R$ is an R-linear map, when does ϕ extend to a map ϕ_S? In other words, when can we find a ϕ_S to make the following diagram commute?

$$\begin{tikzcd} F^e_* S \arrow[r, dotted, "\phi_S"] & S \\ F^e_* R \arrow[u, hook] \arrow[r, "\phi"'] & R \arrow[u, hook] \end{tikzcd}$$

We begin with an example.

Example 7.1. Let k be a perfect field of characteristic $p > 2$. Consider the extension $R = k[x] \subseteq k[x^{1/2}] = S$. We will consider three different maps $F_* R \longrightarrow R$ defined by where they send monomials as follows:

$$\begin{array}{ll} \phi_1 : & F_*(x^{1/2})^{(p-1)} \mapsto 1 \text{ and other monomials to zero} \\ \phi_2 : & F_*(x^{1/2})^{(p-1)/2} \mapsto 1 \text{ and other monomials to zero} \\ \phi_3 : & F_* 1 \mapsto 1 \text{ and other monomials to zero} \end{array}$$

Let us begin with ϕ_1. Suppose ϕ_1 extended to a map $\psi_1 : F_* S \longrightarrow S$. Then notice that

$$1 = \psi_1(F_* x^{p-1}) = \psi_1(F_*(x^{1/2})^{2(p-1)}) = \psi_1(F_*(x^{1/2})^p x^{\frac{p-2}{2}}) = (x^{1/2})\psi_1(F_* x^{\frac{p-2}{2}}).$$

Hence $\psi_1(F_* x^{\frac{p-2}{2}}) = x^{-1/2}$, but that is not in S, a contradiction. Thus ϕ_1 does not extend.

For ϕ_2 and ϕ_3, the maps do extend to $F_* S \longrightarrow S$ which we now verify. Indeed, to extend ϕ_j to $\psi_j : F_* S \longrightarrow S$, it suffices to identify where $F_*(x^{1/2})^i$ goes for $i = 1, \ldots, p-1$ and show that this $\psi_j|_{F_* R} = \phi_j$.

We define ψ_3 to be the map which sends $F_* 1 \mapsto 1$ and $F_*(x^{1/2})^i \mapsto 0$ for $i = 1, \ldots, p-1$. We notice that then $F_* x^i = F_*(x^{1/2})^{2i}$ is sent to 1 for $i = 1, \ldots, \frac{p-1}{2}$. Next, notice that $F_* x^{\frac{p+j}{2}} = F_*(x^{1/2})^{p+j} = x^{1/2} F_*(x^{1/2})^j$

is sent to zero for $0 < j < p$. Hence for $i = \frac{p+1}{2}, \ldots, p-1$, we see that $F_* x^i = x^{1/2} F_* (x^{1/2})^{i-p}$ is sent to zero as well. Thus $\psi_3|_{F_* R} = \phi_3$ (both are the canonical toric Frobenius splitting).

For ψ_2, we define it to be the map which sends $F^e_* (x^{1/2})^{p-1} \mapsto 1$ and the other monomials $F_* 1, F_* x^{1/2}, \ldots, F_* (x^{1/2})^{p-2}$ to zero. A computation similar to the one above for ϕ_3, shows that $\psi_2|_{F_* R} = \phi_2$. We do point out that $\phi_2(F_* x^{(p-1)/2}) = 1 = \psi_2(F_*(x^{1/2})^{p-1})$. Notice that ψ_2 generates $\mathrm{Hom}_S(F_* S, S)$ even though ϕ_2 does not generate $\mathrm{Hom}_R(F_* R, R)$.

7.1. Extending Frobenius splittings for field extensions. We begin in the case when R and S are fields. This is not such an unreasonable place to start, if $R \subseteq S$ is a finite extension of integral domains, then taking fraction fields we obtain $\mathcal{K}(R) \subseteq \mathcal{K}(S)$ a finite field extension. It is straightforward to see that if $\phi : F^e_* R \longrightarrow R$ extends to $F^e_* S \longrightarrow S$, then the map induced by localization $\phi : F^e_* \mathcal{K}(R) \longrightarrow \mathcal{K}(R)$ extends to $F^e_* \mathcal{K}(S) \longrightarrow \mathcal{K}(S)$. Thus we think first about the field case. It turns out then our question is completely determined by whether the field extension is separable or not.

Lemma 7.2. *Suppose that $K \subseteq L$ is a finite purely inseparable extension of F-finite fields of characteristic $p > 0$. If $\phi : F^e_* K \longrightarrow K$ is a K-linear map which extends to a L-linear map $\phi_L : F^e_* L \longrightarrow L$, then $\phi = 0$.*

Proof. Choose $x \in L \setminus K$ with $x^{p^e} \in K$. Suppose ϕ_L exists as in the statement of the lemma. Choose $y \in K$. Then since $x^p y \in K$ we have that:

$$\phi_L(F^e_* x^p y) = \phi(F^e_* x^{p^e} y) = x \phi(F^e_* y) \in K.$$

If $\phi(F^e_* y) \neq 0$, then dividing by that element implies that $x \in K$, a contradiction. Hence $\phi(F^e_* y) = 0$. $\square$

On the other hand, p^{-e}-linear maps always extend uniquely across separable field extensions. We already saw and used a version of this in Chapter 1 Proposition 5.9. Recall that a finite extension of fields is separable if and only if it is étale.

Lemma 7.3. *Suppose that $K \subseteq L$ is a finite separable extension of F-finite fields of characteristic $p > 0$. If $\phi : F^e_* K \longrightarrow K$ is a K-linear map, then there is a unique L-linear extension $\phi_L : F^e_* L \longrightarrow L$.*

Proof. Tensor the map $\phi : F^e_* K \longrightarrow K$ by $\otimes_K L$ to obtain

$$\phi \otimes_K L : (F^e_* K) \otimes_K L \longrightarrow L.$$

It follows from Proposition 5.9 in Chapter 1 that $(F^e_* K) \otimes_K L = F^e_* L$. Thus we take $\phi_L = \phi \otimes_K L$. The uniqueness follows from L-linearity. $\square$

Lemma 7.4. *Suppose that $K \subseteq L$ is a finite inseparable extension of F-finite fields of characteristic $p > 0$. If $\phi : F^e_* K \longrightarrow K$ is a map that extends to $\phi_L : F^e_* L \longrightarrow L$ then $\phi = 0$.*

PROOF. Let $K' \subseteq L$ denote the separable closure of K in L. We will show first that $\phi_L(F^e_* K') \subseteq K'$. Since $K' \cdot (F^e_* K) = F^e_* K'$ by the proof of Lemma 7.3, we see that any $F^e_* x \in F^e_* K'$ can be written as

$$F^e_* x = y_1 \cdot (F^e_* x_1) + \cdots + y_t \cdot (F^e_* x_t)$$

for some $y_i \in K'$ and $x_i \in K$. Hence

$$\phi_L(F^e_* x) = \phi_L(y_1 \cdot (F^e_* x_1) + \cdots + y_t \cdot (F^e_* x_t)) = y_1 \phi(F^e_* x_1) + \cdots + y_t \phi(F^e_* x_t) \in K'.$$

Thus ϕ_L also extends $\phi_L|_{K'}$, but $K' \subseteq L$ is purely inseparable and we are done by Lemma 7.2. $\square$

We immediately obtain the following generalization of Proposition 4.1.

Corollary 7.5. *Suppose X is normal proper variety over an F-finite field k with $H^0(X, \mathcal{O}_X) = k$, and pick non-zero $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ with associated $\Delta = \Delta_\phi$. Further suppose that $x \in X$ is a closed point with $k \subseteq k(x)$ separable, that is compatible with ϕ, and such that the induced $\overline{\phi} : F^e_*\mathcal{O}_X/\mathfrak{m}_x \longrightarrow \mathcal{O}_X/\mathfrak{m}_x$ is nonzero. Then (X, Δ) is globally Frobenius split.*

Conversely, if a F-splitting ϕ is compatible with some closed point $x \in X$, then $k \subseteq k(x)$ is separable.

PROOF. The proof of Proposition 4.1 goes through with little change for the first statement. If $H^0(X, F^e_*\mathcal{O}_X) \xrightarrow{\overline{\phi}} H^0(X, \mathcal{O}_X)$ is zero, so is the induced map $F^e_* k(x) \longrightarrow k(x)$ by Lemma 7.3. On the other hand, if $\overline{\phi}$ is nonzero, it is surjective.

For the second statement, apply Lemma 7.4 to the diagram (4.1.1). $\square$

If $K \subseteq L$ is a finite field extension, then viewing L as a finite-dimensional K-vector space, each $x \in L$ then becomes a K-linear transformation $T_x : L \longrightarrow L$ by multiplication. We can then take the trace of this linear transformation to obtain any element of K. That is, we define:

$$\mathrm{Tr} : L \longrightarrow K$$

by setting $\mathrm{Tr}(x)$ to be the trace of T_x. This map is non-zero if and only if $K \subseteq L$ is separable [**Sta19**, Tag 0BIL].

We need to relate extensions of ϕ along field extensions with the field trace. For this we observe the following.

Lemma 7.6. *Suppose that $K \subseteq L$ is a finite extension of F-finite fields. Fix a non-zero K-linear map $T : L \longrightarrow K$. Then for every K-linear map $\phi : F^e_* K \longrightarrow K$ there is a unique L-linear map $\phi_L : F^e_* L \longrightarrow L$ such that the following diagram commutes:*

$$\begin{tikzcd} F^e_* L \arrow[r, dotted, "\phi_L"] \arrow[d, "F^e_* T"'] & L \arrow[d, "T"] \\ F^e_* K \arrow[r, "\phi"'] & K. \end{tikzcd}$$

Proof. Notice that since $0 \neq \phi$ we have that ϕ generates $\mathrm{Hom}_K(F^e_* K, K)$ as a $F^e_* K$-module. Likewise T generates $\mathrm{Hom}_K(L, K)$ hence $\phi \circ (F^e_* T)$ generates $\mathrm{Hom}_K(F^e_* L, K) \cong F^e_* L$ as an $F^e_* L$-module. For any map $\psi_L : F^e_* L \longrightarrow L$, the composition $T \circ \psi_L$ will be a pre-multiple of $\phi \circ (F^e_* T)$ by some non-zero element $F^e_* z \in F^e_* L$. Setting $\phi_L = \psi_L \circ (\cdot F^e_* z^{-1})$ makes the diagram commute. The uniqueness follows since $\mathrm{Hom}_L(F^e_* L, L) \cong F^e_* L$ and $\mathrm{Hom}_K(F^e_* L, K) \cong F^e_* L$. $\square$

Definition 7.7 (**[ST10a]**)**.** The map ϕ_L in Lemma 7.6 is called the **T-transpose of ϕ**.

Theorem 7.8. *Suppose $K \subseteq L$ is a finite separable field extension. Suppose $\phi_K : F^e_* K \longrightarrow K$ is K-linear and $\phi_L : F^e_* L \longrightarrow L$ is L-linear. Then $\phi_L|_{F^e_* K} = \phi_K$ (i.e. ϕ_L extends ϕ_K) if and only if the following diagram commutes:*

(7.8.1)
$$\begin{tikzcd} F^e_* L \arrow[r, "\phi_L"] \arrow[d, "F^e_* \mathrm{Tr}"'] & L \arrow[d, "\mathrm{Tr}"] \\ F^e_* K \arrow[r, "\phi_K"'] & K. \end{tikzcd}$$

That is, that $\mathrm{Tr} \circ \phi_L = \phi_K \circ (F^e_* \mathrm{Tr})$.

Proof. We begin with the following claim.

Claim 7.9. *Using the inclusion $L \subseteq F^e_* L$, we have that $(F^e_* \mathrm{Tr})|_L = \mathrm{Tr}$.*

Proof of claim. Take $f_1, \dots, f_m$ a basis for L over K. Since

$$F^e_* L = L \cdot (F^e_* K),$$

their images in $F^e_* L$ are also a basis for $F^e_* L$ over $F^e_* K$. In particular, for any element $y \in L$, the matrix expression for multiplication by y, (the linear transformation $\cdot y$ on L), agrees with the matrix expression for multiplication by its image in $F^e_* L$. $\square$

Suppose first that ϕ_L extends ϕ_K and choose $F^e_* z \in F^e_* L$. Similar to the argument in the claim, since $L \supseteq K$ is separable, we see that a basis $\{F^e_* x_1, \ldots, F^e_* x_k\}$ for $F^e_* K$ over K is also a basis for $F^e_* L \cong L \otimes_K F^e_* K$ over L. Write

$$F^e_* z = \sum_{i=1}^{k} y_i F^e_* x_i$$

for some $y_i \in L$. Hence, using the claim, we have that

$$\begin{array}{rcl} (\phi_K \circ (F^e_* \operatorname{Tr}))(F^e_* z) & = & \phi_K\Big(\sum_{i=1}^k (F^e_* x_i) \cdot \operatorname{Tr}(y_i)\Big) \\ & = & \sum_{i=1}^k (\phi_K(F^e_* x_i)) \cdot \operatorname{Tr}(y_i) \\ & = & \sum_{i=1}^k \operatorname{Tr}\left(\phi_K(F^e_* x_i) \cdot y_i\right) \\ & = & \sum_{i=1}^k \operatorname{Tr}\left(\phi_L(F^e_* x_i) \cdot y_i\right) \\ & = & \operatorname{Tr}\Big(\sum_{i=1}^k \phi_L(F^e_* y_i^{p^e} x_i)\Big) \\ & = & \operatorname{Tr} \circ \phi_L(F^e_* z). \end{array}$$

Thus the diagram commutes. This proves the $(\Rightarrow)$ direction.

For the reverse $(\Leftarrow)$ direction, suppose the diagram (7.8.1) commutes. On the other hand, by Lemma 7.3, we know there exists a ϕ'_L extending ϕ_K and so if it we place ϕ'_L in the diagram (7.8.1) replacing ϕ_L, the new diagram commutes as well by the $(\Rightarrow)$. In other words, we have the following equality:

$$\operatorname{Tr} \circ \phi'_L = \phi_K \circ (F^e_* \operatorname{Tr}) = \operatorname{Tr} \circ \phi_L \in \mathrm{Hom}_K(F^e_* L, K).$$

Now, $\phi_L \cdot (F^e_* v) = \phi'_L$, or in other words that $\phi_L(F^e_* -) = \phi'_L(F^e_* v \cdot -)$, for some $v \in L$ since $\mathrm{Hom}_L(F^e_* L, L) \cong F^e_* L$. Thus we have that

$$\operatorname{Tr} \circ \phi_L \cdot (F^e_* v) = \operatorname{Tr} \circ \phi_L \in \mathrm{Hom}_K(F^e_* L, K) \cong F^e_* L$$

proving that $v = 1$. This completes the proof. $\square$

7.2. Extending p^{-e}-linear maps for normal integral domains. Suppose $R \subseteq S$ is a finite generically separable[16] extension of F-finite normal Noetherian domains and $\phi : F^e_* R \longrightarrow R$ is an R-linear map. We are interested in when ϕ extends to a map $F^e_* S \longrightarrow S$. It turns out that the obstruction to extending such a map is the ramification of $R \subseteq S$. A brief description of (wild and tame) ramification can be found in Appendix B Section 8. We return to the problem at hand.

In the previous subsection, we studied when, given some field extension $K \subseteq L$, $\phi : F^e_* K \longrightarrow K$ extended to some $\phi_L : F^e_* L \longrightarrow L$. Now suppose that $R \subseteq S$ is a finite extension of normal domains and $\phi_R : F^e_* R \longrightarrow R$ is a

[16] This means that $\mathcal{K}(R) \subseteq \mathcal{K}(S)$ is separable.

non-zero map. If $K = \mathcal{K}(R)$ and $L = \mathcal{K}(S)$ and $K \subseteq L$ is separable, then we have a diagram

$$\begin{tikzcd} F^e_* L \arrow[r, "\phi_L"] \arrow[d, "F^e_* \mathrm{Tr}"'] & L \arrow[d, "\mathrm{Tr}"] \\ F^e_* K \arrow[r, "\phi_K"'] & K \end{tikzcd}$$

by Theorem 7.8 where ϕ_K is the localization of ϕ_R at (0) and ϕ_L is the unique map extending ϕ_K which exists by Lemma 7.3. It follows immediately that if $\phi_S : F^e_* S \longrightarrow S$ extends ϕ_R, then we have the following commutative diagram:

$$\begin{tikzcd} F^e_* S \arrow[r, "\phi_S"] \arrow[d, "F^e_* \mathrm{Tr}"'] & S \arrow[d, "\mathrm{Tr}"] \\ F^e_* R \arrow[r, "\phi_R"'] & R. \end{tikzcd} \tag{7.9.1}$$

We will use this diagram to understand Δ_{ϕ_L} and compare it to Δ_ϕ.

First we consider what happens in a slightly more general setting.

Proposition 7.10. *Suppose that $R \subseteq S$ is a finite extension of F-finite Noetherian normal domains with $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ the corresponding map of schemes. Suppose that $T \in \operatorname{Hom}_R(S, R)$ is non-zero and so corresponds to a divisor $D_T \sim K_S - f^* K_R$. Suppose that $\phi : F^e_* R \longrightarrow R$ and $\psi : F^e_* S \longrightarrow S$ are such that the following diagram commutes:*

$$\begin{tikzcd} F^e_* S \arrow[r, "\psi"] \arrow[d, "F^e_* T"'] & S \arrow[d, "T"] \\ F^e_* R \arrow[r, "\phi"'] & R. \end{tikzcd}$$

Then

$$\Delta_\psi = f^* \Delta_\phi - D_T.$$

Proof. The statement is about divisors and so may be checked in dimension one on R. Thus we may assume that R is a DVR and S is semi-local, regular and of dimension 1. In this case, we can choose

$$\begin{array}{rcl} \Phi_R & \in & \operatorname{Hom}_R(F^e_* R, R) \\ \Phi_S & \in & \operatorname{Hom}_S(F^e_* S, S) \\ \Phi_{S/R} & \in & \operatorname{Hom}_R(S, R) \end{array}$$

generating their respective Hom-sets as an $F^e_* R$-module, $F^e_* S$-module, and S-module respectively. Since $\Phi_R \circ (F^e_* \Phi_{S/R})$ and $\Phi_{S/R} \circ \Phi_S$ both generate $\operatorname{Hom}_R(F^e_* S, R) \cong F^e_* S$ they are unit multiples of each other: $\Phi_R \circ$

$(F^e_*\Phi_{S/R}) = \Phi_{S/R} \circ \Phi_S \star u$. Now, writing

$$\begin{array}{rcl} \phi & = & \Phi_R \star x \\ \psi & = & \Phi_S \star y \\ T & = & \Phi_{S/R} \cdot z \end{array}$$

we have that:

$$\begin{array}{rl} T \circ \psi = & \Phi_{S/R} \cdot z \circ \Phi_S \cdot (F^e_* y) \\ = & \Phi_{S/R} \circ \Phi_S \cdot (F^e_* y z^{p^e}) \end{array}$$

which equals

$$\begin{array}{rl} \phi \circ (F^e_* T) = & \Phi_R \cdot (F^e_* x) \circ F^e_*(\Phi_{S/R} \cdot z) \\ = & \Phi_R \circ F^e_* \Phi_{S/R} \cdot (F^e_* xz) \\ = & \Phi_{S/R} \circ \Phi_S \cdot (F^e_* uxz) \end{array}$$

Thus $yz^{p^e} = uxz$ and so $\mathrm{div}(yz^{p^e}) = \mathrm{div}(uxz) = \mathrm{div}(xz)$ where we are computing divisors on $\mathrm{Spec}\, S$. Thus since $\mathrm{div}(x) = f^*(p^e - 1)\Delta_\phi$, $\mathrm{div}(y) = (p^e - 1)\Delta_\psi$, and $\mathrm{div}(z) = D_T$ we see that

$$(p^e - 1)\Delta_\psi + p^e D_T = (p^e - 1) f^* \Delta_\phi + D_T.$$

Reorganizing the equation and dividing by $p^e - 1$ completes the proof. $\square$

To translate this into our setting of extending maps, we need to understand what the ramification divisor corresponds to.

Lemma 7.11. *Suppose $R \subseteq S$ is a finite generically separable extension of normal Noetherian domains with corresponding map of schemes f. Then the map* $\mathrm{Tr} \in \mathrm{Hom}(S, R) \cong S(K_S - f^*K_R)$ *defines a non-zerodivisor $D_{\mathrm{Tr}} \sim K_S - f^*K_R$ which agrees with the ramification divisor* Ram *of f on* $\mathrm{Spec}\, S$.

Proof. We will black box this result as it is well known. It can be found in [**Mor53**], [**SS75**] or [**ST10a**, Proposition 4.8]. Also see in [**Sta19**, Tag 0BTC] $\square$

We thus obtain the following as an immediate corollary of Proposition 7.10 and Lemma 7.11.

Corollary 7.12. *Suppose $R \subseteq S$ is a finite generically separable extension of F-finite Noetherian normal domains. Further suppose that $\phi : F^e_* R \longrightarrow R$ extends to a map $\psi : F^e_* S \longrightarrow S$. Then*

$$\Delta_\psi = f^* \Delta_\phi - \mathrm{Ram}$$

where Ram *is the ramification divisor.*

Since Δ_ψ is effective, if $\phi : F^e_* R \longrightarrow R$ extends to a map $F^e_* S \longrightarrow S$, we necessarily must have that $f^*\Delta_\phi - \mathrm{Ram} \geq 0$, or in other words that $f^*\Delta_\phi \geq \mathrm{Ram}$. Indeed, it turns out that this is a sufficient condition for extending maps as well.

Theorem 7.13 ([**ST10a**]). *Suppose $R \subseteq S$ is a finite extension of F-finite normal domains and $0 \neq T \in \mathrm{Hom}_R(S,R)$ with $D_T \geq 0$ the associated divisor on $\operatorname{Spec} S$. Suppose $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is the induced map of schemes. Fix $\phi : F^e_* R \longrightarrow R$. Then there exists $\psi : F^e_* S \longrightarrow S$ making the following diagram commute:*

$$\begin{tikzcd} F^e_* S \arrow[r, dotted, "\psi"] \arrow[d, "F^e_* T"'] & S \arrow[d, "T"] \\ F^e_* R \arrow[r, "\phi"'] & R \end{tikzcd}$$

if and only if $f^*\Delta_\phi \geq D_T$. In this case, $\Delta_\psi = f^*\Delta_\phi - D_T$. In particular, if $R \subseteq S$ is generically separable, then ϕ extends to a map $F^e_* S \longrightarrow S$ if and only if $f^*\Delta_\phi \geq \mathrm{Ram}$.

Proof. If there exists a ψ making the diagram commute, then we have that $f^*\Delta_\phi \geq D_T$ by Proposition 7.10 and our observations above. Thus we can assume that $f^*\Delta_\phi \geq D_T$.

Let $K = \mathcal{K}(R)$ and $L = \mathcal{K}(S)$. Tensoring with $\mathcal{K}(R)$, we find that ψ_S, if it exists, must be the restriction to $F^e_* S$, of the unique ϕ_L, the T-transpose of $\phi_K = \phi \otimes_R \mathcal{K}(R)$, by Lemma 7.6. Therefore it suffices to show that $\phi_L(F^e_* S) \subseteq S$. Since $S, F^e_* S$, and $F^e_* R$ are all S2 as R-modules, we may assume that R is a DVR and so S is semi-local and regular. We choose

$$\begin{array}{rcl} \Phi_R & \in & \mathrm{Hom}_R(F^e_* R, R) \\ \Phi_S & \in & \mathrm{Hom}_S(F^e_* S, S) \\ \Phi_{S/R} & \in & \mathrm{Hom}_R(S, R) \end{array}$$

generating their respective Hom-sets as an $F^e_* R$-module, $F^e_* S$-module, and S-module respectively and write

$$\begin{array}{rcl} \phi & = & \Phi_R \star x \\ \psi_L & = & (\Phi_S \otimes L) \star y \\ T & = & \Phi_{S/R} \cdot z \end{array}$$

where $x \in R$, $z \in S$ and $y \in L = \mathcal{K}(S)$. If we can show that $y \in S$ then we are done since then we must have $\psi_L = (\Phi_S \star y) \otimes L$ and we set $\psi = \Phi_S \cdot (F^e_* y)$. Arguing and using notation as in the proof of Proposition 7.10, we see that $yz^{p^e} = uxz$ and so

$$\mathrm{div}(yz^{p^e}) = \mathrm{div}(uxz) = \mathrm{div}(xz)$$

for some unit $u \in S$. Thus

$$\mathrm{div}(y) = (p^e - 1) f^*\Delta_\phi - (p^e - 1) D_T = (p^e - 1)(\Delta_\phi - D_T) \geq 0$$

and so $y \in S$ as desired.

For the final statement, we set $T = \mathrm{Tr}$ and recall that ϕ extends to ψ if and only if ψ is the Tr-transpose of ϕ by Theorem 7.8. $\square$

If T generates $\mathrm{Hom}_R(S, R)$ as an S-module, then $D_T = 0$ and so we immediately obtain:

Corollary 7.14. *With notation as in Theorem 7.13, if T generates $\mathrm{Hom}_R(S, R)$ as an S-module, then for* every *$\phi : F^e_* R \longrightarrow R$ there exists a $\psi : F^e_* S \longrightarrow S$ making the diagram commute:*

$$\begin{tikzcd} F^e_* S \arrow[r, dotted, "\psi"] \arrow[d, "F^e_* T"'] & S \arrow[d, "T"] \\ F^e_* R \arrow[r, "\phi"'] & R \end{tikzcd}$$

Example 7.15. Suppose $p \neq 2$, that k is a perfect field of characteristic $p > 0$, and consider the extension $R = k[x] \subseteq k[x^{1/2}] = S$ of Example 7.1. In this case the ramification divisor is $\mathrm{div}(x^{1/2})$.

The map ϕ_1 from Example 7.1 has $\Delta_{\phi_1} = 0$ and so since $f^* \Delta_{\phi_1} - \mathrm{Ram} \leq 0$ we see that ϕ_1 does not extend.

The map ϕ_2 has $\Delta_{\phi_2} = \frac{1}{2} \mathrm{div}(x)$ so that $f^* \Delta_{\phi_2} - \mathrm{Ram} = \frac{1}{2} f^* \mathrm{div}(x) - \mathrm{Ram} = \frac{1}{2} \cdot 2 \cdot \mathrm{div}(x^{1/2}) - \mathrm{Ram} = \mathrm{div}(x^{1/2}) - \mathrm{Ram} = 0$. Thus ϕ_2 extends to ψ_2 with $\Delta_{\psi_2} = 0$, in other words, ϕ_2 extends to a map generating $\mathrm{Hom}_S(F_* S, S)$ as an $F_* S$-module.

The map ϕ_3 has $\Delta_{\phi_3} = \mathrm{div}(x)$. Thus $f^* \Delta_{\phi_3} - \mathrm{Ram} = 2 \, \mathrm{div}(x^{1/2}) - \mathrm{Ram} = \mathrm{div}(x^{1/2})$. Which is exactly what we found by explicit computation.

The previous example was tamely ramified, and so it was easy to compute the ramification divisor Ram. Since the extension had ramification index 2 (at the origin), we see that the ramification divisor Ram had coefficient $2 - 1 = 1$ at the origin in $\mathrm{Spec}\, S$. For wild ramification, the coefficient of the ramification divisor is always strictly larger than the ramification index minus 1.

Example 7.16. Suppose k is a perfect field of characteristic 2 and consider the extension $R = k[x^2(1 + x^5)] \subseteq k[x] = S$. This extension is generically separable since if $t = x^2(1 + x^5)$, then x is a root of the polynomial $f(X) = X^7 + X^2 + t = 0$ and that is a separable polynomial. After localizing at $Q = (t) \in R$, we see that $t = ux^2$, and so the ramification index at the origin is 2. Thus $R \subseteq S$ is wildly ramified.

We can compute the ramification divisor by taking the derivative with respect to X, which is X^6, so the ramification divisor is $6 \, \mathrm{div}(x)$.

Now suppose that $\phi : F^e_* R \longrightarrow R$ satisfies $\Delta_\phi = \lambda \operatorname{div}(t) + \text{other terms}$. Then we see that ϕ extends to a map $\phi : F^e_* S \longrightarrow S$ if and only if $f^*\Delta_\phi = 2\lambda \operatorname{div}(x) + f^* \text{ other terms} \geq \operatorname{Ram} = 6 \operatorname{div}(x)$, which occurs if and only if $\lambda \geq 3$.

As a consequence, we see that there is no Frobenius splitting on R that extends to a Frobenius splitting on S, since for such a ϕ to be a Frobenius splitting on R, we need $\lambda \leq 1$.

7.3. Applications to singularities of pairs. Suppose $R \subseteq S$ is a finite extension of F-finite normal Noetherian domains. We saw previously in Chapter 1 Proposition 3.5 that if $R \subseteq S$ splits, and if S is F-split, then so is R. In many cases, the splitting of $R \longrightarrow S$ is actually accomplished by the normalized trace map $\frac{1}{[\mathcal{K}(S):\mathcal{K}(R)]} \operatorname{Tr}$. Thus, suppose that Tr is surjective (for instance if the normalized trace map splits) and $\phi : F^e_* R \longrightarrow R$ is such that $f^*\Delta_\phi \geq \operatorname{Ram}$ where $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is the induced map. Then we have the following commutative diagram:

$$\begin{array}{ccc} F^e_* S & \xrightarrow{\psi} & S \\ {\scriptstyle F^e_* \operatorname{Tr}}\downarrow & & \downarrow{\scriptstyle \operatorname{Tr}} \\ F^e_* R & \xrightarrow[\phi]{} & R \end{array}$$

where ψ is the extension of ϕ. Hence if ψ is surjective, so is $\operatorname{Tr} \circ \psi = \phi \circ (F^e_* \operatorname{Tr})$, and thus so is ϕ. In other words, if $(S, f^*\Delta_\phi - \operatorname{Ram}) = (S, \Delta_\psi)$ is sharply F-split, so is (R, Δ_ϕ).

On the other hand, suppose ϕ is surjective so that $1 \in R$ is in its image. But then its extension $\psi : F^e_* S \longrightarrow S$ also has 1 in its image and thus surjects as well. We have just proven the following.

Proposition 7.17. *Suppose that $R \subseteq S$ is an extension of F-finite normal Noetherian domains with induced map $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. Assume that $\operatorname{Tr} : S \longrightarrow R$ is surjective. Suppose that Δ is a $\mathbb{Q}$-divisor on $\operatorname{Spec} R$ such that $(p^e - 1)(K_R + \Delta)$ is Cartier for some $e > 0$ and that $f^*\Delta \geq \operatorname{Ram}$. Then (R, Δ) is F-split (meaning that ϕ_Δ is surjective) if and only if $(S, f^*\Delta - \operatorname{Ram})$ is F-split.*

This should be viewed as an analog of the standard result in characteristic zero that a normal log-$\mathbb{Q}$-Gorenstein pair (X, Δ), that $f : Y \longrightarrow X$ is a finite surjective morphism of normal varieties, and $f^*\Delta \geq \operatorname{Ram}$, then (X, Δ) is log canonical if and only if $(Y, f^*\Delta - \operatorname{Ram})$ is log canonical, see [**KM98**, Proposition 5.20].

For a generalization of this result to the inseparable case, or simply the case when Tr is not surjective, see Exercise 7.4. This is particularly important for cyclic covers of index divisible by $p > 0$, *cf.* [**CR22**].

Question 7.18. It is unknown if Proposition 7.17 holds without the hypothesis that $K_R + \Delta$ has (Cartier-)index not divisible by p.

We next explore the behavior of the test ideal under finite maps.

Theorem 7.19 ([**ST10a, Spe20**])**.** *Suppose that $R \subseteq S$ is a finite extension of F-finite normal Noetherian domains with f the induced map between the associated affine schemes. Suppose $T \in \mathrm{Hom}_R(S, R)$ is non-zero with corresponding divisor D_T (for instance, if $R \subseteq S$ is separable, we can pick $T = \mathrm{Tr}$ and so $D_T = \mathrm{Ram}$). Suppose that $\Delta \geq 0$ is a $\mathbb{R}$-divisor on $\mathrm{Spec}\, R$ such that $f^*\Delta \geq D_T$. Then*

$$T(\tau(S, f^*\Delta - D_T)) = \tau(R, \Delta).$$

Proof. We begin by proving the containment $\supseteq$. It is sufficient to show that $T(\tau(S, f^*\Delta - D_T))$ is compatible with ϕ for every $\phi \in \mathrm{Hom}_R(F^e_*R, R)$ with $\Delta_\phi \geq \Delta$ since, by definition, $\tau(R, \Delta)$ is the smallest non-zero J compatible with all such ϕ (see Setting 5.17). But for any such ϕ, we have $f^*\Delta_\phi \geq f^*\Delta \geq D_T$ and so we have the following commutative diagram:

$$\begin{tikzcd} F^e_*S \arrow[r, "\psi"] \arrow[d, "F^e_*T"'] & S \arrow[d, "T"] \\ F^e_*R \arrow[r, "\phi"'] & R. \end{tikzcd}$$

By definition $\tau(S, f^*\Delta - D_T)$ is compatible with ψ and so the diagram guarantees that its trace is compatible with ϕ.

For the converse direction, we make the following simplifying assumption, and leave the general case to the reader in Exercise 7.7, or see [**ST10a**, Proposition 6.24] for a proof in an even more general case. Assume that $\Delta = \Delta_\phi$ for some $\phi \in \mathrm{Hom}_R(F^e_*R, R)$ (in other words, assume that $(p^e - 1)(K_R + \Delta) \sim 0$). We choose $\psi : F^e_*S \longrightarrow S$ sitting in a commutative diagram as above. Observe that

$$T \circ \psi^n = T \circ \psi \circ (F^e_*\psi^{n-1}) = \phi \circ (F^e_*T) \circ (F^e_*\psi^{n-1}) = \cdots = \phi^n \circ (F^{ne}_*T).$$

Next choose $d \neq 0$ a strong test element for $(S, f^*\Delta - D_T)$ (that is, $d \in \tau(S, f^*\Delta - D_T) = \tau(S, \Delta_\psi))$ so that $T(dS) \subseteq \tau(R, \Delta)$ (indeed, if $T(dS)$ was not in $\tau(R, \Delta_\phi)$, simply multiply d by something in $\tau(R, \Delta_\phi)$ so that it is). Note that

$$\tau(S, \Delta_\psi) = \sum_{n>0} \psi^n(F^{ne}_*dS)$$

so that if we apply trace, we obtain that

$$\begin{aligned} T(\tau(S, \Delta_\phi)) &= \sum_{n>0} T\big(\psi^n(F^{ne}_* dS)\big) \\ &= \sum_{n>0} \phi^n(F^e_* T(dS)) \\ &\subseteq \sum_{n>0} \phi^n(F^e_* \tau(R, \Delta)) \\ &= \tau(R, \Delta). \end{aligned}$$

□

Remark 7.20. One can find a much more general results, weakening the normality hypothesis and even generalizing the pairs to Cartier algebras (Chapter 8), in [**CRS23**, Theorem B].

Corollary 7.21. *Suppose that $R \subseteq S$ is a finite extension of F-finite normal Noetherian domains with $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ the induced map. Suppose that $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on $\operatorname{Spec} R$ such that $f^*\Delta \geq \mathrm{Ram}$.*

(a) If (R, Δ) is strongly F-regular, then so is $(S, f^*\Delta - \mathrm{Ram})$.
(b) Suppose that $\mathrm{Tr} : S \longrightarrow R$ surjects. If $(S, f^*\Delta - \mathrm{Ram})$ is strongly F-regular, then so is (R, Δ).

Proof. Choose $c \in R \setminus \{0\}$ a strong test element for (R, Δ) that is also a strong test element for $(S, f^*\Delta - \mathrm{Ram})$. Then a ϕ with $\Delta_\phi \geq \mathrm{Ram}$ such that $\phi(F^e_* c) = 1$ extends to a $\psi : F^e_* S \longrightarrow S$ also sending $F^e_* c \mapsto 1$. This proves the first statement.

The second statement is an immediate consequence of Theorem 7.19 since (R, Δ) is strongly F-regular if and only if $\tau(R, \Delta) = R$. □

We can rephrase Theorem 7.19 in terms of parameter test modules of pairs in the sense of Definition 6.11. This particular rephrasing has the advantage that it makes no assumption on the $\mathbb{Q}$-divisor.

Corollary 7.22. *Suppose $R \subseteq S$ is a finite extension of F-finite normal Noetherian domains with f the induced map between the associated affine schemes. Let $T : \omega_S \longrightarrow \omega_R$ denote the Grothendieck dual of $R \subseteq S$. For any effective $\mathbb{Q}$-divisor $\Gamma \geq 0$, we have that*

$$T(\tau(\omega_S, f^*\Gamma)) = \tau(\omega_R, \Gamma).$$

Proof. Without loss of generality, we may pick an embedding $R \subseteq \omega_R$ (picking $K_R \geq 0$). Choose a free S-module $xS = M \subseteq \omega_S$ such that $T(M) \subseteq R$ (if $g\omega_R \subseteq R$, then any free submodule of $g\omega_S$ will work). Identifying M

with S gives us a diagram:

$$\begin{tikzcd} \omega_S \arrow[r, "T"] & \omega_R \\ S \arrow[u, hook] \arrow[r, "T"'] & R \arrow[u, hook] \end{tikzcd}$$

By Proposition 5.26 (b) and the fact that T is R-linear, it is harmless to replace Γ by $\Gamma + \mathrm{div}(f)$. Hence, we may assume that $\Gamma \geq K_R$ and so write $\Delta = \Gamma - K_R$. We may also assume that $f^*\Delta \geq D_T$ and so $f^*\Gamma \geq K_S$. Now by Theorem 7.19

$$\begin{aligned} \tau(\omega_R, \Gamma) = & \quad \tau(R, \Delta) \\ = & \quad T(\tau(S, f^*\Delta - D_T)) \\ = & \quad T(\tau(S, f^*(\Gamma - K_R) - K_S + f^*K_R)) \\ = & \quad T(\tau(S, f^*\Gamma - K_S)) \\ = & \quad T(\tau(\omega_S, f^*\Gamma)) \end{aligned}$$

as desired. □

Remark 7.23. If $R \subseteq S$ is a finite and generically separable extension of normal domains. Suppose we have an embedding $\omega_R = R(K_R) \subseteq \mathcal{K}(R)$ of the canonical module in the fraction field and set $\omega_S = S(f^*K_R + \mathrm{Ram}) \subseteq \mathcal{K}(S)$. Then we may identify the map $T : \omega_S \longrightarrow \omega_R$ with the restriction of the field trace $\mathrm{Tr} : \mathcal{K}(S) \subseteq \mathcal{K}(R)$ up to a unit. We briefly sketch the argument.

Since $\mathrm{Tr} \in \mathrm{Hom}_R(S, R)$ corresponds to the ramification divisor by Lemma 7.11, it is not difficult to see that

$$\mathrm{Tr} \cdot S = \mathrm{Hom}_R(S(\mathrm{Ram}), R)$$

as submodules of $\mathrm{Hom}_R(S, R)$. Now, we see that

$$\begin{aligned} \mathrm{Hom}_R(S(\mathrm{Ram}), R) \cong & \quad \mathrm{Hom}_R(S(\mathrm{Ram}) \otimes_R R(K_R), R(K_R)) \\ \cong & \quad \mathrm{Hom}_R(S(\mathrm{Ram} + f^*K_R), R(K_R)) \\ = & \quad \mathrm{Hom}_S(\omega_S, \omega_R). \end{aligned}$$

Since all these isomorphisms are equalities at the generic point (fraction field level), we see that Tr must also generate $\mathrm{Hom}_S(\omega_S, \omega_R)$. On the other hand, the evaluation-at-1 map $T : \omega_S = \mathrm{Hom}_R(S, \omega_R) \longrightarrow \omega_R$ also generates $\mathrm{Hom}_R(\omega_S, \omega_R)$ by Appendix C Proposition 5.7, we see that Tr and T agree up to a unit.

7.4. Applications to surjectivity of trace and wild ramification. Consider a finite extension of DVRs $(R, (r)) \subseteq (S, (s))$ with $r = us^n$ for some unit $u \in S$. Since $\mathrm{Tr} \in \mathrm{Hom}_R(S, R)$ determines the ramification divisor, we see that Tr generates $\mathrm{Hom}_R(S(\mathrm{Ram}), R) \subseteq \mathrm{Hom}_R(S, R)$. Write $S(\mathrm{Ram}) = s^{-m}S$ as a fractional ideal. Suppose that $R \subseteq S$ is wildly ramified, which means that $m \geq n$ (in the tame case, $m = n-1$). Notice that $\mathrm{Tr} : S(\mathrm{Ram}) \longrightarrow R$ surjects since Tr generates $\mathrm{Hom}_R(S(\mathrm{Ram}), R)$ and $S(\mathrm{Ram})$ is a free R-module[17] (it is abstractly isomorphic to S). Thus since Tr is R-linear we have that

$$(r) = rR = \mathrm{Tr}\left(rS(\mathrm{Ram})\right) = \mathrm{Tr}(s^{n-m}S) \supsetneq \mathrm{Tr}(S).$$

Hence Tr is not surjective. Conversely, if $R \subseteq S$ is tamely ramified, then the same computation shows that Tr is surjective. Indeed, more generally one can show the following well-known lemma:

Lemma 7.24. *Suppose R is a DVR and $R \subseteq S$ is a finite generically Galois extension with S normal. Then* $\mathrm{Tr} : S \longrightarrow R$ *surjects if and only if $R \subseteq S$ is tame.*

For the non-generically Galois case, having Tr surjective means that at least one of the $R \subseteq S_{\mathfrak{n}_i}$ is tame where $\mathfrak{n}_i$ are the finitely many maximal ideals of S lying over $\mathfrak{m}$, see Exercise 7.10. However, this suggests that we might make declare that a finite generically Galois extension of normal domains $R \subseteq S$ to be *tame* if $\mathrm{Tr} : S \longrightarrow R$ is surjective, indeed this is the strongest notion of tameness in higher dimensions considered in the paper [**KS10**].

Example 7.25. Consider the extension $R = \mathbb{F}_2[x^2(1+x)] \subseteq k[x] = S$. This is a variant of Example 7.16. As an R-module S has basis $1, x, x^2$ and it is easy to see that

$$\mathrm{Tr}(1) = \mathrm{Tr}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3 \equiv 1$$

and so trace is surjective. However, this extension is not generically Galois. Indeed, there are two points lying over the origin $P = (x^2(1+x))$ of $\operatorname{Spec} R$; namely $Q_1 = (x)$ and $Q_2 = (x+1)$. Note $R_P \subseteq S_{Q_1}$ is wildly ramified, but $R_P \subseteq S_{Q_2}$ is tame (it is in fact unramified). It follows that $\mathcal{K}(R) \subseteq \mathcal{K}(S)$ cannot be Galois as the Galois group would be able to send these points to each other.

If instead we complete the extension at the two origins $\widehat{R} = \mathbb{F}_2[\![t]\!] \subseteq k[\![x]\!] = \widehat{S}$ where $t \mapsto x^2(1+x)$. In this case $1, x$ form a basis for $\widehat{S}$ over $\widehat{R}$. In this case, $\mathrm{Tr}(1) = 2 \equiv 0$. Notice that in terms of this basis, we have that $x^2 = 1 \cdot f + x \cdot g$ where $f = t + t^2 + t^3 + t^5 + t^6 + t^9 + \dots$ and $g = t + t^3 + t^5 + t^9 + \dots$. Thus the multiplication $\cdot x$ is represented by the matrix

$$\begin{bmatrix} 0 & f \\ 1 & g \end{bmatrix}$$

and one sees that $\mathrm{Tr}(x) = g$, which is not a unit. It follows that Tr is not surjective.

[17] Note that R is a PID and S and hence $S(\mathrm{Ram})$ is a finitely generated and torsion free R-module.

Example 7.26. Suppose $R \subseteq S$ is a finite generically Galois inclusion of normal domains and $X \longrightarrow \operatorname{Spec} R$ is a proper birational map with X normal and with Y the normalization of X in $\mathcal{K}(S)$ making a commutative diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} R. \end{tikzcd}$$

Suppose that E is an exceptional divisor on X with generic point η so that $\mathcal{O}_{X,\eta} \subseteq (f_* \mathcal{O}_Y)_\eta$ is not tame, then $\mathrm{Tr} : (f_* \mathcal{O}_Y)_\eta \longrightarrow \mathcal{O}_{X,\eta}$ is not surjective and so it does not have 1 in its image. It follows that $\mathrm{Tr} : S \longrightarrow R$ cannot have 1 in its image either since $S \subseteq (f_* \mathcal{O}_Y)_\eta$.

On the other hand, there are finite extensions $R \subseteq S$ where there is no divisorial obstruction to tameness as above, but where Tr is not surjective. See for instance [**KS10**] or Exercise 7.11.

A famous result of Zariski-Nagata on purity of the branch locus ([**Zar58, Nag59**], see [**Sta19**, Tag 0BMB]) says that if R is regular and $R \subseteq S$ is finite and étale in codimension 1, then $R \subseteq S$ is étale. We obtain the following related result for strongly F-regular rings.

Theorem 7.27. *Suppose $R \subseteq S$ is a finite extension of F-finite normal Noetherian domains that is generically Galois. Suppose that $R \subseteq S$ is étale in codimension 1 and R is strongly F-regular. Then $\mathrm{Tr} : S \longrightarrow R$ is surjective and hence tamely ramified even in the strongest sense.*

PROOF. Since $R \subseteq S$ is étale in codimension 1 we see that $\mathrm{Ram} = 0$ and so we see that $\mathrm{Tr} \in \mathrm{Hom}_R(S, R)$. Now, we know that $\mathrm{Tr}(\tau(S)) = \tau(R) = R$ and so we have $\mathrm{Tr}(S) = R$ as desired. □

7.5. An application to cyclic covers. It is well known that if $(R, \mathfrak{m})$ is a Noetherian normal local domain, and D is a Weil divisor on $\operatorname{Spec} R$ of index n (that is $R(nD) \cong R$), then a cyclic cover associated to D,

$$S = R \oplus R(-D) \oplus \cdots \oplus R(-(n-1)D)$$

need not be normal or reduced, see for instance Appendix B Example 9.3.

However, when R is a F-split, this pathology is avoided, as first shown by Carvajal-Rojas.

Theorem 7.28 ([**CR22**])**.** *Suppose $(R, \mathfrak{m})$ is a Noetherian F-finite normal local domain and D is a Weil divisor of index n. Let $R \subseteq S$ be a cyclic cover using an isomorphism $R(nD) \cong R$ (that is, so that $D_z = 0$ in the notation of Appendix B Section 9).*

(a) If R is F-split, so is S. In particular S is reduced.
(b) If R is strongly F-regular, so is R. In particular R is normal.

Proof. Let $\phi : F_* R \longrightarrow R$ be a surjective map. Even though the local[18] ring S need not be normal, we can do a construction similar to the ones earlier in this section. Let $T : S \longrightarrow R$ be the projection onto the degree 0 (which we know generates $\mathrm{Hom}_R(S, R)$ as an S-module by Appendix B (a)).

Consider the following diagram

$$\begin{tikzcd} \mathrm{Hom}_{F^e_* R}(F^e_* S, F^e_* R) \arrow[r, "\nu"] \arrow[d] & \mathrm{Hom}_R(S, R) \arrow[d] \\ F^e_* R \arrow[r, "\phi"'] & R \end{tikzcd}$$

where vertical maps are evaluation-at-1 and the horizontal map ν applied to $\psi : F^e_* S \longrightarrow F^e_* R$ is defined to be the composition:

$$S \hookrightarrow F^e_* S \xrightarrow{F^e_* \psi} F^e_* R \xrightarrow{\phi} R.$$

We observe that $\phi(\psi(F^e_* 1_S)) = (\nu(\psi))(1_S)$ and so the diagram commutes. Now, since $S \cong \mathrm{Hom}_R(S, R)$ (with generator T) we may identify this diagram with a diagram:

$$\begin{tikzcd} F^e_* S \arrow[r, "\phi'"] \arrow[d, "F^e_* T"'] & S \arrow[d, "T"] \\ F^e_* R \arrow[r, "\phi"'] & R \end{tikzcd}$$

and so we have constructed ϕ' the T-transpose of ϕ (even though we do not know S is a domain or even reduced).

Since $T(\mathfrak{m}_S) \subseteq \mathfrak{m}$ (see Proposition 9.4 in Appendix B), if ϕ' is not surjective, its image lies in $\mathfrak{m}_s$, and so $T \circ \phi'$ is not surjective. But $T \circ \phi' = \phi \circ F^e_* T$ and the latter is a composition of surjective maps, a contradiction.

This proves that S is Frobenius split and hence reduced.

For the second part, we already know that S is reduced, and hence we can find a nonzero $c \in R$ whose image in S is also a strong test element for the finite extension S. Fix $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* c) = 1$. As above, the T-transpose ϕ' exists. If $\phi'(F^e_* c)$ is a unit, we are done so we may assume $\phi'(F^e_* c) \in \mathfrak{m}_S$. Thus, as above, we see that $\mathfrak{m} \supseteq T(\mathfrak{m}_S)$ contains $T(\phi'(F^e_* c)) = \phi(F^e_* T(c)) = \phi(F^e_* c) = 1$, a contradiction. The result follows. $\square$

[18]by Appendix B (a)

The proof above actually proves something stronger.

Corollary 7.29 ([**CR22**])**.** *Suppose $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ is a finite extension of Noetherian F-finite normal local rings. Further suppose that there exists $T : S \longrightarrow R$ satisfying the following.*

(a) $T(\mathfrak{n}) \subseteq \mathfrak{m}$
(b) T *generates* $\mathrm{Hom}_R(S, R)$ *as an S-module.*
(c) $T(S) = R$ *(that is, T is surjective).*

Then if R is F-split (respectively strongly F-regular) then so is R.

In the case that $T = \mathrm{Tr}$, the second condition is simply that $\mathrm{Ram} = 0$, and so our corollary becomes a variant of Corollary 7.21. If R is strongly F-regular, then the third condition is implied by the second, see Exercise 7.12.

7.6. Application to F-jumping numbers.

In Chapter 4 Theorem 6.14 we proved that in a quasi-Gorenstein ring, that the F-jumping numbers of (R, f^t) are all rational and have no accumulation points. In this section, we generalize this to the case when R is $\mathbb{Q}$-Gorenstein.

We first make the following definition.

Definition 7.30. Suppose that R is a Noetherian F-finite normal domain, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor, and $0 \neq g \in R$. We say a number $t > 0$ is an **F-jumping number for** (R, Δ, g) if

$$\tau(R, \Delta + (t - \epsilon)\,\mathrm{div}(g)) \neq \tau(R, \Delta + t\,\mathrm{div}(g))$$

for all $1 \gg \epsilon > 0$.

Theorem 7.31. *Suppose that R is a Noetherian F-finite normal domain, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier, and $0 \neq g \in R$. Then the set of F-jumping numbers for (R, Δ, g) is a set of rational numbers with no limit points.*

Setting $\Delta = 0$, this generalizes Chapter 4 Theorem 6.14 to the $\mathbb{Q}$-Gorenstein case.

We prove this in two cases. The first is when $\Delta = 0$ and the index of the $\mathbb{Q}$-Cartier divisor K_R is not divisible by p, as this case is immediate.

Proof of Theorem 7.31 when $\Delta = 0$ and $p \not| \, \mathrm{ind}(K_R)$. Since R is $\mathbb{Q}$-Gorenstein, we can use Appendix B Proposition 9.4 (e) to find a cyclic cover $S \supseteq R$ that is quasi-Gorenstein (a **canonical cover**). Since the index of K_R is not divisible by p, we see that S remains normal by Appendix B Lemma 9.2 (c).

Since $R \subseteq S$ is étale-in-codimension-1 (Appendix B Lemma 9.2 (e)) and so $\mathrm{Tr} \in \mathrm{Hom}_R(S, R)$ is an S-module generator, we see that

$$\mathrm{Tr}(\tau(S, f^* t \operatorname{div}(g))) = \tau(R, t \operatorname{div}(g))$$

for all $t > 0$ by Theorem 7.19. We know that the F-jumping numbers of $\tau(S, tf^* \operatorname{div}(g)) = \tau(S, g^t)$ are discrete without limit points by Chapter 4 Theorem 6.14 and thus the same holds for their Tr-images in R. $\square$

To prove the general case, we recall the following result which was an exercise in Chapter 4 and is convenient for our purpose (rephrased since $\tau(\omega_S, g^t) = \tau(\omega_S, t \operatorname{div}(g))$ by Exercise 6.10).

Theorem 7.32 (Chapter 4 Exercise 6.17)**.** *Suppose that S is a Noetherian normal F-finite domain and $g \in S$ is nonzero, then the set of numbers $t > 0$ such that*

$$\tau(\omega_S, (t - \epsilon) \operatorname{div}(g)) \neq \tau(\omega_S, t \operatorname{div}(g))$$

is a set of rational numbers without accumulation points.

Our proof is now quite similar in spirit to the one we gave in the special case above.

Proof of Theorem 7.31, general case. Fix a canonical divisor $K_R \geq 0$ (that is, fix an embedding $R \subseteq \omega_R$). By Appendix B Lemma 9.5 there exists a finite generically separable extension $R \subseteq S$ normal domains such that if f is the induced map on Specs, we have that $f^*(K_R + \Delta) = \operatorname{div}_S(h)$ is Cartier.

We know by Theorem 6.14

$$\tau(R, \Delta + t \operatorname{div}(g)) = \tau(\omega_R, K_R + \Delta + t \operatorname{div}(g))$$

and so it suffices to show that $\tau(\omega_R, K_R + \Delta + t \operatorname{div}(g))$ has jumps only at rational numbers without accumulation points.

Next we know that

$$\begin{aligned} \tau(\omega_R, K_R + \Delta + t \operatorname{div}(g)) = {} & T(\tau(\omega_S, f^*(K_R + \Delta + t \operatorname{div}(g)))) \\ = {} & T(\tau(\omega_S, \operatorname{div}_S(h) + t \operatorname{div}_S(g))) \\ = {} & T(h\, \tau(\omega_S, t \operatorname{div}_S(g))). \end{aligned}$$

The result follows by Theorem 7.32 above. $\square$

7.7. Exercises.

Exercise 7.1. Find an example of an inseparable extension of fields $K \subseteq L$ so that the zero map $\phi : F^e_* K \longrightarrow K$ extends to a non-zero map $\phi : F^e_* L \longrightarrow L$.

Exercise 7.2. Suppose that k is a perfect field of characteristic $p > 0$ and $n > 0$ is an integer relatively prime to p and consider the ring extension $R = k[\![x]\!] \subseteq k[\![x^{1/n}]\!] = S$. Classify, up to pre-multiplication by units, which maps $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ extend to maps $F_* S \longrightarrow S$.

Exercise 7.3. Suppose that $R \subseteq S$ is a finite generically separable extension of normal domains. For any radical ideal $I \subseteq R$, show that $\mathrm{Tr}(\sqrt{IS}) \subseteq I$. In particular, if $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ is a finite local extension of normal domains, then $\mathrm{Tr}(\mathfrak{n}) \subseteq \mathfrak{m}$.

Hint: First reduce to the case where $\mathcal{K}(S)$ is Galois over $\mathcal{K}(R)$ by taking the normalization of R in the Galois closure of $\mathcal{K}(S)$. In that case, use the fact that $\mathrm{Tr}(y) = \sum_{\sigma \in G} \sigma(y)$ where $G = \mathrm{Gal}(\mathcal{K}(S)/\mathcal{K}(R))$.

Exercise 7.4. Suppose that $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ is a finite extension of F-finite normal local domains and that you are given a surjective $T \in \mathrm{Hom}_R(S, R)$. Further suppose that $T(\mathfrak{n}) \subseteq \mathfrak{m}$. Suppose that (R, Δ) is a pair where $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $(p^e - 1)(K_R + \Delta)$ is Cartier and that $f^*\Delta - D_T$ is effective. Show that if $(S, f^*\Delta - D_T)$ is F-split if and only if (R, Δ) is F-split. Note this works even in the case of an inseparable $R \subseteq S$.

Exercise 7.5. Given an example to show that, without the hypothesis that Tr is surjective, that Proposition 7.17 is false.

Hint: You can find an example in [**ST10a**] if you get stuck.

Exercise 7.6. Suppose $R \subseteq S$ is a finite, split, and étale in codimension 1 extension of F-finite normal domains and R is Frobenius split (respectively strongly F-regular). Prove that S is Frobenius split (respectively strongly F-regular).

Exercise 7.7. Complete the proof of the general case of the $\subseteq$ containment for Theorem 7.19.

Hint: Show that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_S(F^e_* S, S) \arrow[r] \arrow[d, "\nu"'] & S \arrow[d, "T"] \\ \mathrm{Hom}_R(F^e_* R, R) \arrow[r] & R \end{tikzcd}$$

where ν is induced by T on the second coordinate and by the inclusion $F^e_* R \subseteq F^e_* S$ on the first. Let $\Delta_S = f^*\Delta - D_T$. Then show that ν sends $\mathrm{Hom}_S(F^e_* S(\lfloor (p^e - 1)\Delta_S \rfloor), S)$ into $\mathrm{Hom}_R(F^e_* R, (\lfloor (p^e - 1)\Delta \rfloor), R)$.

Exercise 7.8. Suppose that R is a normal Noetherian domain and Γ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $\operatorname{Spec} R$. Show that there exists a finite extension $R \subseteq S$ where S is normal and where $f^*\Gamma$ is Cartier where $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is the induced map.

Exercise 7.9. Suppose that R is an F-finite Noetherian domain and choose $K_R \geq 0$. Suppose that $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier, say with $\operatorname{div}(f) = n(K_R + \Delta)$. Prove that there exists a finite extension $R \subseteq S$ containing $f^{1/n}$ so that

$$\tau(R, \Delta) = T(f^{1/n} \cdot \tau(\omega_S))$$

where $T : \omega_S \longrightarrow \omega_R = R(K_R) \supseteq R$ is the Grothendieck dual of the inclusion $R \subseteq S$ (again see Appendix C Proposition 5.6).

Exercise 7.10. Suppose that R is a DVR and $R \subseteq S$ is a finite extension with S a normal domain. Prove that $\operatorname{Tr} : S \longrightarrow R$ surjects if and only if there exists some maximal ideal Q of S such that $R \subseteq S_Q$ is tame.

Hint: One approach is can reduce to the complete case, in which case S becomes a product of domains and the Tr map is the sum of the traces. Then each $R \subseteq S_Q$ is finite.

Exercise 7.11. Fix a base perfect field k. Suppose R is the section ring over an ordinary elliptic curve E with respect to an ample A. Let $f : E' \longrightarrow E$ be an étale p-to-1 cover (see for instance [**Sil09**, Section V.3]) and let R' denote the section ring of E' with respect to f^*A. If X (respectively X') is the blowup of the irrelevant ideal of $\operatorname{Spec} R$ (respectively $\operatorname{Spec} R'$), show that $X' \longrightarrow X$ is étale but that $\operatorname{Tr} : R' \longrightarrow R$ is not surjective. This example, taken from [**KS10**], shows that a divisorial notion of tameness does not imply stronger notions of tameness.

Hint: For the second statement, notice that $H^0(E', \mathcal{O}_{E'}) \xrightarrow{\operatorname{Tr}} H^0(E, \mathcal{O}_E)$ is zero by showing it is multiplication by p on k.

Exercise 7.12. Suppose that $R \subseteq S$ is a finite extension of Noetherian domains with R strongly F-regular. Suppose that $T \in \operatorname{Hom}_R(S, R)$ generates $\operatorname{Hom}_R(S, R)$ as an S-module. Prove that $T(S) = R$.

Hint: Use Exercise 6.20 in Chapter 1

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CHAPTER 6

Frobenius and connections with characteristic zero

In this chapter we study the relations between F-split, strongly F-regular, F-rational and F-injective rings (as well as the test ideal and test module), and their characteristic zero counterparts coming out of birational algebraic geometry. This is summarized in the following diagram.

$$\begin{tikzcd}
\text{Kawamata Log Terminal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[rr, Leftrightarrow, bend left, red] & \text{Rational} \arrow[l, Rightarrow, bend right, "+\text{Gor.}"'] \arrow[d, Rightarrow] \arrow[rr, Leftrightarrow, bend left, red] & \text{Strongly } F\text{-Regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-Rational} \arrow[l, Rightarrow, bend right, "+\text{Gor.}"'] \arrow[d, Rightarrow] \\
\text{Semi Log Canonical} \arrow[r, Rightarrow] \arrow[rr, Leftrightarrow, bend right, red] & \text{Du Bois} \arrow[l, Rightarrow, bend left, "+\text{quasi-Gor.}"] \arrow[rr, Leftrightarrow, bend right, red] & F\text{-Split} \arrow[r, Rightarrow] & F\text{-Injective} \arrow[l, Rightarrow, bend left, "+\text{quasi-Gor.}"] \\
\text{Multiplier Ideals} \arrow[rrr, Leftrightarrow, red] & & & \text{Test Ideals}
\end{tikzcd}$$

In the above diagram, the red arrows correspond to reduction modulo p, which is the first topic of this chapter.

1. Reduction modulo p

Reduction to prime characteristic is an ancient technique, with roots as old as the Chinese Remainder Theorem.[1] In an undergraduate algebra course, students learn how to verify that a monic polynomial with integer coefficients is irreducible over $\mathbb{Q}$ by checking irreducibility after reducing its coefficients modulo some prime. Much more substantially, there are important theorems about complex varieties, such as Mori's *Bend and Break* theorem [**Mor79**], whose proof relies on reduction to characteristic p.

[1]at least as old as the third century where its use appeared in the book *Sunzi Suanjing* by the Chinese mathematician Sunz.

In this section, we explain how one can view any algebraic variety[2] over a field of characteristic zero as one member of a family of related varieties over fields of different *finite* characteristics, and how this can be used to verify or understand basic geometric properties such as reducedness, Cohen-Macaulayness, smoothness, etc.

1.1. The main idea in a simple case. Let us first discuss the idea of reduction to characteristic p in its simplest setting. Given some finitely generated $\mathbb{Q}$-algebra $R_{\mathbb{Q}}$, we can choose a presentation

$$R_{\mathbb{Q}} = \frac{\mathbb{Q}[x_1, \dots, x_n]}{(f_1, \dots, f_m)} \tag{1.0.1}$$

where, without loss of generality, the polynomials f_i have *integer* coefficients. Thus we can define a $\mathbb{Z}$-algebra

$$R_{\mathbb{Z}} = \frac{\mathbb{Z}[x_1, \dots, x_n]}{(f_1, \dots, f_m)} \tag{1.0.2}$$

together with the corresponding **family of models**

$$\operatorname{Spec} R_{\mathbb{Z}} \longrightarrow \operatorname{Spec} \mathbb{Z} \tag{1.0.3}$$

corresponding to the natural ring map $\mathbb{Z} \longrightarrow R_{\mathbb{Z}}$. In this way, we can think of $R_{\mathbb{Q}}$ as just one of many of the different members (fibers) of this family: the fiber over the generic point is our original "characteristic zero" affine scheme

$$\operatorname{Spec} \mathbb{Q} \times \operatorname{Spec} R_{\mathbb{Z}} \quad = \quad \operatorname{Spec} R_{\mathbb{Q}},$$

while the fiber over a closed point $(p) \in \operatorname{Spec} \mathbb{Z}$ is the spectrum of the characteristic p ring

$$R_{\mathbb{F}_p} := \mathbb{F}_p \otimes_{\mathbb{Z}} R_{\mathbb{Z}} \cong \frac{\mathbb{F}_p[x_1, \dots, x_n]}{(f_1, \dots, f_m)}. \tag{1.0.4}$$

(Here, and elsewhere in this chapter, the notation f_i denotes a polynomial in $\mathbb{Z}[x_1, \dots, x_n]$, as well as its image in $\mathbb{Q}[x_1, \dots, x_n]$ and $\mathbb{F}_p[x_1, \dots, x_n]$, depending on the context.)

The idea of "reduction to prime characteristic" is that, often, we can infer nice properties of the generic fiber from the same—or perhaps some analogous—nice property of *most* of the closed fibers.[3]

To see how this works in practice, we state and carefully prove the following theorem reflecting this general principle for the property of *Cohen-Macaulayness:*

---

[2]or any finitely generated algebra

[3]We can not expect good properties to be inherited by *all* fibers in general. For example, the algebra $k[x, y, z]/(x^3 + y^3 + z^3)$ is obviously quite special and different when k has characteristic three than in any other characteristic, including characteristic zero.

Theorem 1.1. *Let $R_{\mathbb{Q}}$ be a domain finitely generated over $\mathbb{Q}$ as in (1.0.1), and let*

$$\operatorname{Spec} R_{\mathbb{Z}} \longrightarrow \operatorname{Spec} \mathbb{Z} \tag{1.1.1}$$

be a family of models as in (1.0.3). Then the ring $R_{\mathbb{Q}}$ is Cohen-Macaulay if and only if, for all but finitely many p, the characteristic p model

$$R_{\mathbb{F}_p} := \mathbb{F}_p \otimes_{\mathbb{Z}} R_{\mathbb{Z}} \tag{1.1.2}$$

is Cohen-Macaulay. In other words, the property of Cohen-Macaulayness for fibers of the map (1.0.3) is open in $\operatorname{Spec}\mathbb{Z}$.

Before proving Theorem 1.1, we introduce a more uniform (and more general) notation: for any finitely generated $\mathbb{Z}$-algebra $R_{\mathbb{Z}}$ and an arbitrary ring L, let

$$R_L := L \otimes_{\mathbb{Z}} R_{\mathbb{Z}}.$$

Its spectrum can be considered as the fiber over the "point" $\operatorname{Spec} L \longrightarrow \operatorname{Spec}\mathbb{Z}$. When $L = \mathbb{Q}$ or $L = \mathbb{F}_p$, we recover the ring $R_{\mathbb{Q}}$ or $R_{\mathbb{F}_p}$ above.

To prove Theorem 1.1, we use the fact that R_L is Cohen-Macaulay if and only if its dualizing complex $\omega^{\bullet}_{R_L}$ has non-trivial cohomology in exactly one cohomological degree; see Lemma 3.14 in Appendix C. The point is that we can construct a dualizing complex $\omega^{\bullet}_{R_{\mathbb{Z}}}$ for the $\mathbb{Z}$-algebra $R_{\mathbb{Z}}$ with the property that *after inverting a single non-zero integer* b, for all regular $\mathbb{Z}[b^{-1}]$-algebras L, we have

(i) An isomorphism $L \otimes_{\mathbb{Z}} \omega^{\bullet}_{R_{\mathbb{Z}}} \cong \omega^{\bullet}_{R_L}$; and
(ii) The cohomology of the complex $\omega^{\bullet}_{R_{\mathbb{Z}[b^{-1}]}}$ is free over $\mathbb{Z}[b^{-1}]$, and commutes with base change to L—that is, $L \otimes_{\mathbb{Z}} H^i(\omega^{\bullet}_{R_{\mathbb{Z}}}) \cong H^i(\omega^{\bullet}_{R_L})$ for all i.

In particular, the dualizing complex $\mathbb{Q} \otimes_{\mathbb{Z}} \omega^{\bullet}_{R_{\mathbb{Z}}}$ for $R_{\mathbb{Q}}$ will have one non-trivial cohomology group if and only if, for all primes p except possibly those dividing the integer b, the dualizing complex

$$\mathbb{F}_p \otimes_{\mathbb{Z}} \omega^{\bullet}_{R_{\mathbb{Z}}} \cong \omega^{\bullet}_{R_{\mathbb{F}_p}} \tag{1.1.3}$$

has one non-trivial cohomology group. In other words, $R_{\mathbb{Q}}$ is Cohen-Macaulay if and only if $R_{\mathbb{F}_p}$ is Cohen-Macaulay for all but finitely many primes p.

Both points (i) and (ii) follow from the very important *Lemma of Generic Freeness*, which we state in a simple case:

Lemma 1.2. *Let A be a Noetherian domain, and let R_A be a finitely generated A-algebra. For any finite set*[4] *$M_1, \ldots, M_r$ of finitely generated R_A*

[4]equivalently, we can consider just one finitely generated R_A-module $M_1 \oplus \cdots \oplus M_r$

modules, there exists a non-zero $b \in A$ such that the $R_{A[b^{-1}]}$-modules

$$A[b^{-1}] \otimes_A M_1, \ldots, A[b^{-1}] \otimes_A M_r$$

are free over $A[b^{-1}]$.

Assuming this lemma of generic freeness for a moment (see Theorem 1.8 for the proof), let us now prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let $S_{\mathbb{Z}}$ be the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ so that $R_{\mathbb{Z}}$ is the quotient of $S_{\mathbb{Z}}$ by the ideal generated by $f_1, \ldots, f_m$. Remembering that

$$\omega^{\bullet}_{R_{\mathbb{Z}}} = \mathbf{R}\operatorname{Hom}_{S_{\mathbb{Z}}}(R_{\mathbb{Z}}, S_{\mathbb{Z}}),$$

we construct an explicit (non-normalized) dualizing complex for $R_{\mathbb{Z}}$ by taking a free $S_{\mathbb{Z}}$-module resolution

$$F^{\bullet}_{\mathbb{Z}} := \ 0 \longrightarrow F^{-t} \longrightarrow F^{-t+1} \longrightarrow \ldots \longrightarrow F^0 \longrightarrow 0$$

of the $S_{\mathbb{Z}}$-module $R_{\mathbb{Z}}$ by finitely generated free $S_{\mathbb{Z}}$-modules, and then applying the functor $\operatorname{Hom}(-, S_{\mathbb{Z}})$. Thus

$$\omega^{\bullet}_{R_{\mathbb{Z}}} = \operatorname{Hom}_{S_{\mathbb{Z}}}(F^{\bullet}_{\mathbb{Z}}, S_{\mathbb{Z}}) \tag{1.2.1}$$

is a (non-normalized) dualizing complex for $R_{\mathbb{Z}}$.

Of course, for any *flat* map $\mathbb{Z} \longrightarrow L$, base change produces

$$L \otimes_{\mathbb{Z}} \omega^{\bullet}_{R_{\mathbb{Z}}} \xrightarrow{\cong} \operatorname{Hom}_{L \otimes S_{\mathbb{Z}}}(L \otimes_{\mathbb{Z}} F^{\bullet}_{\mathbb{Z}}, L \otimes_{\mathbb{Z}} S_{\mathbb{Z}}) = \operatorname{Hom}_{S_L}(F^{\bullet}_L, S_L),$$

which is a (non-normalized) dualizing complex $\omega^{\bullet}_{R_L}$ for R_L (provided that L a regular domain[5]). But the residue map $\mathbb{Z} \longrightarrow \mathbb{F}_p$ is decidedly non-flat, and the natural map

$$L \otimes_{\mathbb{Z}} \operatorname{Hom}_{S_{\mathbb{Z}}}(F^{\bullet}_{\mathbb{Z}}, S_{\mathbb{Z}}) \longrightarrow \operatorname{Hom}_{S_L}(F^{\bullet}_L, S_L) \tag{1.2.2}$$

is not necessarily isomorphism for an *arbitrary* base change $\mathbb{Z} \longrightarrow L$.

On the other hand, if $R_{\mathbb{Z}}$ is *free* over $\mathbb{Z}$, then for any ring L, the complex $L \otimes_{\mathbb{Z}} F^{\bullet}_{\mathbb{Z}}$ is a resolution of R_L over S_L—the point is that the relevant maps in $F^{\bullet}_{\mathbb{Z}}$ are *all split over* $\mathbb{Z}$ in this case (see Exercise 1.1). Furthermore, if each of the cohomology modules of the complex $\operatorname{Hom}_{S_{\mathbb{Z}}}(F^{\bullet}_{\mathbb{Z}}, S_{\mathbb{Z}})$ is *free* over $\mathbb{Z}$, then the map (1.2.2) is an isomorphism for *any* L (again by Exercise 1.1). Although we can not expect this freeness over $\mathbb{Z}$ to hold in general, by the Lemma of Generic Freeness (Lemma 1.2), we can assume that there is a non-zero integer b such that, *after tensoring with* $\mathbb{Z}[b^{-1}]$, both $R_{\mathbb{Z}}$ and each of the cohomology groups of $\operatorname{Hom}_{S_{\mathbb{Z}}}(F^{\bullet}_{\mathbb{Z}}, S_{\mathbb{Z}})$ are *free* over $\mathbb{Z}[b^{-1}]$. So the map

[5] or a Gorenstein domain, so that S_L is Gorenstein; see Lemma 3.7 or Theorem 4.3 in Appendix C

(1.2.2) becomes an **isomorphism** if we replace $\mathbb{Z}$ by $\mathbb{Z}[b^{-1}]$, or equivalently, if we assume that b is invertible in L under the natural map $\mathbb{Z} \longrightarrow L$.

When L is a field, the isomorphism (1.2.2) yields an isomorphism[6]

$$L \otimes_{\mathbb{Z}} \omega^{\bullet}_{R_{\mathbb{Z}}} \cong \omega^{\bullet}_{R_L}$$

provided that b is invertible in L. In particular, for all primes p not dividing b, we have

$$\mathbb{F}_p \otimes_{\mathbb{Z}} \omega^{\bullet}_{R_{\mathbb{Z}}} \cong \omega^{\bullet}_{R_{\mathbb{F}_p}}.$$

Futhermore, because each $H^i(\omega^{\bullet}_{R_{\mathbb{Z}}})$ becomes *free* over $\mathbb{Z}[b^{-1}]$ after tensoring with $\mathbb{Z}[b^{-1}]$, the natural maps

$$L \otimes_{\mathbb{Z}} H^i(\omega^{\bullet}_{R_{\mathbb{Z}}}) \longrightarrow H^i(L \otimes_{\mathbb{Z}} \omega^{\bullet}_{R_{\mathbb{Z}}})$$

are isomorphisms for all $\mathbb{Z}[b^{-1}]$-algebras L and all indices i (Exercise 1.1 yet again).

Now that we have created the desired complex $\omega^{\bullet}_{R_{\mathbb{Z}}}$ satisfying (i) and (ii), we conclude that for any field L, the rank of the free L-module $H^i(\omega^{\bullet}_{R_L})$ is independent of L, as long as b is invertible in L. In particular, $H^i(\omega^{\bullet}_{R_{\mathbb{Q}}}) = 0$ if and only if $H^i(\omega^{\bullet}_{R_{\mathbb{F}_p}}) = 0$ for any p not dividing b. This shows that— *assuming the Lemma on Generic Freeness*— the ring $R_{\mathbb{Q}}$ is Cohen-Macaulay if and only if $R_{\mathbb{F}_p}$ is Cohen-Macaulay for infinitely many p. $\square$

Remark 1.3. Different presentations of R as a finitely generated $\mathbb{Q}$-algebra will produce different algebras $R_{\mathbb{Z}}$ and perhaps different finite sets of primes p for which the map (1.2.2) might fail to be an isomorphism. This does not affect the proof of Theorem 1.1.

1.2. Adapting to finitely generated algebras over $\mathbb{C}$. The reader will easily discern that, other than defining the family of modules $\mathbb{Z} \longrightarrow R_{\mathbb{Z}}$, the fact that $R_{\mathbb{Z}}$ is finitely generated over $\mathbb{Z}$ was not particularly essential in the proof of Theorem 1.1. The entire argument adapts easily to any finitely generated algebra over any field k of characteristic zero.

Indeed, given some finitely generated k-algebra R_k, we can choose a presentation

$$R_k = \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_m)} \tag{1.3.1}$$

where the polynomials $f_i \in k[x_1, \dots, x_n]$. Each of the finitely many polynomials f_i has finitely many coefficients λ_{ij} from the field k. We can then

[6]As $\omega^{\bullet}_R = R\operatorname{Hom}_S(R, S)$ when S is Gorenstein; see Lemma 3.7 or Theorem 4.3 in Appendix C

define a finitely generated $\mathbb{Z}$-algebra

$$A = \mathbb{Z}[\{\lambda_{ij}\} \mid i = 1, \ldots, m] \subseteq k \tag{1.3.2}$$

so that the polynomials f_i can be viewed as elements of the ring $A[x_1, \ldots, x_n]$. This allows us to define a finitely generated A-algebra

$$R_A = \frac{A[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)} \tag{1.3.3}$$

together with the corresponding **family of models**

$$\operatorname{Spec} R_A \longrightarrow \operatorname{Spec} A \tag{1.3.4}$$

arising from the natural ring map $A \longrightarrow R_A$. The base ring A is called a **coefficient ring** for R_k.

Remark 1.4. The coefficient ring A may be assumed *regular*. Indeed, because the regular locus of a domain finitely generated over $\mathbb{Z}$ is a non-empty open[7] set, we can invert one element of A to get a regular ring. Then replacing A by this localization, we can assume that A is regular.

Likewise, by adjoining the inverse of another non-zero element of A if necessary, we may assume that R_A is *free* over A (Lemma 1.2). In particular, with no loss of generality, we may assume $\operatorname{Spec} R_A \longrightarrow \operatorname{Spec} A$ is a flat family.

As before, the base change $A \hookrightarrow k$ is flat, and the fiber over the "point" $\operatorname{Spec} k \longrightarrow \operatorname{Spec} A$ is the spectrum of the original k-algebra

$$k \otimes_A R_A = R_k.$$

For a maximal ideal $\mu \subseteq A$, the base change $A \longrightarrow A/\mu$ is *not flat*, but as we range over the different maximal ideals of A, we get finite fields of different positive characteristics (see Exercise 1.6). Each closed fiber

$$A/\mu \otimes_A R_A$$

is a finitely generated algebra $R_{A/\mu}$ over some *finite* field A/μ. Again, using the Lemma of Generic Freeness, we easily prove:

Theorem 1.5. *Let R_k be a domain finitely generated over a field k as in (1.3.1), let A be a choice of coefficient ring for R_k as in (1.3.2), and let*

$$\operatorname{Spec} R_A \longrightarrow \operatorname{Spec} A \tag{1.5.1}$$

be a family of models as in (1.5.1). Then the ring R_k is Cohen-Macaulay if and only if for all μ in some open set of m-Spec$A$*, the prime characteristic model*

$$R_{A/\mu} := A/\mu \otimes_A R_A \tag{1.5.2}$$

is Cohen-Macaulay.

[7]This is because finitely generated $\mathbb{Z}$-algebras are excellent [**Sta19**, Tag 07QW], so their regular locus is open.

The proof essentially the same as before:

PROOF. With notation as above, assume without loss of generality that the coefficient ring A is regular (Remark 1.4). Let $F_A^\bullet$ be a free resolution for R_A as a module over $S_A = A[x_1, \ldots, x_n]$. Define the complex

$$\omega_{R_A}^\bullet = \mathrm{Hom}_{S_A}(F_A^\bullet, S_A), \tag{1.5.3}$$

which is a (non-normalized) dualizing complex for R_A (by Lemma 3.7 or Theorem 4.3 in Appendix C). By the Lemma of Generic freeness (Lemma 1.2), there is some non-zero $b \in A$ so that R_A and the cohomology of $\omega_{R_A}^\bullet$ become free over $A[b^{-1}]$ after the base change to $A[b^{-1}]$. Then, arguing exactly as in the proof of Theorem 1.1, it follows that for all regular $A[b^{-1}]$-algebras L, there are natural isomorphisms

$$L \otimes_A H^i(\omega_{R_A}^\bullet) \longrightarrow H^i(L \otimes_A \omega_{R_A}^\bullet) \longrightarrow H^i(\omega_{R_L}^\bullet) \tag{1.5.4}$$

for all indices $i \geq 0$ (the key is Exercise 1.1 yet again). Furthermore, since $H^i(\omega_{R_{A[b^{-1}]}}^\bullet)$ is *free* over $A[b^{-1}]$, the rank of $H^i(\omega_{R_L}^\bullet)$ is independent of L. In particular, since we can take $L = k$, the dualizing complex $\omega_{R_k}^\bullet$ of R_k has one non-trivial cohomology group if and only if the dualizing complex of $R_{A/\mu}$ has one non-trivial cohomology group for all $\mu \in$ m-Spec$A$ in the open set of maximal ideals not containing b. $\square$

Remark 1.6. The coefficient ring A is highly non-unique—a different choice of presentation produces a different A; furthermore, we can always enlarge A by adjoining finitely many additional elements of k. None-the-less, if R_k is Cohen-Macaulay, our proof shows that, regardless of the choice of A, for an open set of maximal ideals $\mu \in \mathrm{Spec}\, A$, the prime characteristic model $R_{A/\mu}$ is Cohen-Macaulay as well, and conversely. In particular, there will be infinitely many different prime numbers p for which there exists a characteristic p model of R_k that is Cohen-Macaulay if and only if R_k itself is Cohen-Macaulay (see Exercise 1.4).

1.3. Modeling and flattening other features to prove other properties. Let k be a field of characteristic zero, and let R_k be a finitely generated k-algebra. We saw how to check Cohen-Macaulayness of R_k by viewing R_k as just one member of a flat family over some mixed characteristic base A over which sufficiently many relevant associated objects are *free*. More precisely, we created a family of models where, not only was the ring R_k "modeled" by some flat family $A \longrightarrow R_A$, but also some extra data—namely, the dualizing complex $\omega_{R_k}^\bullet$— was modeled over A. We deployed the Lemma of Generic Freeness to assume that (after inverting an element of A, if necessary) the relevant maps in the complex $\omega_{R_A}^\bullet$ all *split* over A, so that for *every* point P of $\mathrm{Spec}\, A$, the base change to the residue field $\kappa(P)$ of P

$$\kappa(P) \otimes_A \omega_{R_A}^\bullet \to \omega_{R_{\kappa(P)}}^\bullet$$

is an *isomorphism* and the cohomology commutes with base change. In particular, by restricting to closed points in $\operatorname{Spec} A$, we get **prime characteristic models** of the entire set-up for infinitely many different characteristics p.

In other contexts, we may want to model different features—such as a resolution of singularities, say—of a variety over a field of characteristic zero, or some map between cohomology groups. In various contexts, we may want to model finite-type information such as:

(a) a finite collection of ideals of R_k;
(b) a finite collection of finitely generated algebras over k, together with finitely many maps between them and/or finitely many ideals in them;
(c) a finite collection of projective schemes over these finitely generated k-algebras, and finitely many closed subschemes in them.
(d) a finite collection of finitely generated R_k-modules and finitely many maps between them.
(e) a finite collection of coherent sheaves of modules on the finitely many projective schemes in (c), as well as finitely many maps between them.

Because all the objects above are described in terms of finite-type information, we can model them (simultaneously!) over some finitely generated $\mathbb{Z}$-algebra A: we simply adjoin all the elements of k needed to describe a presentation for R_k as well as generators of the ideals in (a), presentations for the algebras and maps in (b), the graded algebras defining the projective schemes in (c) as well as the homogeneous ideals defining the subschemes, presentations for the modules and maps in (d), and presentations for the finitely generated graded modules and maps between them for the coherent sheaves in (e). All this can be done with finitely many elements from k.

For example, say that M_k is a finitely generated module over R_k. Fix a presentation

$$R_k^N \xrightarrow{\alpha} R_k^d \longrightarrow M_k \longrightarrow 0$$

where α_k is a $d \times N$ matrix with entries in R_k. Adjoining enough elements of k to A to describe (polynomials representing) the entries of α, we can consider the entries of α to be in R_A. Define M_A be the cokernel of the map

$$R_A^N \xrightarrow{\alpha} R_A^d,$$

inverting an element of A (if needed, by Lemma 1.2) to make M_A free over A. Base change to k recovers the original module M_k over R_k, where as base change to A/μ for some maximal ideal μ of A produces a module $M_{A/\mu}$ over $R_{A/\mu}$—a **characteristic p model of M_k.**

In the presentations of the objects (a) through (e) above we need to refer only *finitely many* elements of k. So throwing them all into A, we can construct a finitely generated $\mathbb{Z}$-algebra A contained in k, together with a finitely generated *flat*[8] A-algebra R_A and

(i) a finite collection of ideals of R_A;
(ii) a finite collection of finitely generated algebras over A, together with finitely maps between them and/or finitely many ideals in them;
(iii) a finite collection of projective schemes over these finitely generated A-algebras, and finitely many closed subschemes in them.
(iv) a finite collection of finitely generated R_A-modules and finitely many maps between them.
(v) a finite collection of coherent sheaves of modules on the finitely many projective schemes in (c), as well as finitely maps between them.

Notation 1.7. We indicate the original objects over k with the subscript k—R_k, T_k, M_k, π_k, etc—and the corresponding objects over A with the subscript A—R_A, T_A, M_A, π_A, etc. We refer the objects over A as a **family of models** for the original objects or say that they are **spread out to mixed characteristic** from the original objects over k. For any A-algebra L, we denote the base change to L (that is, the result of applying the functor $L \otimes_A -$) with the subscript L—R_L, T_L, M_L, π_L, etc. When L is the residue field of a maximal ideal of A, we will call these objects **reductions to prime characteristic** of the original objects over k.

Replacing A by its localization at one nonzero element, we may assume that A is regular. The base change $A \longrightarrow k$ is flat and in all cases above, recovers the original collection of objects and data. Furthermore, we may assume that all the objects and cokernels of maps are *free* over A by Generic Freeness. To get this A-freeness in (1.1) in the discussion after Theorem 1.1, we need the following stronger form of Generic Freeness:

Theorem 1.8 (Hochster-Roberts Generic Freeness). *Let A be a Noetherian domain. Let $R_A \longrightarrow T_A$ be a map of finitely generated A-algebras, and suppose E_A is a finitely generated T_A-module. Let M_A be a finitely generated R_A-submodule of E_A and let N_A be a finitely generated A-submodule of E_A. Let $D_A = E_A/(M_A + N_A)$. Then there is a non-zero element $a \in A$ such that $A[a^{-1}] \otimes D_A$ is a free $A[a^{-1}]$-module.*

Proof. See [**Mat89**, Thm 24.1] □

Remark 1.9. Enough freeness over A implies that kernels and cohomology commute with *arbitrary* base change. For example, modeling an ideal $J_k \subseteq R_k$ with the ideal $J_A \subseteq R_A$, we can assume that the cokernel is *free* over A,

[8] in fact free!

so the inclusion $J_A \hookrightarrow R_A$ splits over A. Thus tensoring with any A-algebra L, the map

$$L \otimes_A J_A \hookrightarrow L \otimes_A R_A$$

also splits over A, and in particular is injective. In particular, we can identify $L \otimes_A J_A = J_L$ with an ideal in $L \otimes_A R_A = R_L$. Likewise, it is easy to see that cohomology commutes with base change in the presence of enough freeness; see Exercise 1.1.

When we have modeled a collection of data in this way, we call the map $A \longrightarrow R_A$ a **family of prime characteristic models** for the data. All objects in our data are defined over A and can be assumed *free* over A, courtesy of Generic Freeness (Theorem 1.8).

Remark 1.10. We can freely adjoin more elements to A if we discover we need them to model some additional auxiliary (finite-type) objects in the course of our study. By Generic Freeness, we can assume all objects are *free* over A and all cokernels of maps are free over A after inverting finitely many[9] elements of A.

Remark 1.11. Each time we invert an element of A to guarantee some freeness, we are effectively choosing a smaller open subset of m- Spec$A$ since m- Spec$A[a^{-1}] \subseteq$ m- Spec$A$ is an open dense set.

1.4. Some technical lemmas. We record some of the technical steps we used in the proof of Theorem 1.5 for future reference.

Lemma 1.12. *Let A be a Noetherian domain, and let R_A be a finitely generated A algebra. Given a map $\phi_A : M_A \longrightarrow N_A$ of finitely generated R_A-modules, or of finitely generated A-algebras, there exists non-zero $b \in A$ such that the following are equivalent*

- *(a) The map $\phi_{A[b^{-1}]}$ is injective (resp. surjective);*
- *(b) The map ϕ_L is injective (resp. surjective) for every $A[b^{-1}]$-algebra L.*

Proof. This follows easily from the lemma of Generic freeness (Lemma 1.2). The stronger version (Theorem 1.8) is needed for the statement about the surjectivity of algebra maps, since the cokernel of an algebra map is typically not an finitely generated module over a finitely generated A-algebra. $\square$

The following corollary is immediate:

[9] inverting several elements of A is the same as inverting their product

Corollary 1.13. *Let k be a field of characteristic zero and let R_k be a finitely generated k-algebra. Suppose that*

$$0 \longrightarrow M_k \longrightarrow N_k \longrightarrow Q_k \longrightarrow 0$$

is a sequence of R_k modules. This sequence is exact, if and only if, after spreading out to mixed characteristic, there exists a dense open set $U \subseteq$ m-Spec$A$ such that the induced

$$0 \longrightarrow M_{A/\mu} \longrightarrow N_{A/\mu} \longrightarrow Q_{A/\mu} \longrightarrow 0$$

is exact for all $\mu \in U$.

This corollary comes up often because of the following fact, again courtesy of Generic Freeness:

Lemma 1.14. *Let A be a Noetherian domain, and let R_A be a finitely generated flat A-algebra. Let $\mathscr{M}_A$ be a coherent sheaf, flat over A, on a flat A-scheme X_A projective over* $\operatorname{Spec} R_A$. *Then there is a non-zero $b \in A$ such that the natural maps*

$$L \otimes_A H^i(X_A, \mathscr{M}_A) \longrightarrow H^i(X_L, \mathscr{M}_L)$$

are isomorphisms for every $A[b^{-1}]$-algebra L. In particular,

$$A/\mu \otimes_A H^i(X_A, \mathscr{M}_A) \cong H^i(X_{A/\mu}, \mathscr{M}_{A/\mu})$$

for all maximal ideals μ in a dense open set of $\operatorname{Spec} A$.

Proof. Since $X_A \longrightarrow \operatorname{Spec} R_A$ is proper, the cohomology modules are finitely generated over R_A. So $H^i(X_A, \mathscr{M}_A)$ is finitely generated over R_A, and can be assumed free over A (after inverting an element of A) by the Lemma of Generic Freeness. The module $H^i(X_A, \mathscr{M}_A)$ can be computed from the Čech complex for a finite cover of X_A by open affine sets. If U is an open affine subset of X_A, then each $\mathscr{M}_A(U)$ is a finitely generated module over the ring $\mathcal{O}_{X_A}(U)$, which is a finitely generated A-algebra since X_A has finite type over A. Thus the Čech complex for a finite affine cover computing $H^i(X_A, \mathscr{M}_A)$ consists of finitely many modules, each finitely generated over a finitely generated A-algebra, as its terms are direct sums of finitely many terms of the form $\mathscr{M}_A(U)$ for different open affine sets U. Since there are only finitely many in total, we can invert *one* element of A to assume that all are free over A, by the Lemma of Generic Freeness. The desired isomorphism now follows from Exercise 1.1. $\square$

Corollary 1.15. *Let k be a field of characteristic zero, and let R_k be a finitely generated k algebra. Fix a projective scheme X_k over* $\operatorname{Spec} R_k$, *and let $\mathscr{M}_k$ be a coherent sheaf on X_k. Then $H^i(X_k, \mathscr{M}_k) = 0$ if and only if, after spreading out to mixed characteristic, there is a dense open set of maximal ideals μ of the coefficient ring A such that the prime characteristic model $H^i(X_{A/\mu}, \mathscr{M}_{A/\mu})$ is zero.*

We proved carefully that Cohen-Macaulayness can be checked by reduction to positive characteristic. The next proposition summarizes many properties—well beyond Cohen-Macaulayness— that can be checked by reduction to characteristic p:

Proposition 1.16. *Let k be a field of characteristic zero and let R_k be a finitely generated k algebra. Choose any finitely generated $\mathbb{Z}$-algebra A over which R_k is defined, so that $R_k \cong R_A \otimes_A k$. Then R_k satisfies any of the following properties if and only, for a dense set of maximal ideals $\mu \in \operatorname{Spec} A$, the ring $R_{A/\mu}$ has the same property:*

(a) is reduced;
(b) is regular;
(c) is normal;
(d) is Cohen-Macaulay;
(e) satisfies Serre's S_n condition;
(f) is R_n, meaning that the localization at each height n prime is regular;
(g) is unmixed;[10].
(h) is geometrically integral;[11].
(i) $\operatorname{Spec} R$ is geometrically connected;[12].
(j) is quasi-Gorenstein;[13] (respectively Gorenstein).
(k) is normal and $\mathbb{Q}$-Gorenstein of index m.
(l) has dimension m.

Proof. Invert a single non-zero element of A to assume that R_A is flat over A. Likewise, assume that A is regular and has trivial class group. Thus without loss of generality, A has all of the listed properties (except the one about dimension).

For statement (l) on dimension: see [**Sta19**, Tag 05F7].

Now let **P** be one of the properties (a) through (k), say **P**, listed above. We use the fact that if $A \longrightarrow R_A$ is a flat finite-type map of Noetherian rings, the set

$$\{Q \in \operatorname{Spec} A \mid \text{ the fiber ring } k(Q) \otimes_A R_A \text{ has property } \mathbf{P}\} \tag{1.16.1}$$

is a *constructible* subset of $\operatorname{Spec} A$. See [**GW10**, Appendix E] for list of properties, including (a) through (k) for which this holds, together with

[10] Meaning that every associated prime is minimal

[11] A ring finitely generated over a field L is geometrically integral if $R \otimes_L \overline{L}$ is a domain where $\overline{L}$ is the algebraic closure of L

[12] A scheme of finite type over a field L is geometrically connected if $\operatorname{Spec}(R \otimes_L \overline{L})$ is connected.

[13] This means that K_R is Cartier, or in other words that $R(K_R)$ is locally free. Recall that a ring is Gorenstein if it is Cohen-Macaulay and quasi-Gorenstein.

precise references to EGA where they are proved. Only property (k) is not on that list; this is left as an exercise for the reader Exercise 1.12.

Now, a constructible subset of an integral scheme is dense if and only if it contains the generic point, or equivalently, if and only if it contains an open set [**Sta19**, Tag 005K]. And because finitely generated $\mathbb{Z}$-algebras are Jacobson, any dense subset contains a dense subset of m- Spec$A$. Thus property $\mathbf{P}$ holds for the generic fiber $R_{\mathcal{K}(A)}$ (where $\mathcal{K}(A)$ is the fraction field of A) if and only if $\mathbf{P}$ holds for a dense set of closed fibers—that is, for a dense set of characteristic p models.

It remains only to check that the generic fiber $R_{\mathcal{K}(A)}$ has property $\mathbf{P}$ if and only if R_k does. All of the properties are preserved by faithfully flat base change, so all "ascend" from $R_{\mathcal{K}(A)}$ to $R_k \cong R_{\mathcal{K}(A)} \otimes_{\mathcal{K}(A)} k$. Likewise, again using that $R_{\mathcal{K}(A)} \hookrightarrow R_k$ is faithfully flat, all of the properties $\mathbf{P}$ "descend" from R_k to the generic fiber $R_{\mathcal{K}(A)}$. For references, see [**Sta19**, Tag 033D] when $\mathbf{P}$ is one of the properties reduced, Cohen-Macaulay and S_n, regular, R_n or normality. When $\mathbf{P}$ is geometrically connected or geometrically integral, this is part of the definition as $\mathcal{K}(A) \subseteq k$ is a field extension. When $\mathbf{P}$ is quasi-Gorenstein or $\mathbb{Q}$-Gorenstein, this follows from the fact the canonical module, and its reflexive powers, commute with this base change. For unmixedness, see [**Sta19**, Tag 0CUB]. $\square$

Remark 1.17. In fact, the properties properties (a)-(f),(j),(k) can all be assumed to hold for R_A, after replacing A by a localization at a single nonzero element. For most of these properties, this can even be directed deduced from the positive characteristic models. Indeed, inverting one more element of A if needed, we may assume that A is regular (and so is $\mathbf{P}$) and furthermore that all fibers have property $\mathbf{P}$. Now since the base and fibers of the flat map $\operatorname{Spec} R_A \longrightarrow \operatorname{Spec} A$ have property $\mathbf{P}$, we can conclude that the total space $\operatorname{Spec} R_A$ does too. For a reference, see [**Sta19**, Tag 0339] for S_n and [**Sta19**, Tag 033A] for R_n, whence the statements for normality (which is R_1 and S_2), regularity (which is R_n for all n) and reducedness (which is R_0 and S_1) all follow.

Remark 1.18. See Exercise 1.11 below for an example illustrating the need for the "geometric" hypothesis in in (h) and (i).

1.5. Reducing a local ring to characteristic $p > 0$. Let k be a field of characteristic zero. Suppose that $(D, \mathfrak{m})$ is a local ring obtained by localizing a finitely generated k-algebra R_k at a maximal ideal $\mathfrak{m}_k$. We can spread out the pair $(R_k, \mathfrak{m}_k)$ to mixed characteristic to get an ideal $\mathfrak{m}_A$ in the A-algebra R_A. The ideal $\mathfrak{m}_A$ is prime but not maximal, since $R_A/\mathfrak{m}_A \cong A$.

Note that for some $\mu \in$ m- Spec$A$, we may have that $\mathfrak{m}_A + \mu = R_A$. On the other hand for most $\mu \in$ m- Spec$A$, since $R/\mathfrak{m}$ has dimension 0 and is reduced, the ring $\frac{R_A}{\mathfrak{m}_A+\mu}$ is dimension zero and is reduced Proposition 1.16 (a). Thus $\mathfrak{m}_A + \mu$ is a finite intersection of maximal ideals. If $R_k/\mathfrak{m}_k$ is geometrically integral over k, then most $\mathfrak{m}_A + \mu$ consist of just one maximal ideal Proposition 1.16 (h).

Now defining the $D_{A/\mu}$ to be the semi-local (or local as appropriate) ring $R_{A/\mu}$ localized at $\mathfrak{m}_{A/\mu} = \mathfrak{m}_A + \mu$. This is a prime characteristic model of the local ring D.

1.6. Dense and open F-type and choice of model. Consider a family of prime characteristic models $A \longrightarrow R_A$ for a ring R_k finitely generated over a field k of characteristic zero.

An interesting phenomenon occurs when the closed fibers satisfy a property— such as strong F-regularity— that does not make sense in characteristic 0. This leads to the following definition.

Definition 1.19. Suppose that $\mathbf{P}$ is a property of Noetherian rings of prime characteristic (such as Frobenius splitting, strong F-regularity, F-rationality or F-injectivity). The characteristic zero ring R_k is said to have **dense P type** if, for some family of prime characteristic models $A \longrightarrow R_A$, there exists a dense subset $T \subseteq \text{m-Spec}A$ such that $R_{A/\mu}$ has property $\mathbf{P}$ for all $\mu \in T$.

Similarly, the ring R_k has **open P-type** if there exists a dense open subset $U \subseteq \text{m-Spec}A$ such that $R_{A/\mu}$ has property $\mathbf{P}$ for all $\mu \in U$.

Example 1.20. Let $R_{\mathbb{C}} = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$. The obvious family of prime characteristic models is the $\mathbb{Z}$-algebra $R_{\mathbb{Z}} = \mathbb{Z}[x, y, z]/(x^3 + y^3 + z^3)$. Reducing modulo p—that is, looking at the closed fibers—we have the $\mathbb{F}_p$-algebras $R_{\mathbb{F}_p} = \mathbb{F}_p[x, y, z]/(x^3 + y^3 + z^3)$. Because we have seen that for any finite field L, the algebra R_L is Frobenius split when p is congruent to 1 modulo 3 but *not* when p is congruent to 2 modulo 3, the ring $R_{\mathbb{C}}$ is *dense* Frobenius split type but not *open* Frobenius split type. See Subsection 1.1.1 in Chapter 4.

Remark 1.21. If R_k has dense F-rational or dense strongly F-regular type, then R_k is normal and Cohen-Macaulay by Proposition 1.16, and the fact that F-rational and strongly F-regular rings are always normal and Cohen-Macaulay. See Theorems 4.30 and 7.4 in Chapter 1.

Likewise, we have the following global definitions for schemes X_k over a field k on characteristic zero.

Definition 1.22. Suppose that $\mathbf{P}$ is a property of schemes of characteristic $p > 0$ (such as being globally Frobenius split or globally F-regular). We say that X_k has **dense P type** if for some choice of positive characteristic models $X_A \longrightarrow \operatorname{Spec} A$, there exists a dense subset $T \subseteq \text{m-}\operatorname{Spec} A$ such that $X_{A/\mu}$ has property $\mathbf{P}$ for all $\mu \in T$.

Similarly, X_k has **open P-type** if there exists a dense open subset $U \subseteq \text{m-}\operatorname{Spec} A$ such that $X_{A/\mu}$ has property $\mathbf{P}$ for all $\mu \in U$.

One can also make definitions for pairs (X, Δ) or $(R, \mathfrak{a}^t)$, analogously.

1.7. Independence of model. The choice of family of models is highly non-unique. We may begin by forming R_A over A but later discover we need to enlarge A. Or, we might have two apparently unrelated mixed characteristic models for R_k.

Fortunately, for the properties we study, whether or not a finite type scheme or variety of characteristic zero has dense (or open) $\mathbf{P}$ type is independent of the choice of family of models used to model it:

Proposition 1.23. *Fix a characteristic zero field k, and let R_k be a finitely generated k-algebra. Suppose that A_1 and A_2 are two different finitely generated $\mathbb{Z}$-algebras contained in k that each are the base of a free family of models $A_1 \longrightarrow R_{A_1}$ and $A_2 \longrightarrow R_{A_2}$ for R_k. Let $\mathbf{P}$ be one of the properties strongly F-regular, Frobenius split, F-injective or F-rational. Then a dense (respectively, open) set of fibers of $A_1 \longrightarrow R_{A_1}$ has property $\mathbf{P}$ if and only if a dense (respectively, open) set of fibers of $A_2 \longrightarrow R_{A_2}$ has property $\mathbf{P}$.*

PROOF. Let $A \subseteq k$ be the finitely generated $\mathbb{Z}$-algebra $A_1[A_2]$ (equivalently, $A_2[A_1]$) obtained by adjoining to $\mathbb{Z}$ the union of the two finite sets of elements in k used to construct A_1 and A_2. By Generic Freeness, for $i = 1, 2$, we may assume (after adjoining the inverse of some element of A_i Theorem 1.8) that A is free over A_i and the map $A_i \hookrightarrow A$ splits over A_i. So

$$R_{A_i} \hookrightarrow A \otimes_{A_i} R_{A_i} = R_A$$

is split injective (and free) as well. Because R_{A_i} is free over A_i, also R_A is free over A. It suffices to show that a dense (respectively open) set of the fibers of $A_i \longrightarrow R_{A_i}$ have property $\mathbf{P}$ if and only if a dense (respectively open) set of the fibers of $A \longrightarrow R_A$ have property $\mathbf{P}$.

For any $\mu \in \text{m-}\operatorname{Spec} A$, let μ_i be its contraction $\mu \cap A_i$ to $\text{m-}\operatorname{Spec} A_i$. The induced map

$$A_i/\mu_i \hookrightarrow A/\mu$$

is a finite separable extension of fields (as both are finite fields). So tensoring over A_i with R_{A_i},

$$A_i/\mu_i \otimes_{A_i} R_{A_i} \hookrightarrow A/\mu \otimes_{A_i} R_{A_i} \cong A/\mu \otimes_A (A \otimes_{A_i} R_{A_i}) \cong A/\mu \otimes_A R_A$$

produces a *split* finite étale extension

$$R_{A_i/\mu_i} \hookrightarrow R_{A/\mu}.$$

Thus for any of the properties $\mathbf{P}$ (strong F-regularity, Frobenius splitting, F-rationality, or F-injectivity), the ring R_{A_i/μ_i} has property $\mathbf{P}$ if and only if $R_{A/\mu}$ has property $\mathbf{P}$. See Proposition 5.8 and Exercise 5.11 in Chapter 1, as well as Exercise 1.15 and Exercise 1.16.

Finally, the inclusion $A_i \hookrightarrow A$ of finitely generated $\mathbb{Z}$-algebras induces a map $\operatorname{Spec} A \longrightarrow \operatorname{Spec} A_i$ whose image is constructible and dense since the zero ideal contracts to the zero ideal. So the image contains an open set of $\operatorname{Spec} A_i$. As both rings are finitely generated over $\mathbb{Z}$, there is an induced map

$$\text{m-}\operatorname{Spec} A \longrightarrow \text{m-}\operatorname{Spec} A_i$$

whose image contains an open set as well []. It follows that dense (respectively, open) $\mathbf{P}$ type for our family of models over A_i is the same as having dense (respectively, open) $\mathbf{P}$ type for our family of models over A. Thus the condition that a k-algebra R_k has dense (or open) $\mathbf{P}$-type is independent of the choice of models. $\square$

1.8. Exercises.

Exercise 1.1. Let $P^\bullet$ be right bounded complex of (not necessarily finitely generated) free modules over some commutative ring A

$$\dots P^{-t} \xrightarrow{\partial^t} P^{-t+1} \xrightarrow{\partial^{-t+1}} \dots \xrightarrow{\partial^{-2}} P^1 \xrightarrow{\partial^{-1}} P^0 \longrightarrow 0$$

with the property that $H^i(P^\bullet)$ are free for all i. Then for any commutative A-algebra L, the natural maps

$$L \otimes_{\mathbb{Z}} H^i(P^\bullet) \xrightarrow{\cong} H^i(L \otimes_{\mathbb{Z}} P^\bullet) \tag{1.23.1}$$

are isomorphisms for all $i \geq 0$.

Hint: Induce on i and use the fact that the complex breaks up into short exact sequences

$$0 \longrightarrow \ker \partial^i \longrightarrow P^i \longrightarrow \operatorname{im} \partial^i \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \operatorname{im} \partial^{i+1} \longrightarrow \ker \partial^i \longrightarrow H^i(P^\bullet) \longrightarrow 0.$$

Exercise 1.2. Let $R_{\mathbb{Z}}$ be a homomorphic image of the polynomial ring $S_{\mathbb{Z}} = \mathbb{Z}[x_1, \dots, x_n]$, and suppose that $F_{\mathbb{Z}}^\bullet$ is a free resolution of $R_{\mathbb{Z}}$ as an $S_{\mathbb{Z}}$-module. Assuming that $\mathbb{Z}[b^{-1}] \otimes_{\mathbb{Z}} R_{\mathbb{Z}}$ is a free $\mathbb{Z}[b^{-1}]$-module, show that for *any* commutative $\mathbb{Z}[b^{-1}]$-algebra L, the complex $L \otimes_{\mathbb{Z}} F_{\mathbb{Z}}^\bullet$ is a free resolution of R_L as an S_L-module (where $S_L = L[x_1, \dots, x_n]$ and $R_L = L \otimes_{\mathbb{Z}} R_{\mathbb{Z}}$).

Hint: Use Exercise 1.1.

Exercise 1.3. Let $R_{\mathbb{Z}}$ be a homomorphic image of the polynomial ring $S_{\mathbb{Z}} = \mathbb{Z}[x_1, \dots, x_n]$, and suppose that $F_{\mathbb{Z}}^{\bullet}$ is a free resolution of $R_{\mathbb{Z}}$ as an $S_{\mathbb{Z}}$-module. Suppose that both $R_{\mathbb{Z}}$ and the cohomology of the complex

$$\operatorname{Hom}_{S_{\mathbb{Z}}}(F_{\mathbb{Z}}^{\bullet}, S_{\mathbb{Z}})$$

become free (over $\mathbb{Z}[b^{-1}]$) after tensoring with $\mathbb{Z}[b^{-1}]$. Prove that the natural map

$$L \otimes_{\mathbb{Z}} \operatorname{Hom}_{S_{\mathbb{Z}}}(F_{\mathbb{Z}}^{\bullet}, S_{\mathbb{Z}}) \longrightarrow \operatorname{Hom}_{S_L}(F_L^{\bullet}, S_L) \tag{1.23.2}$$

is an isomorphism for all $\mathbb{Z}[b^{-1}]$-algebras L. Deduce that for all p not dividing b,

$$\mathbb{F}_p \otimes_{\mathbb{Z}} \omega_{R_{\mathbb{Z}}}^{\bullet} \cong \omega_{R_{\mathbb{F}_p}}^{\bullet}.$$

Hint: Use Exercise 1.1 and Exercise 1.2.

Exercise 1.4. Let A be a domain finitely generated over $\mathbb{Z}$ and contained in a field k of characteristic zero. Prove that if $\mathcal{U}$ is a non-empty open set of $\operatorname{maxSpec} R$, then among the fields $\{A/\mu \mid \mu \in \mathcal{U}\}$, there are fields of every positive characteristic p except for finitely many p.

Hint: For $b \in A \setminus \{0\}$, consider $bA \cap \mathbb{Z}$.

Exercise 1.5. Let M be a non-zero finitely generated module over a Noetherian ring. Prove that for all but finitely many primes p, M is a p-torsion free $\mathbb{Z}$-module.

Hint: Use the fact that M has finitely many associated primes in R.

Exercise 1.6. Let A be a finitely generated $\mathbb{Z}$-algebra. Prove that if $\mu \subseteq A$ is a maximal ideal, then the residue field A/μ is *finite.* Conversely, prove that if $Q \in \operatorname{Spec} A$ has a finite residue field, then Q is maximal.

Exercise 1.7. With notation as in Notation 1.7, suppose that R_i is a finite S-algebra (in other words, that R_i is finite as an S-module). Show that we may also choose $R_{A,i}$ to be a finite S_A-algebra.

Hint: Make sure to choose, among the relations presenting $R_{A,i}$, monic relations that make each algebra generator integral of $R_{A,i}$ integral over S_A.

Exercise 1.8. With notation as in Notation 1.7, prove that if M is a locally free R-module of finite rank then we can spread it out to a locally free R_A-module M_A (possibly enlarging A if necessary) and then base change to obtain $R_{A/\mu}$-modules $M_{A/\mu}$. Further show that we may assume that the $M_{A/\mu}$ are locally free for all $\mu \in U$, a dense open subset of m-Spec$A$.

Exercise 1.9. With notation as in Notation 1.7 suppose that R is a domain and M is a finitely generated Cohen-Macaulay R-module. Show that we may spread out M to a Cohen-Macaulay R_A-module M_A.

Exercise 1.10. With notation as in Notation 1.7, suppose that $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of finitely generated R-modules. Show we may spread out these modules and maps between them to a *short exact sequence* $0 \longrightarrow L_A \longrightarrow M_A \longrightarrow N_A \longrightarrow 0$ of finitely generated modules over R_A.

Exercise 1.11. Consider $R = \mathbb{Q}[x]/(x^2+1)$ and observe that R is a domain (although it not geometrically a domain over $\mathbb{Q}$). Show that we may take $A = \mathbb{Z}$ in such a way that the reduction modulo p fibers, $R_{\mathbb{F}_p}$ is not a domain (in fact not even connected), for a dense set of $(p) \in \text{m-}\operatorname{Spec}\mathbb{Z}$.

Exercise 1.12. With notation as in the section, show that $R_{A/\mu}$ is quasi-Gorenstein for a Zariski-dense set of $\mu \in \text{m-}\operatorname{Spec}A$ if and only if R_k is also quasi-Gorenstein. More generally, show that m is an integer such that $mK_{R_{A/\mu}}$ is Cartier for all $\mu \in U$ a Zariski-dense subset of m-Spec$A$ if and only if mK_{R_k} is Cartier in characteristic zero. This finishes the proof of Proposition 1.16.

Hint: A slick way to deal with this it to show that the R-module M is locally free of rank 1 if and only if the map $M \otimes_R \operatorname{Hom}_R(M, R) \longrightarrow R$ induced by $m \otimes_R \phi \mapsto \phi(m)$, is an isomorphism. To show this fact, work locally.

Exercise 1.13. With notation as in the section, suppose that $R_{A/\mu}$ is seminormal for a dense set of $\mu \in \text{m-}\operatorname{Spec}A$. Show that R_k is seminormal. See Chapter 1 Exercise 4.14 for a brief introduction to seminormality or see the exercises later in this chapter for a more complete introduction, in particular see Definition 5.19.

Hint: If R is not seminormal, then there exists a finite ring extension $R \subseteq S = R[x] \subseteq \mathcal{K}(R)$ where $x^2, x^3 \in R$.

Exercise 1.14. Suppose that $(D, \mathfrak{m}_D)$ is a local ring of essentially finite type over a field of characteristic zero obtained by localizing a finitely generated k-algebra at a maximal ideal. Recall we may reduce this local ring to characteristic $p > 0$ as in Subsection 1.5. Show that the following are equivalent.

(a) D is Cohen-Macaulay.
(b) $D_{A/\mu}$ is Cohen-Macaulay for a dense set of $\mu \in \text{m-}\operatorname{Spec}A$.
(c) $D_{A/\mu}$ is Cohen-Macaulay for an open dense set of $\mu \in \text{m-}\operatorname{Spec}A$.

Exercise 1.15. Suppose that X_k is a variety of finite type over a perfect field k of characteristic $p > 0$ and that $k \subseteq K$ is a finite extension. Show that X_k is F-rational (respectively F-injective) if and only if the base change $X_K = X_k \times_k K$ is F-rational (respectively F-injective). In fact, show that we may canonically identify $\tau(\omega_{X_k}) \otimes_k K$ with $\tau(\omega_{X_K})$.

Hint: Assume $X = \operatorname{Spec} R$ is affine. For the F-rational case, since k perfect, $k^p \cong k$. Use this to show that the base change of Frobenius $R \longrightarrow F_* R$ can be identified with $R' \longrightarrow F_* R'$ and the deduce that the map $T : F_* \omega_R \longrightarrow \omega_R$ base changes to $T : F_* \omega_{R'} \longrightarrow \omega_{R'}$. Finally use a common strong test element.

Exercise 1.16. Suppose that X is a normal variety of finite type over a perfect field k of characteristic $p > 0$, that $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on X and that $k \subseteq K$ is a finite extension. Show that X is strongly F-regular (respectively locally F-split) if and only if the base change (X_K, Δ_K) is strongly F-regular (respectively locally F-split). In fact, show that $\tau(X, \Delta) \otimes_k K$ is canonically identified with $\tau(X_K, \Delta_K)$.

Hint: Again assume X is affine. Show that the formation of $\operatorname{Hom}_R(F^e_* R, R)$ commutes with our base change (likewise with the divisorial version).

2. Resolution of singularities and rational singularities

Thanks to Hironaka's Fields medal-winning work [**Hir64**], resolution singularities for complex varieties is now a well-known tool. In this section, we recall the definitions of several variants of resolution of singularities. We also discuss rational singularities— singularities which, cohomologically at least, are negligible in the sense that they can be replaced by a resolution with little consequence.

2.1. Resolution of singularities. We recall three variants— resolutions, embedded resolutions, and log resolutions— of singularities.

Definition 2.1. Let X be a reduced Noetherian scheme. A **resolution of singularities of** X is a proper birational[14] map $\pi : Y \longrightarrow X$ with Y a regular scheme.[15]

We say that π is **strong resolution** of singularities if π is an isomorphism over the regular locus of X.

Caution 2.2. There are schemes—even one dimensional schemes of characteristic zero— that do not admit resolutions of singularities. For example,

[14]Birational means that π induces a bijection on components, and for each component Y_i of Y, the restriction $\pi|_{Y_i} : Y_i \longrightarrow X_i$ is birational, meaning an isomorphism at the generic point.

[15]meaning each stalk is a regular local ring.

in Caution 2.4 in Chapter 1, we constructed a one dimensional Noetherian domain R whose normalization $\widetilde{R}$ was *not finite*. The scheme $\operatorname{Spec} R$ therefore cannot admit a resolution of singularities, because by the universal property of normalization, any such resolution would factor through $\operatorname{Spec} \widetilde{R} \longrightarrow \operatorname{Spec} R$, which is not proper [**Sta19**, Tag 01WN].

Example 2.3. Every *excellent* reduced Noetherian scheme of dimension one admits a resolution of singularities, namely, its normalization. Note that by definition, normalization of an excellent scheme is finite, hence proper, eliminating the pathology of Caution 2.2.

Remark 2.4. If $\pi : Y \longrightarrow X$ is a proper birational map, then the locus of points of Y where π fails to be an isomorphism is a closed set, called the **exceptional set** of π, and denoted E_{exc}. Note that if π is a strong resolution, then $\pi(E_{\text{exc}})$ is the singular set of X. Typically, for us, this set will be a divisor, in which case it is called the **exceptional divisor**.

Embedded resolutions resolve (certain) subschemes of X:

Definition 2.5. Let X be a reduced Noetherian scheme, and suppose $Z \subseteq X$ is a reduced closed subscheme X, none of whose components are contained in the singular locus[16] of X. A map $\pi : Y \longrightarrow X$ is an **embedded resolution of singularities of Z in X** if π is a resolution of singularities of X which, when restricted to the strict transform $\widetilde{Z}$ of Z, induces a resolution of singularities $\pi|_{\widetilde{Z}} : \widetilde{Z} \longrightarrow Z$ of Z.

Finally, log resolutions resolve collections of closed subschemes—typically divisors— on X:l

Definition 2.6. Let X be a reduced Noetherian scheme, and $\{Z_1, \dots, Z_n\}$ a finite collection of closed subschemes. A **log resolution** of $(X, Z_1, \dots, Z_n)$ is a resolution of singularities $\pi : Y \longrightarrow X$ such that

(a) the locus of points in Y where π is not an isomorphism— the exceptional set E of π— is a simple normal crossings divisor;[17]
(b) the scheme theoretic inverse image $\pi^{-1}(Z_i)$ of each closed subscheme is a divisor with simple normal crossing support; and
(c) The sum of all the divisors in (a) and (b) form a divisor on Y with simple normal crossing support.

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[16] equivalently, such that X is non-singular at each of the generic points of Z

[17] This means that it is a divisor whose irreducible components are non-singular, and at any point where two or more components intersect, the equations defining the irreducible components of the divisor are a product of part of a regular system of parameters (part of minimal set of generators of the maximal ideal). In particular, these intersections are all non-singular as well.

Remark 2.7. If π is a *projective* log resolution—meaning that the log resolution π is a projective morphism—then π is given by blowing up some sheaf of ideals $J \subseteq \mathcal{O}_X$. In this case, the expansion $J\mathcal{O}_Y$ has the form $\mathcal{O}_Y(-G)$ for some effective divisor G on Y which has simple normal crossing support and contains all the components of the exceptional divisor. Note that the support of G also has normal crossings with the support of sum of the divisors in the $\pi^{-1}(Z_i)$.

Caution 2.8. If Z is a reduced divisor and X is normal, then any log resolution of (X, Z) will obviously be an embedded resolution of the subscheme Z in X. However, if Z is not a divisor, then a log resolution of (X, Z) is never an embedded resolution of singularities: in this case, π is not an isomorphism over the generic points of Z. See, however, Exercise 2.2.

The three variants of resolutions of singularities are closely related. For example, if $\pi : Y \longrightarrow X$ is an embedded resolution of some irreducible closed scheme Z whose exceptional set is a simple normal crossing divisor, then blowing up the proper transform of Z produces a log resolution of (X, Z), provided the newly created exceptional divisor has normal crossings with the (proper transform of) the previous exceptional divisor. Conversely, embedded resolutions can often be constructed from log resolutions; see Exercise 2.2. See also Exercise 2.1.

All three types of resolutions of singularities were understood to exist for *complex surfaces* (complex varieties of dimension two) by the Italian school of algebraic geometry in the early twentieth century.[18] By mid-century, Hironaka had addressed the issue completely for varieties of characteristic zero in any dimension:

Theorem 2.9. [**Hir64**] *Let X be a reduced scheme essentially of finite type over a field of characteristic zero. Then*

(a) X admits a strong resolution of singularities π, with π projective;
(b) Any reduced closed subscheme Z of X admits an embedded resolution of its singularities π, with π projective; and
(c) Any finite collection of closed subschemes $\{Z_1, \ldots, Z_n\}$ of X admits a projective log resolution of singularities.

Hironaka's proof shows that such resolutions can be constructed by a carefully chosen sequence of blow ups along non-singular subvarieties. Simplified presentations of his proof can be found in [**BM97, BEV05, Wło05, Kol07**] among other sources, also *cf.* [**dJ96, AdJ97, BP96**] for other proofs.

[18] settled definitely by Walker [**Wal35**] and Zariski [**Zar39a**]

2.2. Resolving $\mathbb{Q}$-divisors. There is an important variant of log resolutions for $\mathbb{Q}$-divisors on normal varieties:

Definition 2.10. Let X be a normal variety and Γ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a X. A **log resolution** of the pair (X, Γ) is a proper birational map $\pi : Y \longrightarrow X$, with Y non-singular, such that, picking any natural number n so that $n\Gamma$ is a Cartier divisor, the divisor $\pi^* n\Gamma$ has simple normal crossings support and its union with the exceptional set is also in simple normal crossings.

The existence of log resolutions as in Definition 2.10—at least for varieties of characteristic zero— follows from the existence of log resolutions as defined in Definition 2.6. The details are worked out in Exercise 2.5.

Often, one encounters a pair (X, Δ) where Δ is a $\mathbb{Q}$-divisor such that $\Gamma = K_X + \Delta$ is $\mathbb{Q}$-Cartier. In this case, a **log resolution** of (X, Δ) is a log resolution of (X, Γ), provided that K_X and Δ have disjoint support (otherwise, if $\operatorname{Supp} \Delta \not\subseteq \operatorname{Supp}(K_X + \Delta)$ we also require that the strict transform $\pi_*^{-1}\Delta$ of Δ has simple normal crossing support, and its union with the exceptional set has simple normal crossings support).

Remark 2.11. Resolutions of singularities (and their various variants) are expected to exist quite generally, including prime characteristic, under mild conditions. Indeed, Grothendieck introduced the notion of *excellence* as notion of well-behaved rings, and suggested it may be the setting for the existence of resolutions of singularities for reduced Noetherian schemes. Indeed, he showed that existence of resolution of singularities for schemes of finite type ove X implies the quasi-excellence[19] of X [**Gro65**, Proposition 7.9.5 and Remarque 7.9.6].

Hironaka's proof shows that resolution of singularities exists for schemes of finite type over a complete local reduced ring containing $\mathbb{Q}$ [**Hir64**]. Later, Temkin wrote down proofs that resolutions exist for schemes of finite type over quasi-excellent reduced schemes of equicharacteristic zero [**Tem12, Tem18**].

Remark 2.12. Resolution of singularities for varieties of prime characteristic remains a vexing open problem. In characteristic $p > 0$, Abhyankar proved this for surfaces [**Abh56a, Abh66**]. It was done for threefolds by Abhyankar when $p > 5$ [**Abh66**] (see also Cutkosky's simplification [**Cut09**]), and Cossart and Piltant for $p = 2, 3, 5$, [**CP09a, CP09b**].

Progress has also been made under suitable hypothesis for mixed characteristic (or "arithmetic") surfaces [**Lip78**] and threefolds [**CP19**].

[19] Quasi-excellence is the same as excellence but with the catenary condition dropped.

Despite our lack of understanding of resolution in prime and mixed characteristic, we often do work with such resolutions because they arise as models of characteristic zero resolutions when we reduce to prime characteristic. Indeed, given a variety X_k over a field k of characteristic zero, a projective resolution of singularities $Y_k \longrightarrow X_k$ can be modeled over some mixed characteristic base A. That is, we can spread out to a projective map $Y_A \longrightarrow X_A$ of flat finite type A-schemes with Y_A regular in such away that base-change to any field L—including the closed fibers A/μ which all have prime characteristic—produces a projective resolution $Y_L \longrightarrow X_L$ of the model variety over L. See Subsection 3.1

2.3. Resolutions and the canonical module. Resolution of singularities for varieties of characteristic zero have *built in vanishing theorems.* One important such is the Grauert-Riemenschneider vanishing theorem, concerning the canonical module of a resolution:

Theorem 2.13. *Let X be a reduced scheme essentially of finite type over a field of characteristic zero, and suppose that $\pi : Y \longrightarrow X$ is a resolution of singularities. If ω_Y is a canonical sheaf on Y, then $\mathbf{R}^i\pi_*\omega_Y = 0$ for all $i > 0$.*

Proof. See [**GR70**] for the original proof, or deduce it as a special case of relative Kawamata-Viehweg vanishing. □

Remark 2.14. The assumption that X is essentially finite type over a field is unnecessary in Theorem 2.13; for a scheme X over $\mathbb{Q}$, we need only that X has a dualizing complex [**Mur21**].

Theorem 2.13 begs the question: what about zero-th cohomology $\mathbf{R}^i\pi_*\omega_Y$? Note that if $\pi : Y \longrightarrow X$ is a resolution of singularities of a normal variety X, then there is an obvious natural map

$$\pi_*\omega_Y \longrightarrow \omega_X.$$

Namely, letting $\mathcal{U} \subseteq Y$ and $\mathcal{U}' \subseteq X$ be open sets where π is an isomorphism, there is a natural map

$$\pi_*\omega_Y \xrightarrow{\text{restrict to } \mathcal{U}'} \omega_{\mathcal{U}'} \cong \pi_*\omega_{\mathcal{U}}$$

which extends uniquely to a map

$$\pi_*\omega_Y \hookrightarrow \omega_X \tag{2.14.1}$$

because ω_X is reflexive and the complement of $\mathcal{U}'$ can be assumed of codimension two or more. The map is injective, as we can view it inside the function field $\mathcal{K}(X)$ of X.

In fact, the map (2.14.1) is defined much more generally. Before defining it, we recall our definition and conventions about canonical modules from Chapter 2:

Definition 2.15. For a normal variety X generically smooth over a field k, we define the **canonical sheaf** ω_X to be the reflexive hull of the rank one coherent sheaf $\bigwedge^{\dim X} \Omega_{X/k}$. More generally, if X is essentially of finite type over k and locally equidimensional, and h is the structural morphism $h : X \longrightarrow k$, then the complex $h^! k = \omega_X^{\bullet}$ is a (not-necessarily normalized) dualizing complex, and we define the **canonical sheaf** ω_X *locally* to be the first non-vanishing cohomology of $\omega_X^{\bullet}$ on each component; In particular, if X is finite type[20] over k and equidimensional, then $\omega_X = \mathcal{H}^{-\dim X}(\omega_X^{\bullet})$. See Remark 3.11 in Appendix C.

Lemma 2.16. *Suppose $\pi : Y \longrightarrow X$ is a proper generically finite map of locally equidimensional schemes over a field k. By applying Grothendieck duality to $\mathcal{O}_X \longrightarrow \mathbf{R}\pi_* \mathcal{O}_Y$ and then taking cohomology of the resulting map, we obtain a map*

$$\pi_* \omega_Y \longrightarrow \omega_X.$$

Proof. Consider $\mathcal{O}_X \longrightarrow \mathbf{R}\pi_* \mathcal{O}_Y$. We apply the Grothendieck duality functor $\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(-, \omega_X^{\bullet})$ and obtain

$$\omega_X^{\bullet} \cong \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^{\bullet}) \leftarrow \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathbf{R}\pi_* \mathcal{O}_Y, \omega_X^{\bullet}).$$

By Grothendieck duality (see Appendix C) the right side is quasi isomorphic to $\mathbf{R}\pi_* \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y^{\bullet}) \cong \mathbf{R}\pi_* \omega_Y^{\bullet}$ and so we consider the map

$$\omega_X^{\bullet} \leftarrow \mathbf{R}\pi_* \omega_Y^{\bullet} \tag{2.16.1}$$

dual to $\mathcal{O}_X \longrightarrow \mathbf{R}\pi_* \mathcal{O}_Y$. Set $-d$ to be the smallest integer such that $\omega_X^{\bullet}$ (and so likewise $\omega_Y^{\bullet}$) has cohomology in degree $-d$. Since π_* is left exact, and $\omega_Y^{\bullet}$ is zero below degree $-d$, we see that

$$\pi_* \omega_Y = \mathcal{H}^{-d}(\mathbf{R}\pi_* \omega_Y^{\bullet}).$$

Thus taking $-d$th cohomology of (2.16.1) we obtain the map

$$\pi_* \omega_Y \longrightarrow \omega_X \tag{2.16.2}$$

as desired. □

Remark 2.17. The map $\pi_* \omega_Y \longrightarrow \omega_X$ of Lemma 2.16 is always injective when π is birational; see Exercise 2.3.

[20] If X is not finite type but is the localization of some locally equidimensional finite type scheme W over k, our dualizing complex $\omega_X^{\bullet}$ (respectively, canonical module ω_X) is the corresponding localization of $\omega_W^{\bullet}$ (respectively, ω_W). In this case, the complex $\omega_X^{\bullet}$ is not normalized: its first non-zero cohomology will be in degree $-\dim W$, which may not equal $-\dim X$; see Caution 3.10 in Appendix C.

It turns out that, restricting to the case where $\pi : Y \longrightarrow X$ is a resolution of singularities in characteristic zero, the image of the map (2.16.2) is independent of the choice of resolution and is an important invariant of the singularities of X. Assuming this independence for a moment, we define:

Definition 2.18. Let X be a reduced locally equidimensional Noetherian scheme essentially of finite type over a field of characteristic zero. Suppose $\pi : Y \longrightarrow X$ is a resolution of singularities. Then **multiplier submodule of** X is the image of the natural map

$$\pi_* \omega_Y \longrightarrow \omega_X,$$

or, equivalently, the subsheaf $\pi_* \omega_Y$ of ω_X. We denote the multiplier submodule by $\mathcal{J}(\omega_X)$.

Remark 2.19. The object $\mathcal{J}(\omega_X)$ is also frequently called the Grauert-Riemenschneider canonical sheaf, and denoted ω_X^{GR}, since it was first studied in [**GR70**]. We will use the term "multiplier submodule" however to emphasize the relation to the *multiplier ideals*; see Subsection 4.3.

To prove the independence of the image of map (2.16.2), we need the following lemma:

Lemma 2.20 ([**LT81**, Section 5])**.** *Suppose that X is a non-singular*[21] *Noetherian scheme with canonical module ω_X. Then for every proper birational map $\pi : Y \longrightarrow X$ the map $\pi_* \omega_Y \longrightarrow \omega_X$ is an isomorphism.*

Proof. By working on components, we can assume X is irreducible. We may also assume that Y is normal. Otherwise, say $Y' \longrightarrow Y$ is the normalization with composition $\pi' : Y' \longrightarrow X$. Then there is a factorization $\pi'_* \omega_{Y'} \hookrightarrow \pi_* \omega_Y \hookrightarrow \omega_X$, where all maps are inclusions since they are maps of rank-1 torsion free modules (since π and π' are isomorphisms on an open set).

We need only show that the map $\pi_* \omega_Y \longrightarrow \omega_X$ is surjective (Remark 2.17).

We first prove the surjectivity in the case that X is smooth over a field k, where it is straightforward. Working locally on X, take $w = g dx_1 \wedge \cdots \wedge dx_d \in \omega_X \in \bigwedge^d \Omega_{X/k}$. Its pullback

$$\pi^* w = \pi^*(g dx_1 \wedge \cdots \wedge dx_d) = \pi^* g d(\pi^* x_1) \cdots d(\pi^* x_d)$$

is in $\bigwedge^d \Omega_{Y/k}$. The reflexification map $\bigwedge^d \Omega_{Y/k} \longrightarrow \omega_Y$ gives a map

$$\pi^* \omega_X \longrightarrow \omega_Y \quad \text{and hence} \quad \omega_X \longrightarrow \pi_* \omega_Y.$$

Then, as these maps are an isomorphism where π is an isomorphism, it follows that $\pi_* \omega_Y \longrightarrow \omega_X$ is an isomorphism, as desired.

[21]Meaning that every local ring is regular

For the general case, we work locally and so assume that ω_X is trivial. The map $\pi_* \omega_Y \longrightarrow \omega_X$ can be written

$$\pi_* \mathcal{O}_Y(K_Y) \longrightarrow \mathcal{O}_X \tag{2.20.1}$$

where the divisor K_Y is supported on the exceptional set. Thus the map (2.20.1) is surjective if and only if $K_Y \geq 0$.

To check $K_Y \geq 0$, we should check this at each of the prime (exceptional) divisors D in its support. Let $Z_0 = \pi(D)$, the center of the discrete valuation v_D corresponding to D on X. Let $\pi_1 : X_1 \longrightarrow X$ denote the blowup of X_0 along Z_0. Since X is non-singular at the generic point of Z_0, the blowup X_1 is non-singular over the generic point of Z_0. Hence if we set Z_1 to be the center of v_D on X_1, the generic point of Z_1 is non-singular in X_1 (it lies over the generic point of Z_0). Let $\pi_2 : X_2 \longrightarrow X_1$ denote the blowup of X_1 at Z_1. Again X_2 is non-singular generically over Z_1, we define Z_2 as above and continue on. We stop the procedure when Z_m has codimension 1 in X_m, and indeed this occurs eventually by [**Abh56b**, Proposition 3] or [**Art86**, Theorem 5.2]; also see [**Zar39b**]. In that case, the stalk $\mathcal{O}_{X_m,z}$ at z, the generic point of Z_m is the valuation ring associated to v_D. It follows that Y and X_m agree in a neighborhood of the generic point of Z_m and D respectively.

We next assert that *at least* over the generic point of Z_i, that

$$K_{X_{i+1}} = K_{X_i} + (d_i - 1)E_{i+1}$$

where d_i is the codimension of Z_i in X_i. For varieties, this is [**Har77**, Chapter II, Exercise 8.5]. However, that proof works in our generality. The point is that over the generic point η_i of Z_i we have that the exceptional divisor is $E_{i+1} = \mathbb{P}^{d_i-1}_{\eta_i}$. Using the adjunction map $\mathcal{O}_{X_{i+1}}(K_{X_{i+1}} + E_{i+1}) \longrightarrow \mathcal{O}_{E_{i+1}}(K_{E_{i+1}})$ (at least over the nonsingular loci) and the fact that $\mathcal{O}_{E_{i+1}}(K_{E_{i+1}})_{\eta_i} \cong \mathcal{O}_{\mathbb{P}^{d_i-1}_{\eta_i}}(-d_i)$ one obtains the formula above.

Iterating the formula, we see the coefficient of K_{X_m} along Z_m is positive. It follows that K_Y has positive coefficient at D, since Y and X_m agree in a neighborhood of the generic point of D. The result is proven. $\square$

Corollary 2.21. *Let X be a reduced locally equidimensional scheme of finite type over a field of characteristic zero. Then for any resolution of singularities $\pi : Y \longrightarrow X$, the image of the map $\pi_* \omega_Y \longrightarrow \omega_X$ is independent of the choice of resolution.*

Proof. Suppose that $\pi : Y \longrightarrow X$ and $\pi' : Y \longrightarrow X$ are two resolution of singularities. Let Y'' denote a resolution of singularities of the subscheme consisting of irreducible components of $Y \times_X Y'$ that dominate a component of X. It follows that $\pi'' : Y'' \longrightarrow X$ is a resolution of singularities X.

Likewise the induced maps $g : Y'' \longrightarrow Y$ and $h : Y'' \longrightarrow Y$ are resolution of singularities of Y and Y'. It thus suffices to show that $\pi''_* \omega_{Y''} = \pi_* \omega_Y$ as by symmetry it will also follow that $\pi_* \omega_{Y''} = \pi'_* \omega_{Y'}$. We have maps

$$\pi''_* \omega_{Y''} \longrightarrow \pi_* \omega_Y \longrightarrow \omega_X$$

where the first map is π_* applied to $g_* \omega_{Y''} \longrightarrow \omega_Y$. But that map is an isomorphism by Lemma 2.20. The result follows. $\square$

Remark 2.22. Corollary 2.21 and its proof hold much more generally: it suffices if X is a reduced locally equidimensional scheme essenetially of finite type over a quasi-excellent scheme of characteristic zero, and X admits dualizing complex $\omega_X^{\bullet}$.

2.4. Rational singularities.

Now we introduce rational singularities.

Definition 2.23. Suppose that X is a reduced scheme essentially of finite type over a field of characteristic zero. The scheme X has **rational singularities** if for every resolution of singularities $\pi : Y \longrightarrow X$,

(i) $\pi_* \mathcal{O}_Y \cong \mathcal{O}_X$ (this is equivalent to requiring that X is normal, see [**Har77**, III, Exercise 11.4]), and
(ii) $\mathbf{R}^i \pi_* \mathcal{O}_Y = 0$ for all $i > 0$.

These two conditions may be more compactly written as $\mathbf{R}\pi_* \mathcal{O}_Y \cong \mathcal{O}_X$.

A reduced Noetherian ring R has **rational singularities** if $\operatorname{Spec} R$ has rational singularities.

Because rational singularities are normal, they are a disjoint union of irreducible components, each with rational singularities. Thus we usually stick to the integral case in dealing with rational singularities.

The following is a dual characterization, often called Kempf's criterion [**KKMSD73**]:

Theorem 2.24. *Suppose that X is a connected locally equidimensional reduced scheme essentially of finite type over a field of characteristic zero. Then X has rational singularities if and only if the following two conditions are satisfied:*

(a) X is Cohen-Macaulay.
(b) For every resolution of singularities $\pi : Y \longrightarrow X$, the induced map $\pi_* \omega_Y \longrightarrow \omega_X$ of Lemma 2.16 is an isomorphism.

In other words, X has rational singularities if and only if X is Cohen-Macaulay and it multiplier submodule $\mathcal{J}(\omega_X)$ is all of ω_X.

PROOF. Suppose first that X has rational singularities. So $\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y$ is an isomorphism. Applying the Grothendieck duality functor, also (b) is an isomorphism.

Thus we need to show that X is Cohen-Macaulay. Using that Y is non-singular and hence Cohen-Macaulay, we see that

$$\omega_X^\bullet \cong \mathbf{R}\pi_*\omega_Y^\bullet \cong \mathbf{R}\pi_*\omega_Y[d] \cong \pi_*\omega_Y[d]$$

for some shift [22] d. Thus $\omega_X^\bullet$ has non-zero cohomology only in one cohomological degree, so X is Cohen-Macaulay as well.

Conversely, if X satisfies (a) and (b), then the map $\pi_*\omega_Y[d] \longrightarrow \omega_X[d]$ is a quasi-isomorphism, in other words:

$$\mathbf{R}\pi_*\omega_Y^\bullet \longrightarrow \omega_X^\bullet.$$

is a quasi-isomorphism (using Grauert-Riemenschneider vanishing Theorem 2.13 and the fact that X is Cohen-Macaulay). Applying Grothendieck duality $\mathbf{R}\mathscr{H}om_{\mathcal{O}_X}(-, \omega_Y^\bullet)$ we obtain that

$$\mathcal{O}_X \longrightarrow \mathbf{R}\pi_*\mathcal{O}_Y$$

is also a quasi-isomorphism, and so X has rational singularities. Here we used that applying Grothendieck duality twice is naturally isomorphic to the identity. □

Example 2.25. Every non-singular variety of characteristic zero has rational singularities, by Lemma 2.20.

Remark 2.26. It is enough to check conditions (a) and (b) in Definition 2.23 of rational singularities for *one* resolution of singularities. This follows from the dual criterion Theorem 2.24 using Corollary 2.21. Generalizations of this fact beyond characteristic zero can be found in [**CR11, CR15, Kov17**].

Finally, we observe that having rational singularities is a local and open property.

Lemma 2.27. *Let X be a reduced Noetherian scheme of essentially finite type over a field of characteristic zero. The subset*

$$\{P \in X \mid \mathcal{O}_{X,P} \ \text{ has rational singularities}\}$$

is open in X, and constitutes the largest open subscheme of X that has rational singularities.

PROOF. This is left to the reader in Exercise 2.10. □

[22] in fact, d is such that $-d$ is the lowest degree where the dualizing complex of Y has non-zero cohomology (and hence likewise with X) but this is not so important for the argument

2.5. Pseudo-rational singularities. Let $(R, \mathfrak{m})$ be a reduced local Noetherian ring and let $\pi : Y \longrightarrow X = \operatorname{Spec} R$ be a proper birational map. Suppose X has a normalized[23] dualizing complex $\omega_X^\bullet$. Grothendieck dual to $\mathcal{O}_X \longrightarrow \mathbf{R}\pi_* \mathcal{O}_Y$ we have the map

$$\mathbf{R}\pi_* \omega_Y^\bullet \longrightarrow \omega_X^\bullet \cong \mathbf{R}\,\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathbf{R}\pi_* \mathcal{O}_Y, \omega_X^\bullet)$$

whose first non-vanishing cohomology is $\pi_* \omega_Y \longrightarrow \omega_X$ (Lemma 2.16). Let E denote the injective hull of the residue field and apply the exact functor $\operatorname{Hom}_R(-, E)$ to obtain:

$$\operatorname{Hom}(\omega_X^\bullet, E) \longrightarrow \operatorname{Hom}(\mathbf{R}\pi_* \omega_Y^\bullet, E).$$

The left side is quasi-isomorphic to $\mathbf{R}\Gamma_\mathfrak{m}(\mathcal{O}_X)$ by local duality Appendix C Section 6 and the right side is $\mathbf{R}\Gamma_\mathfrak{m}(\mathbf{R}\pi_* \mathcal{O}_Y)$ also by local duality. The map $\pi_* \omega_Y \longrightarrow \omega_X$ is thus dual to $H^d_\mathfrak{m}(R) \longrightarrow \mathcal{H}^d \mathbf{R}\Gamma_\mathfrak{m}(\mathbf{R}\pi_* \mathcal{O}_Y)$.

This motivates the following definition:

Definition 2.28. [**Lip69, LT81**] Let $(R, \mathfrak{m})$ be a reduced Noetherian d-dimensional local ring. Then R has **pseudo-rational** singularities if the following two conditions are satisfied:

(a) R is Cohen-Macaulay, and
(b) For every proper birational map $\pi : Y \longrightarrow X = \operatorname{Spec} R$ we have that $H^d_\mathfrak{m}(R) \longrightarrow H^d_\mathfrak{m}(\mathbf{R}\pi_* \mathcal{O}_Y) = H^d_{\pi^{-1}(\mathfrak{m})}(\mathcal{O}_Y)$ is injective.

A ring R is **pseudo-rational** if all its localizations at prime ideals R_Q are pseudo-rational. A scheme is **pseudo-rational** if all its stalks are pseudo-rational local rings.

When R has a dualizing complex, condition (b) is dual to the surjectivity of $\pi_* \omega_Y \longrightarrow \omega_X$. Thus if R is essentially of finite type over a field of characteristic zero, it follows that R has rational singularities if and only if R is pseudo-rational.

Pseudo-rationality, however, makes sense even if $\operatorname{Spec} R$ does not have a resolution of singularities or even if it does not have a dualizing complex. A disadvantage is that it is not clear whether pseudo-rationalityis an open condition. See [**Kov17**] for evidence that pseudo-rational singularities are much closer to rational singularities than is immediately obvious.

Lemma 2.29. *Suppose that $(R, \mathfrak{m})$ is a local ring. If the completion $\widehat{R}$ is pseudo-rational, then so is R.*

[23]We normalize so that $\mathcal{H}^{-\dim R} \omega_X^\bullet = \omega_R$; See Definition 3.5 in Appendix C.

Proof. If $\widehat{R}$ is Cohen-Macaulay, then so is R, since $H^i_{\mathfrak{m}}(R) \cong H^i_{\mathfrak{m}}(\widehat{R})$ for all i. On the other hand, if $\pi : Y \longrightarrow X = \operatorname{Spec} R$ is a proper birational map, then its base change to the completion $\widehat{\pi} : Y \times X^{\widehat{X}} \longrightarrow \widehat{X} = \operatorname{Spec} \widehat{R}$ is also proper and birational. We then see that

$$H^d_{\mathfrak{m}}(R) \cong H^d_{\mathfrak{m}}(\widehat{R}) \hookrightarrow H^d_{\mathfrak{m}}(\mathbf{R}\pi_* \mathcal{O}_Y \otimes_R \widehat{R}) \cong H^d_{\mathfrak{m}}(\mathbf{R}\pi_* \mathcal{O}_Y)$$

injects as desired. □

We also have the following variant of Lemma 2.20.

Corollary 2.30. *If $(R, \mathfrak{m})$ is a Noetherian regular local ring, then R is pseudo-rational.*

Proof. Since R is regular, it is Gorenstein and thus also Cohen-Macaulay. Thus it has a canonical module $\omega_R \cong R$. Now apply Lemma 2.20. □

2.6. F-rational implies pseudo-rational. For schemes of prime characteristic, the test submodule is closely related to the multiplier module discussed above.

Theorem 2.31 ([**Smi97a**]). *Let X be an F-finite reduced locally equidimensional scheme with canonical module ω_X. Let $T : F_* \omega_X \longrightarrow \omega_X$ be the Grothendieck dual to Frobenius.*[24] *For any proper dominant generically finite map from a locally equidimensional scheme $\pi : Y \longrightarrow X$ (for instance, a proper birational map), we have*

$$\tau(\omega_X) \subseteq \operatorname{Image}(\pi_* \omega_Y \longrightarrow \omega_X).$$

As an immediate corollary, we recover the main theorem [**Smi97a**]:

Corollary 2.32. *An F-finite*[25] *F-rational Noetherian scheme is pseudo-rational.*

Proof of Corollary. This follows immediately from the theorem using Chapter 2 Corollary 5.14 and the fact fact that F-rationality is equivalent to being Cohen-Macaulay and satisfying $\tau(\omega_X) = \omega_R$. □

Proof of Theorem 2.31. The hypotheses imply that over a dense open subset U of X, π is finite.

We first claim that $\operatorname{Image}(\pi_* \omega_Y \longrightarrow \omega_X)$ is non-zero at every generic point of X. To see this, observe that over each generic point η of X, $X_\eta =$

[24]See Chapter 2 add dmore precise reference.

[25]This hypothesis can be removed.

$\operatorname{Spec} \mathcal{K}(\eta)$ is the prime spectrum of a field. Replacing X by X_η, we see that $\operatorname{Spec} S = Y \longrightarrow X = \operatorname{Spec} \mathcal{K}(\eta)$ is finite and $\pi_* \omega_Y \longrightarrow \omega_X$ is identified with

$$\operatorname{Hom}(S, \mathcal{K}(\eta)) \xrightarrow{\text{eval@1}} \mathcal{K}(\eta)$$

which is nonzero, as claimed.

Now consider the diagram

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"] \\ X \arrow[r, "F"'] & X \end{tikzcd}$$

which induces a diagram

$$\begin{tikzcd} F_* \pi_* \omega_Y \arrow[r, "\pi_* T_Y"] \arrow[d] & \pi_* \omega_Y \arrow[d] \\ F_* \omega_X \arrow[r, "T_X"'] & \omega_X. \end{tikzcd}$$

where T_Y and T_X are dual to Frobenius as described in Chapter 2. This diagram shows that image of $\pi_* \omega_Y \longrightarrow \omega_X$ is compatible with $T_X : F_* \omega_X \longrightarrow \omega_X$. Since $\tau(\omega_X)$ is the smallest such compatible module non-zero at every generic point of X, we see that

$$\tau(\omega_X) \subseteq \operatorname{Image}(\pi_* \omega_Y \longrightarrow \omega_X).$$

as desired. □

Remark 2.33. In fact, the F-finite hypothesis is unnecessary in Corollary 2.32 if R is local we define F-rationality appropriately in the non-F-finite case [**Smi97a**]. We will explore this later using tight closure Chapter 7.

Remark 2.34. In a fixed positive characteristic there are pseudo-rational singularities that are not F-rational, even for Gorenstein surface singularities, see Exercise 3.3.

2.7. Exercises.

Exercise 2.1.

Exercise 2.2. Show that one can construct an embedded resolution from a log resolution as follows. Let W be a closed integral subscheme of a reduced Noetherian scheme X, not contained in its singular locus, and let $\phi : \tilde{X} \longrightarrow X$ be the normalized blowup of X along W. Let $\mathcal{I}_W \subseteq \mathcal{O}_X$ be the sheaf of ideals defining W on X, and let $Z \subseteq \tilde{X}$ be the closed codimension one subscheme of $\tilde{X}$ cut out by its expansion $\mathcal{I}_W \mathcal{O}_{\tilde{X}}$ to $\mathcal{O}_X$. Now, if $\pi : Y \longrightarrow \tilde{X}$ is a log resolution of $(\tilde{X}, Z)$, show that the restriction of $\phi \circ \pi$ to the strict transform $\tilde{W}$ of W is an embedded resolution of W.

Hint: You will want to intersect the strict transform of a suitable component of Z with exceptional divisors of π.

Exercise 2.3. Show that if $\pi : Y \longrightarrow X$ is a resolution of singularities of schemes with canonical sheaves, then the natural map $\pi_* \omega_Y \longrightarrow \omega_X$ is injective.

Hint: This can be checked on affine charts U. So we want to check $\omega_Y(\pi^{-1}(U)) \subseteq \omega_X(U)$.

Exercise 2.4. Suppose that X is a reduced Noetherian scheme and $\pi : Y \longrightarrow X$ is a proper birational map with Y normal. Prove that $\mathcal{O}_X \cong \pi_* \mathcal{O}_Y$ if and only if X is normal. Furthermore, show that the object $\pi_* \mathcal{O}_Y$ is independent of π and Y and the sheafy Spec, $\mathbf{Spec}\, \pi_* \mathcal{O}_Y$, is the normalization of X.

Exercise 2.5. Suppose that X is a normal variety and Γ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Show that there exists a log resolution of (X, Γ) in the sense of Definition 2.10.

Hint: If $\Gamma \geq 0$ (respectively anti-effective), one may simply work with Z, the scheme associated to a Cartier divisor $n\Gamma$ (respectively to $-n\Gamma$). We wish to reduce to this case. First take a resolution to make the ambient space smooth, and pull back the divisor there (keeping track of the Cartier exceptional divisor as well). Replace X by this resolution. By Chow's lemma, we may assume that X is quasi-projective over a field (further resolving singularities if necessary) and hence there exists a ample effective divisors whose components include those of Γ (the new Γ). Use these.

Exercise 2.6. Suppose Y is a scheme essentially of finite type over a field of characteristic zero and $\pi : Y \longrightarrow X$ is a resolution of singularities. Let $\mathscr{M}$ be a finite rank locally free sheaf on X. Suppose that X has rational singularities and use that to show that

$$H^i(X, \mathscr{M}) \cong H^i(Y, \pi^* \mathscr{M})$$

for every integer i.

Hint: Observe that $\mathscr{M} \cong \mathscr{M} \otimes_{\mathcal{O}_X} \mathbf{R}\pi_* \mathcal{O}_Y$ and then use the derived projection formula.

Exercise 2.7. Suppose that $(R, \mathfrak{m})$ is a reduced equidimensional local ring essentially of finite type over a field of characteristic zero. Suppose that $f \in \mathfrak{m}$ is a non-zerodivisor such that $R/(f)$ has rational singularities. Prove that R has rational singularities. This was first shown by R. Elkik in [**Elk78**].

Hint: First show that R is Cohen-Macaulay. Let $\pi : Y \longrightarrow X = \operatorname{Spec} R$ be a resolution of singularities that is also an embedded resolution of the divisor $D = \operatorname{div}(f)$. Let $\widetilde{D}$ denote the strict transform of D. Push forward the short exact sequence $0 \longrightarrow \omega_Y \longrightarrow \omega_Y(\widetilde{D}) \longrightarrow \omega_{\widetilde{D}} \longrightarrow 0$, using Grauert-Riemenschneider vanishing, compare with $0 \longrightarrow \omega_X \longrightarrow \omega_X(D) \longrightarrow \omega_D \longrightarrow 0$ and use the snake lemma, or argue similarly to Theorem 7.14 after Matlis duality.

Exercise 2.8. Suppose that X is a smooth projective scheme over a field of characteristic zero and $\mathscr{L}$ is a big a nef line bundle on Y. One special case of Kawamata-Viehweg vanishing says that $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ for all $i > 0$. Prove that that vanishing also holds when X has rational singularities. More generally, show that

$$H^i(X, \mathcal{J}(\omega_X) \otimes \mathscr{L}) = 0$$

for all $i > 0$.

Hint: Notice that $H^i(X, \mathcal{J}(\omega_X) \otimes \mathscr{L}) = \mathcal{H}^i \mathbf{R}\Gamma(X, \mathbf{R}\pi_*(\omega_Y \otimes \pi^*\mathscr{L}))$ and then compose the derived functors, or use a spectral sequence. Notice also that the pullback of a big and nef line bundle under a proper birational map is still big and nef.

Exercise 2.9. Suppose that $X \subseteq \mathbb{P}^n$ is a projective variety over an algebraically closed field of characteristic zero with rational singularities. Suppose that $H \subseteq \mathbb{P}^n$ is a general hyperplane. Show that $H \cap X$ also has rational singularities.

Hint: Pull back the linear system $|\mathcal{O}_X(1)|$ to a resolution of singularities $\pi : Y \longrightarrow X$. This linear system is still base point free, so we can use Bertini's theorem to show that $H_Y = \pi^*(H \cap X)$ is non-singular.

Exercise 2.10. Prove Lemma 2.27.

Exercise 2.11. Suppose that X is a normal variety and $D \subseteq X$ is a reduced[26] divisor. Fix $\pi : Y \longrightarrow X$ an embedded resolution of (X, D) with $\widetilde{D} := \pi_*^{-1} D$ the strict transform of D. Show that $\pi_* \omega_Y(\widetilde{D}) \subseteq \omega_X(D)$ is independent of the choice of embedded resolution Y.

Exercise 2.12. Show that any pseudo-rational local ring is normal.

Hint: First reduce to the complete case by [**Sta19**, Tag 033G] and recall that a complete local ring always has a dualizing complex. Choose your proper birational map to be the normalization map. Use Appendix C Proposition 6.9.

Exercise 2.13. Suppose that X is a variety of characteristic zero and that $\pi : Y \longrightarrow X$ is a resolution of singularities. Suppose that there is a map $\phi : \mathbf{R}\pi_* \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ in the derived category such that the composition with

[26] Meaning all its coefficients are 1.

the canonical map $\mathcal{O}_X \longrightarrow \mathbf{R}\pi_*\mathcal{O}_Y \xrightarrow{\phi} \mathcal{O}_X$ is the identity. Prove that X has rational singularities. This is sometimes known as Kovács' rationality criterion, see [**Kov00**].

Hint: Apply the Grothendieck duality functor $\mathbf{R}\mathscr{H}\mathrm{om}_X(-, \omega_X^{\bullet})$ and recall that $\mathbf{R}\mathscr{H}\mathrm{om}_X(\mathcal{O}_Y, \omega_X^{\bullet}) = \mathbf{R}\pi_*\omega_Y^{\bullet}$ by Grothendieck duality. Use Grauert-Riemenschneider vanishing and the fact that $\omega_Y[\dim Y] = \omega_Y^{\bullet}$ to prove that X is Cohen-Macaulay.

3. F-rational singularities vs rational singularities

Our goal in this section is to prove the following theorem.

Theorem 3.1. *Suppose X is a variety over a field of characteristic zero. The following are equivalent.*

- *(a) X has open F-rational type.*
- *(b) X has dense F-rational type.*
- *(c) X has rational singularities.*

In fact, modeling the multiplier submodule $\mathcal{J}(\omega_X) \subseteq \omega_X$ over a mixed characteristic ring A, the prime characteristic models in a non-empty open set of m-Spec$A$ *all reduce to the test submodule in $\omega_{X_{A/\mu}}$.*

Certainly (a) $\Rightarrow$ (b). We will then show that (b) $\Rightarrow$ (c) and finally that (c) $\Rightarrow$ (a).

3.1. Modeling a resolution of singularities. A resolution of singularities for a variety over a field of characteristic zero can be reduced to prime characteristic using a family of models.

Given a projective birational map $\pi : Y_k \longrightarrow \operatorname{Spec} R_k$ with Y_k smooth over k, we can build a coefficient ring— a (regular) domain A finitely generated over $\mathbb{Z}$ and contained in k— together with a projective birational map

$$\pi_A : Y_A \longrightarrow \operatorname{Spec} R_A,$$

of schemes over A such that the flat base change $A \hookrightarrow k$ recovers the resolution π. The point is that, because π is finite type, only finitely many elements of k are needed to describe π (and Y_k) completely, so we can just adjoin them all to A.[27] Indeed, the map π_k is the blowup up some ideal $J_k \subseteq R_k$, with finitely many generators $g_1, \ldots, g_r$. Adjoining to A a set of

[27]In fact, by Hironaka's theorem, once we have picked an A containing enough elements of k to define R_k, if we let K be the fraction field of A, then we know that $\operatorname{Spec} R_K$

finitely many elements of k needed to describe polynomials in $k[x_1, \ldots, x_n]$ representing these generators, we can define an ideal $J_A \subseteq R_A$ generated by the (images of the) corresponding polynomials, also denoted $g_1, \ldots, g_r$ in $A[x_1, \ldots, x_n]$. Define T_A to be the Rees ring

$$T_A := R_A[J_A t] \cong R_A \oplus J_A \oplus J_A^2 \oplus \cdots \subseteq R_A[Jt],$$

which is finitely generated over A because there is a surjection of $\mathbb{N}$-graded domains

$$A[x_1, \ldots, x_n, Y_1, \ldots, Y_r] \twoheadrightarrow T_A,$$

sending each x_i to its image in R_A and Y_j to the degree one element $g_j t$ in T_A. Now define Y_A to be $\operatorname{Proj} T_A$. There is a natural projective map

$$\pi_A : \operatorname{Proj} T_A \longrightarrow \operatorname{Spec} R_A \tag{3.1.1}$$

of schemes of finite type over A. Base change to k recovers π. The Lemma of Generic Freeness allows us to assume—at the expense of possibly adding one more $\mathbb{Z}$-algebra generator b^{-1} to A— that the finitely generated A-algebra T_A (as well as R_A) is free over A, and also that A is regular.

Remark 3.2. The scheme Y_A is covered by affine charts $D_+(y_j)$, where y_j is the image of Y_j in T_A. Note that $D_+(y_j)$ is the spectrum of the ring

$$\left[T_A[\frac{1}{Y_j}]\right]_0,$$

which is finite type and (without loss of generality) free over A (again, by inverting one element of A if needed). In particular, for each j, the localization $T_A[\frac{1}{y_j}]$, which is the Laurent extension

$$\left[T_A[\frac{1}{y_j}]\right]_0 [y_j, y_j^{-1}]$$

of $\left[T_A[\frac{1}{y_j}]\right]_0$, is free over A as well. Base change to k produces

$$\left[T_k[\frac{1}{y_j}]\right]_0 = \mathcal{O}_{Y_k}(D_+(y_j)),$$

where here we use the same notation y_j for the image in T_k.

The map π_A models our resolution of singularities for $\operatorname{Spec} R_k$, at least on a dense open subset of $\operatorname{Spec} A$:

Theorem 3.3. *Let R_k be a domain finitely generated over a field k as in (1.3.1) and let $\pi : Y_k \longrightarrow \operatorname{Spec} R_k$ be projective birational map of k-schemes with Y_k smooth over k. Let A be a choice of coefficient ring for R_k as in the proceeding paragraph, and define*

$$\pi_A : \operatorname{Proj} T_A \longrightarrow \operatorname{Spec} R_A \tag{3.3.1}$$

---

has a resolution of singularities defined over K, so we don't even need to adjoin any further elements from k to A beyond possibly the inverses of some elements in A.

be as in (3.1.1). Then there exists non-zero $b \in A$ such that for all $A[b^{-1}]$-algebras L, the base change

$$\pi_L : \operatorname{Proj} T_L \longrightarrow \operatorname{Spec} R_L \tag{3.3.2}$$

is a projective birational map with $\operatorname{Proj} T_L$ *smooth over L. In particular, for closed points μ in some open set of* $\operatorname{Spec} A$*, the map*

$$\pi_{A/\mu} : Y_{A/\mu} \longrightarrow \operatorname{Spec} R_{A/\mu}$$

is a resolution of singularities for the prime characteristic scheme $R_{A/\mu}$.

Proof Sketch. The coherent sheaf $\Omega_{Y_A/A}$ of Kähler differentials for Y_A over A has the property that base change to k produces the locally free sheaf of $\mathcal{O}_{Y_k}$-modules $\Omega_{Y_k/k}$. Because Y_k is smooth over k, we can chose a sufficiently fine affine cover of Y_k so that $\Omega_{Y_k/k}$ is a *free* $\mathcal{O}_{Y_k}$-module on each chart. These charts can be assumed of the form $D_+(h) = \operatorname{Spec}\left[T_k[h^{-1}]\right]_0$, where h is a degree one element of T_A. Adding finitely many elements[28] (if needed) to A, we may assume that these affine charts are obtained by base change from corresponding affine charts $D_+(h)$ of Y_A, all of which are spectra of finitely generated A-algebras free over A. Now, $\Omega_{Y_k/k}(D_+(h))$ is free over $\left[T_k[h^{-1}]\right]_0$ if and only if $\Omega_{Y_A/A}(D_+(h))$ is free over $\left[T_A[h^{-1}]\right]_0$, and in this case, they have the same rank. Thus, base changing to any A-algebra L, also $\Omega_{Y_L/L}(D_+(h))$ is free over $\left[T_L[h^{-1}]\right]_0 = \mathcal{O}_{Y_L}(D_+(h))$. In other words, $\Omega_{Y_L/L}$ is locally free of rank equal to the relative dimension of $\dim Y_L$ over L if and only if $\Omega_{Y_k/k}$ is locally free of rank $\dim Y_k$. $\square$

3.2. Modeling the multiplier module. Now we reduce to characteristic p from characteristic zero.

Recall Theorem 2.31, which ensures that the test submodule is contained in the multiplier module for a variety over an F-finite field of prime characteristic. Using this, we can state

Theorem 3.4 ([**Smi97a**]). *Suppose that X is a reduced equidimensional scheme of finite type over a field of characteristic zero. Let $\mathcal{J}(\omega_X) \subseteq \omega_X$ be the multiplier submodule. Then after reduction to characteristic p from any chosen family of models $X_A \longrightarrow \operatorname{Spec} A$, we have that*

$$\tau(\omega_{X_{A/\mu}}) \subseteq \mathcal{J}(\omega_X)_{A/\mu}$$

for an open dense set of $\mu \in$ m-Spec$A$. Therefore if R has dense F-rational type, R has rational singularities.

[28]any elements of k involved in the finitely many polynomials h, and possibly the inverse of some nonzero element of A. Here, we abusively use h to denote an element of T_A or its image in any T_L.

PROOF. Without loss of generality we may assume that $X = \operatorname{Spec} R$ is affine. Fix a projective resolution of singularities $\pi : Y \longrightarrow X = \operatorname{Spec} R$ and spread it out to mixed characteristic to obtain $\pi_A : Y_A \longrightarrow R_A$ over A as in Theorem 3.3. Define a $\mathcal{J}(\omega_{R_A}) = (\pi_A)_* \omega_{Y_A} \subseteq \omega_{R_A}$, and invert an element of A to assume the inclusion splits over A.

Reduce to characteristic $p > 0$ by base changing to A/μ. Cohomology, and in particular zeroth cohomology commutes with base change for almost all $\mu \in \text{m-}\operatorname{Spec} A$, so

$$((\pi_A)_* \omega_{Y_A})_{A/\mu} \cong (\pi_{A/\mu})_* \omega_{Y_{A/\mu}}$$

by Lemma 1.14. The first part of theorem follows by Theorem 2.31.

It remains to justify why if R has dense F-rational type, then R has rational singularities. First, if R has dense F-rational type then R is Cohen-Macaulay by (d). On the other hand, if $\mathcal{J}(\omega_R) \neq \omega_R$, then this inequality will be preserved via reduction to characteristic $p > 0$ for most μ and so $\tau(\omega_{R_{A/\mu}}) \neq \omega_{R_{A/\mu}}$ as well by Lemma 1.14. This completes the proof. $\square$

3.3. Hara's surjectivity theorem. We now begin to tackle the converse direction, showing that rational singularities have open F-rational type. Our goal in this section is to prove the following result, *cf.* the work of Mehta-Srinivas [**MS97**].

Lemma 3.5 (Hara's Surjectivity Theorem)**.** *With notation as in Section 1, suppose that $\pi : Y \longrightarrow X = \operatorname{Spec} R$ is a log resolution of singularities in characteristic 0, D is a π-ample $\mathbb{Q}$-divisor with simple normal crossings support. We reduce this setup to characteristic $p > 0$ as in Section 1. Then the dual to Frobenius*

$$T^e_{Y_t} : (\pi_t)_* F^e_* \omega_{Y_t}(\lceil p^e D_t \rceil) \longrightarrow (\pi_t)_* \omega_{Y_t}(\lceil D_t \rceil)$$

surjects for a dense open set of $t \in \text{m-}\operatorname{Spec} A$

We will prove this below in Subsection 3.4, but for now we will use it.

We also need the following lemma which shows us we can find a test element independent of the characteristic.

Lemma 3.6. *Suppose R is finite type over a field of characteristic zero, reduced and equidimensional. Then there exists non-zerodivisor $d \in R$ so that for a sufficiently large finite generated $\mathbb{Z}$-algebra A, the mod p reduction d_t of d is a strong test element in R_t for an open dense set of $t \in \text{m-}\operatorname{Spec} A$.*

PROOF. By Noether normalization, we may take $C \subseteq R$ where C is smooth over k and $C \subseteq R$ is finite. Choose $\mathbf{x} = x_1, \ldots, x_n \in R$ that form a basis for $\mathcal{K}(R) = R \otimes_C \mathcal{K}(C)$ over $\mathcal{K}(C)$. By inverting an element $g \in C$ we may even assume that $R[g^{-1}]$ is a free $C[g^{-1}]$-module with basis $\mathbf{x}$. All this structure may be preserved via reduction to characteristic $p > 0$ at least for an open dense set of m-Spec$A$ and it follows by construction that the discriminant $D_{\mathbf{x}}$ of Chapter 6 Subsection 7.3 reduces to characteristic $p > 0$ as well since we may do all computations over $C[g^{-1}]$ instead of over $\mathcal{K}(C)$. In particular, if $\mathbf{x}_t$ is the mod-p reduction of $\mathbf{x}$, we have that

$$D_{\mathbf{x}_t} = (D_{\mathbf{x}})_t.$$

But now by Chapter 1 Theorem 7.12 we see that $D_{\mathbf{x}_t}$ is a strong test element. The theorem follows setting $d = D_{\mathbf{x}}$. $\square$

We are finally in a position to combine these results to show that rational singularities have open F-rational type and even better that the multiplier submodule reduces to the test submodule.

Theorem 3.7 (*cf.* **[Har98a, MS97]**). *Suppose that R is an equidimensional reduced ring over a field of characteristic zero. Suppose that $\{R_t\}$ is a family of characteristic p models of R as in Section 1 for a sufficiently large A. Then*

$$\mathcal{J}(\omega_R)_t = \tau(\omega_{R_t})$$

for all $t \in U$ where U is an open dense subset of m-Spec$A$. *In particular, if R has rational singularities, then R has open F-rational type.*

Proof. Choose a projective log resolution of singularities $\pi : Y \longrightarrow X = \operatorname{Spec} R$ of the pair $(R, V(d))$ where d is as in Lemma 3.6, a test element after reduction modulo p. We assume that π is the blow-up of some ideal $J \subseteq R$ and hence that $J \cdot \mathcal{O}_Y = \mathcal{O}_Y(-G)$ is relatively ample and G is an effective divisor.

Set $D = \varepsilon(-G - \operatorname{div}_Y(d))$ for some $1 \gg \varepsilon > 0$. It is a π-ample because $-G$ is π-ample and $\operatorname{div}_Y(d) = \pi^* \operatorname{div}_X(d)$ is π-trivial. Since $\varepsilon > 0$ is sufficiently small, we also have that $\lceil D \rceil = 0$. We reduce this setup to characteristic $p > 0$ and obtain the following diagram (at least assuming $p^e \epsilon \operatorname{div}_X d_t \geq \operatorname{div}_X d_t$).

$$\begin{tikzcd}
(\pi_t)_* F^e_* \omega_{Y_t}(\lceil p^e D_t \rceil) \arrow[r, two heads, "T^e_{Y_t}"] \arrow[d] & (\pi_t)_* \omega_{Y_t}(\lceil D_t \rceil) \arrow[r, equal] \arrow[d] & (\pi_t)_* \omega_{Y_t}. \\
F^e_*(d_t \cdot \omega_{R_t}) \arrow[r, "T^e_{R_t}"'] & \omega_{R_t}. &
\end{tikzcd}$$

For t in a dense open subset of m-Spec$A$ we see that the map $T^e_{Y_t}$ is surjective by Hara's surjectivity theorem Lemma 3.5. Restricting our t further if necessary, we may assume that d_t is a strong test element and therefore $T^e_{R_t}(F^e_* d\omega_{R_t})$ is contained in $\tau(\omega_{R_t})$. The diagram implies that

$$(\pi_t)_*\omega_{Y_t} \subseteq T^e_{R_t}(F^e_* d\omega_{R_t}) \subseteq \tau(\omega_{R_t}).$$

The reverse containment follows from Theorem 2.31. Now because

$$\mathcal{J}(\omega_R)_t = (\pi_*\omega_Y)_t = (\pi_t)_*\omega_{Y_t}$$

for an open dense set of t, the first statement of the theorem is proven.

For the statement about rational singularities, simply recall R is rational if and only if $\mathcal{J}(\omega_R) = \omega_R$ and R is Cohen-Macaulay and likewise remember that R is F-rational if and only $\tau(\omega_R) = \omega_R$ and R is Cohen-Macaulay. Since R is Cohen-Macaulay if and only if R_t is for an open set of t, the theorem is proven. □

3.4. The Cartier isomorphism, vanishing theorems, the setup to the proof of Hara's surjectivity theorem. Before we prove Hara's surjectivity theorem Lemma 3.5, we need to build some tools. First however, we recall how the (log) de Rham complex behaves in characteristic $p > 0$, at least for (log) smooth varieties. What follows is an expanded version of [**Har98a**] (also see [**MS97**]).

Suppose that Y is a smooth variety over a perfect field (of characteristic 0 or $p > 0$) and E is a reduced simple normal crossings divisor. Let $U = Y \setminus E$ with inclusion map $\nu : U \longrightarrow Y$. We define

$$\Omega^i_Y(\log E)$$

to be the subsheaf $\nu_*\Omega^i_U$ made up of sections α where α and $d\alpha \in \nu_*\Omega^{i+1}_U$ have simple (degree 1) poles along E. Explicitly, working locally at a stalk $\mathcal{O}_{Y,x}$ where $E_x = \operatorname{div}(f_1 \cdots f_t)$ for f_i part of a regular system of parameters[29] $f_1, \ldots, f_n$ for $\mathfrak{m}_x$. Then

$$(\Omega^1_Y(\log E))_x = \Big(\frac{df_1}{f_1}, \frac{df_2}{f_2}, \ldots, \frac{df_t}{f_t}, df_{t+1}, \ldots, df_n\Big)$$

is the free $\mathcal{O}_{Y,x}$-module generated by $\frac{df_1}{f_1}, \frac{df_2}{f_2}, \ldots, \frac{df_t}{f_t}, df_{t+1}, \ldots, df_n$. Because

$$\Omega^i_Y(\log E) = \bigwedge^i \Omega^1_Y(\log E)$$

we see that $(\Omega^i_Y(\log E))_x$ can be generated by ith wedge powers of the same forms. Notice that $\Omega^0_Y(\log E) = \mathcal{O}_Y$ since the wedge of no sheaves is $\mathcal{O}_Y$.

Remark 3.8. It is important to note that $\Omega^1_Y(\log E)$ is *not* the same as $\Omega^1_Y \otimes \mathcal{O}_Y(E)$. Notice that $\frac{df_{t+1}}{f_1}$ is in the latter, but not the former.

[29] In other words, the f_i define the non-singular irreducible components of E that pass through x.

The $\Omega^i_Y(\log E)$ can be put into a complex, the *logarithmic de Rham complex* but care must be taken because the differentials are *not* morphisms of $\mathcal{O}_Y$-modules. Indeed, if our base field is $\mathbb{C}$ and then the associated analytic object $(\Omega^\bullet_Y)^{\mathrm{an}}$ is a resolution of the constant sheaf $\mathbb{C}$. Hence, by GAGA, we can use $H^j(Y, \Omega^i_Y)$ to compute singular cohomology $H^{i+j}(Y^{\mathrm{an}}, \mathbb{C})$. In fact, the associated Hodge-to-de Rham spectral sequence degenerates at the E^1 page if Y is projective.

We are interested in what happens in characteristic $p > 0$ however. In that case, if $f \in \Gamma(U, \mathcal{O}_Y) = \Gamma(U, \Omega^0_Y)$ is a pth power $f = g^p$, we have that $df = 0$. In particular, $\Omega^\bullet_Y(\log E)$ is not exact like it is in characteristic 0. On the other hand, while in characteristic 0, the differential maps are not $\mathcal{O}_Y$-linear, they are only k-linear. However, in characteristic $p > 0$, since $df^p = 0$, the differential maps are $\mathcal{O}_Y^p$-linear. It immediately follows that we have the following:

Lemma 3.9. *For Y as smooth variety over a field of characteristic $p > 0$, the differential maps*

$$(F_*d) : F_*\Omega^i_Y(\log E) \longrightarrow F_*\Omega^{i+1}_Y(\log E)$$

are $\mathcal{O}_Y$-linear.

The same argument also implies that $\Omega^0_Y = \mathcal{O}_Y \subseteq \ker(F_*\Omega^0_Y \xrightarrow{F_*d} F_*\Omega^1_Y)$. It turns out that this completely describes the non-exactness of the log de Rham complex in characteristic $p > 0$.

Theorem 3.10 (The log Cartier isomorphism)**.** *Suppose that Y is a smooth variety over a perfect field of characteristic $p > 0$ and E is a reduced simple normal crossings divisor on Y. Then there is an isomorphism for all i*

$$C^{-1} : \Omega^i_Y(\log E) \cong \mathcal{H}^i(F_*\Omega^\bullet_Y(\log E)).$$

In other words, the cohomology sheaves of the complex $F_*\Omega^\bullet_Y(\log E)$ are sheaves of log differentials.

Proof. We will not work out the details, see **[EV92]** or **[BK05]** for a more detailed presentation. However, it will be useful to know that locally C^{-1} is given by $c df \mapsto [F_*c^p \cdot f^{p-1} \cdot df]$ (here $[-]$ means the equivalence class of $(-)$) for $i = 1$ and more generally for $i = n$ by

$$c \cdot df_1 \wedge \cdots \wedge df_n \mapsto [F_*c^p \cdot f_1^{p-1} \cdots f_n^{p-1} \cdot df_1 \wedge \cdots \wedge df_n].$$

$\square$

In view of the fact that the log de Rham complex is a complex of $\mathcal{O}_Y^p$-modules, there is another way to describe the De Rham complex, in the local setting, which we will find useful. First recall the following definition.

Definition 3.11. Suppose that k is a perfect field of characteristic $p > 0$ and $K \supseteq k$ is a finitely generated field extension. We say that $x_1, \dots, x_n \in K$ are a *p-basis* or a **differential basis** if $dx_1, \dots, dx_n \in \Omega^1_{K/k}$ freely generate $\Omega^1_{K/k}$ as an K-module. This is equivalent to requiring that $x_1^{a_1} \cdots x_n^{a_n}$, for $0 \leq a_i < p$, generate K over K^p by [**Sta19**, Tag 07P0].

Suppose Y is a variety over a perfect field k and that $(R = \mathcal{O}_{Y,Q}, \mathfrak{m}_Q)$ is the stalk of a closed point of $Q \in Y$. Suppose now that $\mathcal{O}_{Y,Q}$ is regular with $x_1, \dots, x_n \in \mathfrak{m}_Q$ a minimal system of generators. Since $x_1^{a_1}, \dots, x_n^{a_n}$ freely generate F_*R over R, for $0 \leq a_i < p$, we see that the x_i form a p-basis for $\mathcal{K}(Y)$ over k. Likewise, the dx_i generate $\Omega_{R/k}$ (which we can see directly as well).

Lemma 3.12. *Suppose R is a ring, smooth over a perfect field k of characteristic $p > 0$. Suppose that Ω^1_R is a free R-module generated by $dx_1, \dots, dx_n$ where $x_1, \dots, x_n \in Q \subseteq R$ are such that $E = \sum_{i=1}^t E_i = \sum_{i=1}^t \operatorname{div}(x_i)$ is a simple normal crossing divisor, here $t \leq n$. For each $j = 1, \dots, t$ define the complex of R^p-modules*

$$K_j^\bullet = \left(0 \longrightarrow \bigoplus_{i=0}^{p-1} x_j^i R^p \xrightarrow{\ d\ } \bigoplus_{i=0}^{p-1} x_j^i \frac{dx_j}{x_j} R^p \longrightarrow 0 \right)$$

and for $j = t+1, \dots, n$ write

$$K_j^\bullet = \left(0 \longrightarrow \bigoplus_{i=0}^{p-1} x_j^i R^p \xrightarrow{\ d\ } \bigoplus_{i=0}^{p-1} x_j^i dx_j R^p \longrightarrow 0 \right)$$

Then

$$\Omega^\bullet_R(\log E) \cong \bigotimes_{j=1}^{n} K_j^\bullet$$

where the tensor product is taken over R^p.

Proof. This is left as an exercise to the reader in Exercise 3.6. $\square$

For the top cohomology $n = \dim Y$, when $E = 0$ the Cartier isomorphism is stated as

$$\omega_Y = \Omega^n_Y \cong \operatorname{coker}(F_*\Omega^{n-1}_Y \longrightarrow F_*\omega_Y)$$

and we notice that it gives a surjective map $C : F_*\omega_Y \longrightarrow \omega_Y$. This map can be identified, up to isomorphism, with our dual to Frobenius $T : F_*\omega_Y \longrightarrow \omega_Y$, and we will use this in what follows. But notice that it has given us the kernel of T.

Lemma 3.13. *The map $C : F_*\omega_Y \longrightarrow \omega_Y$ induced by the Cartier isomorphism agrees, up to isomorphism, with the map $T : F_*\omega_Y \longrightarrow \omega_Y$ Grothendieck dual to Frobenius.*

Proof. Suppose $\dim Y = n$. Recall that

$$\mathscr{H}\mathrm{om}_{\mathcal{O}_Y}(F_*\omega_Y, \omega_Y) \cong F_* \mathscr{H}\mathrm{om}_{\mathcal{O}_Y}(\omega_Y, \omega_Y) \cong F_*\mathcal{O}_Y$$

and T locally generates this Hom-sheaf. It will suffice to show that $C : F_*\omega_Y \longrightarrow \omega_Y$ also generates this Hom-sheaf, and thus it suffices to show it locally agrees with T.

Working locally, suppose that $x_1, \dots, x_n$ are a regular system of parameters generating the maximal ideal $\mathcal{O}_{Y,x}$. Since $dx_1 \wedge \dots \wedge dx_n$ locally generates ω_Y, we see that we may assume that T is given by the map which sends, for $0 \leq a_i \leq p-1$,

$$F_* x_1^{a_1} \dots x_n^{a_n} dx_1 \wedge \dots \wedge dx_n \mapsto \begin{cases} dx_1 \wedge \dots \wedge dx_n & \text{if all } a_i = p-1, \text{ and} \\ 0 & \text{otherwise} \end{cases}.$$

For C, notice that $C^{-1}(dx_1 \wedge \dots \wedge dx_n) = [F_* x_1^{p-1} \dots x_n^{p-1} dx_1 \wedge \dots d \wedge dx_n]$ and hence $C : F_*\omega_Y \longrightarrow \omega_Y$ also sends $F_*(x_1^{p-1} \dots x_n^{p-1} dx_1 \wedge \dots d \wedge dx_n)$ to $dx_1 \wedge \dots \wedge dx_n$. By Exercise 2.8, this shows that C must also (locally) generate the Hom-sheaf. This completes the proof. $\square$

We now follow a modified version of Hara's proof. We first prove one of Hara's lemmas [**Har98a**, Lemma 3.3], the proof is essentially as in *loc. cit.*.

Lemma 3.14. *Suppose that Y is a smooth projective variety over a perfect field of characteristic $p > 0$ and $E = \sum_{j=1}^t E_j$ is a reduced simple normal crossings divisor with irreducible components E_j with complement $U = Y \setminus E \xrightarrow{\nu} Y$. Suppose that $B = \sum_{j=1}^t r_j E_j$ is another divisor with $0 \leq r_j \leq p-1$. We have inclusions*

$$\Omega_Y^i(\log E) \subseteq \Omega_Y^i(\log E) \otimes \mathcal{O}_Y(B) \subseteq \nu_*\Omega_U^i.$$

The differential from $\nu_*\Omega_U^i$ restricted to $\Omega_Y^i(\log E) \otimes \mathcal{O}_Y(B)$ makes $\Omega_Y^{\bullet}(\log E) \otimes \mathcal{O}_Y(B)$ into a complex and the induced map of complexes

$$\Omega_Y^{\bullet}(\log E) \longrightarrow \Omega_Y^{\bullet}(\log E) \otimes \mathcal{O}_Y(B) \tag{3.14.1}$$

is a quasi-isomorphism.

Proof. First we need to show that the differential $d : \nu_*\Omega_U^i \longrightarrow \nu_*\Omega_U^{i+1}$ restricts to

$$\Omega_Y^i(\log E) \otimes \mathcal{O}_Y(B) \longrightarrow \Omega_Y^{i+1}(\log E) \otimes \mathcal{O}_Y(B).$$

Fix a point $x \in Y$ and work locally using that E is a simple normal crossings divisor in Y. We may suppose that at x, the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{Y,x}$ is generated by $x_1, \ldots, x_n$ where $E_j = V(x_j)$ for $j = 1, \ldots, t \leq n$, the E_j passing through $x \in Y$. In this case $\Omega^i_Y(\log E)_x$ is generated by a wedge of i distinct forms from $\frac{dx_1}{x_1}, \ldots, \frac{dx_t}{x_t}, dx_{t+1}, \ldots, dx_n$. Furthermore $\Omega^0_Y(\log E) \otimes \mathcal{O}_Y(B) \cong \mathcal{O}_Y(B)$. By direct computation, one sees that $d : \mathcal{O}_Y(B) \longrightarrow \Omega^1_Y(\log E) \otimes \mathcal{O}_Y(B)$ is well defined since $\frac{1}{x_i^r} \mapsto \frac{-r}{x_i^{r+1}} dx_i$ for $i = 1, \ldots, t$. For degree 1, notice that if $1 \leq j \leq t$, then for any $1 \leq l \leq n$,

$$d\left(\frac{1}{x_j^r} dx_l\right) = \frac{-r}{x_j^{r+1}} dx_j \wedge dx_l \in \Omega^0_Y(\log E) \otimes \mathcal{O}_Y(B).$$

The higher degree computations are similar.

To show that (3.14.1) is a quasi-isomorphism we work locally, say on $\operatorname{Spec} R$. We can then describe $\Omega^\bullet_Y(\log E)$ as in Lemma 3.12. Set $L^\bullet_j = K^\bullet_j \cdot {x_j}^{-r_j}$ for $1 \leq j \leq t$, where the differential is still chosen by restricting the differential of $\nu_* \Omega^\bullet_U$ (and *not* post multiplying the differential by $x_j^{-r_j}$). For $j > t$, we set $L^\bullet_j = K^\bullet_j$. This is chosen so that

$$\Omega^\bullet_R(\log E) \otimes R(B) \cong L^\bullet_1 \otimes \cdots \otimes L^\bullet_t \otimes K^\bullet_{t+1} \otimes \cdots \otimes K^\bullet_n.$$

where the tensor products are over R^p.

Claim 3.15. *$K^\bullet_j \longrightarrow L^\bullet_j$ is a quasi-isomorphism.*

Proof of claim. We only need consider ourselves with $1 \leq j \leq t$ since the objects defined to be the same for $j > t$. The kernel of the differential of $K^\bullet_j$ is $R^p \subseteq R$. Since in $K^\bullet_j$, we send $x_j^{a_j} \mapsto x_j^{a_j} \frac{dx_j}{x_j}$, we see that the cokernel is the R^p summand of K^1_j generated by $\frac{dx_j}{x_j}$.

Now, for $L^\bullet_j$, our basis for L^0_j is $x_j^{-r_j}, x_j^{-r_j+1}, \ldots, x_j^{-r_j+p-1}$ and the kernel of the differential is still the summand $R^p \subseteq L^0_j$ (notice we are using the $0 \leq r_j < p$). Likewise, the cokernel of the differential is the R^p-summand generated by $\frac{dx_j}{x_j}$. The map $K^\bullet_j \longrightarrow L^\bullet_j$ is a quasi-isomorphism. $\square$

The map $\Omega^\bullet_Y(\log E) \longrightarrow \Omega^\bullet_Y(\log E) \otimes \mathcal{O}_Y(B)$ is then simply induced by the tensor product of the $K^\bullet_j \longrightarrow L^\bullet_j$. The lemma follows. $\square$

3.5. The proof of Hara's surjectivity theorem. Our first result shows that our desired surjectivity can be reduced to certain vanishing theorems. This follows [**Har98a**, Proposition 3.6].

Proposition 3.16. *Suppose that Y is a smooth n-dimensional variety over a perfect field k of characteristic $p > 0$ and E is a simple normal crossings divisor on Y. Suppose that D is a $\mathbb{Q}$-divisor on Y and suppose that $\{D\} := D - \lfloor D \rfloor$, the fractional part of D, satisfies* $\operatorname{Supp}\{D\} \subseteq \operatorname{Supp} E$. *Further suppose that we have the following vanishing of cohomologies:*

(a) $H^j(Y, \Omega^i_Y(\log E) \otimes \mathcal{O}_Y(-E + \lceil D \rceil)) = 0$ *if* $j > 1$ *and* $i + j = n + 1$, *and*
(b) $H^j(Y, \Omega^i_Y(\log E) \otimes \mathcal{O}_Y(-E + \lceil pD \rceil)) = 0$ *if* $j > 0$ *and* $i + j = n$.

Then

$$T : H^0\big(Y, F_* \mathcal{O}_Y(K_Y + \lceil pD \rceil)\big) \longrightarrow H^0\big(Y, \mathcal{O}_Y(K_Y + \lceil D \rceil)\big).$$

is surjective.

Before proving this, notice that the T map is the one we want to show is surjective (at least when only taking F^1_*).

Proof. Set $B := (p-1)E - p\lceil D \rceil + \lceil pD \rceil$ and notice that $\operatorname{Supp} B \subseteq \operatorname{Supp} E$ and that the coefficients of B are between 0 and $p - 1$. Combining the Cartier isomorphism Theorem 3.10 with Lemma 3.14, we have a variant of the log Cartier isomorphism:

$$\Omega^i_Y(\log E) \cong \mathcal{H}^i\Big(F_*\big(\Omega^\bullet_Y(\log E) \otimes \mathcal{O}_Y(B)\big)\Big).$$

We tensor by $\mathcal{O}_Y(-E + \lceil D \rceil)$ to obtain a twisted log Cartier isomorphism:

$$\begin{aligned} & \Omega^i_Y(\log E) \otimes \mathcal{O}_Y(-E + \lceil D \rceil) \\ \cong\ & \mathcal{H}^i\Big(F_*\big(\Omega^\bullet_Y(\log E) \otimes \mathcal{O}_Y(B - pE + p\lceil D \rceil)\big)\Big) \\ \cong\ & \mathcal{H}^i\Big(F_*\big(\Omega^\bullet_Y(\log E) \otimes \mathcal{O}_Y(-E + \lceil pD \rceil)\big)\Big) \end{aligned}$$

Hence, for every $i = 0, \ldots, n$, we have two short exact sequences, the first breaking up our twisted log de Rham complex into cycles and boundaries:

$$0 \longrightarrow Z^i \longrightarrow F_*\big(\Omega^i_Y(\log E) \otimes \mathcal{O}_Y(-E + \lceil pD \rceil)\big) \longrightarrow B^{i+1} \longrightarrow 0 \tag{3.16.1}$$

and the second exhibiting the twisted log Cartier isomorphism

$$0 \longrightarrow B^i \longrightarrow Z^i \longrightarrow \Omega^i_Y(\log E) \otimes \mathcal{O}_Y(-E + \lceil D \rceil) \longrightarrow 0. \tag{3.16.2}$$

We take global sections and notice that for $i = n$, $\Omega^n_Y(\log E) = \omega_Y(E) \cong \mathcal{O}_Y(K_Y + E)$ and so

$$Z^n = F_*\big(\Omega^n_Y(\log E) \otimes \mathcal{O}_Y(-E + \lceil pD \rceil)\big) \cong F_* \mathcal{O}_Y(K_Y + \lceil pD \rceil).$$

Therefore (3.16.2) provides us with the exact sequence

$$H^0\big(Y, F_* \mathcal{O}_Y(K_Y + \lceil pD \rceil)\big) \to H^0(Y, \mathcal{O}_Y(K_Y + \lceil D \rceil)) \to H^1(Y, B^n).$$

The first map is the one we are trying to show is surjective, therefore it suffices to show that $H^1(Y, B^n) = 0$. Using now (3.16.1), we have the exact sequence

$$H^1\big(F_*\big(\Omega_Y^{n-1}(\log E) \otimes \mathcal{O}_Y(-E + \lceil pD \rceil)\big)\big) \longrightarrow H^1(Y, B^n) \longrightarrow H^2(Y, Z^{n-1}).$$

The first term vanishes by (b) and so it suffices to show that $H^2(Y, Z^{n-1}) = 0$. Returning to (3.16.2) we have the exact sequence

$$H^2(Y, B^{n-1}) \longrightarrow H^2(Y, Z^{n-1}) \longrightarrow H^2(Y, \Omega_Y^{n-1}(\log E) \otimes \mathcal{O}_Y(-E + \lceil D \rceil))$$

where the right term vanishes by (a) and so it suffices to show that $H^2(Y, B^{n-1}) = 0$. Continuing in this way, we see that it suffices to show that $H^3(Y, Z^{n-2})$ vanish, or that $H^3(Y, B^{n-2}) = 0$ etc. But $H^{n+1}(Y, Z^0) = 0$ since $n = \dim Y$. This proves the proposition. $\square$

We thus need to arrange that (a) and (b) hold after reduction to characteristic $p > 0$. In characteristic zero, in our case where Y is a resolution of singularities, (a) is a relative variant of Kodaira-Nakano-Akizuki vanishing, now stated.

Theorem 3.17 ([**Har98a, AMPWo23**], *cf.* [**Kod53, Nak73, Nak75, AN54, DI87**]). *Suppose that $\pi : Y \longrightarrow X = \operatorname{Spec} R$ is a resolution of singularities of an affine variety in characteristic zero and E is a reduced simple normal crossings divisor on Y. Further suppose that D is a π-relatively ample divisor such that $\operatorname{Supp}(\{D\}) = \operatorname{Supp}(D - \lfloor D \rfloor) \subseteq E$. Then*

$$H^j\big(Y, \Omega_Y^i(\log E) \otimes \mathcal{O}_Y(-E + \lceil D \rceil)\big) = R^j \pi_* \Big(\Omega_Y^i(\log E) \otimes \mathcal{O}_Y(-E + \lceil D \rceil)\Big) = 0$$

as long as $i + j > \dim Y$.

In view of Lemma 1.14, this vanishing still holds after reduction to characteristic $p \gg 0$, as long as we know that $(\Omega_X)_t = (\Omega_{X_t})$ in our modulo p reduction. We show this below in Exercise 3.7.

On the other hand (b) will hold for $p \gg 0$ by a version of Serre vanishing. We use relative Castelunovo-Mumford regularity.

Definition 3.18. Suppose that $\pi : Y \longrightarrow X = \operatorname{Spec} R$ is a projective morphism and $\mathscr{L}$ is an ample globally generated line bundle on Y. We say that a coherent sheaf $\mathscr{M}$ on Y is **m-regular (with respect to $\mathscr{L}$)**, or simply *m-regular*, if

$$H^i(Y, \mathscr{M} \otimes \mathscr{L}^{m-i}) = 0$$

for all $i > 0$.

By Serre vanishing, every coherent sheaf is m-regular (with respect to any ample $\mathscr{L}$) for some $m \gg 0$.

Theorem 3.19 (Relative Castelnuovo-Mumford Regularity)**.** *Suppose that* $\pi : Y \longrightarrow X = \operatorname{Spec} R$ *is a projective morphism and* $\mathscr{L}$ *is an ample globally generated line bundle on* Y*. Suppose* $\mathscr{M}$ *is an* m*-regular coherent sheaf on* Y*, then:*

(a) $\mathscr{M}$ *is* $m+1$*-regular.*
(b) $\mathscr{M} \otimes \mathscr{L}^m$ *is generated by global sections.*
(c) *The map induced by multiplication*

$$H^0(Y, \mathscr{M} \otimes \mathscr{L}^m) \otimes_R H^0(Y, \mathscr{L}) \longrightarrow H^0(Y, \mathscr{M} \otimes \mathscr{L}^{m+1})$$

is surjective.

Proof. Castelnuovo-Mumford regularity is normally stated for X projective over a field, see for instance [**Laz04a**, Theorem 1.8.5]. However, it holds for X projective over a Noetherian ring as well when $\mathscr{L}$ is relatively very ample, see [**Ooi82**] for a statement in terms of local cohomology on a graded ring and [**Kee03**, Proposition 4.9] for the geometric phrasing. In the case that $\mathscr{L}$ is only ample and globally generated, we may choose N global generators of $H^0(Y, \mathscr{L})$ to induce a map

$$\kappa : Y \longrightarrow \mathbb{P}^N_X.$$

Since $\mathscr{L}$ is globally generated and ample, this map is finite and so κ_* is exact. Thus for all $i \geq 0$,

$$H^i(Y, \mathscr{M} \otimes \mathscr{L}^m) \cong H^i\big(\mathbb{P}^N_X, \kappa_*(\mathscr{M} \otimes \mathscr{L}^m)\big) \cong H^i\big(\mathbb{P}^N_X, (\kappa_*\mathscr{M}) \otimes \mathcal{O}_{\mathbb{P}^N_X}(m)\big)$$

and so $\mathscr{M}$ is m-regular with respect to $\mathscr{L}$ if and only if $\kappa_*\mathscr{M}$ is m-regular with respect to $\mathcal{O}_{\mathbb{P}^N_X}(1)$. The consequences (a), (b), (c) of Castelnuovo-Mumford regularity may likewise be translated back to Y and so we may cite the above references. □

Lemma 3.20. *Suppose that* Y *is a variety of characteristic zero and* $\pi : Y \longrightarrow X = \operatorname{Spec} R$ *is a projective morphism. Further suppose that* $\mathscr{M}$ *is a coherent sheaf on* Y *and* $\mathscr{L}$ *is a* π*-ample line bundle. We reduce this setup to characteristic* $p > 0$*. Then for each* $i > 0$*, there exist an* n_0 *and an open dense* $U \subseteq \operatorname{m-Spec} A$ *so that for all* $n \geq n_0$ *and* $t \in U$*, we have that*

$$H^i(Y_t, \mathscr{M}_t \otimes \mathscr{L}_t^n) = R^i(\pi_t)_*\big(\mathscr{M}_t \otimes \mathscr{L}_t^n\big) = 0.$$

Proof. First suppose that $\mathscr{L}$ is π-very ample. In this case we see that if $\mathscr{M}$ is n_0-regular for some n_0, then by Lemma 1.14, so is $\mathscr{M}_t$. Hence

$$H^i(Y_t, \mathscr{M}_t \otimes \mathscr{L}_t^{n-i}) = 0$$

for all $n \geq n_0$ by Theorem 3.19(a). The result follows for all $t \in U_0$. Now, suppose that $\mathscr{L}$ is not very ample but that $\mathscr{L}' = \mathscr{L}^N$ is very ample. Then we may apply our above work and Theorem 3.19(a) for $\mathscr{L}'$ and simultaneously for the sheaves

$$\mathscr{M}, \mathscr{M} \otimes \mathscr{L}, \mathscr{M} \otimes \mathscr{L}^2, \dots, \mathscr{M} \otimes \mathscr{L}^{N-1}$$

obtaining $U_0, U_1, \dots, U_{N-1}$ and $n_{0,0}, \dots, n_{0,N-1}$. Now set $U = \bigcap U_i$ and $n_0 = \max n_{0,i}$, the result follows. □

Finally, we can combine our results to prove Hara's surjectivity.

Proof of Lemma 3.5. We reduce our setup to characteristic $p > 0$, using the notation of Section 1. Enlarging A if necessary, we may assume that components of E are geometrically irreducible, which means that they will not split up into new components in our reduction to characteristic $p > 0$. We must show that

$$T^e_{Y_t} : F^e_* \omega_{Y_t}(\lceil p^e D_t \rceil) \longrightarrow \omega_{Y_t}(\lceil D_t \rceil) \tag{3.20.1}$$

surjects for all $e > 0$. Choose m such that mD is a $\mathbb{Z}$-divisor. As n varies, we notice that we may write $\lceil nD \rceil = \lfloor \frac{n}{m} \rfloor mD + R_n$ where R_n takes on only finitely many values. Therefore, using Lemma 3.20 for finitely many modules, we may restrict to an open set $U \subset \text{m-Spec} A$, where all $p = \text{char}(A/t)$ satisfy $\lfloor \frac{p}{m} \rceil \geq n_0$, we may assume that

$$\begin{aligned} & H^j(Y_t, \Omega^i_{Y_t}(\log E_t) \otimes \mathcal{O}_{Y_t}(-E_t + \lceil p^e D_t \rceil)) \\ = \; & H^j(Y_t, \Omega^i_{Y}(\log E_t) \otimes \mathcal{O}_{Y_t}(-E_t + \lfloor \tfrac{p^e}{m} \rfloor m D_t + (R_n)_t)) \\ = \; & 0 \end{aligned} \tag{3.20.2}$$

for all $j > 0$, $e > 0$ and all $t \in U$.

From Theorem 3.17, we may also assume that

$$H^j(Y_t, \Omega^i_{Y_t}(\log E_t) \otimes \mathcal{O}_{Y_t}(-E_t + \lceil D_t \rceil)) = 0$$

for $i + j > \dim Y_t$, since the same vanishing holds in characteristic zero.

By Proposition 3.16, we thus have that

$$T_1 : H^0\big(Y_t, F_* \mathcal{O}_{Y_t}(K_{Y_t} + \lceil p D_t \rceil)\big) \longrightarrow H^0\big(Y_t, \mathcal{O}_{Y_t}(K_{Y_t} + \lceil D_t \rceil)\big)$$

is surjective. Even more, we also have that

$$T_e : H^0\big(Y_t, F_* \mathcal{O}_{Y_t}(K_{Y_t} + \lceil p^{e+1} D_t \rceil)\big) \longrightarrow H^0\big(Y_t, \mathcal{O}_{Y_t}(K_{Y_t} + \lceil p^e D_t \rceil)\big)$$

surjects for all $e > 0$ by using (3.20.2) for (a) in Proposition 3.16.

Hence $T^e_{Y_t}$ is composition of surjective maps

$$T^e_{Y_t} = T_1 \circ (F_* T_2) \circ \dots \circ F^{e-1}_* T_{e-1}.$$

This completes the proof of Lemma 3.5. □

3.6. Exercises.

Exercise 3.1. Suppose that $\pi : Y \longrightarrow X$ is a proper dominant generically finite map of locally equidimensional schemes of finite type over a field of characteristic zero. We shall show that there exists a further proper dominant generically finite $\pi' : Y' \longrightarrow X$ that factors through π and such that there exists a finite map $h : Y' \longrightarrow X'$ with $X' \longrightarrow X$ proper birational and $X' \longrightarrow X$ is a resolution of singularities of X. We do this in the following steps.

(a) Show that we may assume that X is normal and integral, Y is normal and integral, and that $\mathcal{K}(X) \subseteq \mathcal{K}(Y)$ is Galois with Galois group G. Further use Chow's Lemma [**Har77**, Chapter II, Exercise 4.10] to show that we may assume that Y is projective.
(b) Show that we can factor $\pi : Y \xrightarrow{\nu} S \longrightarrow X$ where $S \longrightarrow X$ is finite and $Y \longrightarrow S$ is a blowup of some ideal sheaf J. Show that the quotient of S by G is X.
(c) Let $J'' = \prod_{\sigma \in G} \sigma(J)$ and show that $\nu'' : Y'' \longrightarrow S$, the blowup of J'', factors through the blowup of J.
(d) Show that Y'' has a G action as well and let X'' denote the quotient of Y' by G. Finally, resolve the singularities X'' by $X' \longrightarrow X''$ and take Y' to be the normalization of X' in $\mathcal{K}(Y)$. Show that $Y' \longrightarrow X'$ satisfies the desired properties.

Exercise 3.2. Let $(R, \mathfrak{m})$ be a local d-dimensional Noetherian domain with normalized dualizing complex $\omega_R^\bullet$. Suppose that $\pi : Y \longrightarrow X = \operatorname{Spec} R$ is a proper dominant map of integral schemes with dualizing complex $\omega_Y = \pi^! \omega_R^\bullet$. Show that the map

$$\mathcal{H}^{-d}\mathbf{R}\pi_*\omega_Y^\bullet \longrightarrow \mathcal{H}^{-d}\omega_R^\bullet =: \omega_R$$

is nonzero. In the case that R is F-finite and of characteristic $p > 0$, deduce that

$$\tau(\omega_R) \subseteq \operatorname{Image}(\mathcal{H}^{-d}\mathbf{R}\pi_*\omega_Y^\bullet \longrightarrow \mathcal{H}^{-d}\omega_R^\bullet).$$

Show that this also implies, and hence generalizes, Theorem 2.31.

Hint: Reduce to the case that R is a field and use Grothendieck-Serre duality.

Exercise 3.3. Let $S = \mathbb{F}_2[x, y, z]$ and let $f = z^2 + x^2y + xy^2$. Show that $R = S/(f)$ is not F-pure by Fedder's criterion and deduce that it is not F-rational. Then show that $\operatorname{Spec} R$ *does* have rational singularities.

Hint: For the second part, it suffices to find a resolution of singularities $\pi : Y \longrightarrow X = \operatorname{Spec} R$ such that $K_Y \geq 0$ (notice we may chose $K_X = 0$). Compute an embedded resolution of singularities of X in $\operatorname{Spec} S$ by blowing up points. Next compute the relative canonical divisor of that map over a non-singular base, and use the adjunction formula to compute K_Y.

Exercise 3.4. Prove the Cartier isomorphism Theorem 3.10 in the special case that $X = \mathbb{P}^1_{\mathbb{F}_p}$ and $E = 0$.

Exercise 3.5. For $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$, show that the map $C^{-1} : \Omega^1_X \mapsto \mathcal{H}^1(F_*\Omega^{\bullet}_X)$ given by $f \mapsto [f^{p-1}df]$. Show that this map is well defined and additive. Then show it is injective. Finally show it is surjective.

Hint: To show it is additive, write $h = \frac{1}{p}((f+g)^p - f^p - g^p) = \sum_{i=1}^{p-1} \binom{p}{i}/pf^ig^{p-i}$. Compute dh and use it to show that $C^{-1}(df + dg) = C^{-1}(df) + C^{-1}dg$. For surjectivity, first verify it for $\mathbb{A}^1$. For a detailed writeup, see for instance [**BK05**, Theorem 1.3.4].

Exercise 3.6. Prove Lemma 3.12.

Exercise 3.7. Suppose that X is a smooth quasi-projective variety of a field k of characteristic zero and E is a SNC divisor on X. With notation as in Section 1, show that

$$(\Omega^1_{X/k}(\log E))_t \cong \Omega^1_{X_t/k(t)}(\log E_t)$$

and conclude that also $(\Omega^i_{X/k})_t \cong \Omega^i_{X_t/k(t)}$ for all i. Notice that we may *not* reduce the differentials of the de Rham complex to characteristic $p > 0$ as they are not $\mathcal{O}_X$-linear.

Hint: The most direct route is to work on charts and keep track of transition maps in the reduction mod p process. Notice that while $\nu_*\Omega_U \supseteq \Omega^1_{X/k}(\log E)$ cannot be reduced modulo p, since it is only quasi-coherent, we can reduce the intermediate sheaf $\Omega^1_{X/k} \otimes \mathcal{O}_X(E)$. There is a quick way to reduce the *non*-log differentials. Suppose $X \longrightarrow X \times_k X$ denotes the diagonal map with kernel ideal sheaf I, then I/I^2 may be identified with $\Omega_{X/k}$.

4. Log terminal, log canonical singularities and multiplier ideals and their characteristic $p > 0$ analogs

Now that we have handled rational and F-rational singularities, we turn our attention to log canonical and log terminal singularities, the characteristic zero analog of locally F-split and and strongly F-regular singularities. We will also study the multiplier ideal, a characteristic zero analog of the test ideal.

4.1. Log terminal and log canonical singularities. Our first goal is to define log terminal and log canonical singularities. For a much more complete treatment, see for instance [**KM98, Kol13**].

We first explain how to pull back $\mathbb{Q}$-Cartier divisors. Suppose $\pi : Y \longrightarrow X$ is a dominant map between normal integral Noetherian schemes. If D is a Cartier divisor on X, then we define π^*D to be the divisor obtained by pulling back the local defining equations of D: if locally on $U \subseteq X$, $D|_U = \mathrm{div}_U(f)$, then $(\pi^*D)|_{\pi^{-1}U} = \mathrm{div}_{\pi^{-1}U}(f)$. Notice that we have $\mathcal{O}_Y(\pi^*D) = \pi^*\mathcal{O}_X(D)$.

If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, meaning that mD is a Cartier divisor (with integer coefficients) for some $m > 0$, we define

$$\pi^*D := \frac{1}{m}\pi^*(mD). \tag{4.0.1}$$

See Appendix B Subsection 6.1 and Exercise 4.1 for additional discussion.

Remark 4.1. If D is not $\mathbb{Q}$-Cartier, there are also ways to define the pullback of D. See **[DH09]** for two options.

Setting 4.2 (Choosing canonical divisors somewhat canonically)**.** Suppose that $\pi : Y \longrightarrow X$ is a birational map of normal Noetherian integral schemes with dualizing complexes. Suppose first that π is proper. If one fixes a canonical divisor K_Y, then as in the proof of Lemma 2.20, $K_X := \pi_*K_Y$ is a canonical divisor on X (recall, we simply remove any components of K_Y that are contracted to non-divisors). Conversely, if K'_X is another canonical on X, then $K'_X - K_X = \mathrm{div}_X(f)$ for some $f \in \mathcal{K}(X)$. Hence we can define $K'_Y = K_Y + \mathrm{div}_Y(f)$ as another canonical divisor on Y and we still have $\pi_*K'_Y = K'_X$. Of course, if π is not proper then we may compactify π via Nagata's compactification $\overline{\pi} : \overline{Y} \longrightarrow X$ **[Nag63]**, normalizing if needed. We may then pick $K_{\overline{Y}}$ and restrict it to the open set U to obtain K_Y.

Going forward, for any birational map π, we always pick our canonical divisors on Y and X compatibly as described above.

Definition 4.3 (Discrepancies)**.** Suppose that $\pi : Y \longrightarrow X$ is a birational map of normal Noetherian integral schemes. Let Δ be a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (we call (X, Δ) a *log $\mathbb{Q}$-Gorenstein pair*). Choose K_Y and K_X as in Setting 4.2. Write

$$K_Y - \pi^*(K_X + \Delta) = \sum_{i=1}^{t} a_i E_i$$

where the E_i are prime divisors. The number a_i is called the **discrepancy of** (X, Δ) **along** E_i. The number $a_i + 1$ is called the **log discrepancies of** (X, Δ) **along** E_i. If $\Delta = 0$, in which case X is $\mathbb{Q}$-Gorenstein,we simply call a_i's the **discrepancy of** X **along** E_i.

Lemma 4.4. *The discrepancies defined in Definition 4.3 are independent of the choice of canonical divisor K_X.*

Proof. Left to the reader in Exercise 4.2. $\square$

Example 4.5. Suppose that $X = \operatorname{Spec} k[x, y, z]/(xy - z^2)$ is the affine quadric cone. Let $Y \longrightarrow X$ be the blowup of the origin (x, y, z). We embed $X \subseteq A = \mathbb{A}^3$ and notice that if we blowup the origin $\pi : P \longrightarrow A$ then Y is the strict transform of X in P. Let $E \cong \mathbb{P}^2 \subseteq P$ denote the exceptional divisor and notice that $E \cap Y$ is the exceptional divisor of $\pi : Y \longrightarrow X$.

We may take $K_A = 0$. In this case $K_P = 2E$ by [**Har77**, Chapter II, Exercise 8.5]. We notice that $\pi^* X = Y + 2E$ since $xy - z^2$ has multiplicity two at the origin. By the adjunction formula

$$K_Y = (K_P + Y)|_Y = (2E + \pi^* X - 2E)|_Y = (\pi^* X)|_Y \sim_P 0|_Y = 0.$$

Notice here that $\pi^* X \sim_P 0$ since $X \sim_A 0$. We've shown that $K_Y \sim 0$ and thus we may choose $K_X = \pi_* K_Y = 0$ as well (recall, pushing forward divisors discards any exceptional components).

It follows that the discrepancy of $(X, 0)$ along E is 0. Now, let $D = \operatorname{div}_X((x, z))$, this is a prime divisor with $2D = \operatorname{div}_X(x)$, see [**Har77**, Chapter II, Example 6.11.3], and hence $K_X + D = D$ is $\mathbb{Q}$-Cartier. Abusing notation and writing $\pi = \pi|_Y$, we have that

$$\pi^*(K_X + D) = \frac{1}{2} \operatorname{div}_Y(x).$$

We have that $\operatorname{div}_Y(x) = E + 2D_Y$ where D_Y is the strict transform of D. Thus $K_Y - \pi^*(K_X + D) = -\frac{1}{2}E - D_Y$ and so (X, D) has discrepancy $-\frac{1}{2}$ along E. Notice that (X, D) has a rational discrepancy even though every coefficient of D is an integer.

We now return to the general case where we have written:

$$K_Y - \pi^*(K_X + \Delta) = \sum_{i=1}^{t} a_i E_i$$

Suppose that $\eta_i \in Y$ is the generic point of E_i, so that $\mathcal{O}_{Y,\eta_i} \subseteq \mathcal{K}(Y) = \mathcal{K}(X)$ is a discrete valuation ring. The next lemma says that the numbers a_i depend only on this valuation ring. For instance, if one starts with a resolution singularities $Y \longrightarrow X$, and then takes a further log resolution $Y' \longrightarrow Y \longrightarrow X$, the discrepancies from Y need not be recomputed on Y'.

Lemma 4.6. *Suppose that $\pi : Y \longrightarrow X$ and $\pi' : Y' \longrightarrow X$ are two birational maps between normal integral Noetherian schemes and $E_i \subseteq Y$ and $E'_j \subseteq Y'$ are two divisors whose corresponding valuation rings are the same inside the fraction field $\mathcal{K}(X)$. Suppose now that (X, Δ) is a log $\mathbb{Q}$-Gorenstein pair. Then the discrepancy of (X, Δ) along E_i is the same as its discrepancy along E'_j.*

Proof. Left as an exercise in Exercise 4.3. □

In view of this, we frequently work independently of the birational model and simply talk about discrepancies along a divisorial[30] discrete valuation ring (in fact, there are even generalizations of discrepancies to non-divisorial valuation rings, see for instance [**FJ04, JM12, BdFFU15, Can20**]). Or more commonly, simply talk about the discrepancy of (X, Δ) along some divisor E without specifying the birational model.

Definition 4.7 (Log canonical and log terminal singularities)**.** Suppose that $(X, \Delta \geq 0)$ is a log $\mathbb{Q}$-Gorenstein pair.

(a) We say that (X, Δ) is **log canonical (LC)** if every discrepancy is ≥ -1.
(b) We say that (X, Δ) is **Kawamata log terminal (KLT)** if every discrepancy is > -1.
(c) We say that (X, Δ) is **purely log terminal (PLT)** if every discrepancy along an exceptional[31] E is > -1.
(d) We say that (X, Δ) is **canonical** if every discrepancy along an exceptional E is ≥ 0.
(e) We say that (X, Δ) is **terminal** if every discrepancy along an exceptional E is > 0.

If $\Delta = 0$, we ascribe those same definitions to X itself.

Remark 4.8. We will not study the notions of *canonical* or *terminal* singularities in any real way, but they are key notions within the minimal model program [**KM98**]. However, it is still an open question whether there are any natural notions of singularities in characteristic $p > 0$ that correspond to to canonical or terminal singularities, see also [**TW04**, Proposition 3.5] in the three-dimensional case.

Remark 4.9. In the case that X is quasi-Gorenstein (that is K_X is Cartier), we see that X is KLT if and only if it is canonical. The point is that $K_Y - \pi^* K_X$ is a divisor with integer coefficients. Hence if it's coefficients are > -1 they are also ≥ 0.

Fortunately, we do not need to compute all possible birational maps $\pi : Y \longrightarrow X$. A sufficiently good resolution of singularities will suffice.

Proposition 4.10. *Suppose that $\pi : Y \longrightarrow X$ is a log resolution of singularities of a log-$\mathbb{Q}$-Gorenstein pair (X, Δ). Write*

$$K_Y - \pi^*(K_X + \Delta) = \sum_{i=1}^{t} a_i E_i.$$

[30]meaning it appears as the local ring of a divisor on some birational model

[31]meaning that E is an exceptional divisor in some birational model

If all $a_i \geq -1$, then (X, Δ) is log canonical and if all $a_i > -1$, then (X, Δ) is Kawamata log terminal.

Further suppose that the strict transform $\pi_*^{-1}\Delta$ of Δ non-singular (for instance, if Δ has only one component). Finally, if all exceptional discrepancies along exceptional E_i are > -1 (respectively ≥ 0, > 0) we have that (X, Δ) is purely log terminal (respectively canonical, terminal).

For the log terminal and log canonical case, this is contained in [**Kol13**, Corollary 2.13] and the idea is very similar to that of Lemma 2.20. For the other cases the condition that the strict transform is nonsingular is necessary. For example, if $\Delta = \mathrm{div}(x) + \mathrm{div}(y) \subseteq \operatorname{Spec} k[x, y] = \mathbb{A}^2$, then $(\mathbb{A}^2, \Delta)$ is already in simple normal crossings and so the identity map $\mathbb{A}^2 \longrightarrow \mathbb{A}^2$ is its own log resolution. But this pair is not canonical, terminal, or purely log terminal as the blowup of the origin creates an exceptional divisor with discrepancy -1. It is however divisorially log terminal, a notion we are not introducing [**Kol13**].

Proof of Proposition 4.10. We leave the details to the reader in Exercise 4.5. $\square$

Example 4.11. In Example 4.5 our blowup was a log resolution. Hence we see immediately that the pair $(X, 0)$ is canonical, and hence also KLT. However, the pair (X, D) of the same example is PLT, and hence LC, but it is not KLT. The reason it is not KLT is that the discrepancy of (X, D) along D itself, is -1.

Example 4.12. Suppose that $H \subseteq \mathbb{P}^n_k$ is a non-singular hypersurface of degree d. Let $X \subseteq \mathbb{A}^{n+1}$ denote the affine cone over H. Then we see that

(a) X is LC if and only if $d \leq n + 1$.
(b) X is KLT (equivalently canonical since X is Gorenstein) if and only if $d \leq n$.
(c) X is terminal if and only if $d \leq n - 1$.

Let $\pi : Z \longrightarrow \mathbb{A}^{n+1}$ denote the blow up of the origin (the cone point) with exceptional divisor $E \cong \mathbb{P}^n$. This blowup also provides an (embedded) log resolution of X because H itself was non-singular. Hence if $Y \subseteq Z$ is the strict transform of X, we need to compute K_Y. Observe that we may take $K_{\mathbb{A}^{n+1}} = 0$ and hence $K_Z = nE$. Notice that $\pi^*X = Y + dE$. We compute K_Y as follows.

$$K_Y \sim (K_Z + Y)|_Y = (nE + \pi^*X - dE)|_Y = (n - d)E|_Y$$

The linear equivalence is the adjunction formula and the final equality is because $(\pi^*X)|_E = 0$ (since up to linear equivalence, X can be moved away from the cone point). In this case, $E|_Y$ is the reduced exceptional divisor of $\pi|_Y : Y \longrightarrow X$ and so our only discrepancy is exactly $n-d$. Notice that $-1 \leq n-d$ is equivalent to the assertion that $d \leq n+1$, hence the LC case is done. The other cases are the same.

4.2. F-split implies log canonical. Our next goal to prove the following of Hara and Watanabe [**HW02**].

Theorem 4.13. *Suppose that $(X = \operatorname{Spec} R, \Delta)$ is an F-split pair as in Chapter 5 Definition 3.1. Then (X, Δ) is also log canonical. Furthermore, if (X, Δ) is strongly F-regular, then (X, Δ) is Kawamata log terminal.*

As a corollary, we obtain a way to prove varieties are log terminal or log canonical via reduction to characteristic $p \gg 0$.

Corollary 4.14. *Suppose that $(X = \operatorname{Spec} R, \Delta)$ is a variety over a field of characteristic zero. Suppose that (X, Δ) has dense F-split (respectively dense strongly F-regular type). Then (X, Δ) is LC (respectively KLT).*

Proof. Being log canonical or Kawamata log terminal may be checked on a log resolution $\pi : Y \longrightarrow X$, which we may reduce to characteristic $p \gg 0$. The condition that the coefficients of the divisor $K_Y - \pi^*(K_X + \Delta)$ are < -1 or ≤ -1 may be preserved by reduction to characteristic $p \gg 0$. The corollary follows. $\square$

Before proving this, we need the following key lemma, explaining how maps $\phi \in \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ can be lifted to maps on Y where $\pi : Y \longrightarrow X$ is birational.

Lemma 4.15 (Lifting Frobenius splittings to birational maps)**.** *Suppose that X is a normal F-finite integral Noetherian scheme and $\pi : Y \longrightarrow X$ is a birational map from a normal integral scheme. Suppose that $\phi \in \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$. Form the $\mathbb{Q}$-divisor Δ_ϕ associated to ϕ as in Chapter 5 Definition 2.1 and write $K_Y + \Delta_Y = \pi^*(K_X + \Delta_\phi)$ where K_Y and K_X are chosen as in Setting 4.2. Then there is a map*

$$\phi_Y : F^e_* \mathcal{O}_Y \longrightarrow \mathcal{O}_Y$$

that agrees with ϕ on $F^e_* \mathcal{K}(X) = F^e_* \mathcal{K}(Y)$ (in other words, at the generic point of X or Y) if and only if we have that $\Delta_Y \geq 0$. In this case, $\Delta_Y = \Delta_{\phi_Y}$.

Proof. We begin with the following claim.

Claim 4.16. *We may choose isomorphisms of $F^e_* \mathcal{O}_Y$ and $F^e_* \mathcal{O}_X$-modules respectively,*

$$\begin{array}{ccc} \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_Y, \mathcal{O}_Y) & \cong & F^e_* \mathcal{O}_Y((1-p^e)K_Y) \\ & \textit{and} & \\ \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_X, \mathcal{O}_X) & \cong & F^e_* \mathcal{O}_X((1-p^e)K_X) \end{array}$$

which agree where π is an isomorphism.

Proof of claim. First assume that π is proper and fix an isomorphism on Y. Pushing forward, we obtain an isomorphism:

$$\pi_* \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_Y, \mathcal{O}_Y) \cong \pi_* F^e_* \mathcal{O}_Y((1-p^e)K_Y)$$

whose reflexification (S_2-ification) is the desired isomorphism on X, since π is an isomorphism outside a set of codimension 2 on X. Notice that in doing this, we may also fix K_Y and K_X that agree where π is an isomorphism. This proves the claim in the case that π is proper. If π is not proper, then we may embed Y in a normal proper scheme over X, and so reduce to the proper case by [**Nag63**] and normalizing. $\square$

Now that we have the claim in place, suppose $\phi_Y : F^e_* \mathcal{O}_Y \longrightarrow \mathcal{O}_Y$ is a map that agrees with ϕ at their respective generic points. Certainly $\Delta_{\phi_Y} \geq 0$, we need to show that $K_Y + \Delta_{\phi_Y} = \pi^*(K_X + \Delta_\phi)$. Since X is not $\mathbb{Q}$-Gorenstein, we cannot simply pullback $\mathcal{O}_X((1-p^e)K_X)$ as it is not a line bundle.

However, we may view ϕ as a global section which generates, as an $F^e_* \mathcal{O}_X$-module, the top row of the following diagram:

$$\begin{tikzcd} F^e_* \mathcal{O}_X((1-p^e)(K_X + \Delta_\phi)) \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook] & \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((p^e-1)\Delta_\phi), \mathcal{O}_X) \arrow[d, hook] \\ F^e_* \mathcal{O}_X((1-p^e)K_X) \arrow[r, leftrightarrow, "\sim"] & \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_X, \mathcal{O}_X). \end{tikzcd} \tag{4.16.1}$$

Likewise ϕ_Y generates, as an $F^e_* \mathcal{O}_Y$-module, the top row of the following diagram:

$$\begin{tikzcd} F^e_* \mathcal{O}_Y((1-p^e)(K_Y + \Delta_{\phi_Y})) \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook] & \mathscr{H}\mathrm{om}_{\mathcal{O}_Y}(F^e_* \mathcal{O}_Y((p^e-1)\Delta_{\phi_Y}), \mathcal{O}_Y) \arrow[d, hook] \\ F^e_* \mathcal{O}_Y((1-p^e)K_Y) \arrow[r, leftrightarrow, "\sim"] & \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_Y, \mathcal{O}_Y). \end{tikzcd}$$

Using the claim shows that the diagrams agree where π is an isomorphism. Since ϕ and ϕ_Y correspond to the same section generically, we then see that $\mathcal{O}_X((1-p^e)(K_X+\Delta_\phi)) \cong \mathcal{O}_X$, from (4.16.1), pulls back to $\mathcal{O}_Y((1-p^e)(K_Y+\Delta_{\phi_Y}))$. Hence $\pi^*(1-p^e)(K_X+\Delta) = (1-p^e)(K_Y+\Delta_{\phi_Y})$. It follows that $\Delta_{\phi_Y} = \Delta_Y$ as desired.

Conversely, if $K_Y + \Delta_Y = \pi^*(K_X + \Delta_\phi)$ then Δ_Y corresponds to a (generating) global section of $\mathcal{O}_Y((1-p^e)(K_Y + \Delta_Y))$ pulled back from $\mathcal{O}_X((1-p^e)(K_X+\Delta))$. We apply the compatible isomorphisms of the claim to complete the proof. $\square$

We can now prove our result.

Proof of Theorem 4.13. First suppose that (X, Δ) is locally F-split. The statement is local so we may assume $X = \operatorname{Spec} R$ and thus choose a surjective $\phi \in \mathscr{H}om(F^e_* \mathcal{O}_X(\lceil (p^e-1)\Delta \rceil), \mathcal{O}_X)$ with $\Delta_\phi \geq \Delta$. It follows that (X, Δ_ϕ) is also F-split and hence we may assume that $\Delta = \Delta_\phi$ using Exercise 4.4.

Choose $\pi : Y \longrightarrow X$ a birational map, write $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ and let $U \subseteq Y$ be the open set where the coefficients of Δ_Y non-negative. Note we are trying to show that all coefficients of Δ_Y are ≤ 1 for LC (and < 1 for KLT), so we may restrict to U. Replacing Y by U, we may thus assume that $\Delta_Y \geq 0$ and thus ϕ lifts to $\phi_Y : F^e_* \mathcal{O}_Y \longrightarrow \mathcal{O}_Y$.

Now, in the F-split case, ϕ is surjective, so has 1 in its image, and thus so does ϕ_Y. Hence (Y, Δ_Y) is locally F-split. Thus the coefficients of Δ_Y are ≤ 1 by Chapter 5 Exercise 2.7. This proves that (X, Δ) is log canonical.

Now suppose that $(X = \operatorname{Spec} R, \Delta)$ is strongly F-regular and fix $\pi : Y \longrightarrow X$ birational. As above, we may assume that $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ with $\Delta_Y \geq 0$. Fix some prime divisor E on Y and choose $f \in R$ that vanishes on $\pi(E)$. By Chapter 5 Proposition 3.14, $(X, \Delta + \epsilon \operatorname{div}_X(f))$ is also strongly F-regular (in the affine case, strongly F-regular is the same as globally F-regular). In particular, $(X, \Delta + \epsilon \operatorname{div}_X(f))$ is F-split, and hence if we write $K_Y + \Delta'_Y = \pi^*(K_X + \Delta + \epsilon \operatorname{div}_X(f))$ we see that the coefficients of Δ'_Y are also ≤ 1. But

$$\Delta'_Y = \Delta_Y + \epsilon \operatorname{div}_Y(f) = \Delta_Y + \epsilon \pi^* \operatorname{div}_X(f)$$

and so since the E-coefficient of $\operatorname{div}_Y(f)$ is > 0, we see that E-coefficient of Δ_Y is < 1. This proves that (X, Δ) is KLT as claimed. $\square$

4.3. Multiplier ideals and vanishing theorems. The multiplier ideal of a pair (X, Δ) is a sheaf that measures how KLT the pair is. It will correspond to the test ideal under reduction to characteristic $p \gg 0$.

Definition 4.17. Suppose that X is a normal integral Noetherian scheme in characteristic zero with a dualizing complex, that $(X, \Delta \geq 0)$ is a log $\mathbb{Q}$-Gorenstein pair and $\pi : Y \longrightarrow X$ is a log resolution of (X, Δ). We define the **multiplier ideal** of (X, Δ) to be:

$$\mathcal{J}(X, \Delta) = \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil).$$

Here K_Y and K_X are chosen to agree where π is an isomorphism.

This is independent of the choice of resolution and it canonically embeds inside $\mathcal{O}_X$ (and so really is an ideal). For a detailed account of this, see for instance [**Laz04b**]. One can define a multiplier ideal without having a resolution of singularities. In that case one takes the intersection of the objects $\pi_* \mathcal{O}_X(\lceil K_Y - \pi^*(K_X + \Delta)\rceil)$ as we run over all proper birational maps $\pi : Y \longrightarrow X$. However, it is not immediately clear if such an intersection is even a quasi-coherent sheaf, for additional discussion and proofs in some cases, see [**Can20**, Section 7].

Lemma 4.18. *(X, Δ) is KLT if and only if $\mathcal{J}(X, \Delta) = \mathcal{O}_X$.*

Proof. This holds even if we do not have a resolution of singularities. In that case we see that $\mathcal{J}(X, \Delta) = \mathcal{O}_X$ if and only if $\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil) = \mathcal{O}_X$ for every proper birational map $\pi : Y \longrightarrow X$.

We may assume that X is affine. Notice that $\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil) = \mathcal{O}_X$ if and only if the divisor $\lceil K_Y - \pi^*(K_X + \Delta)\rceil$ is effective. That occurs if and only if the coefficients of $K_Y - \pi^*(K_X + \Delta)$ are > -1, which is the definition of KLT. $\square$

The power of multiplier ideals are tied up with vanishing theorems, which are known when X is finite type over a field of characteristic zero. What follows is a special case of the relative Kawamata-Viehweg vanishing theorem.

Theorem 4.19 (Local vanishing for multiplier ideals)**.** *Suppose that (X, Δ) is a log $\mathbb{Q}$-Gorenstein pair. If $\pi : Y \longrightarrow X$ is a log resolution of (X, Δ) then*

$$0 = \mathbf{R}^i \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil)$$

for all $i > 0$. In other words, we can compactly write

$$\mathcal{J}(X, \Delta) = \mathbf{R}\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil).$$

As an immediate corollary, we obtain the following.

Theorem 4.20. *If (X, Δ) is KLT and X is a variety of characteristic zero, then X has rational singularities.*

Proof. We will use Exercise 2.13. Fix $\pi : Y \longrightarrow X$ a log resolution of (X, Δ). Since (X, Δ) is KLT, we see that $\lceil K_Y - \pi^*(K_X + \Delta)\rceil \geq 0$ and so $\mathcal{O}_Y \subseteq \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil)$. Thus we have the following composition

$$\mathcal{O}_X \longrightarrow \mathbf{R}\pi_* \mathcal{O}_Y \longrightarrow \mathbf{R}\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil) = \mathcal{J}(X, \Delta) = \mathcal{O}_X.$$

Since it is the identity at the generic point of X, it is the identity. Hence Exercise 2.13 implies that X has rational singularities. $\square$

4.4. Multiplier ideals and test ideals. We first need a lemma, which can be viewed as a generalization (or variant) of Lemma 4.15.

Lemma 4.21. *Suppose that R is a normal domain, $\pi : Y \longrightarrow X = \operatorname{Spec} R$ is a proper birational map with Y normal. Then for every $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ we have that ϕ induces a map*

$$\phi_Y : F^e_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil) \longrightarrow \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil)$$

that agrees with ϕ generically.

Proof. Since $\mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_X(\Delta_\phi), \mathcal{O}_X) \cong F^e_* \mathcal{O}_X((1-p^e)(K_X + \Delta_\phi))$, we may view ϕ as a global section of the latter. We pull this back via $(F^e_* \pi)^*$ to obtain a section

$$\phi_Y \in (F^e_* \pi)^* \mathcal{O}_X((1-p^e)(K_X + \Delta_\phi)).$$

Writing $\pi^*(K_X + \Delta_\phi) = K_Y + \Delta_Y$ (notice that Δ_Y need not be effective), we see that we may view ϕ_Y as a global section of

$$F^e_* \mathcal{O}_Y((1-p^e)(K_Y + \Delta_Y)) \cong \mathscr{H}\mathrm{om}_{\mathcal{O}_Y}(F^e_* \mathcal{O}_Y((p^e-1)\Delta_Y), \mathcal{O}_Y).$$

Hence we obtain

$$\phi_Y : F^e_* \mathcal{O}_Y((p^e-1)\Delta_Y) \longrightarrow \mathcal{O}_Y \tag{4.21.1}$$

a map that agrees with ϕ generically (on the locus where π is an isomorphism, we have done nothing). This is not quite the map we want, but it's close. We now twist both sides of (4.21.1) by $\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil$ and reflexify to make everything a (Frobenius pushforward of a) divisorial sheaf to obtain

$$\begin{array}{rcl} \phi_Y & : & F^e_* \mathcal{O}_Y((p^e-1)\Delta_Y + p^e\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil) \\ & \longrightarrow & \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil). \end{array} \tag{4.21.2}$$

Since

$$\begin{array}{rl} & (p^e-1)\Delta_Y + p^e\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil \\ \geq & (p^e-1)\Delta_Y + p^e K_Y - p^e\pi^*(K_X + \Delta_\phi) \\ = & (p^e-1)\Delta_Y + p^e K_Y - p^e(K_Y + \Delta_Y) \\ = & -\Delta_Y \\ = & K_Y - \pi^*(K_X + \Delta_\phi) \end{array}$$

we see that

$$(p^e-1)\Delta_Y + p^e\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil \geq \lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil$$

as both sides have integer coefficients. Hence, restricting ϕ_Y to $F^e_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil)$ and calling the resulting map ϕ_Y as well, we obtain our desired map. $\square$

We will show the following result in characteristic $p > 0$, which will immediately imply that the modulo-$p \gg 0$ reduction of the multiplier ideal is at least as big as the test ideal:

$$\tau(R_t, \Delta_t) \subseteq \mathcal{J}(R, \Delta)_t.$$

Proposition 4.22. *Suppose that* $(X = \operatorname{Spec} R, \Delta)$ *is a log* $\mathbb{Q}$*-Gorenstein pair and that* $\pi : Y \longrightarrow X$ *is a proper birational map with* Y *normal. Then for every effective Weil divisor* $D \geq 0$ *on* Y*, we have that*

$$\pi_* \mathscr{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil + D)$$

is compatible with every $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ *such that* $\Delta_\phi \geq \Delta$*. In particular, it contains* $\tau(R, \Delta)$*.*

There is a complication in the above statement that may be missed in a first reading. Notice that if D contains a non-exceptional $d_i D_i$ such that the coefficient d_i is larger than the corresponding coefficient of Δ (at D_i), then $\pi_* \mathscr{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil + D)$ is not contained in $\mathscr{O}_X$. In that case it is only a fractional ideal, but it is still compatible with ϕ in the sense it is compatible with the induced map $\phi_{\mathcal{K}(X)} : F^e_* \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$ on the fraction field.

PROOF. Fix $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. We have a map from Lemma 4.21

$$\phi_Y : F^e_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil) \longrightarrow \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil)$$

that agrees with ϕ generically. If we twist both sides by D and reflexify, we obtain a map

$$\phi_Y : F^e_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil + p^e D) \longrightarrow \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil + D).$$

Since $p^e D \geq D$ (since D is effective) we may restrict the domain of ϕ_Y to obtain

$$\phi_Y : F^e_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil + D) \longrightarrow \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil + D).$$

Pushing forward, we obtain a map that agrees with ϕ generically. Hence, $\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta_\phi)\rceil + D)$ is compatible with ϕ as claimed. $\square$

We now prove our main result of the section.

Theorem 4.23 ([**Tak04b**], *cf.* [**Smi00b, Har01, HY03**]). *Suppose that* $(X = \operatorname{Spec} R, \Delta \geq 0)$ *is a log* $\mathbb{Q}$*-Gorenstein pair of finite type over a field of characteristic zero where* R *is a normal domain. Suppose that* $\{(R_t, \Delta_t)\}$ *is a family of characteristic* $p > 0$ *models of* (R, Δ) *as in Section 1 for a sufficiently large* A*. Then there exists an open dense subset* U *of* m-Spec$A$ *such that*

$$\mathcal{J}(X, \Delta)_t = \tau(X_t, \Delta_t)$$

for all $t \in U$*. In particular,* (R, Δ) *has KLT singularities if and only if it has open strongly* F*-regular type.*

Proof. By Proposition 4.22, we automatically obtain the containment $\supseteq$, hence we need the reverse.

Fix $0 \neq d \in R$ a test element after reduction to characteristic $p \gg 0$ by Lemma 3.6. We may also assume that $\mathrm{div}(d) \geq \Delta$ and so we in fact have that

$$d_t^2 \in d_t \cdot \tau(R_p) = \tau(R_p, \mathrm{div}(d_t)) \subseteq \tau(R_t, \Delta_t)$$

after reduction modulo p, where the first equality is Skoda's theorem for test ideals Chapter 4 Theorem 5.20.

We fix a resolution of singularities $\pi : Y \longrightarrow X$ in characteristic zero which is the blowup of some ideal $J \subseteq R$ and hence that $J \cdot \mathcal{O}_Y = \mathcal{O}_Y(-G)$ where G is effective and $-G$ is relatively ample. Write $D = \varepsilon(-G - \mathrm{div}_Y(d^2)) - \pi^*(K_X + \Delta)$ for some rational number $1 \gg \varepsilon > 0$ so that D is π-ample and notice we have that $\lceil D \rceil = \lceil -\pi^*(K_X + \Delta) \rceil$.

Since $K_X + \Delta$ is $\mathbb{Q}$-Cartier, there exists a $n > 0$ such that $n(K_X + \Delta)$ is Cartier. We may also assume that $n\varepsilon \in \mathbb{Z}$ by replacing n by a multiple. Replacing R by finitely many localizations at a single element, we may even assume that $n(K_X + \Delta) \sim 0$. By shrinking U, we may restrict ourselves to characteristics $p > 0$ such that n and p are relatively prime and hence there exists an e (depending on p) so that $(p^e - 1)(K_X + \Delta) \sim 0$ and $(p^e - 1)\varepsilon \in \mathbb{Z}$.

By Hara's Surjectivity Theorem, Lemma 3.5, for some open dense $U \subseteq$ m-Spec$A$, we see that we have surjections

$$T^e_{Y_t} : \pi_* F^e_* \mathcal{O}_{Y_t}(K_{Y_t} + \lceil p^e D_t \rceil) \longrightarrow \pi_* \mathcal{O}_{Y_t}(K_{Y_t} + \lceil D_t \rceil) \tag{4.23.1}$$

for all $t \in U$ and all $e > 0$. The target of (4.23.1) is $\mathcal{J}(X, \Delta)_t$ since $\lceil D \rceil = \lceil -\pi^*(K_X + \Delta) \rceil$.

The source of (4.23.1) can be rewritten as

$$\begin{array}{rl} & \pi_* F^e_* \mathcal{O}_{Y_t}(K_{Y_t} + (p^e - 1)D_t + \lceil D_t \rceil) \\ = & F^e_* \Big(\mathcal{O}_{X_t}\big((1-p^e)(K_{X_t} + \Delta_t) + (p^e - 1)\varepsilon \,\mathrm{div}_{X_t}(d_t^2)\big) \otimes \pi_* F^e_* \mathcal{O}_{Y_t}(K_{Y_t} + \lceil D_t \rceil) \Big) \\ = & F^e_* \Big(\mathcal{O}_{X_t}\big((1-p^e)(K_{X_t} + \Delta_t) + (p^e - 1)\varepsilon \,\mathrm{div}_{X_t}(d_t^2)\big) \otimes \mathcal{J}(X, \Delta)_t \Big) \end{array}$$

by the projection formula.

Next we observe that if we twist $F^e_* \omega_{X_t} \longrightarrow \omega_{X_t}$ by $-K_{X_t}$ and use that $-\Delta$ is effective, we obtain a map $\nu : F^e_* \mathcal{O}_{X_t}((1 - p^e)(K_{X_t} + \Delta_t)) \longrightarrow \mathcal{O}_X$. Next if we apply the isomorphism the $\mathcal{O}_{X_t}((1 - p^e)(K_{X_t} + \Delta_t)) \cong \mathcal{O}_{X_t}$ we may identify our map ν with a

$$\phi_t : F^e_* \mathcal{O}_{X_t} \longrightarrow \mathcal{O}_{X_t}$$

so that $\Delta_{\phi_t} = \Delta_t$.

Putting all this together with the surjectivity of (4.23.1), we see that ϕ_t restricts to a surjection:

$$\phi_t : F^e_* d_t^{2(p^e-1)\varepsilon} \cdot \mathcal{J}(X, \Delta)_t \longrightarrow \mathcal{J}(X, \Delta)_t.$$

Since $d_t^2 \in \tau(X_t, \Delta_t)$, so is the image of $F^e_* d_t^{2(p^e-1)\varepsilon} \cdot \mathcal{J}(X, \Delta)_t$. Hence $\mathcal{J}(X, \Delta)_t \subseteq \tau(X_t, \Delta_t)$. This completes the proof of the first statement.

The second statement follows from the first since (X, Δ) is KLT if and only if $\mathcal{J}(X, \Delta) = \mathcal{O}_X$ and (X_t, Δ_t) is strongly F-regular if and only if $\tau(X_t, \Delta_t) = \mathcal{O}_{X_t}$. □

It is natural to expect an analogous result to Theorem 4.23 for LC singularities and F-split type. Note we cannot expect open F-split type as the following example shows:

Example 4.24. Suppose that R is the affine cone over a smooth elliptic curve, for instance we could take $R = \mathbb{Q}[x, y, z]/(x^3 + y^3 + z^3)$. Then R is log canonical by Example 4.5. However, R is only F-split when the associated elliptic curve is ordinary, see Chapter 4 Subsection 1.1. For the Fermat cubic above, that happens if and only if $p \equiv_3 1$.

The following conjecture is open and considered to be quite difficult.

Conjecture 4.25. *Suppose that (X, Δ) is log canonical, then it has dense F-split type.*

See [**Sri91, Har98b, ST09, FT13, Her15, Her16**], for a small sample of the special cases that are known.

It is known that Conjecture 4.25 follows from the weak ordinarity conjecture of Mustaţă-Srinivas, see [**MS11**], *cf.* [**BK86, JR03, Mus10, BST17, ST17, DS17, Bit20, Stfrm-e1**].

Conjecture 4.26 (Weak Ordinarity)**.** *Suppose X is a d-dimensional smooth variety over a field of characteristic zero. After reduction to characteristic $p \gg 0$, for a dense set $T \subseteq \text{m-Spec}A$, we have that the Frobenius action on $H^d(X_t, \mathcal{O}_{X_t})$ is bijective for all $t \in T$.*

For some cases where this latter conjecture is known see [**Ogu82, JR03, BZ09**].

4.5. Examples and connections with the F-pure threshold. It is particularly important to note that the open set U constructed in Theorem 4.23 depends on both X *and* Δ. If we change the coefficients of Δ, then the open set U changes as well. Indeed, consider the following example.

Example 4.27. Let $R = \mathbb{Q}[x, y]$ with $X = \operatorname{Spec} R$, fix $f = y^2 - x^3$, and set $\Delta = s \operatorname{div}(f)$ for varying s. Recall from Chapter 4 Example 3.9 that the F-pure threshold of (R, f) is $\frac{5}{6} - \frac{1}{6p}$ for $p \equiv_6 5$ (that is for $p = 5, 11, 17, 23, \dots$). In particular $\tau(R_p, f_p^s) \neq R_p$ for any such p and any $s \in (\frac{5}{6} - \frac{1}{6p}, \frac{5}{6})$.

On the other hand, a straightforward computation shows that $\mathcal{J}(X, \Delta) = \mathcal{O}_X$ if and only if $t < \frac{5}{6}$. Putting this together, we obtain the following.

Fix $0 < s < \frac{5}{6}$. Then

$$R_p = \mathcal{J}(X, s \operatorname{div}(f))_p = \tau(R_p, s \operatorname{div}(f_p))$$

if and only if $\frac{5}{6} - s > \frac{1}{6p}$. That is, if and only if

$$p > \frac{1}{5 - 6s}.$$

Analogous to the F-pure threshold is the log canonical threshold (indeed, the F-pure threshold of Chapter 4 Section 3 was inspired by the following definition).

Definition 4.28 (Log canonical thresholds)**.** Suppose that R is a $\mathbb{Q}$-Gorenstein log canonical domain and $0 \neq f \in R$. We define the **log canonical threshold of** f to be

$$\operatorname{lct}(R, f) = \sup\{s > 0 \mid (\operatorname{Spec} R, s \operatorname{div} f) \text{ is log canonical}\}.$$

At this point, we recall the F-pure threshold from Chapter 4.

Definition 4.29 (F-pure threshold)**.** Suppose that R is a F-pure domain in characteristic $p > 0$ and $0 \neq f \in R$. We define the F**-pure threshold of** f to be

$$\operatorname{fpt}(R, f) = \sup\{s > 0 \mid (\operatorname{Spec} R, s \operatorname{div} f) \text{ is } F\text{-split}\}.$$

We then immediately obtain the following result, which is frequently colloquially described as:

$$\lim_{p \longrightarrow \infty} \operatorname{fpt}(R_p, f_p) = \operatorname{lct}(R, f).$$

Theorem 4.30. *Suppose R is a KLT and $\mathbb{Q}$-Gorenstein domain of finite type over a characteristic zero and $0 \neq f \in R$. Reduce this setup to mixed characteristic for a sufficiently large A. Then for every $\epsilon > 0$, there exists an open set $U \subseteq$ m-Spec$A$ such that for every $t \in U$, we have that*

$$\operatorname{lct}(R, f) - \epsilon < \operatorname{fpt}(R_t, f_t) \leq \operatorname{lct}(R, f)$$

PROOF. Since R is KLT, we have by Exercise 4.8 that

$$\operatorname{lct}(R, f) = \sup\{s > 0 \mid (\operatorname{Spec} R, s \operatorname{div} f) \text{ is KLT}\}.$$

We may use Theorem 4.23 to restrict to t so that R_t is strongly F-regular. Hence we have that

$$\mathrm{fpt}(R_t, f_t) = \sup\{s > 0 \mid (\operatorname{Spec} R, s \operatorname{div} f) \text{ is } F\text{-split}\}.$$

The inequality $\leq$ in the statement can be deduced from the fact that F-split pairs are log canonical Theorem 4.13.

If we set $c = \mathrm{lct}(R, f)$, then we see that $(R, (c - \epsilon) \operatorname{div}(f))$ is KLT (if $c - \epsilon < 0$, the statement is vacuous). Hence there exists an open set $U \subseteq \text{m-}\operatorname{Spec} A$ so that if $t \in U$, then $(R_t, (c - \epsilon) \operatorname{div}(f_t))$ is strongly F-regular. The statement follows. $\square$

A variant of Conjecture 4.25 says the following.

Conjecture 4.31. *With notation as in Theorem 4.30, there exists a dense set of $t \in \text{m-}\operatorname{Spec} A$ such that*

$$\mathrm{fpt}(R_t, f_t) = \mathrm{lct}(R, f).$$

Remark 4.32 (Small primes)**.** It is natural also to ask how things like the log canonical threshold itself behave under mod-p reduction. While it is not difficult to see that $\mathrm{lct}(R_t, f_t) = \mathrm{lct}(R, f)$ for $t \in U$, an open dense subset of $\text{m-}\operatorname{Spec} A$, it is less clear what happens for all t (ie, for small primes). This has been studied in the papers [] where roughly speaking, it is shown that $\mathrm{lct}(R_t, f_t) \leq \mathrm{lct}(R, f)$ for all $t \in \text{m-}\operatorname{Spec} A$ under various hypotheses.

4.6. Calabi-Yau and Log Fano varieties. The theory we have so far developed also informs us about analogs of globally F-regular varieties.

Definition 4.33. Suppose that $(X, \Delta \geq 0)$ is a pair where X is a normal projective variety over a field. We say that (X, Δ) is **log Calabi-Yau** if (X, Δ) has log canonical singularities and $n(K_X + \Delta) \sim 0$ for some $n > 0$. We say that (X, Δ) is **log Fano** if (X, Δ) has KLT singularities and $-K_X - \Delta$ is ample.

As a consequence of our work in Chapter 5 Theorem 3.20, we immediately obtain the following.

Theorem 4.34. *Suppose that X is a variety over an F-finite field of characteristic $p > 0$. Suppose that X is globally F-split, then there exists a Δ such that (X, Δ) is log Calabi-Yau. Furthermore, if X is globally F-regular, then there exists a Δ such that (X, Δ) is log Fano.*

Proof. If X is globally F-split, then using Chapter 5 Proposition 3.6 we see that there exists a $\Delta \geq 0$ such that (X, Δ) is globally F-split with $n(K_X + \Delta) \sim 0$ for some $n > 0$. Since (X, Δ) is F-split and $K_X + \Delta$ is $\mathbb{Q}$-Cartier, we see that (X, Δ) is log canonical by Theorem 4.13.

Next if X is globally F-regular, by Chapter 5 Theorem 3.20, we see that there exists a $\Delta \geq 0$ such that (X, Δ) is globally F-regular and $-K_X - \Delta$ is ample. In particular, (X, Δ) is KLT by Theorem 4.13, which completes the proof. □

Based on this, we might expect the following conjecture.

Conjecture 4.35. *Suppose that X is a projective variety of characteristic zero that has open globally F-regular type (respectively, dense globally F-split type). Then there exists a $\Delta \geq 0$ such that (X, Δ) is log Fano (respectively, log Calabi-Yau).*

In Conjecture 4.35, if X is globally F-regular type, as we reduce to characteristic $p > 0$ we can find a Δ_t on each X_t making (X_t, Δ_t) log Fano, see Chapter 5 Subsection 3.5. However, these Δ_t vary as t varies, and in fact the denominators of its coefficients might very well increase as the characteristic increases. There has been some progress on this conjecture however, see for instance [**GOST15, GT16, Oka17, Yos22**].

We can prove the following however.

Theorem 4.36. *If (X, Δ) is log Fano, then (X, Δ) has open globally F-regular type.*

Proof. Left to the reader in Exercise 4.9. □

4.7. Triples. We conclude this section with a brief discussion of another variant of multiplier ideals, those associated to *triples*. This generality was introduced for test ideals in Chapter 5 Exercise 5.16.

Definition 4.37. A **triple** $(X, \Delta, \mathfrak{a}^s)$ consists of a normal integral scheme X, an effective $\mathbb{Q}$-divisor $\Delta \geq 0$, an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ and a real number $s \geq 0$. A triple is called **log $\mathbb{Q}$-Gorenstein** if $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

In characteristic zero, associated to any triple, we can also define a multiplier ideal as follows.

Definition 4.38 (Multiplier ideals of triples)**.** Suppose $(X, \Delta, \mathfrak{a}^s)$ is a log $\mathbb{Q}$-Gorenstein triple where X is finite type over a field of characteristic zero. Fix $\pi : Y \longrightarrow X$ a log resolution of X, Δ and $V(\mathfrak{a}) \subseteq X$ where we write $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_X(-H)$. We define the **multiplier ideal of** $(X, \Delta, \mathfrak{a}^s)$ to be

$$\mathcal{J}(X, \Delta, \mathfrak{a}^s) = \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) - sH \rceil) \cong \mathbf{R}\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) - sH \rceil).$$

Note implicitly used here is the fact that the higher direct images are zero.

The multiplier ideal in this generality still reduces to the test ideal (the proof is left to the exercises).

Theorem 4.39. *Suppose that $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^s)$ is a log $\mathbb{Q}$-Gorenstein triple of finite type over a field of characteristic zero where R is a normal domain. Suppose that $\{(R_t, \Delta_t, \mathfrak{a}_t)\}$ is a family of characteristic $p > 0$ models of $(R, \Delta, \mathfrak{a})$ as in Section 1 for a sufficiently large A. Then there exists an open dense subset U of* m-Spec$A$ *such that*

$$\mathcal{J}(X, \Delta, \mathfrak{a}^s)_t = \tau(X_t, \Delta_t, \mathfrak{a}_t^s)$$

for all $t \in U$.

Proof. See Exercise 4.10 and Exercise 4.11. □

4.8. Exercises.

Exercise 4.1. Show that our definition of the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor from (4.0.1) does not depend on the choice of m.

Exercise 4.2. Prove Lemma 4.4. That is, show the definition of discrepancy is independent of the choice of canonical divisor K_X.

Exercise 4.3. Prove Lemma 4.6. That is, show that the discrepancy of (X, Δ) along E depends only on the valuation associated to E and not the particular choice of $\pi : Y \longrightarrow X$.

Exercise 4.4. Suppose that X is a normal Noetherian integral scheme and that $\Delta' \geq \Delta \geq 0$ are $\mathbb{Q}$-divisors such that $K_X + \Delta$ and $K_X + \Delta'$ are both $\mathbb{Q}$-Cartier. Show that if (X, Δ') is KLT (respectively LC, PLT, canonical, terminal), then (X, Δ) is also KLT (respectively LC, PLT, canonical terminal).

Exercise 4.5. Prove Proposition 4.10 for the properties Kawamata log terminal and purely log terminal.

Definition 4.40. Suppose $\pi : Y \longrightarrow X$ is a proper birational map between normal varieties and consider a log $\mathbb{Q}$-Gorenstein pair (X, Δ). We can always write

$$K_Y - \pi^*(K_X + \Delta) = \sum_{i=1}^{t} a_i E_i$$

for some prime divisors E_i on Y. We say that a closed subvariety $Z \subseteq X$ is a *non-KLT-center of* (X, Δ) if there exists a Y and $E_i \subseteq Y$ as above such that $Z = \pi(E_i)$ and such that the associated $a_i \leq -1$.

Exercise 4.6. Now suppose that $\phi : F^e_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ is a nonzero map and set $\Delta = \Delta_\phi$. Suppose that $Z \subset X$ is a non-KLT-center of (X, Δ). Prove that the Z is compatible with ϕ.

Exercise 4.7. Suppose X is a normal Gorenstein variety in characteristic zero and D is a normal prime Cartier divisor on X. Define the *adjoint ideal* of (X, D) to be

$$\mathrm{adj}_D(X, D) := \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + D) + \pi_*^{-1} D \rceil)$$

where $\pi : Y \longrightarrow X$ is a log resolution of (X, D) and $\pi_*^{-1} D$ is the strict transform of D.

(a) Show that $\mathrm{adj}_D(X, D) = \mathscr{O}_X$ if and only if (X, D) is PLT.
(b) Prove that there is a surjection $\mathrm{adj}_D(X, D) \twoheadrightarrow \mathcal{J}(D)$.
(c) Conclude that D has KLT singularities if and only if (X, D) is PLT near D.

The hypotheses that X is Gorenstein and that D is Cartier can be weakened substantially. The general statement is typically called *inversion of adjunction for log terminal singularities.*

Exercise 4.8. Suppose that R is KLT and $\mathbb{Q}$-Gorenstein and of characteristic zero (so that log resolutions exist). Show that

$$\mathrm{lct}(R, f) = \sup\{s > 0 \mid (\operatorname{Spec} R, s \operatorname{div} f) \text{ is KLT}\}.$$

Next suppose that R is strongly F-regular of characteristic $p > 0$. Show that

$$\mathrm{fpt}(R, f) = \sup\{s > 0 \mid (R, s \operatorname{div} f) \text{ is strongly } F\text{-regular}\}.$$

Exercise 4.9. Prove Theorem 4.36.

Hint: Suppose that (X, Δ) is log Fano. Form a section ring S with respect to an ample Cartier divisor $-n(K_X + \Delta)$ for n sufficiently divisible. Let Δ_S be a divisor on S corresponding to Δ and show that (S, Δ_S) is KLT. Now reduce to positive characteristic.

To show that (S, Δ_S) is KLT, it suffices to blowup at the irrelevant ideal and check that the discrepancy on that exceptional divisor E (abstractly isomorphic to X is > -1). To do that, use that $E \cong X$ and $E|_E \sim n(K_X + \Delta)$.

Exercise 4.10. With notation as in Theorem 4.39, show that

$$\mathcal{J}(X, \Delta, \mathfrak{a}^s)_t \subseteq \tau(X_t, \Delta_t, \mathfrak{a}_t^s)$$

for all $t \in U$, where U is a dense open subset of $\text{m-Spec} A$.

Hint: We may replace s by $s/n < 1$ and $\mathfrak{a}$ by $\mathfrak{a}^n$ for some n. By the argument of [**Laz04b**, Proposition 9.2.28], we may find a principal $f \in \mathfrak{a}$ such that $\mathcal{J}(X, \Delta, \mathfrak{a}^s) = \mathcal{J}(X, \Delta + s \operatorname{div}(f))$. In characteristic $p > 0$, show that $\tau(X_t, \Delta_t + s \operatorname{div}(f_t)) \subseteq \tau(X_t, \Delta_t, \mathfrak{a}_t^s)$.

Exercise 4.11. With notation as in Theorem 4.39, show that

$$\mathcal{J}(X, \Delta, \mathfrak{a}^s)_t \supseteq \tau(X_t, \Delta_t, \mathfrak{a}_t^s)$$

for all $t \in U$, where U is a dense open subset of m- Spec$A$.

Hint: The trick used in Exercise 4.10 does not work here. Instead show that $\mathcal{J}(X, \Delta, \mathfrak{a}^s)_t$ is compatible with every element of $(F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil}) \cdot \mathscr{C}^e(\Delta)$ as in Chapter 5 Exercise 5.16. Use the fact that for $\mathscr{M}$ on Y, that $\mathfrak{a} \cdot \pi_* \mathscr{M} \subseteq \pi_*(\mathscr{M} \otimes \mathcal{O}_Y(-H))$ where $\mathfrak{a} \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(-H)$.

5. Du Bois and F-injective singularities

In this section we describe the relationship between Du Bois and F-injective singularities. We will show that singularities of Dense F-injective type are Du Bois. The converse direction is open, and is equivalent to the weak ordinarity conjecture of Conjecture 4.26, as we shall see. Our first goal is to describe Du Bois singularities.

5.1. Du Bois singularities. Suppose that X is essentially finite type over a field of characteristic zero and $X \subseteq Y$ where Y is a smooth variety. Take a log resolution $\pi : Y' \longrightarrow Y$ of (Y, X) and let $\overline{X} = \pi^{-1}(X)_{\mathrm{red}}$ denote the reduced inverse image of X. Then define

$$\underline{\Omega}^0_X = \mathbf{R}\pi_* \mathcal{O}_{\overline{X}}$$

an object in $D^b_{\mathrm{coh}}(X)$.

This object is independent of the choice of log resolution (up to quasi-isomorphism) or the embedding $X \subseteq Y$. This construction is not the usual way to describe this object, we normally use hypercovers or hyperresolutions, see [**GNPP88, PS08, KS11b**]. However, this is the quickest way to get to this object for us.

The more general construction does not require an embedding into a non-singular scheme (although, of course that always exists locally for schemes essentially of finite type, and exists globally for quasi-projective varieties). We will therefore assume that there is an object $\underline{\Omega}^0_X \in D^b_{\mathrm{coh}}(X)$ for any scheme essentially of finite type over a field of characteristic zero which agrees with the one described above locally. It also admits a canonical map $\mathcal{O}_X \longrightarrow \underline{\Omega}^0_X$ that agrees (locally) with the one that comes from our description above.

Remark 5.1. In fact, instead of taking a log resolution of (Y, X), one may take an embedded resolution $\pi : Y' \longrightarrow Y$ of $X \subseteq Y$ such that the reduced exceptional divisor E is a simple normal crossing divisor that meets $\widetilde{X}$, the strict transform of X, in a simple normal crossing divisor. If we let $\overline{X} = \pi^{-1}(X)_{\mathrm{red}}$, then we still have $\underline{\Omega}^0_X = \mathbf{R}\pi_* \mathcal{O}_{\overline{X}}$. Note in this case $\overline{X}$

Definition 5.2 (*cf.* [**Ste85, Esn90, Sch07**])**.** Suppose that X is a scheme essentially of finite type over a field of characteristic zero. We say that X has **Du Bois singularities** if the canonical map

$$\mathcal{O}_X \longrightarrow \underline{\Omega}^0_X$$

is a quasi-isomorphism.

Remark 5.3. Note that a scheme has Du Bois singularities if it can be covered by Du Bois charts (equivalently, if its stalks have Du Bois singularities).

There is a striking similarity between this definition and rational singularities. For Du Bois singularities, we use $\underline{\Omega}^0_X$ instead of $\mathbf{R}\pi_* \mathcal{O}_{\widetilde{X}}$ where $\widetilde{X} \longrightarrow X$ is a resolution of singularities.

In characteristic zero, one reason rational singularities have been so useful is because of the Grauert-Riemenschneider vanishing theorem Theorem 2.13. There is a result for Du Bois singularities that plays the same role. We fix the following notation

$$\underline{\omega}^{\bullet}_X := \mathbf{R}\,\mathscr{H}\mathrm{om}(\underline{\Omega}^0_X, \omega^{\bullet}_X)$$

which when E is as above, agrees with $\mathbf{R}\pi_* \omega^{\bullet}_{\overline{X}}$ by Grothendieck duality.

Theorem 5.4 ([**KS11a**])**.** *Suppose X is a scheme essentially of finite type over a field of characteristic zero. Then for every i, the Grothendieck dual map to $\mathcal{O}_X \longrightarrow \underline{\Omega}^0_X$,*

$$\mathcal{H}^i \underline{\omega}^{\bullet}_X \longrightarrow \mathcal{H}^i \omega^{\bullet}_X$$

is injective on cohomology. Dually, for every closed point $x \in X$, we have that

$$H^i_x(X, \mathcal{O}_X) \longrightarrow H^i_x(X, \underline{\Omega}^0_X)$$

surjects for every i.

Remark 5.5 (Injectivity implies some vanishing)**.** Since $\mathcal{H}^i \omega^{\bullet}_X = 0$ for $i < -\dim X$, we immediately see that $\mathcal{H}^i \underline{\omega}^{\bullet}_X = 0$ for $i < -\dim X$ as well.

Notice that X is a Cohen-Macaulay variety if and only if

$$\mathcal{H}^i \underline{\omega}^{\bullet}_X = 0$$

for all $i \neq \dim X$. Hence, using the notation $\overline{X} = \pi^{-1}(X)_{\mathrm{red}}$ as above, we have that

$$\mathbf{R}^i \pi_* \omega^{\bullet}_{\overline{X}} = 0$$

for all $i \neq -\dim X$. If we replace $\overline{X}$ by $\widetilde{X}$, this is exactly Grauert-Riemenschneider vanishing.

Our next goal is to explain how Du Bois singularities are related to the other singularities we have defined so far. Notice from our diagram at the start of Chapter 6, we expect that Du Bois singularities should be related to rational and log canonical singularities.

First we prove a criterion for Du Bois singularities analogous to Kovács criterion for rational singularities Exercise 2.13, see [**Kol95, Kov99**].

Lemma 5.6. *Suppose X is a scheme of finite type over a field of characteristic zero and that there exists a map $\underline{\Omega}^0_X \longrightarrow \mathscr{O}_X$ in the derived category such that the composition $\mathscr{O}_X \longrightarrow \underline{\Omega}^0_X \longrightarrow \mathscr{O}_X$ is an isomorphism. Then X has Du Bois singularities.*

Proof. We apply Grothendieck duality to our composition to obtain

$$\omega_X^{\bullet} \leftarrow \mathscr{H}\mathrm{om}(\underline{\Omega}^0_X, \omega_X^{\bullet}) \leftarrow \omega_X^{\bullet}$$

an isomorphism in the derived category. By Theorem 5.4 the left arrow is injective on cohomology. But our displayed composition also implies it is surjective on cohomology. Hence $\omega_X^{\bullet} \leftarrow \mathscr{H}\mathrm{om}(\underline{\Omega}^0_X, \omega_X^{\bullet})$ is a quasi-isomorphism. Dualizing again proves that $\mathscr{O}_X \longrightarrow \underline{\Omega}^0_X$ is an isomorphism. The result follows. $\square$

This immediately will show that rational singularities are Du Bois (first shown by Kovács and Saito [**Kov99, Sai00**]).

Theorem 5.7 ([**Kov99, Sai00**]). *Suppose X is a scheme essentially of finite type over a field of characteristic zero which has rational singularities, then X has Du Bois singularities.*

Proof. Working locally, we may assume that $X \subseteq Y$ where Y is nonsingular. We may then take an embedded resolution $\pi : Y' \longrightarrow Y$ where $\widetilde{X} = \pi_*^{-1}X$ is the strict transform and $E = \pi^{-1}(X)_{\mathrm{red}}$ is the reduced inverse image. We may also assume that $\underline{\Omega}^0_X = \mathbf{R}\pi_*\mathscr{O}_E$ as in Remark 5.1 by choosing our embedded resolution carefully enough. Notice now that $\widetilde{X}$ is a closed subscheme of E. Consider the induced composition:

$$\mathscr{O}_X \longrightarrow \underline{\Omega}^0_X = \mathbf{R}\pi_*\mathscr{O}_E \longrightarrow \mathbf{R}\pi_*\mathscr{O}_{\widetilde{X}}.$$

Since X has rational singularities, this is a quasi-isomorphism. Hence Lemma 5.6 applies and the proof is complete. $\square$

The argument used in the previous proof can also tell us about the $-d$th cohomology of $\mathscr{H}\mathrm{om}(\underline{\Omega}^0_X, \omega_X^{\bullet})$.

Lemma 5.8. *Suppose that X is a normal d-dimensional integral scheme of finite type over a field of characteristic zero. Let $\pi : \widetilde{X} \longrightarrow X$ be a log* *resolution of X with reduced simple normal crossings exceptional divisor F. Then*

$$\pi_*\omega_{\widetilde{X}}(F) \cong \mathcal{H}^{-d}\,\mathscr{H}\mathrm{om}(\underline{\Omega}^0_X, \omega^\bullet_X).$$

PROOF. It can be shown that $\pi_*\omega_{\widetilde{X}}(F)$ is independent of the choice of log resolution, see Exercise 5.1. Hence, working locally on X if necessary, we may assume that $X \subseteq Y$ with Y non-singular, $\pi : Y' \longrightarrow Y$ an embedded resolution of X where $\widetilde{X} = \pi_*^{-1}X$ is the strict transform, E is the reduced exceptional divisor of π and $F = E \cap \widetilde{X}$. Since we may assume that π is an isomorphism away from the singular locus of X, we may assume that $S = \pi(E)$ has codimension ≥ 2 and hence $\dim S \leq \dim X - 2$ since X is normal.

Notice we may write

$$\overline{X} = \pi^{-1}(X)_{\mathrm{red}} = \widetilde{X} \cup E.$$

The map $\mathcal{O}_{\overline{X}} \twoheadrightarrow \mathcal{O}_E$ is an isomorphism away from $\widetilde{X}$ and hence we have that the kernel is $\mathcal{O}_{\widetilde{X}}(-F)$ (the functions on $\mathcal{O}_{\overline{X}}$ which vanish on E). Hence we have the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(-F) \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

We apply the Grothendieck duality functor and push forward to obtain the exact triangle:

$$\mathbf{R}\pi_*\omega^\bullet_E \longrightarrow \mathbf{R}\pi_*\omega^\bullet_{\overline{X}} \longrightarrow \mathbf{R}\pi_*\big(\omega^\bullet_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(F)\big) \xrightarrow{+1} . \tag{5.8.1}$$

Notice that E, $\widetilde{X}$ are equidimensional (although $\overline{X}$ is not, unless $X \subseteq Y$ is a hypersurface).

We next observe that $\mathbf{R}\pi_*\mathcal{O}_E = \underline{\Omega}^0_S$ and so $\mathbf{R}\pi_*\omega^\bullet_E = \mathbf{R}\,\mathscr{H}\mathrm{om}(\mathbf{R}\pi_*\mathcal{O}_E, \omega^\bullet_Y)$ has zero cohomology in degrees $< -\dim S \leq -(\dim X - 2)$. Hence, taking cohomology in (5.8.1), we have an isomorphism

$$\mathcal{H}^{-d}\mathbf{R}\pi_*\omega^\bullet_{\overline{X}} \longrightarrow \mathcal{H}^{-d}\mathbf{R}\pi_*\big(\omega^\bullet_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(F)\big) = \pi_*\omega_{\widetilde{X}}(F)$$

as claimed. $\square$

Hence in the Cohen-Macaulay case, we have the following criterion for Du Bois singularities.

Corollary 5.9. *Suppose that X is a normal Cohen-Macaulay integral scheme of finite type over a field of characteristic zero. Let $\pi : \widetilde{X} \longrightarrow X$ be a log resolution with reduced exceptional divisor F. Then X has Du Bois singularities if and only if the canonical map*

$$\pi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + F) = \pi_*\omega_{\widetilde{X}}(F) \longrightarrow \omega_X = \mathcal{O}_X(K_X)$$

is an isomorphism.

PROOF. Since X is Cohen-Macaulay, we have that $0 = \mathcal{H}^i(\omega_X^\bullet) = \mathcal{H}^i(\underline{\Omega}_X^0, \omega_X^\bullet)$ for $i \neq \dim X$. Hence we only need to show that $\square$

Finally, we relate Du Bois and log canonical singularities, although for one direction we restrict to the case that X is Cohen-Macaulay (the general case is quite a bit more difficult, see for instance [**KK10, FL22**]).

Theorem 5.10 ([**Kov99, KK10**], *cf.* [**Kol13**], [**FL22**]). *Suppose (X, Δ) has log canonical singularities. Then X is Du Bois. Conversely, if X is normal, quasi-Gorenstein, and Du Bois, then X is log canonical.*

PROOF. Without loss of generality, we may assume that X is affine. We use the notation of Corollary 5.9,

We prove the second statement first. If X is Du Bois, normal and quasi-Gorenstein, the isomorphism $\pi_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + F) \cong \mathcal{O}_X(K_X)$ can be rewritten as

$$\pi_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} - \pi^* K_X + F) \cong \mathcal{O}_X$$

by the projection formula. This implies that the coefficients of $K_{\widetilde{X}} - \pi^* K_X + F$ are ≥ 0 and hence that the coefficients of $K_{\widetilde{X}} - \pi^* K_X$ are ≥ -1 since F is reduced. This proves the second statement.

As mentioned above, we only prove the first statement when X is Cohen-Macaulay. Hence it is sufficient to show that $\pi_* \omega_{\widetilde{X}}(F) = \omega_X$. Suppose $f \in \Gamma(X, \mathcal{O}_X(K_X))$ is a section. We must show that $f \in \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + F))$, or in other words that $\operatorname{div}_{\widetilde{X}}(f) + K_{\widetilde{X}} + F \geq 0$.

Since (X, Δ) is log canonical, the coefficients of $K_{\widetilde{X}} - \pi^*(K_X + \Delta)$ are all ≥ -1. It follows that all the coefficients of

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta) + F + \pi_*^{-1}\Delta$$

are ≥ 0. Since $\operatorname{div}_X(f) + K_X \geq 0$, we also have that $\operatorname{div}_X(f) \geq -(K_X + \Delta)$ and so $\operatorname{div}_{\widetilde{X}}(f) \geq -\pi^*(K_X + \Delta)$. Thus we also have that all coefficients of

$$K_{\widetilde{X}} + \operatorname{div}_{\widetilde{X}}(f) + F + \pi_*^{-1}\Delta$$

are also ≥ 0. The term $\pi_*^{-1}\Delta$ is not even needed since where π is an isomorphism $K_{\widetilde{X}}$ and K_X agree and so there is nothing to check. This completes the proof of the first statement. $\square$

5.2. F-injective versus Du Bois singularities. Now that we have some intuition and basic results about Du Bois singularities, we come to our main goal: showing that varieties of dense F-injective type have Du Bois singularities.

The key observation is as follows.

Lemma 5.11. *Suppose that $X \subseteq Y = \operatorname{Spec} S$ is a subscheme of a normal integral Noetherian scheme Y in characteristic $p > 0$. Further assume that $\pi : Y' \longrightarrow Y$ is a projective birational map with ample anti-effective divisor $-E$ such that $\overline{X} = \pi^{-1}(X)_{\mathrm{red}}$ has the same support as E. Suppose finally that $\mathcal{O}_Y \longrightarrow \mathbf{R}\pi_* \mathcal{O}_{Y'}$ is an isomorphism. Then Frobenius acts nilpotently on $\mathcal{H}^i(\mathbf{R}\pi_* \mathcal{O}_{\overline{X}})$ for every $i > 0$. Furthermore, Frobenius acts nilpotently on the quotient $\pi_* \mathcal{O}_{\overline{X}} / \mathcal{O}_X$.*

Proof. For the first statement, by considering the long exact sequence

$$\ldots \longrightarrow \mathbf{R}^i \pi_* \mathcal{O}_{Y'}(-\overline{X}) \longrightarrow \mathbf{R}^i \pi_* \mathcal{O}_{Y'} \longrightarrow \mathbf{R}^i \pi_* \mathcal{O}_{\overline{X}} \longrightarrow \ldots,$$

we see that $\mathbf{R}^i \pi_* \mathcal{O}_{Y'}(-\overline{X}) \cong \mathbf{R}^{i-1} \pi_* \mathcal{O}_{\overline{X}}$ for $i > 1$. Additionally, since $\pi_* \mathcal{O}_{Y'} = \mathcal{O}_Y$, we have the isomorphism $\pi_* \mathcal{O}_{\overline{X}} / \mathcal{O}_X \cong \pi_* \mathcal{O}_{\overline{X}} / \operatorname{Image}(\pi_* \mathcal{O}_{Y'})$ and so to prove the lemma, it suffices to show that Frobenius acts nilpotently on $\mathbf{R}^i \pi_* \mathcal{O}_{Y'}(-\overline{X})$ for $i > 0$.

The e-iterated Frobenius map $\mathcal{O}_Y(-\overline{X}) \longrightarrow F^e_* \mathcal{O}_Y(-\overline{X})$ factors as

$$\mathcal{O}_Y(-\overline{X}) \longrightarrow F^e_* \mathcal{O}_Y(-p^e \overline{X}) \subseteq F^e_* \mathcal{O}_Y(-\overline{X}).$$

That second inclusion factors through $F^e_* \mathcal{O}_Y(-nE)$ where n can be made arbitrarily large when $e \gg 0$. Hence by Serre vanishing applied to the ample divisor $-E$, we see that F^e acts as zero on $\mathbf{R}^i \pi_* \mathcal{O}_Y(-\overline{X})$ for $e \gg 0$. This completes the proof. $\square$

Now we come to our main theorem of the section.

Theorem 5.12. *Suppose X of finite type over a field of characteristic zero has dense F-injective type. Then X has Du Bois singularities.*

Proof. We may assume that $X = \operatorname{Spec} R$ is affine and so $X \subseteq Y = \operatorname{Spec} S$ where S is a polynomial ring. We may fix a log resolution $\pi : Y' \longrightarrow Y$ of (Y, X) that is the blowup of an ideal sheaf with the same support as X by Exercise 5.2 (in particular, there is a ample divisor on Y with the same support as $\overline{X}$). Because Y is non-singular, it has rational singularities and so $\mathbf{R}^i \pi_* \mathcal{O}_{Y'} = 0$ for all $i > 0$. Next notice that $\mathcal{H}^i \mathbf{R}\pi_* \omega^{\bullet}_{\overline{X}} \hookrightarrow \mathcal{H}^i \omega^{\bullet}_X$ injects for all i by Theorem 5.4. Our goal is to show they are surjective as well. Suppose that X is not Du Bois and so fix a prime $Q \in X = \operatorname{Spec} R$ minimal with respect to the condition that R_Q is not Du Bois.

We reduce this entire setup to a family of positive characteristic models. Chose a minimal $i \geq 0$ so there is a short exact sequence

$$0 \longrightarrow \mathcal{H}^{-i} \mathbf{R}(\pi_t)_* \omega^{\bullet}_{\overline{X_t}} \hookrightarrow \mathcal{H}^{-i} \omega^{\bullet}_{X_t} \longrightarrow C_t \longrightarrow 0$$

where $C_t \neq 0$, and it is non-zero at every minimal prime of Q_t, for all $t \in U$ an open dense set of m-Spec$A$. Since X has dense F-injective type, there exist $t \in U$ such that $X_t = \operatorname{Spec} R_t$ is F-injective. We fix such a t. We localize at Q_t (or a minimal prime over Q_t if Q_t is not prime) to obtain a local ring R_t' with maximal ideal $\mathfrak{m}_t$. Taking Matlis duality $-^\vee$ we obtain the following short exact sequence:

$$0 \longrightarrow C_t^\vee \longrightarrow H^i_{\mathfrak{m}_t}(R_t') \longrightarrow H^i_{\mathfrak{m}_t}(R_t' \otimes \mathbf{R}(\pi_t)_*\mathcal{O}_{\overline{X_t}}) \longrightarrow 0.$$

Frobenius acts injectively on $H^i_{\mathfrak{m}_t}(R_t')$ by hypothesis and so it acts injectively $C_t^\vee$ as well.

On the other hand, we can also construct $C_t^\vee$ as follows. Consider the exact triangle:

$$R_t' \longrightarrow \mathbf{R}(\pi_t)_*\mathcal{O}_{\overline{X_t}} \longrightarrow D^\bullet \xrightarrow{+1}.$$

We see that $H^{i-1}_{\mathfrak{m}_t}(D^\bullet) \cong C_t^\vee$. On the other hand, by Lemma 5.11, Frobenius acts nilpotently on the cohomology of $D^\bullet$. By construction, the cohomology of $D^\bullet$ is supported at $\mathfrak{m}_t$, and hence $C_t^\vee \cong H^{i-1}_{\mathfrak{m}_t}(D^\bullet) = \mathcal{H}^{i-1}D^\bullet$. Thus Frobenius acts injectively and nilpotently on $C_t^\vee \neq 0$, a contradiction. □

5.3. Du Bois singularities and weak ordinarity. We first deduce some consequences of the weak ordinarity conjecture (which hence are equivalent to the conjecture).

Lemma 5.13 ([**MS11**, Remark 5.2])**.** *If Conjecture 4.26 holds, then for a finite set of smooth varieties $X^{(1)}, \ldots, X^{(r)}$, an dense set $T \subset$ m-Spec$A$ can be chosen so that Conjecture 4.26 holds for all $X^{(i)}$ simultaneously for all $t \in T$.*

Proof. See Exercise 5.3. □

Lemma 5.14. *Assuming Conjecture 4.26, then for any finite collection of varieties $X^{(1)}, \ldots, X^{(r)}$, there exists a dense set $T \subseteq$ m-Spec$A$ so that the Frobenius action on $H^i(X_t^{(j)}, \mathcal{O}_{X_t^{(j)}})$ is bijective for all i, all $j = 1, \ldots, r$, and all $t \in T$.*

Proof. Proceed by induction with the following inductive hypothesis. For any finite collection of varieties $X^{(j)}$ and any integer m we may find a $T \subseteq$ m-Spec$A$ so that Frobenius action on $H^i(X_t^{(j)}, \mathcal{O}_{X_t^{(j)}})$ is bijective for all i if $\dim X^{(j)} \leq m$ and such that the Frobenius action on $H^{\dim X^j}(X_t^{(j)}, \mathcal{O}_{X_t^{(j)}})$ is bijective. We will perform ascending induction on m. The base case when $m = 0$ is trivial as it follows from Lemma 5.13.

Suppose then we are given a set of varieties S and our inductive hypothesis holds for some m. Let S_{m+1} denote the set of varieties $X^{(j)}$ where $\dim X^{(j)} = m+1$. For each $X^{(j)} \in S_{m+1}$, choose a smooth ample divisor $D^{(j)}$ such that $H^i(X^{(j)}, \mathcal{O}_X(-D^{(j)})) = 0$ for all $i < \dim X^{(j)}$ (this vanishing is preserved even on an open dense subseteq of m-Spec$A$). We include the components of the D_j to form a new set S and apply our inductive hypothesis. We now have that Frobenius acts bijectively on $H^i(X_t^{(j)}, \mathcal{O}_{D_t^{(j)}})$ for all $t \in T$ and all i. But we now have that

$$H^i(X_t^{(j)}, \mathcal{O}_{X_t^{(j)}}) \hookrightarrow H^i(X_t^{(j)}, \mathcal{O}_{D_t^{(j)}})$$

injects for all $i < \dim X^{(j)}$. Hence Frobenius acts injectively and hence bijectively (since $k(t)$ is perfect) on the left side, as desired. $\square$

This bijectivity can also be generalized to simple normal crossings varieties as well.

Proposition 5.15. *Suppose that Y is a smooth projective variety over a field of characteristic zero and $X \subseteq Y$ is an SNC divisor. Assume that Conjecture 4.26 holds. Then, in reduction to positive characteristic, there exists a dense set $T \subseteq$ m-Spec$A$ such that Frobenius acts bijectively on $H^i(X_t, \mathcal{O}_{X_t})$ for all i and all $t \in T$ and furthermore for those t we may assume that Frobenius acts bijectively on $H^i(Y_t, \mathcal{O}_{Y_t})$.*

Proof. This is assigned to the reader in Exercise 5.4. $\square$

We then obtain the following.

Corollary 5.16. *Suppose Y is a smooth projective variety over a field of characteristic zero and $X \subseteq Y$ is an SNC divisor. Assume that Conjecture 4.26 holds. Then, in reduction to positive characteristic, there exists a dense set $T \subseteq$ m-Spec$A$ such that Frobenius acts bijectively on $H^i(Y_t, \mathcal{O}_{Y_t}(-X_t))$ and on $H^i(Y_t, \mathcal{O}_{Y_t})$ for all i and all $t \in T$.*

Proof. Consider the long exact sequence

$$\dots \longrightarrow H^{i-1}(Y_t, \mathcal{O}_{Y_t}) \longrightarrow H^{i-1}(Y_t, \mathcal{O}_{X_t}) \longrightarrow$$

$$\longrightarrow H^i(Y_t, \mathcal{O}_{Y_t}(-X_t)) \longrightarrow H^i(Y_t, \mathcal{O}_{Y_t}) \longrightarrow H^i(Y_t, \mathcal{O}_{X_t}).$$

We may choose t so that Frobenius acts bijectively on all terms except $H^i(Y_t, \mathcal{O}_{Y_t}(-X_t))$ by Proposition 5.15. But now the 5-Lemma implies that Frobenius acts bijectively on that term as well. $\square$

Another consequence of weak ordinarity is the following.

Corollary 5.17. *Suppose $\pi : Y \longrightarrow V$ is a projective morphism of schemes where V is finite type over a field of characteristic zero, Y is smooth and $X \subseteq Y$ is a reduced SNC divisor. Assume that Conjecture 4.26 holds. Then, in reduction to positive characteristic, there exists a dense set $T \subseteq \text{m-Spec} A$ such that map, induced by the Grothendieck dual to Frobenius on $\mathcal{O}_{Y_t}(-X_t)$,*

$$\mathbf{R}^i F_* \pi_* \omega_{Y_t}(X_t) \longrightarrow \mathbf{R}^i \pi_* \omega_{Y_t}(X_t)$$

is surjective for every $i \geq 0$ and $t \in T$.

Proof. We first assume that V is projective over a field. By Serre vanishing, we choose a very ample line bundle $\mathscr{L}$ on V so $\mathscr{L} \otimes \mathbf{R}^i \pi_* \omega_Y(X)$ is 0-regular with respect to $\mathscr{L}$ for all i (see Definition 3.18), and hence that

$$H^j(V, \mathscr{L}^n \otimes \mathbf{R}^i \pi_* \omega_Y(X)) = 0$$

for all $j > 0$, all $n > 0$ and all i and so that $\mathscr{L} \otimes \mathbf{R}^i \pi_* \omega_Y(X)$ is globally generated for all i. By Bertini's theorem, we may fix $D \in |\mathscr{L}|$ so that $X + \pi^* D$ is a reduced SNC divisor in Y. We reduce this setup to characteristic $p > 0$.

By Serre duality applied to the conclusion of Corollary 5.16, we may restrict to a T so the composition

$$H^i(Y_t, F_* \omega_{Y_t}(X_t + \pi^* D_t)) \longrightarrow H^i(Y_t, F_* \omega_{Y_t}(X_t + p \cdot \pi^* D_t)) \longrightarrow H^i(Y_t, \omega_{Y_t}(X_t + \pi^* D_t))$$

is bijective. Hence the second map is surjective. But that second map is

$$H^i(Y_t, F_*(\pi^* \mathscr{L}_t^p \otimes \omega_{Y_t}(X_t))) \longrightarrow H^i(Y_t, \mathscr{L}_t \otimes \omega_{Y_t}(X_t)). \tag{5.17.1}$$

Our Castelnuovo-Mumford regularity hypothesis, when combined with a spectral sequence, also guarantees that

$$H^i(Y_t, \mathscr{L}_t^n \otimes \omega_{Y_t}(X_t)) \cong H^0(Y_t, \mathbf{R}^i \pi_* \mathscr{L}_t^n \otimes \omega_{Y_t}(X_t))$$

for all i and all $n > 0$. The right side of (5.17.1), remains globally generated and by our 0-regularity. Therefore we obtain the desired result.

For the case of a general V, compactify V to a projective V' (and $X', Y' \longrightarrow V'$). Then the resolution of singularities algorithms in fact imply the existence of a resolution of Y' that is an isomorphism over the locus where Y' is smooth and X' is SNC. Hence we may assume that V is projective. ☐

We can use this to prove the following.

Theorem 5.18. *Assume the weak ordinarity conjecture Conjecture 4.26. Suppose that X is a variety in characteristic zero with Du Bois singularities, then X has dense F-injective type.*

PROOF. The statement is local and so we embed X in a non-singular Y (and we may assume that $X \subseteq Y$ is not a divisor). Fix $\pi : Y' \longrightarrow Y$ a log resolution of (Y, X) with $\overline{X} = \pi^{-1}(X)_{\mathrm{red}}$ and SNC divisor in Y'. Notice that $\mathbf{R}^i\pi_*\omega_{Y'} = 0$ for all $i > 0$ by Theorem 2.13 and that $\pi_*\omega_{Y'} \subseteq \pi_*\omega_{Y'}(\overline{X}) \subseteq \omega_Y$ is an isomorphism since Y is non-singular (we use here that X is not codimension 1 in Y). Since X has Du Bois singularities we also observe that $\omega_X^{\bullet} \cong \mathbf{R}\pi_*\omega_{\overline{X}}^{\bullet} = \mathbf{R}\pi_*\omega_{\overline{X}}[\dim \overline{X}]$.

By reducing to positive characteristic and applying Corollary 5.17 we see that $\mathbf{R}^i F_*\pi_*\omega_{Y_t}(X_t) \longrightarrow \mathbf{R}^i\pi_*\omega_{Y_t}(X_t)$ surjects for every $i \geq 0$. From the short exact sequence

$$0 \longrightarrow \omega_{Y_t'} \longrightarrow \omega_{Y_t'}(\overline{X}_t) \longrightarrow \omega_{\overline{X}_t} \longrightarrow 0$$

and the associated long exact sequence obtained by applying $\mathbf{R}\pi_*$, we see that the dual Frobenius acts surjectively on $\mathbf{R}^i\pi_*\omega_{\overline{X}_t}$ for every i, and so a shift by $[\dim \overline{X}]$ does not change this. Hence the dual Frobenius is surjective on the (isomorphic) cohomology of $\omega_X^{\bullet}$ and so X has F-injective singularities. □

Interestingly enough, the conjecture that Du Bois implies dense F-injective type implies (and so is equivalent to) the weak ordinarity conjecture. See Exercise 5.6 and Exercise 5.7.

5.4. Exercises.

Exercise 5.1. Suppose that $\pi : \widetilde{X} \longrightarrow X, \pi' : \widetilde{X}' \longrightarrow X$ are two log resolutions of an equidimensional scheme X of finite type over a field of characteristic zero, with reduced exceptional divisors F, F' respectively. Show that $\pi_*\omega_{\widetilde{X}}(F) = \pi'_*\omega_{\widetilde{X}'}(F')$.

Hint: Show that we may assume that we may factor our resolutions as follows: $\pi' : \widetilde{X}' \xrightarrow{\rho} \widetilde{X} \xrightarrow{\pi} X$. Notice that we do not necessarily have $F' = \rho^{-1}(F)_{\mathrm{red}}$ but we do have $F' \supseteq \rho^{-1}(F)_{\mathrm{red}}$. Now use the projection formula and the fact that $\mathcal{J}(\widetilde{X}, (1-\epsilon)F) \cdot \omega_{\widetilde{X}} = \omega_{\widetilde{X}}$ for $1 \gg \epsilon > 0$ (which can be deduced either directly, or via reduction to characteristic p).

Exercise 5.2. Working over a field of characteristic zero, if $X \subseteq Y$ is a closed subscheme and $Y = \operatorname{Spec} S$ is non-singular and affine, show that there exists a log projective resolution $\pi : Y' \longrightarrow Y$ of (Y, X) with an anti-effective ample divisor $-E$ on Y' so that $\pi^{-1}(X)$ has the same support as E. In particular, π may be taken to be the blowup of an ideal sheaf with the same support as X.

Hint: There are two cases, when X is a hypersurface and when X is not a hypersurface.

Exercise 5.3. Prove Lemma 5.13.

Hint: Apply the conjecture to $X^{(1)} \times \cdots \times X^{(r)}$ and use the Künneth formula.

Exercise 5.4. Prove Proposition 5.15.

Hint: Write $X = \bigcup_{i=1}^{r} X_i$ the union of irreducible components. For each $I \subseteq \{1, \dots, r\}$ write $X_I = \bigcap_{i \in I} X_i$. Now, form an acyclic complex where $C^0 = X$ and $C^i = \bigoplus_{|I|=i} \mathcal{O}_{X_I}$,

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \dots \xrightarrow{d^{r-1}} C^r \xrightarrow{d^r} 0$$

where the maps are the obvious ones. Perform reduction to characteristic p where Frobenius acts bijectively on all $H^i((X_I)_t, \mathcal{O}_{(X_I)_t})$. Set $Z^i = \ker d_i$ and use descending induction to prove that Frobenius acts bijectively on the cohomology of the $\mathcal{O}_{Z^i}$ and eventually on X.

Exercise 5.5. Suppose that $X \subseteq Y$ is a closed embedding of varieties over a field k of characteristic zero. Further suppose X is non-singular except at a single closed point $x \in X$, that $\pi : Y' \longrightarrow Y$ is an embedded log resolution of singularities of X which is an isomorphism except over x and $E = \pi^{-1}(x)_{\mathrm{red}}$ is a SNC divisor. Further set $\widetilde{X}$ be the strict transform of X. Prove that there exists an exact triangle in the derived category:

$$\underline{\Omega}^0_X \longrightarrow \mathbf{R}\pi_* \mathcal{O}_{\widetilde{X}} \oplus k(x) \longrightarrow \mathbf{R}\pi_* \mathcal{O}_{E \cap \widetilde{X}} \xrightarrow{+1} .$$

In fact, there are a number of generalizations of this when x is not a point.

Hint: Recall that $\pi^{-1}(X)_{\mathrm{red}} = \overline{X} = \widetilde{X} \cup E$ is such that $\mathbf{R}\pi_* \mathcal{O}_{\overline{X}} = \underline{\Omega}^0_X$. Construct a related short exact sequence on Y' and then push it down.

Exercise 5.6. Suppose that Z is a smooth projective variety in characteristic zero. Then for any ample line bundle $\mathscr{L}$ on Z and any sufficiently large integer $m \gg 0$, we have that the section ring with respect to $\mathscr{L}^m$,

$$S := \bigoplus_{n \in \mathbb{Z}} H^0(Z, \mathscr{L}^{nm}),$$

has Du Bois singularities. In fact, this holds under the weaker hypothesis that Z has Du Bois singularities even if Z is not smooth.

Hint: Use the previous exercise. Note that if $x = \operatorname{Spec} S$ and $\pi : \widetilde{X} \longrightarrow X$ is the blowup of the cone point with exceptional divisor $F \cong Z$, then it suffices to show that $\mathbf{R}^i \pi_* \mathcal{O}_{\widetilde{X}} \longrightarrow \mathbf{R}^i \mathcal{O}_F$ is an isomorphism for $i > 0$. Prove this isomorphism degree by degree (the degree zero piece is the only one which requires real computation).

Exercise 5.7. Suppose that it is shown that every variety in characteristic zero with Du Bois singularities has dense F-injective type. Show that the Weak Ordinarity Conjecture Conjecture 4.26 must hold.

Hint: Use the previous exercise.

5.5. Exercises on seminormality.

See Chapter 2 Subsection 4.5 for a related discussion of weak normality. The following definition will be used in the exercises that follow.

Definition 5.19 (Seminormal extensions and seminormal normal rings)**.** An integral extension of reduced rings $R \subseteq S$ is called **subintegral** if

(a) $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is a bijection and,
(b) For every prime $Q \in \operatorname{Spec} R$ with corresponding $Q' \in \operatorname{Spec} S$, the inclusion of residue fields $k(Q) \subseteq k(Q')$ is an equality.

We say R is **seminormal in an overring** B if every subintegral extension $R \subseteq S \subseteq B$ has the property that $R = S$. We say that R is **seminormal** if it is weakly normal in its total ring of fractions $\mathcal{K}(R)$. A scheme is seminormal if and only if its stalks are seminormal (or equivalently if it can be covered by seminormal affine charts).

Earlier we defined R to be seminormal if any $x \in \mathcal{K}(R)$ such that $x^2, x^3 \in R$ also satisfies $x \in R$. These two notions of seminormality coincide, see [**Ham75, Swa80**] and the exercises below.

Exercise 5.8. For a ring essentially of finite type over a field of characteristic zero, show that weak normality and seminormality coincide.

Exercise 5.9. Suppose that R is excellent reduced Noetherian ring, show that there exists a seminormalization R^{SN} (a finite extension) of R in $\mathcal{K}(R)$. That is, R^{SN} is the unique largest subintegral extension of R.

Exercise 5.10. Suppose that R is reduced $x \in \mathcal{K}(R)$ and $x^2, x^3 \in R$. Show that $R \subseteq R[x] =: R'$ is subintegral.

Hint: Choose a prime $Q \subseteq R$. We need to show that $P = \sqrt{QR'}$ is prime and the residue fields are the same. This can be checked after localizing R at Q. Handle this in cases: either $x \in P$ or $x \notin P$.

Exercise 5.11. Show that R is seminormal if and only if it satisfies the following condition due to Hamann [**Ham75**], also see the work of Swan [**Swa80**]. For any $y, z \in R$ such that $y^2 = z^3$ there exists a unique $x \in R$ with $x^3 = y$ and $x^2 = z$.

Exercise 5.12. Suppose that X is a reduced seminormal scheme. Prove that for any open subseteq $U \subseteq X$, that $\Gamma(U, X)$ is seminormal.

Exercise 5.13. Suppose that D is a simple normal crossings divisor in some scheme Y. Prove that D is seminormal (in fact, it is weakly normal, say if one is in characteristic $p > 0$).

Exercise 5.14. With notation as in Subsection 5.1, assume $X = \operatorname{Spec} R$. Show that $\mathcal{H}^0\underline{\Omega}^0_X = \nu_*\mathcal{O}_{X^{\mathrm{SN}}}$ where $\nu : X^{\mathrm{SN}} \longrightarrow X$ is the seminormalization map (in other words $\Gamma(X, \mathcal{H}^0\underline{\Omega}^0_X) = R^{\mathrm{SN}}$).

Hint: With notation as in the section, we need to show that $\pi_*\mathcal{O}_{\overline{X}}$ is the seminormalization of $\mathcal{O}_X$. Since Y is normal and $\pi : Y' \longrightarrow Y$ is birational, the fibers of π are geometrically connected hence so are the fibers of $\overline{X} \longrightarrow X$. Use this to show that $R \subseteq \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$ is subintegral.

6. Test ideals and quotients by height 1 ideals

We have already seen some of this theory in various exercises, but here we write down some general results which closely mimic similar results in characteristic $p > 0$. First however, we generalize the results of Chapter 5 Section 2 to encompass $\mathbb{Q}$-divisors such that $K_X + \Delta$ that are only *locally* $\mathbb{Q}$-linearly equivalent to zero.

We continue to work in the following setting.

Setting 6.1. Suppose that X is an F-finite normal integral separated Noetherian scheme with a chosen canonical module ω_X such that $\operatorname{Hom}(F^e_*\mathcal{O}_X, \omega_X) \cong F^e_*\omega_X$. We fix a canonical divisor K_X with $\omega_X \cong \mathcal{O}_X(K_X)$.

6.1. Divisor pairs corresponding to twisted p^{-e}-linear maps.

Fix L a line bundle on a normal F-finite Noetherian scheme X satisfying Setting 6.1 and write $\mathscr{L} = \mathcal{O}_X(L)$. Consider a map

$$\phi \in \operatorname{Hom}_{\mathcal{O}_X}(F^e_*\mathscr{L}, \mathcal{O}_X) \cong H^0(X, F^e_*\mathcal{O}_X((1-p^e)K_X - L))$$

In then induces an effective divisor $D_\phi \sim (1-p^e)K_X - L$. We set $\Delta_\phi := \frac{1}{p^e-1}D_\phi$. Notice that $(p^e-1)(K_X + \Delta) \sim -L$ and in particular $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Furthermore, the map $\phi : F^e_*\mathscr{L} \longrightarrow \mathcal{O}_X$ can be self composed in the following sense. We define

$$\phi^2 := \phi \circ (F^e_*(\phi \otimes \mathscr{L})) : F^{2e}_*\mathscr{L}^{1+p^e} \xrightarrow{F^e_*(\phi\otimes\mathscr{L})} F^e_*\mathscr{L} \xrightarrow{\phi} \mathcal{O}_X.$$

and more generally:

$$\phi^n := \phi \circ (F^e_*(\phi \otimes \mathscr{L})) \circ \cdots \circ (F^{(n-1)e}_*(\phi \otimes \mathscr{L}^{\frac{p^{ne}-1}{p^e-1}})). \tag{6.1.1}$$

Locally, restricting to an open affine set U such that $\mathscr{L}|_U \cong \mathcal{O}_U$, this is the same as the composition of maps we have been working with for instance as in Chapter 1 (4.12.5). Indeed, on these charts, the divisors $D_\phi|_U$ and $\Delta_\phi|_U$ are exactly the same same as that described in Chapter 5 Definition 1.1 and Definition 2.1.

We then obtain a bijection as follows: *cf.* Chapter 5 Theorem 2.2.

Proposition 6.2. *With notation as in Setting 6.1 there is a bijection between the two sets:*

$$\left(\begin{array}{c} \mathbb{Q}\text{-divisors } \Delta \geq 0 \\ \text{such that} \\ (p^e - 1)(K_X + \Delta) \\ \text{is Cartier for some } e > 0 \end{array} \right) \leftrightarrow \left(\begin{array}{c} \text{non-zero } \mathcal{O}_X\text{-linear maps} \\ \phi : F^e_* \mathscr{L} \longrightarrow \mathcal{O}_X \\ \text{(for some } e > 0 \text{ and} \\ \text{some line bundle } \mathscr{L}). \end{array} \right) \Big/ \sim$$

where the equivalence relation on the right is generated by the relation $\phi \sim \phi^d$ and also by $\phi(F^e_* -) \sim \phi(F^e_* u \cdot -)$ where $u \in H^0(X, \mathcal{O}_X)$ is a unit.

Proof. This follows as in the case that $\mathscr{L} = \mathcal{O}_X$ which is done in Chapter 5 Theorem 2.2. $\square$

6.2. Global F-adjunction. Consider the following situation. Suppose that X is a normal F-finite Noetherian scheme, $D = \operatorname{Spec} \mathcal{O}_X/I_D$ is a reduced divisor on X with normalization $D^{\mathrm{N}} \longrightarrow D$. We have seen that divisors $\Delta \geq D$ with $K_X + \Delta$ $\mathbb{Q}$-Cartier with index not divisible by p, correspond to line bundles $\mathscr{L} = \mathcal{O}_X(L)$ and non-zero elements in

$$\operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathscr{L}, \mathcal{O}_X).$$

Note here that $L \sim (1 - p^e)(K_X + \Delta)$ for some e. Those elements are compatible with D in the sense that

$$\phi(F^e_* \mathcal{O}_X(L - D)) \subseteq \mathcal{O}_X(-D)$$

by Chapter 5 Proposition 2.7 (this condition can be checked locally after trivializing $\mathcal{O}_X(L)$)), and so induce maps in

$$\operatorname{Hom}_{\mathcal{O}_D}(F^e_* \mathscr{L}|_D, \mathcal{O}_D).$$

Finally, in Chapter 5 Theorem 4.8 and we have seen that these in fact lift to the normalization (locally, and hence globally) to produce a map in

$$\operatorname{Hom}_{\mathcal{O}_{D^{\mathrm{N}}}}(F^e_* \mathscr{L}|_{D^{\mathrm{N}}}, \mathcal{O}_{D^{\mathrm{N}}}).$$

Note we abuse notation as is common in birational algebraic geometry: by $\mathscr{L}|_{D^{\mathrm{N}}}$ we simply mean the pull back of $\mathscr{L}$ to D^{N} via the map $D^{\mathrm{N}} \longrightarrow D \longrightarrow X$. In particular, each map in $\operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X(L + (p^e - 1)D), \mathcal{O}_X)$ then induces a divisor $\Delta_{D^{\mathrm{N}}} \geq 0$ on D^{N} with $\mathcal{O}_{D^{\mathrm{N}}}\big((1-p^e)(K_{D^{\mathrm{N}}} + \Delta_{D^{\mathrm{N}}})\big) \cong \mathscr{L}|_{D^{\mathrm{N}}}$. In other words, we have produced an effective divisor $\Delta_{D^{\mathrm{N}}}$ on D^{N} so that

$$(K_X + D)|_{D^{\mathrm{N}}} \sim_{\mathbb{Q}} K_{D^{\mathrm{N}}} + \Delta_{D^{\mathrm{N}}}.$$

This is called the *F-different of Δ along D^{N}* and denoted by $F\operatorname{Diff}_{D^{\mathrm{N}}}(\Delta)$.

6.3. The different vs the F-different. In birational geometry, there is another way to produce such a $\Delta_{D^{\mathrm{N}}}$ called "the different" or "Shokurov's different". We briefly introduce this concept and prove that the F-different agrees with the different. This was first shown in [**Das15**].

Suppose that X is a normal integral scheme with a normalized[32] dualizing complex $\omega_X^{\bullet}$. Suppose further that $D \subseteq X$ is a reduced divisor on X and that $\Delta = D + B \geq 0$ is a $\mathbb{Q}$-divisor where D and B have no common components and with $K_X + \Delta$ $\mathbb{Q}$-Cartier. Suppose further that we write $K_X = -D + H$ where H and D have no common components (this can be arranged since K_X is only defined up to principal divisors). In this case, we have that

$$K_X + D = -D + H + D = H$$

where H does not contain any component of D. Now, we have the following exact sequence

$$\mathcal{O}_X(K_X) \longrightarrow \mathcal{O}_X(K_X + D) \longrightarrow \omega_D \longrightarrow \mathcal{H}^{-\dim X+1}\omega_X^{\bullet}$$

Since X is normal, it is S2, and so $\mathcal{H}^{-\dim X+1}\omega_X^{\bullet}$ is supported on a codimension ≥ 3 set. The rational section s of $\mathcal{O}_X(K_X + D)$ corresponding to H then yields a rational section of ω_D. Since we have a birational map $\mu : D^{\mathrm{N}} \longrightarrow D$, we obtain an inclusion $H^0(D^{\mathrm{N}}, \omega_{D^{\mathrm{N}}}) \longrightarrow H^0(D, \omega_D)$. Hence our rational section also gives us a rational section of $\omega_{D^{\mathrm{N}}}$. In particular, it selects a particular choice of canonical divisor $K_{D^{\mathrm{N}}}$ on D^{N}. In summary:

our choice of K_X determines a canonical divisor $K_{D^{\mathrm{N}}}$.

It is worth seeing this in an example or two:

Example 6.3. Consider $X = \operatorname{Spec} k[x, y, z]/(xy - z^2)$ the quadric cone and set $D = V((x, z))$. This is not Cartier, although $2D = \operatorname{div}_X(x)$ is, as is $D + E = \operatorname{div}_X(z) = D + V((y, z))$. In our case, K_X is Cartier, so we can choose $K_X = -D - E = -\operatorname{div}_X(z)$ satisfying our condition above. Then $K_X + D = -E = -\operatorname{div}((x, z))$. The divisor $-E$ corresponds to the rational section 1 of $\mathcal{O}_X(-E)$ (notice that 1 is not a global section of $\mathcal{O}_X(-E)$, only a rational section). In our case, X is Cohen-Macaulay and so

$$\omega_D = \mathcal{O}_X(K_X+D)/\mathcal{O}_X(K_X) = \mathcal{O}_X(-E)/\mathcal{O}_X(-D-E) = (y, z)_{\mathcal{O}_X}/(z\cdot\mathcal{O}_X) = y\cdot\mathcal{O}_D.$$

The rational section 1 then corresponds to the non-effective divisor $-Q$ where Q is the origin in $D = \operatorname{Spec} k[y]$. In particular, our choice of $K_X + D = -E$ determines the divisor K_D to be $-Q$.

[32] Meaning that $\omega_X^{\bullet}$ has its first non-zero cohomology in $-\dim X$, and in particular $\mathcal{H}^{-\dim X}\omega_X^{\bullet} = \omega_X$ is the canonical module.

On the other hand, we could have just as easily chosen $K_X = -D + E = \mathrm{div}(z/x)$. In that case, $K_X + D = E = \mathrm{div}((y, z))$. Notice that $\mathcal{O}_X(E) = 1/y \cdot \mathcal{O}_X(-E) = 1/y(y, z) = (1, z/y)$ and $\mathcal{O}_X(-D+E) = x/z \cdot \mathcal{O}_X = z/y \cdot \mathcal{O}_X$

$$\omega_D = \mathcal{O}_X(K_X+D)/\mathcal{O}_X(K_X) = \mathcal{O}_X(E)/\mathcal{O}_X(-D+E) = (1, z/y)_{\mathcal{O}_X}/((z/y \cdot \mathcal{O}_X) = \mathcal{O}_D.$$

Thus we have that the rational section 1 is an actual section, so that $K_D = 0$. This should not be surprising – to pass between the two choices of K_X, we multiplied by y, and $\mathrm{div}_D(y) = Q$.

When D is non-normal, we need to understand the map $\mu_* \omega_{D^{\mathrm{N}}} \longrightarrow \omega_D$ as well (coming from $\mu : D^{\mathrm{N}} \longrightarrow D$). In this case one has the formula

$$\mu_* \omega_{D^{\mathrm{N}}} = \mathscr{H}\mathrm{om}(\mu_* \mathcal{O}_{D^{\mathrm{N}}}, \omega_D)$$

and the map $\mu_* \omega_{D^{\mathrm{N}}} \longrightarrow \omega_D$ is simply evaluation-at-1 (it is also an inclusion since it is a non-generically zero map of rank-1 torsion-free sheaves). In the case that D is is Gorenstein, then the image of this inclusion map is simply the conductor ideal times ω_D. In particular, the conductor on D^{N} will contribute to the rational section in question.

Example 6.4 (A non-normal non-integral D)**.** Within the same ambient space $X = \mathrm{Spec}\, k[x, y, z]/(xy - z^2)$ consider $D = \mathrm{div}(z) = D_1 + D_2 = \mathrm{Spec}\, k[x, y]/(xy)$ where $D_1 = \mathrm{div}_X((x, z))$ and $D_2 = \mathrm{div}_X((y, z))$. In this case, we can take $K_X = -D$, since both are principal divisors. In this case we have from the adjunction sequence that

$$\omega_D = \mathcal{O}_X(K_X + D)/\mathcal{O}_X(K_X) = \mathcal{O}_X/\mathcal{O}_X(-D) = \mathcal{O}_D$$

and the rational section 1 is an honest section and gets mapped to the $1 \in \omega_D$. Thus if D was normal, we could take $K_D = 0$. Now, instead we must consider the normalization map $D^{\mathrm{N}} \longrightarrow D$ which induces

$$\omega_{D^{\mathrm{N}}} \longrightarrow \omega_D = \mathcal{O}_D.$$

Now, D^{N} is just the disjoint union of the two components of D. If we set $R = k[x, y]/(xy)$ and $S = k[x] \times k[y]$ to be the normalization, then this map is identified with the evaluation-at-1 map:

$$\mathrm{Hom}_R(S, R) \longrightarrow R.$$

Notice S as an R-module is generated by $(0, 1))$ and $(1, 0)$. In this case, one can show that $\mathrm{Hom}_R(S, R) \cong S$ and the induced inclusion $\mathrm{Hom}_R(S, R) \longrightarrow R$ is described by saying where $(0, 1)$ and $(1, 0)$ go – to y and x respectively. Hence the section $1 \in R$ corresponds to the rational section $(1/x, 1/y)$.

In conclusion, if Q_1, Q_2 are the two origins of D^{N}, then $K_{D^{\mathrm{N}}} = -Q_1 - Q_2$.

Now, recalling that $\Delta = D + B$ where D and B have no common components, we have that:

$$K_X + \Delta = (-D + H) + (D + B) = H + B$$

does not vanish along D. Hence, working locally on some $U \subseteq X$, we can write $n(K_X + \Delta)|_U = \mathrm{div}(f)$ for some $f \in \Gamma(U, \mathcal{O}_X)$ and some n sufficiently divisible. This function f does not vanish along D and so pulls back to a non-zero function on D^{N}, and so we write

$$\mathrm{div}_{D^{\mathrm{N}}}(f) = nK_{D^{\mathrm{N}}} + G$$

and define the different $\Delta_{D^{\mathrm{N}}}$ to be $\frac{1}{n}G$. Here $K_{D^{\mathrm{N}}}$ is the particular choice of canonical divisor we made above. It is straightforward to see that this $\Delta_{D^{\mathrm{N}}}$ does not depend on the choice of K_X. We have finally come to our definition of the different:

Definition 6.5 (Shokurov's different)**.** With notation as above, $\Delta_{D^{\mathrm{N}}}$ is called the **different of Δ along D^{N}**. It is denoted by $\mathrm{Diff}_{D^{\mathrm{N}}}(\Delta)$ and it satisfies:

$$(K_X + \Delta)|_{D^{\mathrm{N}}} \sim_{\mathbb{Q}} K_{D^{\mathrm{N}}} + \mathrm{Diff}_{D^{\mathrm{N}}}(\Delta).$$

Example 6.6. Using the setup and notation of Example 6.3, we set $X = \mathrm{Spec}\, k[x, y, z]/(xy - z^2)$, $D = V((x, z))$ and $E = V((y, z))$. Finally fix $\Delta = D$. We choose $K_X = -D + E$ and found that $K_D = -Q$ where Q is the point at the origin. Since $2(K_X + \Delta) = -2E = \mathrm{div}(1/y)$. We restrict $1/y$ to D where we get

$$-Q = \mathrm{div}_D(1/y) = 2K_D + G = -2Q + G$$

Hence $G = Q$ and so $\mathrm{Diff}_D(\Delta) = \frac{1}{2}Q$.

We note the following lemma which lets us compute the different in many common cases. We will use this in our proof that the different agrees with the F-different.

Lemma 6.7. *Suppose X is a non-singular integral scheme and suppose D is a reduced, normal, and Cartier divisor. Suppose that $\Delta = D + B$ where B and D have no common components and $B \geq 0$ is a $\mathbb{Q}$-divisor. Then the different $\Delta_D = B|_D$.*

Proof. We work locally, and so can set $K_X = -D$. Since we are working sufficiently locally, we also have $\omega_D = \omega_X(D)/\omega_X = \mathcal{O}_X/\mathcal{O}_X(-D) = \mathcal{O}_D$ and it follows that we chose $K_D = 0$. Next, $K_X + D + B = B$ and so we choose $n > 0$ so that $m(K_X + D + B) = mB = \mathrm{div}_X(f)$ (again working sufficiently locally). Finally, we write $(mB)|_D = \mathrm{div}_D(f) = K_D + G = G$ and so the different $\Delta_D = \frac{1}{m}(mB)|_D = B|_D$. $\square$

We finally describe the way that the different is most often computed in practice, assuming that X is excellent. To compute the different $\mathrm{Diff}_{D^{\mathrm{N}}}(\Delta)$, we must compute its coefficients at codimension 1 points of D^{N}. Those map to codimension 2 points of X and so we may assume that $X = \mathrm{Spec}\, R$ where $(R, \mathfrak{m})$ is a normal 2-dimensional local domain.

Since X is excellent, there exists a resolution of singularities $\pi : Y \longrightarrow X$ that simultaneously resolves the singularities of D, see [**Lip78**]. Let $D' = \pi_*^{-1}D$ denote the strict transform, then $D' = D^{\mathrm{N}}$ is in fact the normalization of D, since D is 1-dimensional. We now have the following commutative diagram:

$$\begin{tikzcd} D' \arrow[r, hook] \arrow[d, two heads, "\mu"'] & Y \arrow[d, two heads, "\pi"] \\ D \arrow[r] & X. \end{tikzcd}$$

A choice of K_X induces a choice of $K_{D'}$, but now we have another way to select $K_{D'}$. Fix K_Y so that K_X and K_Y agree where π is an isomorphism. Then since $K_Y = -D' + H$ where H and D' have no common components, we can write $K_{D'} = (K_Y + D')|_{D'}$.

Lemma 6.8. *The two definitions of $K_{D'}$ agree.*

PROOF. If s is a rational section of ω_X that defines K_X, then we may also view it as a rational section of ω_Y, defining K_Y, using that $\pi_*\omega_Y \subseteq \omega_X$. Because the following diagram commutes:

$$\begin{tikzcd} \mu_*\omega_{D'} \arrow[d, hook] & \pi_*\omega_Y \arrow[l] \arrow[d, hook] \\ \omega_D & \omega_X \arrow[l] \end{tikzcd}$$

we see it does not matter which way we pull back this rational section, and the proof is complete. $\square$

Now, if $m(K_X + D + B) = \mathrm{div}_X(f)$, we can pull this element f back to Y and then restrict it to D', or we can restrict it to D and then pull it back to D'. Either way will give us the same element on D'. Hence this provides the following algorithm for computing the different.

Proposition 6.9. *Suppose $X = \operatorname{Spec} R$ where R is an excellent normal local 2-dimensional domain. Suppose D is a reduced divisor and $\Delta = D + B$ where D, B have no common components and $K_X + D + B$ is $\mathbb{Q}$-Cartier. Next write $K_X = -D + G$ where G has no common components with D. Suppose $\pi : Y \longrightarrow X$ is a resolution of singularities of X which is also an embedded resolution of D (this exists since $\dim R = 2$). Then we have*

$$\pi^*(K_X + D + B) = K_Y + D' + B_Y$$

where D' is the strict transform and hence normalization[33] of D and where K_X and K_Y agree where π is an isomorphism. Then the different of Δ along D is simply $B_Y|_{D'}$.

[33]since $\dim D = 1$

PROOF. This follows from the discussion above. □

Finally, we can use this description to show that F-different agrees with the different. First we prove the analog of Lemma 6.7.

Lemma 6.10. *Suppose that X is a regular F-finite scheme and that D is a normal reduced Cartier divisor on X. Suppose that $\Delta = D + B$ where $B \geq 0$ is a $\mathbb{Q}$ divisor ans where B and D have no common components. Finally, suppose that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p. Then*

$$F\operatorname{Diff}_D(\Delta) = B|_D.$$

PROOF. Working locally with $X = \operatorname{Spec} R$ for some local F-finite regular ring R, write $B = \frac{1}{p^e-1}\operatorname{div}_X(g)$ and $D = \operatorname{div}_X(f)$. The map $\phi : F^e_* R \longrightarrow R$ corresponding Δ is then simply the generating map Φ pre-composed with multiplication by $F^e_* f^{p^e-1} g$. Since Φ pre-composed with multiplication by $F^e_* f^{p^e-1}$ restricts to the generating map for R/I_D by Fedder's Lemma, we see that ϕ restricts the generating map on D pre-composed with multiplication by $F^e_* g$. In other words, $F\operatorname{Diff}_D(\Delta) = B|_D$ as claimed. □

Theorem 6.11 ([**Das15**]). *Suppose X is a normal integral F-finite scheme, D is a reduced divisor on X and $\Delta = D+B$ where D and B have no common components and $B \geq 0$ is a $\mathbb{Q}$-divisor. Further suppose that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p. Then*

$$F\operatorname{Diff}_{D^{\mathrm{N}}}(\Delta) = \operatorname{Diff}_{D^{\mathrm{N}}}(\Delta).$$

PROOF. This may be computed at codimension 2 points of X and so we may assume that $X = \operatorname{Spec} R$ where $(R, \mathfrak{m})$ is a 2-dimensional F-finite normal local domain. Let $\pi : Y \longrightarrow X$ be a log resolution of singularities of (X, Δ). In particular, it is simultaneously is an embedded resolution/normalization of D with $D' = \pi^{-1}_* D$ the strict transform. We will show we can compute the F-different by working on Y. This will complete the proof since our F-different computation will agree with the computation of the different in Proposition 6.9.

Fix $Q \in D \subseteq X$ a codimension 2 point of X and replace X by $\operatorname{Spec} R$ where $R = \mathcal{O}_{X,Q}$. Since we are locally, we may assume that $(p^e - 1)(K_X + \Delta) \sim 0$. Choose $\phi : F^e_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ with $\Delta = \phi_\Delta$.

Let $Q_1, \ldots, Q_t \in D'$ be the points of D' mapping to $Q \in D$. Write $\pi^*(K_X + \Delta) = K_Y + D' + \sum c_j C_j$ where the C_j are divisors on Y distinct from D'. Suppose some $c_j < 0$, then we can find an element $f \in R$ that does not vanish along D, but such that when we write

$$\pi^*(K_X + \Delta + \operatorname{div}(f)) = K_Y + D' + \sum (c_j + v_{C_j}(f)) C_j$$

we have that $c_j + v_{C_j}(f) \geq 0$. The added $\mathrm{div}(f)$ is added to the different and the F-different simply by pullback to D^{N}, hence we may replace $\Delta = D + B$ by $\Delta + \mathrm{div}(f) = D + (B + \mathrm{div}(f))$ and notice that for either the different or the F-different. Notice that we have also implicitly replaced ϕ by premultiplication by f. At this point we can write $\pi^*(K_X + \Delta) = K_Y + \Delta_Y$ where $\Delta_Y \geq 0$.

In particular, we can now apply Lemma 4.15 and see that we obtain a map $\phi_Y : F^e_* \mathcal{O}_Y \longrightarrow \mathcal{O}_Y$ inducing Δ_Y where $\Delta_Y = D' + B'$. Since $\Delta_Y \geq D'$ we see that ϕ_Y is compatible with D' hence it induces a map $\phi_{D'} : F^e_* \mathcal{O}_{D'} \longrightarrow \mathcal{O}_{D'}$. At the generic point of D', this is the same as the map $\phi_D : F^e_* \mathcal{O}_D \longrightarrow \mathcal{O}_D$ induced by ϕ (since Y is an isomorphism there). Thus we see that $\phi_{D'}$ is *the* extension of ϕ_D to $F^e_* \mathcal{O}_{D'}$ and so computes the F-different. This completes the proof. $\square$

6.4. Adjoint test ideals and the restriction theorem. In this subsection we generalize the work done in Chapter 5 Exercise 5.9 to also incorporate normalizations.

Suppose that R is an F-finite normal domain and $D \subseteq X = \operatorname{Spec} R$ is a reduced divisor. Suppose $B \geq 0$ is a $\mathbb{Q}$-divisor with no common components with D and suppose that $(p^e - 1)(K_X + D + B) \sim 0$. Choose $\phi : F^e_* R \longrightarrow R$ corresponding to $\Delta = D + B$. We make the following definition:

Definition 6.12. With notation as above, let $Q_1, \dots, Q_t$ be height one primes corresponding to the components of D. We define

$$\tau_{\not\subseteq Q_1, \dots, Q_t}(R, \Delta) = \tau_D(R, \Delta)$$

to be the smallest ideal J, compatible with ϕ, such that we have $J_{Q_i} = R_{Q_i}$ for $i = 1, \dots t$. This is called the **adjoint test ideal of** (X, Δ) **along** D

This object was originally defined by S. Takagi in [**Tak11**].

Proposition 6.13. *With notation as above, $\tau_D(R, \Delta)$ exists.*

Proof. We must find $c \in R$ where $c \notin Q_i$ for all $i = 1, \dots, t$ but where for each $d \in R$, $d \notin Q_i$ for any i, we have that $c \in \phi^n(F^{ne}_*(d))$. In that case, we may take

$$\tau_D(R, \Delta) = \sum_{n \geq 0} \phi^{ne}(F^{ne}_*(c)).$$

To this end let $I = Q_1 \dots Q_t$ denote the radical ideal defining D. Notice that (R, Δ) is sharply F-pure after localizing at each Q_i since R is normal, D has coefficients 1 and B and D have no common components. We then choose $b \in R$ so that

(a) $b \notin Q_i$ for any $i = 1, \dots, t$.
(b) $(R_b, \Delta|_{R_b})$ is sharply F-pure.
(c) For any ϕ-compatible prime not containing b (in other words, corresponding to a ϕ_b-compatible prime), if $P_{Q_i} \neq R_{Q_i}$, then we have that $P \subseteq Q_i$. Note that this implies that if $\overline{\phi} : R_b/I_b \longrightarrow R_b/I_b$ is the map induced by ϕ, then $(R_b/I_b, \Delta_{\overline{phi}})$ is strongly F-regular. In particular, $R_b/I_b = (R/I)_b$ is normal.
(d) $b \in \phi^n(F^{ne}_* R)$ for all $n \geq 0$ (that is, $b \in \sigma(R, \phi)$).

We claim that $c = b^2$ works. Let $J' = \sum_{n \geq 0} \phi^{ne}(F^{ne}_*(d))$, this is a ϕ-compatible ideal so that $J'_{Q_i} = R_{Q_i}$ for all i. By our third assumption, we have that $J'_b = J'R[b^{-1}] = R[b^{-1}]$. Thus we see that $b^m \in J'$ for some m.

But now for $n \gg 0$ we have that

$$b^2 \in \phi^n(F^{ne}_*(bb^{p^{ne}})) \subseteq \phi^n(F^{ne}_*(b^m)) \subseteq \phi^n(F^{ne}_* J').$$

The result follows. □

Theorem 6.14 ([**Tak11**]). *With notation as in Definition 6.12, let I denote the radical ideal defining D. We have that*

$$\tau_D(R, \Delta) \cdot (R/I)^{\mathrm{N}} = \tau((R/I)^{\mathrm{N}}, \mathrm{Diff}_{D^{\mathrm{N}}}(\Delta)).$$

In fact, the map $R \longrightarrow R/I \longrightarrow (R/I)^{\mathrm{N}}$ induces a surjection:

$$\tau_D(R, \Delta) \longrightarrow \tau((R/I)^{\mathrm{N}}, \mathrm{Diff}_{D^{\mathrm{N}}}(\Delta)).$$

Proof. Set ϕ corresponding to Δ and let $\phi_{R/I} : F^e_* R \longrightarrow R$ be the induced map and let $\phi_{(R/I)^{\mathrm{N}}} : F^e_*(R/I)^{\mathrm{N}} \longrightarrow (R/I)^{\mathrm{N}}$ be the extension to the normalization. The right side is $\tau_D(R, \phi) \cdot (R/I)^{\mathrm{N}}$ and the right side is $\tau((R/I)^{\mathrm{N}}, \phi_{(R/I)^{\mathrm{N}}})$

Notice that we may choose $c \in R$ a strong test element for (R, ϕ) so that $\overline{c} \in R/I \subseteq (R/I)^{\mathrm{N}}$ is also a strong test element for $(R/I, \phi_{R/I})$ and for $((R/I)^{\mathrm{N}}, \phi_{R/I\mathrm{N}})$. Notice that we may assume $\overline{c}$ is in the conductor (in fact, it must be since the conductor is $\phi_{R/I}$-compatible), so that $\overline{c}(R/I)^{\mathrm{N}} \subseteq R/I$.

We then see that

$$\begin{array}{rcl}\tau_D(R, \Delta) \cdot (R/I) & = & \left(\sum_{n \geq 0} \phi^{ne}(F^{ne}_*(c))\right) \cdot (R/I) \\ & = & \sum_{n \geq 0} \phi^{ne}_{R/I}(F^{ne}_*(\overline{c})) \\ & = & \tau(R/I, \phi_{R/I}).\end{array}$$

But we also have that

$$\begin{array}{rcl}\tau((R/I)^{\mathrm{N}}, \phi_{(R/I)^{\mathrm{N}}}) & = & \sum_{n\geq 0} \phi^{ne}_{(R/I)^{\mathrm{N}}}(F^{ne}_* \overline{c}^2 (R/I)^{\mathrm{N}}) \\ & \subseteq & \sum_{n\geq 0} \phi^{ne}_{R/I}(F^{ne}_*(\overline{c})) \\ & \subseteq & \sum_{n\geq 0} \phi^{ne}_{(R/I)^{\mathrm{N}}}(F^{ne}_* \overline{c}(R/I)^{\mathrm{N}}) \\ & = & \tau((R/I)^{\mathrm{N}}, \phi_{(R/I)^{\mathrm{N}}})\end{array}$$

But the second line is equal to $\tau(R/I, \phi_{R/I})$. The result follows. $\square$

Definition 6.15. With notation as in Definition 6.12, we say that (R, Δ) is **purely F-regular (along D)** if $\tau_D(R, \Delta) = R$.

We obtain the following immediate corollary of Theorem 6.14.

Corollary 6.16. *With notation as in Definition 6.12, we have that (X, Δ) is purely F-regular in a neighborhood of D if and only if $(D^{\mathrm{N}}, \mathrm{Diff}_{D^{\mathrm{N}}}(\Delta))$ is strongly F-regular. In either case, D was normal to begin with.*

Proof. We may assume that $X = \operatorname{Spec} R$ where R is local. The first statement follows from the fact that an ideal agrees with a ring if and only if it contains 1. The second fact comes from the surjection:

$$\tau_D(R, \Delta) \longrightarrow \tau((R/I)^{\mathrm{N}}, \mathrm{Diff}_{D^{\mathrm{N}}}(\Delta))$$

induced by $R \longrightarrow R/I \longrightarrow (R/I)^{\mathrm{N}}$ since if $R/I \longrightarrow (R/I)^{\mathrm{N}}$ surjects, then R/I was already normal to begin with. $\square$

Remark 6.17. This normality we deduced above can be seen another way. If R is F-finite and reduced with normalization R^{N} then any $\phi : F^e_* R \longrightarrow R$ lifts to a $\phi_{R^{\mathrm{N}}} : F^e_* R^{\mathrm{N}} \longrightarrow R^{\mathrm{N}}$ which is compatible with the conductor ideal, see Chapter 1 Exercise 6.30. In particular, $(R^{\mathrm{N}}, \phi_{R^{\mathrm{N}}})$ cannot be strongly F-regular unless $R = R^{\mathrm{N}}$.

Remark 6.18. There are numerous generalizations of these results possible. We leave them to the exercises. Note it is also possible to obtain this restriction theorem without assuming that the index of $K_X + \Delta$ is not divisible by p, [**Tak08**].

6.5. Exercises.

Exercise 6.1. Consider the ring $R = k[x, y, z]/(xy - z^n)$ for $n \geq 2$. Let $D = \mathrm{div}((x, z))$ and set $\Delta = D$. Show that $K_R + \Delta$ is $\mathbb{Q}$-Cartier and compute the different $\mathrm{Diff}_D(\Delta)$.

6.5.1. *F-pure centers.* The following exercises will be about the next definition.

Definition 6.19. Suppose that R is a reduced F-finite ring and $\mathscr{C}$ is a Cartier subalgebra of the full Cartier algebra. We say that a reduced irreducible subscheme $Z = V(Q) \subseteq X$ (corresponding to a prime ideal Q) is an **F-pure center of** $(R, \mathscr{C})$ if the following conditions hold.

(a) The localization $(R_Q, \mathscr{C}_Q)$ is F-pure (that is, there exists some homogeneous $\phi \in \mathscr{C}_e$ so that $\phi_Q : F^e_* R_Q \longrightarrow R_Q$ surjects).
(b) For every $\phi \in \mathscr{C}_e$ we have that $\phi(F^e_* Q) \subseteq Q$.

If R is normal, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor and $\mathscr{C} = \mathscr{C}^\Delta$, then we call Z an **F-pure center of** (X, Δ)**.**

Exercise 6.2. Suppose that $(R, \mathscr{C})$ is as in Definition 6.19. Show that an F-pure center $Z \subseteq X$ is minimal (with respect to containment among F-pure centers) if and only if the restricted pair $(R/Q, \mathscr{C}|_{R/Q})$ is strongly F-regular. In particular, in such a case Z is normal.

Exercise 6.3. Suppose that R is an F-finite normal domain and $\Delta = D + B$ is a $\mathbb{Q}$-divisor with D reduced and B having no common components with D. Further suppose that (R, Δ) is purely F-regular along D. Show that D the prime components of D are minimal F-pure centers of (X, Δ).

Exercise 6.4. Consider the ring $R = \overline{\mathbb{F}_p}[x, y, z, t]/(y^2 z - x(x - z)(x - tz))$ with $X = \operatorname{Spec} R$. This is the cone over a family of elliptic curves.

(a) Show that the line $L = V((x, y, z))$ is an F-pure center of $(R, 0) = (R, \mathscr{C}^R)$.
(b) The generating map $\Phi \in \operatorname{Hom}_R(F^e_* R, R)$ induces the F-different Δ_L on L. Show that Δ_L is supported at exactly those $(t - \alpha)$ for $\alpha \in \overline{\mathbb{F}_p}$ where the associated elliptic curve $y^2 z - x(x - z)(x - \alpha z)$ is supersingular.

7. Finding explicit test elements

If R is a reduced F-finite Noetherian ring and $c \in R$ is such that $R[c^{-1}]$ is strongly F-regular, then *some* power of c is in the test ideal by Theorem 5.21. But often it is important to be able to identify an *explicit* strong test element—that is, to understand *what power* of c is a strong test element. For instance, in the ring $\mathbb{F}_p[x, y, z]/(x^4 + y^4 + z^4)$ with $p > 2$, what power of x is a strong test element? Is there a power works for *all* $p > 2$?

Fortunately, there are practical tools for identifying explicit test elements. Indeed, under suitable hypothesis, we will see that certain *Jacobian ideals* and certain *discriminant ideals* are contained in the test ideal. The main results are Theorems 7.2 and 7.4 showing different kinds of Jacobian determinants are strong test elements, as well as Theorem 7.12, showing certain discriminants are strong test elements. Importantly, these theorems construct strong test elements in a manner *independent of characteristic*, which will be important later in Chapter 6 when we apply our F-singularity theory to *complex* varieties and other characteristic zero settings. Interestingly, all three theorems are deduced in a similar way from Theorem 7.8, which is highly dependent on the specific characteristic.

The ideas in this section closely follow the work of Hochster and Huneke in their papers on the existence of tight closure test elements. A theorem of Lipman and Sathaye also plays a critical role.

7.1. Jacobian ideals. We review the definition of Jacobian ideals before stating our theorems connecting them to the test ideal. See [**?**] for more information.

Definition 7.1. Let A be an arbitrary domain. Consider a finitely generated A-algebra

$$R = A[x_1, x_2, \cdots, x_n]/(f_1, \dots, f_t)$$

such that $\mathcal{K}(A) \otimes_A R$ has equidimension d. The **Jacobian ideal** $\mathcal{J}(R/A)$ of R over A is the ideal generated by the (images in R of the) $(n-d)$-sized minors of the $t \times n$ Jacobian matrix:

$$\left[\frac{\partial f_i}{\partial x_j}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_t}{\partial x_1} & \frac{\partial f_t}{\partial x_2} & \cdots & \frac{\partial f_t}{\partial x_n} \end{bmatrix}.$$

The Jacobian ideal $\mathcal{J}(R/A)$ is independent of the presentation of the A-algebra; indeed, it can be described more canonically as a certain Fitting ideal of the module $\Omega_{R/A}$ of Kähler differentials. In general, the Jacobian ideal $\mathcal{J}(R/A)$ cuts out non-smooth locus of the morphism $\operatorname{Spec} R \longrightarrow \operatorname{Spec} A$.

Two special cases of Jacobian ideals $\mathcal{J}(R/A)$, at opposite extremes, play a role in our test element story: one is where A is a field and the other is where A is a regular domain of the same dimension as R. We state two theorems, postponing the proofs until the next subsection.

Theorem 7.2. *Let R be a geometrically reduced*[34] *finitely generated algebra over an F-finite field k. Then the Jacobian ideal $\mathcal{J}(R/k)$ is contained in the test ideal $\tau(R)$.*

[34]This means that R is reduced and also that $R \otimes_k \overline{k}$ is reduced where $\overline{k}$ is the algebraic closure of k.

Example 7.3. For $p > 2$, let $R = \mathbb{F}_p[x,y,z]/(x^4+y^4+z^4)$. The Jacobian ideal $\mathcal{J}(R/k)$ is generated by the partial derivatives of $x^4+y^4+z^4$ with respect to each of the variables x, y, and z. So Theorem 7.2 says that

$$(x^3, y^3, z^3) \subset \tau(R).$$

This is true regardless of the characteristic p (assuming $p \neq 2$).

Theorem 7.2 follows from the following result, in which one should imagine that $A \hookrightarrow R$ is a Noether Normalization of the finitely generated k-algebra R:

Theorem 7.4. *Let A be an F-finite regular domain, and suppose $A \hookrightarrow R$ is a finite, torsion-free, generically étale extension.*[35] *Then the Jacobian ideal $\mathcal{J}(R/A)$ is contained in the test ideal $\tau(R)$.*

Example 7.5. Let $R = \mathbb{F}_p[x,y,z]/(x^4+y^4+z^4)$ where $p > 2$ be the ring of Example 7.3. Viewing R as an extension of the regular subring $A = k[x,y]$, the *Noether normalization* $A \hookrightarrow R$ satisfies the hypothesis of Theorem 7.4. As an A-algebra, the ring R has presentation $A[z]/(f(z))$ where $f(z) = z^4+x^4+y^4$, so its Jacobian ideal is generated by the image of $\frac{\partial f}{\partial z}$ in R. That is,

$$\mathcal{J}(R/A) = (z^3) \subseteq \tau(R),$$

recovering a fact we already knew from Theorem 7.2. Indeed, choosing different Noetherian normalizations—say, one given by the regular subring $k[x,z]$ and another by $k[y,z]$, we can use Theorem 7.4 to recover the fact that $\mathcal{J}(R/k) = (x^3,y^3,z^3) \subset \tau(R)$ from Theorem 7.2. In a similar way, we will deduce Theorem 7.2 from Theorem 7.4 in general.

Proof that Theorem 7.2 follows from Theorem 7.4. Suppose that R has dimension d, and let $k[x_1,x_2,\ldots,x_n]/(f_1,f_2,\ldots,f_t)$ be a finite presentation of R. Note that we can enlarge the ground field k by tensoring with a finite separable extension; this preserves all hypotheses, the Jacobian, and the test ideal (see (c)). A generic choice of Noether normalization $A \subseteq R$ (that is, a generic projection of $\operatorname{Spec} R$ onto a linear subspace of dimension d in $\mathbb{A}^n$) will be generically étale.[36] Thus, enlarging k if needed, after a generic change of variables $g \in GL_n(k)$, we may assume that our coordinates $\{x_1,\ldots,x_n\}$ have the property that *every* subset of cardinality d consists of algebraically independent elements generating a polynomial ring over which R is generically étale. For example, we have a factorization

$$k \hookrightarrow k[x_1,\ldots,x_d] = A \hookrightarrow \frac{k[x_1,\ldots,x_n]}{(f_1,f_2,\ldots,f_t)} = \frac{A[x_{d+1},\ldots,x_n]}{(f_1,f_2,\ldots,f_t)}$$

---

[35] Recall that $A \longrightarrow R$ is generically étale if, letting K denote the fraction field of A, the ring $K \otimes_A R$ is a product of separable field extensions; see Subsection 5.2.

[36] For a detailed discussion of this point, see, *e.g.* [**?**, p44].

where A is a polynomial ring over which R is finite, torsion free and generically étale, and likewise for any other subset $\{x_{i_1}, \ldots, x_{i_d}\}$ of the generators. Thinking about the Jacobian matrix for R/A and for R/k, observe that the former is a submatrix of the later: both have t rows indexed by the f_i, but only the columns indexed by $x_{d+1}, \ldots, x_n$ appear in the Jacobian matrix for R/A. The Jacobian ideal $\mathcal{J}(R/k)$ is generated by the (appropriate-sized) minors of the larger Jacobian matrix, and so we see many of them are contained in $\mathcal{J}(R/A)$. To get the remaining minors generating $\mathcal{J}(R/k)$, we choose different sets $\{x_{i_1}, \ldots, x_{i_d}\}$. Ranging over all $\binom{n}{d}$ choices of $\{x_{i_1}, \ldots, x_{i_d}\}$, we see that

$$\mathcal{J}(A,k) \subseteq \sum_{A} \mathcal{J}(R/A) \subset \tau(R).$$

$\square$

Remark 7.6. Importantly, note that Theorems 7.2 and 7.4 construct strong test ideals that are in some sense *independent of the characteristic.*

Caution 7.7. With hypothesis as in Theorem 7.2, the Jacobian ideal $\mathcal{J}(R/k)$ is typically *strictly* contained in $\tau(R)$. For example, if R is strongly F-regular but not regular, then the Jacobian ideal is a proper ideal (it defines the non-geometrically regular locus) but $\tau(R) = R$.

Theorem 7.4 will follow from Theorem 7.8, which constructs test elements in a highly-characteristic-dependent way, together with a result of Lipman and Sathaye guaranteeing certain Jacobians are in the conductor of a non-normal ring.

7.2. Test Elements from generically étale extensions. The following result guaranteeing the existence of strong test elements is the main ingredient in the proofs of all the other theorems about test elements in this section:

Theorem 7.8. *Let A be an F-finite regular domain, and let $A \hookrightarrow R$ be a finite, torsion free, generically étale extension. Suppose that for all $e \gg 0$, $c \in R$ annihilates the cokernel of the natural map*

$$R \otimes_A F^e_* A \longrightarrow F^e_* R \tag{7.8.1}$$

(or in other words, that $cF^e_* R \subseteq R[F^e_* A]$). Then c is a strong test element for R.

The proof of Theorem 7.8 requires the following lemma:

Lemma 7.9. *Let A be an F-finite regular domain, and let $A \hookrightarrow R$ be a finite, torsion free, generically étale extension. Then the natural map*

$$R \otimes_A F^e_* A \xrightarrow{r \otimes F^e_* a \mapsto r F^e_* a} F^e_* R \tag{7.9.1}$$

is injective for all $e > 0$. Furthermore, identifying $R \otimes_A F^e_* A$ with a subring of $F^e_* R$, the ring $F^e_* R$ is contained in the normalization of $R \otimes_A F^e_* A$.

PROOF. Because R is torsion free over A, and $F^e_* A$ is faithfully flat over A, the module $R \otimes_A F^e_* A$ is torsion free over $F^e_* A$, and hence over A. So if (7.9.1) has a non-zero kernel, it would still have a non-zero kernel after tensoring with the fraction field $K = \mathcal{K}(A)$ of A. Thus it suffices to check that, as a map of A-modules, (7.9.1) is injective at the generic point of $\operatorname{Spec} A$. Since localization commutes with Frobenius, this is equivalent to checking that

$$L \otimes_K F^e_* K \xrightarrow{r \otimes F^e_* a \mapsto r F^e_* a} F^e_* L \tag{7.9.2}$$

is injective, where $L = R \otimes_A K$. But our assumption that R is generically étale over A means precisely that L is étale over K, so the map (7.9.2) is an *isomorphism* by Proposition 5.9. This completes the proof of the first statement.

For the second statement, take arbitrary $F^e_* r \in F^e_* R$. Since $(F^e_* r)^{p^e} = r \otimes F^e_* 1 \in R \otimes F^e_* A$, the extension (7.9.1) is integral. But also the ring $F^e_* R$ and its subring $R \otimes_A F^e_* A$ have the same total quotient ring because inverting the non-zero elements in the subring A already produces the isomorphism (7.9.2). $\square$

PROOF OF THEOREM 7.8. Our hypotheses imply R is F-finite, Noetherian and reduced. Further, since $A \hookrightarrow R$ is generically étale, there exists non-zero $d \in A$ such that $A[d^{-1}] \hookrightarrow R[d^{-1}]$ is étale, and hence $R[d^{-1}]$ is regular. So d has a power that is a strong test element for R; replace d by this power. The element d is not in any minimal prime of R, so it is not a zero-divisor of R, and we can use d to generate $\tau(R)$ as a module over the Cartier algebra (see Corollary 6.16). That is, to show that c is a test element, it suffices to find $e > 0$ and $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* d) = c$.

To this end, because A is regular, we know A is F-regular; so for sufficiently large e, we can find $\psi \in \operatorname{Hom}_A(F^e_* A, A)$ such that $\psi(F^e_* d) = 1$. Tensoring with R, we have a map

$$R \otimes_A F^e_* A \xrightarrow{1 \otimes \psi} R$$

sending $F^e_* d$ to 1. By Lemma 7.9, the image of the map (7.9.1) is isomorphic to $R \otimes_A F^e_* A$, and by our assumption on c, we know c multiplies $F^e_* R$ into this image. So the map

$$F^e_* R \xrightarrow{\text{mult by } c} \operatorname{im}(R \otimes_A F^e_* A \longrightarrow F^e_* R) \cong R \otimes_A F^e_* A \xrightarrow{1 \otimes \psi} R$$

is an R-linear map $F^e_* R \to R$ sending $F^e_* d$ to

$$c(1 \otimes \psi)(1 \otimes F^e_* d) = c\psi(F^e_* d) = c.$$

Thus c is a strong test element for R. $\square$

Finally, we explain how to deduce Theorem 7.4 from Theorem 7.8. For this, we need the following general theorem of Lipman and Sathaye [**LS81**, Theorem 2], which we state without proof[37]:

Theorem 7.10 (Lipman-Sathaye Jacobian Theorem)**.** *Let A be a Noetherian domain, and let $A \hookrightarrow R$ be a finite, torsion-free, generically étale extension. Then the Jacobian ideal of R/A is contained in the conductor of R. That is,*

$$\mathcal{J}(R/A)\, R^{\mathrm{N}} \subset R$$

where R^{N} is the integral closure of R in its total ring of fractions.

Proof of Theorem 7.4. Because $F^e_* A$ is faithfully flat over A, the Jacobian ideal of the extension $F^e_* A \hookrightarrow F^e_* A \otimes_A R$ can be identified with $F^e_* A \otimes_A \mathcal{J}(R/A) \subset F^e_* A \otimes_A R$. In addition, identifying $F^e_* A \otimes_A R$ with a subring of $F^e_* R$, recall that $F^e_* R$ is contained in the normalization of $F^e_* A \otimes_A R$ (Lemma 7.9). So by Theorem 7.10,

$$\mathcal{J}(R/A)\, F^e_* R \subset F^e_* A \otimes_A R. \tag{7.10.1}$$

In other words, $\mathcal{J}(R/A)$ annihilates the cokernel of the natural map $F^e_* A \otimes_A R \to F^e_* R$ for all $e > 0$, and so is contained in the test ideal $\tau(R)$ by Theorem 7.8. $\square$

7.3. Discriminants. As another application of Theorem 7.8, we prove that certain *discriminants* are always in the test ideal. Like Theorems 7.4 and 7.4, Theorem 7.12 produces test elements in a manner *independent of the characteristic*.

7.3.1. *Discriminants and the trace form.* We review discriminants and the trace form, which are used in the statement of Theorem 7.12. See [**Sta19**, Tag 0BVH] for details.

Let A be a domain and let $A \to R$ be a finite map of rings, where R is *free* over A. The *discriminant* of R/A is a principal ideal of A cutting out the non-étale locus of $A \to R$. To define it, recall the trace form

$$R \times R \to A \qquad (r, s) \mapsto \mathrm{trace}(rs), \tag{7.10.2}$$

(where the trace of an element $r \in R$ over A is, by definition, the trace[38] of the A-linear map $R \xrightarrow{\text{mult by } r} R$). The trace form is obviously symmetric and bilinear. In the case where A is a *field*, the trace form is non-degenerate

[37] In the paper [**LS81**], the ring R is assumed a domain; the argument can be adapted to work under the hypothesis stated below; see [**?**, Thm3.1]

[38] in the usual linear algebra sense—for example, the sum of the diagonal entries a matrix representing this map of free module.

if and only if the extension $A \longrightarrow R$ is a finite product of separable field extensions (or, equivalently, R/A is *étale*) [**Sta19**, Tag 0BIE]. The bilinear map (7.10.2) induces a natural A-linear map of free A-modules

$$R \to \operatorname{Hom}_A(R, A), \tag{7.10.3}$$

so taking top exterior powers, there is an induced map of rank one free A-modules

$$A \cong \bigwedge^{\operatorname{rank} R/A} R \longrightarrow \bigwedge^{\operatorname{rank} R/A} \operatorname{Hom}_A(R, A) \cong A. \tag{7.10.4}$$

The map (7.10.4) is therefore (up to unit) multiplication by some element of A, called the **discriminant** of R over A. In particular, when A is a field, the discriminant is non-zero if and only if the extension R/A is étale. More generally, the discriminant cuts out the non-étale locus of the extension $A \longrightarrow R$.

In practice, the discriminant is easy to compute: if $x_1, \ldots, x_n$ is a free basis for R over A, then the matrix of the bilinear form (7.10.2) with respect to this basis is

$$M_{\mathbf{x}} = \begin{pmatrix} \operatorname{Tr}(x_1x_1) & \operatorname{Tr}(x_1x_2) & \ldots & \operatorname{Tr}(x_1x_n) \\ \operatorname{Tr}(x_2x_1) & \operatorname{Tr}(x_2x_2) & \ldots & \operatorname{Tr}(x_2x_n) \\ \ldots & \ldots & \ldots & \ldots \\ \operatorname{Tr}(x_nx_1) & \operatorname{Tr}(x_nx_2) & \ldots & \operatorname{Tr}(x_nx_n) \end{pmatrix}. \tag{7.10.5}$$

The discriminant $D_{\mathbf{x}}$ is then the determinant of $M_{\mathbf{x}}$. This depends on the choice of basis! If $\mathbf{y}$ is a different basis for R/A, with change of basis matrix $g \in GL_n(A)$, then the matrix of the bilinear form transforms by $M_{\mathbf{y}} = g^t M_{\mathbf{x}} g$. So computing the determinants, we see they differ by multiplication by $(\det g)^2 \in A^*$. Thus the discriminant of R/A is defined only up to multiplication by (square) units in A, and should be thought of as an ideal (or divisor) in A.

Remark 7.11. Even if R is not free over A, we can define the matrix $M_{\mathbf{x}}$ and the discriminant $D_{\mathbf{x}}$ using formula (7.10.5) above, whenever $\mathbf{x}$ is a collection of elements in R that *generically* form a basis for R over A (meaning that their images in $\mathcal{K}(A) \otimes R$ form a $\mathcal{K}(A)$-vector space basis.) If A is *normal*, then $\operatorname{Tr}(r) \in A$ for any $r \in R$ [**Sta19**, Tag 032L], so the entries of $M_{\mathbf{x}}$ are in A. In particular, $D_{\mathbf{x}} \in A$.

Theorem 7.12. *Let A be an F-finite regular domain. Suppose $A \hookrightarrow R$ is a finite, generically étale extension. Then*

(a) If R is free over A, then the discriminant ideal of R/A is contained in the test ideal $\tau(R)$;
(b) More generally, if R is torsion free over A, then for any set $\mathbf{x} = \{x_1, \ldots, x_n\} \subseteq R$ that generically form a free basis for R/A, the

discriminant

$$D_{\mathbf{x}} = \det \begin{pmatrix} \mathrm{Tr}(x_1x_1) & \mathrm{Tr}(x_1x_2) & \dots & \mathrm{Tr}(x_1x_n) \\ \mathrm{Tr}(x_2x_1) & \mathrm{Tr}(x_2x_2) & \dots & \mathrm{Tr}(x_2x_n) \\ \dots & \dots & \dots & \dots \\ \mathrm{Tr}(x_nx_1) & \mathrm{Tr}(x_nx_2) & \dots & \mathrm{Tr}(x_nx_n) \end{pmatrix}$$

is in the test ideal $\tau(R)$.

We record a basic fact about discriminants needed to deduce Theorem 7.12 from Theorem 7.8:

Lemma 7.13. *Let S be a normal domain, and let $S \hookrightarrow T$ be a finite, torsion-free extension with T reduced. For any subset $\mathbf{x} := \{x_1, \dots, x_n\} \subseteq T$ which generically form a free basis for T over S, the discriminant $D_{\mathbf{x}}$ is in the conductor of T—that is,*

$$D_{\mathbf{x}}\, T^{\mathrm{N}} \subseteq T$$

where T^{N} denotes the normalization, or integral closure in the total ring of quotients, of T.

Proof. Let $K = \mathcal{K}(S)$ be the fraction field of S. Our assumptions imply that $L = K \otimes_S T$ is a finite product of finite field extensions of K, and that $\mathbf{x}$ is a K-vector space basis for L.

Note that $T^{\mathrm{N}} \subset L$. So any $y \in T^{\mathrm{N}}$ can be written in the basis $\mathbf{x}$ as $y = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ where $\lambda_i \in K$. To show $D_{\mathbf{x}} y \in T$, it suffices to show each $D_{\mathbf{x}} \lambda_i \in S$.

Since $D_{\mathbf{x}} = \det M_{\mathbf{x}}$, we have an equality of $n \times n$ matrices over S

$$D_{\mathbf{x}}\, I_n = \mathrm{adj}(M_{\mathbf{x}})\, M_{\mathbf{x}},$$

where I_n is the $n \times n$ identity matrix and $\mathrm{adj}(M_{\mathbf{x}})$ is the adjugate, or classical adjoint matrix, of $M_{\mathbf{x}}$ (see [**?**]). Note that the entries of $\mathrm{adj}(M_{\mathbf{x}})$ are all in S because the entries of $M_{\mathbf{x}}$ are in S.

To show that $D\lambda_i \in S$, it then suffices to show that the column vector

$$D_{\mathbf{x}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \mathrm{adj}(M_{\mathbf{x}})\, M_{\mathbf{x}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \tag{7.13.1}$$

has all entries in S. But by definition of $M_{\mathbf{x}}$,

$$M_{\mathbf{x}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \mathrm{Tr}(x_1 y) \\ \mathrm{Tr}(x_2 y) \\ \vdots \\ \mathrm{Tr}(x_n y) \end{bmatrix},$$

which is clearly in S^n, since $\mathrm{Tr}(T^{\mathrm{N}}) \subseteq S$ [**Sta19**, Tag 032L]. So the entries of the column vector in equation (7.13.1) are all in S. Thus $D_{\mathbf{x}}\lambda_i \in S$, as needed. $\square$

Proof of Theorem 7.12. By Theorem 7.8, it is enough to show that the discriminants $D_{\mathbf{x}}$ annihilate the cokernel of

$$R \otimes_A F^e_* A \longrightarrow F^e_* R \tag{7.13.2}$$

for all e. Choose $\mathbf{x} = \{x_1, \ldots, x_n\} \subseteq R$ whose image in $R \otimes \mathcal{K}(A)$ is a basis over $\mathcal{K}(A)$. Tensoring the map $A \hookrightarrow R$ with the faithfully flat A-module $F^e_* A$, we have a finite, torsion-free, generically étale extension

$$F^e_* A \hookrightarrow F^e_* A \otimes_A R, \tag{7.13.3}$$

and the image of the set $\mathbf{x}$ in $F^e_* A \otimes_A R$ is generically a basis over $F^e_* A$. Note that the map (7.13.3) satisfies the hypothesis of Lemma 7.13. So because $F^e_* R$ is contained in the normalization of $F^e_* A \otimes_A R$ (see Lemma 7.9), we have

$$D_{\mathbf{x}}\, F^e_* R \subset F^e_* A \otimes_A R$$

by (7.13.3). Thus $D_{\mathbf{x}}$ annihilates the cokernel of the map (7.13.2) for all $e > 0$. By Theorem 7.8, we conclude that $D_{\mathbf{x}}$ is a strong test element. $\square$

7.4. Exercises.

Exercise 7.1. Suppose p does not divide n. Let $R = \mathbb{F}_p[x_1, \cdots, x_d]/(x_1^n + \cdots + x_d^n)$. Show that $x_i^{n-1} \in \tau(R)$ for all i.

Exercise 7.2. For the ring $R = \mathbb{F}_p[x, y, z]/(x^4+y^4+z^4)$ with $p > 2$, compute the discriminant of R over $\mathbb{F}_p[x, y]$ using the basis $\{1, z, z^2, z^3\}$ to find a test element for R. Compare to **??**.

Exercise 7.3. Let A be an F-finite regular domain, and let $A \hookrightarrow R$ be a finite, torsion free, generically étale extension. Suppose that $c \in R$ annihilates the cokernel of the natural map

$$R \otimes_A F_* A \longrightarrow F_* R$$

Then prove that c^2 is a strong test element for R.

Hint: The point is to show that c^2 annihilates the cokernel of all $R \otimes_A F^e_* A \longrightarrow F^e_* R$.

CHAPTER 7

Tight closure

This chapter is missing some content we hope to add in the future, for instance phantom homology.

Tight closure is a closure operation on ideals in a commutative ring of prime characteristic—or more generally, on submodules of R-modules—introduced by Mel Hochster and Craig Huneke in a long series of papers starting with [**HH90**], *cf.* [**HH89, HH91, AHH93, HH93, HH94a, HH94b**]. Tight closure has several deep properties that have led to a better understanding of integral closure, Cohen-Macaulayness and other subtle issues in commutative algebra.

The resulting insight into singularities in prime characteristic, documented in earlier chapters of this book, ultimately helped enable progress in the minimal model program for prime characteristic threefolds and fourfolds; see Chapter 10. Although it arose independently, tight closure is reminiscent of Faltings' theory of "almost ring theory" in arithmetic geometry [**Fal02, GR03**], which is used in Scholze's theory of perfectoid algebras. It is no surprise then, that the methods pioneered in tight closure theory have also appeared in work on the minimal model program in mixed characteristic; see [**TY23, BMP$^+$23, HW23**].

In this chapter, we re-tell Hochster and Huneke's story of tight closure, emphasizing the perspective of F-singularities developed in the previous chapters. The first section defines tight closure, including a variant called "finitistic tight closure" for infinitely generated modules that appears at the center of several important open questions. The second section introduces the *tight closure test ideal*, which of course inspired the what we called the test ideal — a key figure in evolving algebraic and geometric applications and a star of the earlier chapters. We focus mainly on the F-finite case, where the arguments are more natural and transparent. One advantage of tight closure, however, is that the definition does not require the F-finite hypotheses, so we include pointers to the literature where statements have been proved more generally.

Section 3 treats *F-regularity*, outlining the still open question of whether *strong* F-regularity is the same as *weak* F-regularity, the property that all ideals are tightly closed. We then turn to the connection with integral extensions and plus closures in Section 4, and prove Hochster and Huneke's famous theorem that the absolute integral closure of a Noetherian domain of positive characteristic is a "big Cohen-Macaulay algebra" in Section 5. There, we point out how many ideas and techniques about tight closure have been developed in mixed characteristic through work of Y. André, B. Bhatt, Gabber, Heitmann, Ma and others [**And20, BS22, Bha20, HM18**].

We touch only on the basics of tight closure theory (and do not prove things in as great as generality as is possible). For those who are interested in further study, we recommend the excellent notes of Hochster [**Hoc07**] or the short book of Huneke [**Hun96**].

1. The definition of tight closure

Fix a commutative Noetherian ring R of prime characteristic $p > 0$.

Definition 1.1. Fix an ideal $I \subseteq R$. The tight closure of I is the ideal I^* of all elements $z \in R$ with the following property: there exists some $c \in R$, not in any minimal prime of R, and some natural number e_0, such that

$$cz^{p^e} \in I^{[p^e]} \qquad \text{for all } e \geq e_0, \tag{1.1.1}$$

or equivalently,

$$zF^e_* c \in IF^e_* R \qquad \text{for all } e \geq e_0. \tag{1.1.2}$$

It is important to note that, *a priori*, the element c in the definition of tight closure may depend on I and z *but not* e. On the other hand, for many classes of rings R—including all reduced F-finite rings—there are certain distinguished elements c that can be used in tight closure tests (as in (1.1.1)) for *any* I and *any* z. These are called **tight closure test elements** (Definition 2.1) and include the strong test elements we have already met in Chapter 1; see Proposition 2.3 later in this chapter.

Remark 1.2. There is no loss of generality in restricting to reduced rings to compute tight closure: $z \in I^*$ if and only if the image of z in the reduced ring R_{red} is in the tight closure of IR_{red}, where $R_{\text{red}} = R/\sqrt{0}$ is the quotient of R by the ideal of its nilpotent elements; see Exercise 1.1. In the reduced case, the condition that c is not in any minimal prime is equivalent to the condition that c is a non-zerodivisor.

Remark 1.3. Many questions about tight closure reduce to the domain case, where the condition on c is simply that it is non-zero; see Exercise 1.2.

We say that an ideal I of the ring R is **tightly closed** if $I = I^*$. The next proposition, whose proof is left as an exercise, establishes some basic properties of tight closure.

Proposition 1.4. *Let I and J be ideals in a commutative Noetherian ring of prime characteristic. Then*

(a) $I \subseteq I^*$;
(b) $(I^*)^* = I^*$;
(c) If $I \subseteq J$, *then* $I^* \subseteq J^*$;
(d) $(0)^*$ *is the ideal of all nilpotent elements of* R.

Remark 1.5. An element of the tight closure of I is "almost" in I in the following sense. Let R_{perf} be the perfection of a domain R, obtained by adjoining the p^e-roots of every element for all e (see Definition 2.16 in Chapter 1). Then $z \in I^*$ if and only if $c^{1/p^e} z \in IR_{\text{perf}}$ for all e and fixed c not in any minimal prime. Note here that the elements c^{1/p^e} are getting "arbitrarily close to 1" as e approaches infinity in a way that can be made precise using valuations; see [**HH91**]. In other words, z is "almost in I". We will discuss tight closure in the context of Faltings' almost mathematics [**Fal02, Gab04**] in Theorem 5.36.

Remark 1.6. One can define tight closure in characteristic zero by reduction to characteristic $p > 0$ as in Chapter 6. We won't explore this but instead refer the reader to [**HH06**] or the appendix by Hochster to [**Hun96**]. For some interesting examples see [**BK06**].

1.1. Tight closure for modules.

The definition of tight closure extends naturally to submodules in an ambient module:

Definition 1.7. Let M be a module over a commutative ring R of prime characteristic $p > 0$. For any submodule $N \subseteq M$, the **tight closure of N in M** is the set of all elements $z \in M$ with the following property: there exists some $c \in R$, not in any minimal prime, such that the element $F^e_* c \otimes z \in F^e_* R \otimes_R M$ is in the image of the natural map

$$F^e_* R \otimes_R N \to F^e_* R \otimes_R M$$

for all $e \gg 0$. The set of all elements of M in the tight closure of some submodule N forms a submodule of M, denoted N^*_M. We say that N **is tightly closed in** M if $N^*_M = N$.

One readily checks that Definition 1.7 and Definition 1.1 are just two different ways of writing the same thing in the case that $M = R$. So Definition 1.7 gives a natural generalization of tight closure to modules.

Notation 1.8. To make the notation less onerous, we introduce notation matching our notation in the case where $M = R$. For any $m \in M$, we let

m^{p^e} denote the image of m under the natural map $M \to F^e_*R \otimes_R M$ sending m to $F^e_* 1 \otimes m$, and cm^{p^e} for $F^e_* c \otimes m$. Likewise, we write

$$N_M^{[p^e]} \quad \text{for} \quad \operatorname{Image}\left(F^e_*R \otimes_R N \to F^e_*R \otimes_R M\right),$$

the F^e_*R-submodule generated by all elements n^{p^e} for $n \in N$. With this notation, an element $m \in M$ is in the tight closure of a submodule N means that there exists $c \in R$, not in any minimal prime of R, such that

$$cm^{p^e} \in N_M^{[p^e]}$$

for all $e \gg 0$.

Remark 1.9. For a Noetherian local ring $(R, \mathfrak{m})$ of dimension d, Notation 1.8 is consistent with the notation for the Frobenius action on $H^d_{\mathfrak{m}}(R)$ described in Section 7 of Chapter 1. Indeed, we have natural isomorphisms of R-modules,

$$F^e_*R \otimes_R H^d_{\mathfrak{m}}(R) \cong H^d_{\mathfrak{m}}(F^e_*R) \cong F^e_* H^d_{\mathfrak{m}}(R)$$

as explained in Subsection 7.1 of Chapter 1 (see also Lemma 7.2 in Appendix C). With these identifications, η^{p^e} (or $F^e_*\eta^{p^e}$ if we want to emphasize the R-module structure given by Frobenius) is just another notation for $F^e(\eta)$ as in Chapter 1 Section 7. Likewise, $c\eta^{p^e}$ is another notation for $cF^e(\eta)$, for any $\eta \in H^d_{\mathfrak{m}}(R)$.

1.2. Strong F-regularity and F-rationality in terms of tight closure.

With tight closure in mind, we revisit strong F-regularity and F-rationality, both of which have simple interpretations in terms of tight closure.

Theorem 1.10. *Let $(R, \mathfrak{m}, k)$ be an F-finite local Noetherian ring of prime characteristic. Then*

(a) The ring R is strongly F-regular if and only if the zero module is tightly closed in the injective hull of its residue field;
(b) [**Smi94**] *The ring R is F-rational if and only if zero is tightly closed in the local cohomology module $H^d_{\mathfrak{m}}(R)$, and R is Cohen-Macaulay.*

Proof. We first prove (a). First assume that R is strongly F-regular (hence reduced). If $0^*_E \neq 0$, then suppose $\eta \neq 0 \in 0^*_E$. This means that there is some $c \in R$, not in any minimal prime, such that $F^e_*c \otimes \eta = 0$ in $F^e_*R \otimes_R E$ for all $e \gg 0$. But since for large e, the R-module map

$$R \to F^e_*R \quad r \mapsto rF^e_*c$$

splits, we have a split map

$$E \to F^e_*R \otimes E \quad \xi \mapsto F^e_*c \otimes \xi$$

for all $e \gg 0$. In particular, if $\eta \neq 0$, then also $F^e_* c \otimes \eta$ is not zero in $F^e_* R \otimes_R E$. So $0^*_E = 0$.

Conversely, fix any non-zero divisor $c \in R$. To check that

$$R \to F^e_* R \quad r \mapsto r F^e_* c$$

splits for some $e \in \mathbb{N}$, it suffices to check that, after tensoring with E, the map

$$E \to F^e_* R \otimes_R E \quad \xi \mapsto F^e_* c \otimes \xi \tag{1.10.1}$$

is injective (Lemma 7.22 in Chapter 1). Since E has a one-dimensional socle, say generated by η, it suffices to check that η is not in the kernel of (1.10.1) for some e. But otherwise, η is in the kernel of (1.10.1) for *all* e, which means that

$$c\eta^{p^e} := F^e_* c \otimes \eta \quad \text{is zero in} \quad F^e_* R \otimes_R E$$

for all $e \in \mathbb{N}$. This says that $\eta \in 0^*_E$.

We next prove (b). First suppose R is F-rational (Definition 7.9 of Chapter 1). Because R is Cohen-Macaulay, all embedded primes are minimal ([**Sta19**, Tag 031Q] or [**Mat89**, p136]) so the non-zero-divisors of R are precisely the elements not in any minimal prime of R. Now, if $\eta \in H^d_{\mathfrak{m}}(R)$ is in the tight closure of the zero submodule of $H^d_{\mathfrak{m}}(R)$, then there exists a $c \in R$, not in any minimal prime of R (hence a non-zero-divisor) such that the image of $F^e_* c \otimes \eta$ in $F^e_* R \otimes_R H^d_{\mathfrak{m}}(R)$ is zero for all $e \gg 0$. Re-interpreting, we see that η is in the kernel of the map

$$H^d_{\mathfrak{m}}(R) \xrightarrow{F^e_* c \,\circ F^e} F^e_* H^d_{\mathfrak{m}}(R) \tag{1.10.2}$$

obtained by applying the local cohomology functor to the R-module map $R \longrightarrow F^e_* R$ sending 1 to $F^e_* c$. By definition, because R is F-rational, the map (1.10.2) is injective. This means that $\eta = 0$, so the zero submodule is tightly closed in $H^d_{\mathfrak{m}}(R)$.

Conversely, assume that zero is tightly closed in $H^d_{\mathfrak{m}}(R)$ and that R is Cohen-Macaulay. Take any non-zero-divisor $c \in R$. We need to show that the map (1.10.2) is injective for $e \gg 0$. We first observe that Frobenius acts injectively on $H^d_{\mathfrak{m}}(R)$. Indeed, if $\eta \in H^d_{\mathfrak{m}}(R)$ is in the kernel of Frobenius, then $\eta^p = 0$, and so for all $e \in \mathbb{N}$ also $\eta^{p^e} = (\eta^p)^{p^{e-1}} = 0$. Hence $c\eta^{p^e}$ must be zero as well for $e \gg 0$. This says that $\eta \in 0^*$ in $H^d_{\mathfrak{m}}(R)$. So by assumption, $\eta = 0$. That is, Frobenius acts injectively on $H^d_{\mathfrak{m}}(R)$.

We next claim that for each e

$$\ker(cF^{e+1}) \subseteq \ker cF^e. \tag{1.10.3}$$

Indeed, suppose that $\eta \in \ker(cF^{e+1})$. This means that $c\eta^{p^{e+1}} = 0$, so that also $c^p \eta^{p^{e+1}} = (c\eta^{p^e})^p = 0$. But since Frobenius is injective on $H^d_{\mathfrak{m}}(R)$, we can conclude that $c\eta^{p^e} = 0$, proving (1.10.3). Now (1.10.3) implies that we have a descending chain of R-submodules

$$H^d_{\mathfrak{m}}(R) \supseteq \ker(cF) \supseteq \ker(cF^2) \supseteq \ker(cF^3) \supseteq \dots,$$

and since $H^d_{\mathfrak{m}}(R)$ is Artinian, this chain stabilizes. That is, there exists an e_0 such that for all $e \geq e_0$, $\ker(cF^{e_0}) = \ker(cF^e)$. Now, if the map cF^e of (1.10.2) fails to be injective for $e \gg 0$, then there is some non-zero $\eta \in \ker(cF^e)$ for all $e \geq e_0$. For this η, we have that $c\eta^{p^e} = 0$ for all $e \geq e_0$, which says that $\eta \in 0^*_{H^d_{\mathfrak{m}}(R)}$. So in fact $\eta = 0$ by hypothesis. This shows that the maps (1.10.2) are injective for all $e \geq e_0$. $\square$

Remark 1.11. Hochster and Huneke's original definition of F-rationality is different—it is stated in terms of "parameter" ideals being tightly closed. We return to this shortly in Subsection 3.3.

1.3. Finitistic tight closure.

For infinitely generated modules over a Noetherian ring, there is an alternate notion of tight closure that is more closely associated with tight closure for ideals.

Definition 1.12. Let M be a (possibly non-Noetherian) module over a Noetherian ring R of prime characteristic p. The **finitistic tight closure** of a submodule $N \subseteq M$, denoted $N^{*\,\mathrm{fg}}_M$, is the union, over all *finitely generated* submodules M' of M, of the submodules

$$(N \cap M')^*_{M'}.$$

From the definition, we see immediately, for any pair of R-modules $N \subseteq M$, that

$$N^{*\,\mathrm{fg}}_M \subseteq N^*_M,$$

and that equality holds if M is Noetherian. Equality holds in some other important settings:

Proposition 1.13 ([**Smi94**]). *Let $(R, \mathfrak{m})$ be a Cohen Macaulay local ring of dimension d. Then $0^*_{H^d_{\mathfrak{m}}(R)} = 0^{*\,\mathrm{fg}}_{H^d_{\mathfrak{m}}(R)}$.*

Remark 1.14. Proposition 1.13 holds without the Cohen-Macaulay hypothesis, for example, whenever R is F-finite and reduced, or more generally, excellent; See Exercise 2.9.

Proof of Proposition 1.13. To see that an arbitrary $\eta \in 0^*_{H^d_{\mathfrak{m}}(R)}$ is in the tight closure of zero in some finitely generated submodule of $H^d_{\mathfrak{m}}(R)$, recall that, when R is Cohen-Macaulay, the module $H^d_{\mathfrak{m}}(R)$ can be identified with a direct limit of an *injective* system of R-module homomorphisms[1].

[1] See Subsection 10.3 in Appendix A

Specifically, fix a system of parameters $x_1, \ldots, x_d$ for R, then for all $t \in \mathbb{N}$, we have injective maps

$$\frac{R}{(x_1^t, \ldots, x_d^t)} \xrightarrow{\text{multiply by } x_1 x_2 \cdots x_d} \frac{R}{(x_1^{t+1}, \ldots, x_d^{t+1})}, \tag{1.14.1}$$

and $H^d_{\mathfrak{m}}(R)$ is the directed limit of these maps. We leave it to the reader to check that if an element $\eta \in 0^*_{H^d_{\mathfrak{m}}(R)}$ is represented by some z mod $(x_1^t, \ldots, x_d^t)$ in $R/(x_1^t, \ldots, x_d^t)$, then $z \in (x_1^t, \ldots, x_d^t)^*$ (see Exercise 1.11). In this case, $\eta \in 0^*$ in the submodule $R/(x_1^t, \ldots, x_d^t)$. Thus $\eta \in 0^{*\,\mathrm{fg}}_{H^d_{\mathfrak{m}}(R)}$. $\square$

An important open question that will appear in various guises and generalizations is the following. If $(R, \mathfrak{m})$ is an F-finite Noetherian local ring and E is an injective hull of $R/\mathfrak{m}$, is

$$0^*_E = 0^{*\,\mathrm{fg}}_E? \tag{1.14.2}$$

Proposition 1.13 shows that (1.14.2) has a positive answer for *Gorenstein local rings,* even without the assumption that R is F-finite and reduced, since $E = H^d_{\mathfrak{m}}(R)$ in this case; see also [**HH94a**, Prop 4.9]. We'll return to this question, and some related history, in Remark 2.11

1.4. Tight closure versus integral closure. The tight closure of an ideal is much "tighter" than its integral closure— one readily checks that the tight closure is contained in the integral closure for any ideal: $I^* \subseteq \overline{I}$ see Exercise 1.4. In particular, integrally closed ideals, including all radical ideals, are always tightly closed. More substantially, we have an inclusion in the other direction via the so-called Briançon-Skoda Theorem:

Theorem 1.15 ([**HH90**, Thm 5.4])**.** *Let R be a Noetherian ring of prime characteristic. Let I be an ideal of positive height of R generated by n elements (or more generally, integral over an ideal generated by n elements). Then the integral closure*

$$\overline{I^{m+n}} \subseteq (I^{m+1})^*$$

for all integers $m \geq 0$. In particular, $\overline{(I^n)} \subseteq I^*$.

Proof. We prove this only in the main case — when R is a domain and $m = 0$ — leaving the straightforward reduction to this case as Exercise 1.6 (or see [**HH90**]). In general, it is easy to see (Exercise 1.5) that if y is integral over J, then there exists some $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$,

$$(J + y)^{k+m} = J^{m+1}(J + y)^{k-1}. \tag{1.15.1}$$

In particular, if I is integral over $(z_1, \ldots, z_n)$, and $y \in \overline{I^n} = \overline{(z_1, \ldots, z_n)^n}$, then

$$y^{k+m} \in (z_1, \ldots, z_n)^{n(m+1)} \subseteq (z_1^m, z_2^m, \ldots, z_n^m). \tag{1.15.2}$$

Now, assuming R is a domain, $c = y^k$ is not in any minimal prime of R, and so taking $m = p^e$, the equation (1.15.2) for all e says that $y \in I^*$. $\square$

1.5. Exercises.

Exercise 1.1. Let I be an ideal in a commutative ring R of prime characteristic, and let $N \subseteq R$ be the nilradical of R. Prove that the tight closure I^* is the pre-image under the natural surjection $R \to R/N$ of the tight closure of the ideal (I/N) in the reduced ring (R/N).

Exercise 1.2. Let I be an ideal in a commutative ring R of prime characteristic. Show that $z \in I^*$ if and only if the image of z is in the tight closure of the image of I modulo each minimal prime of R.

Exercise 1.3. Prove Proposition 1.4.

Exercise 1.4. Let I be an ideal in a Noetherian commutative ring R of characteristic $p > 0$. Show that $I^* \subseteq \overline{I}$, that is the tight closure is contained in the integral closure.

Hint: Use the following characterization of integral closure in a Noetherian domain R: $z \in \overline{I}$ if and only if there exists a non-zero $c \in R$ such that $cz^n \in I^n$ for all $n \gg 0$ [**SH06**, Corollary 6.8.12].

Exercise 1.5. Assume that y is integral over J, meaning that there is an equation of the form $y^k + a_1 y^{k-1} + \cdots + a_{k-1} y + a_k = 0$ where $a_i \in J^{in}$. Prove that for all $m \in \mathbb{N}$, $(J+y)^{k+m} = J^{m+1}(J+y)^{k-1}$.

Exercise 1.6. Complete the proof of Theorem 1.15 by showing that the statement reduces to the domain case, and deriving an appropriate analog of (1.15.2).

Exercise 1.7. Let R be a commutative ring of prime characteristic p. Show that an element $z \in R$ is in I^* if and only if its image $\overline{x}$ in R/I is in the tight closure of the zero-submodule in R/I. More generally, show that if $N \subseteq M$ are R-modules, then for any $m \in M$, $m \in N^*_M$ if and only if the class $\overline{m}$ of m in M/N is in $0^*_{M/N}$.

Hint: Use the right exactness of tensor.

Exercise 1.8. Let M be a module over an arbitrary commutative R of prime characteristic. Write M as a quotient of a free module, with module of relations K—that is, suppose that

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

is an exact sequence of R modules, with F free. Prove that

$$0^*_M = \pi\left(K^*_F\right).$$

Hint: Use the right exactness of tensor.

Exercise 1.9. [**HH90**, Prop 8.5] Prove that if M is a finitely generated module over a Noetherian ring and $N \subseteq Q \subseteq M$ are submodules, then $N^*_M \subseteq Q^*_M$.

Exercise 1.10 ([**HH90**, Prop 8.9])**.** Let $N \subseteq M$ be finitely generated modules over Noetherian ring R, and suppose that the support of M/N consists of one maximal ideal $\mathfrak{m}$. Then $(N^*_M)_{\mathfrak{m}} = (N_{\mathfrak{m}})^*_{M_{\mathfrak{m}}}$, where the later is computed over $R_{\mathfrak{m}}$.

Exercise 1.11. Let $(R, \mathfrak{m})$ be a Cohen Macaulay[2] for Noetherian local ring of dimension d with system of parameters $x_1, \ldots, x_d$. Show that $z \in (x_1^t, \ldots, x_d^t)^*$ if and only if the element η in $H^d_{\mathfrak{m}}(R)$ represented by the class of z modulo $(x_1^t, \ldots, x_d^t)$ in the limit (1.14.1) is in the tight closure of the zero module.

Hint: Use Exercise 1.7, the fact that (1.14.1) is injective, and the fact that $F^e_* R \otimes_R R/(x_1^t, \ldots, x_d^t) \cong F^e_* \left(R/(x_1^{p^e t}, \ldots, x_d^{p^e t} \right)$ is another module in the limit system (1.14.1).

Definition 1.16 ([**HY03**])**.** Suppose R is a Noetherian domain, $\mathfrak{a} \subseteq R$ is an ideal, and $t \geq 0$ is a real number. For any ideal $J \subseteq R$ define the **$\mathfrak{a}^t$-tight closure of** J, denoted $J^{*\mathfrak{a}^t}$ to be the set of $x \in R$ such that there exists $0 \neq c \in R$ so that

$$c\mathfrak{a}^{\lceil t(p^e-1) \rceil} x^{p^e} \in J^{[p^e]}$$

for all $e > 0$.

Exercise 1.12. Suppose that $(R, \mathfrak{a}^t)$ is strongly F-regular in the sense of Chapter 4 Definition 4.21. Show that $J = J^{*\mathfrak{a}^t}$ for all ideals $J \subseteq R$.

Exercise 1.13. Find an example where $\mathfrak{a}^t$-tight closure is not idempotent. In other words, show that

$$J^{*\mathfrak{a}^t} \neq (J^{*\mathfrak{a}^t})^{*\mathfrak{a}^t}$$

need not hold. When $(R, \mathfrak{m})$ is local and $\mathfrak{a}$ is $\mathfrak{m}$-primary, Vraciu introduced an idempotent variant of $\mathfrak{a}^t$-tight closure, see [**Vra08**].

Hint: Even polynomial rings and principal ideals will work.

[2]The Cohen Macaulay assumption can be removed under mild hypothesis; see Exercise 2.9

2. Test elements and test ideals

Test elements and test ideals were originally defined by Hochster and Huneke differently and in more general settings than we did in Chapter 1. To avoid confusion, we will refer to their test elements as *tight closure test elements*:

Definition 2.1 ([**HH90**, 8.11])**.** Let R be a Noetherian ring of prime characteristic p. An element $c \in R$, not in any minimal prime of R, is a **tight closure test element** for R if, for all finitely generated modules M and all submodules $N \subseteq M$, whenever $m \in N^*_M$, it follows that $cm^{p^e} \in N^{[p^e]}$, or equivalently, that

$$F^e_* c \otimes m \in \operatorname{Image}(F^e_* R \otimes_R N \to F^e_* R \otimes_R M),$$

for *all* integers $e \geq 0$.

Remark 2.2. In deference to Hochster and Huneke, our definition of tight closure test element includes the restriction that a test element is *not in any minimal prime* of the ring. We did not make such a restriction in our definition of *strong test element* (Definition 5.14 in Chapter 1) although we saw that a strong test element is most useful when it is a non-zerodivisor.

Hochster and Huneke proved that, under mild hypothesis on a Noetherian ring R (including the case where R is reduced and F-finite), the ring R always admits a test element. Indeed, they showed that the strong test elements we encountered in Chapter 1 are always tight closure test elements:

Proposition 2.3. *Let R be a reduced Noetherian F-finite ring of prime characteristic. If $c \in R$ is a strong test element for R, then for an arbitrary R-module M, not necessarily finitely generated, if $m \in N^*_M$, then $cm^{p^e} \in N^{[p^e]}_M$ for all integers $e \geq 0$.*

In particular, strong test elements for R are tight closure test elements provided they are not in any minimal prime of R.

Proof. Fix an arbitrary R-module M and suppose that $m \in N^*_M$ for some submodule N. We need to prove that $cm^{p^f} \in N^{[p^f]}$ for all integers $f \geq 0$. Because $m \in N^*_M$, there exists some d, not in any minimal prime of R, such that $dm^{p^e} \in N^{[p^e]}$ for all $e \gg 0$. Since R is reduced, the element d is a non-zerodivisor. Fixing sufficiently large e, we have

$$dm^{p^{e+f}} \in N^{[p^{e+f}]} \tag{2.3.1}$$

for all $E \geq 0$.

Now recall Definition 5.14 from Chapter 1: the element $c \in R$ is a *strong test element* if for all non-zerodivisors $d \in R$ and for all $e \gg 0$, there exists

$\phi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* d) = c$. Fix such an e large enough so that (2.3.1) holds as well. In this case, for all integers $f \geq 0$, the map $F^E_* \phi \in \mathrm{Hom}_R(F^{e+f}_* R, F^f_* R)$ has the property that $F^f_* \phi(F^{e+f}_* d) = F^f_* c$.

Now we can apply $F^f_* \phi$ to the first factor in $F^{e+f}_* R \otimes_R M$ to get an R-linear map $F^{e+f}_* R \otimes_R M \longrightarrow F^f_* R \otimes_R M$. The statement $dm^{p^{e+f}} \in N^{[p^{e+f}]}$ means that

$$F^{e+f}_* d \otimes m \in \mathrm{Image}(F^{e+f}_* R \otimes_R N \to F^{e+f}_* R \otimes_R M),$$

so that after applying $F^f_* \phi$, also

$$F^f_* c \otimes m \in \mathrm{Image}(F^f_* R \otimes_R N \to F^f_* R \otimes_R M).$$

Since this hold for arbitrary $f \geq 0$, we have shown that $cm^{[p^f]} \in N^{[p^f]}_M$ for all $f \geq 0$. This completes the proof. $\square$

Remark 2.4. Hochster and Huneke defined a **tight closure test element for ideals** to be an element c, not in any minimal prime of R, with the property that for all ideals $I \subseteq R$ and all elements $z \in I^*$,

$$cz^{p^e} \in I^{[p^e]} \qquad \text{or, equivalently,} \qquad zF^e_* c \in IF^e_* R$$

for all non-negative integers e. Clearly every tight closure test element is a tight closure test element for ideals, as we can take $M = R$. The converse holds when R is reduced and F-finite, or more generally, when all local rings of R are *approximately Gorenstein*[3] [**HH90**, Prop 8.15].

The following question is open.

Question 2.5. Does every reduced excellent Noetherian ring R have a tight closure test element (which is necessarily a non-zerodivisor)?

Of course, if the ring R is F-finite, the answer is yes (Proposition 2.3). For dimension ≤ 2, see [**Abe93**]. If R is complete and local, or more generally excellent and semi-local, or even essentially finite type over an excellent semi-local ring, then R has a test element by use of the so-called Γ-construction; see [**HH94a**] or [**Hoc07**]. The Γ-construction allows one to reduce the question for complete local rings (or rings of finite type over them) to the F-finite case. The excellent case can be reduced to the complete case since the formal fibers are geometrically regular. Other cases of existence of test elements have can be found in [**Sha12**], including in the F-pure case, and in [**DET23**].

[3]For the definition of approximately Gorenstein, see [**Hoc77**] or Section 11 in Appendix A. Here, what is important is that a reduced locally excellent ring is approximately Gorenstein, so an F-finite Noetherian reduced ring is approximately Gorenstein (invoking [**Kun76**, Thm 2.5].)

2.1. Tight closure test ideals. We now define tight closure test *ideals*:

Definition 2.6. Let R be a Noetherian ring of prime characteristic. The **tight closure test ideal** for R is the ideal

$$\tau_{\mathrm{tc}}^{\mathrm{fg}}(R) = \bigcap_{N \subseteq M} N :_R N_M^*,$$

where the intersection is taken over all pairs $N \subseteq M$ of *finitely generated* R modules. Equivalently (see Exercise 1.7),

$$\tau_{\mathrm{tc}}^{\mathrm{fg}}(R) = \bigcap_{M} \operatorname{ann}_R 0_M^*$$

where M runs through all finitely generated R-modules M.

The subscript tc reminds us that we are defining test ideals via tight closure. As before, the superscript fg reminds us we are using finitistic tight closure.

The elements in $\tau_{\mathrm{tc}}^{\mathrm{fg}}(R)$ are all tight closure test elements, provided they are not in any minimal prime; see Exercise 2.1. Furthermore, as long as R is reduced and has a tight closure test element, $\tau_{\mathrm{tc}}^{\mathrm{fg}}(R)$ is *generated* by tight closure test elements by Corollary 5.23 in Chapter 1.

The tight closure test ideal is, in fact, the annihilator of a *single* module:

Proposition 2.7 ([**HH90**, Prop 8.23])**.** *Let R be a Noetherian ring of prime characteristic. Then*

$$\tau_{\mathrm{tc}}^{\mathrm{fg}}(R) = \operatorname{ann}_R 0_E^{*\,\mathrm{fg}}$$

where $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ is the direct sum of the injective hulls of the residue fields of R over all maximal ideals of R.

Proof. By definition, $\tau_{\mathrm{tc}}^{\mathrm{fg}}(R)$ annihilates $0_E^{*\,\mathrm{fg}}$. Suppose $c \in \operatorname{ann}_R 0_E^{*\,\mathrm{fg}}$. Take any finitely generated R-module M. We need to show that $c\, 0_M^* = 0$. If not, then we can find M violating this statement with $\operatorname{Supp}(M) = \{\mathfrak{m}\}$, where $\mathfrak{m}$ is a maximal ideal of R (Exercise 2.7). Localizing at $\mathfrak{m}$, we may assume that $(R, \mathfrak{m})$ is local with injective hull $E = E_R(R/\mathfrak{m})$ of its residue field, and M is Artinian (Exercise 1.10). Now M embeds in a finite direct sum of copies of E ([**Sta19**, Tag 08Z3]). So

$$m \in 0_M^* \subseteq 0_{\bigoplus E}^{*\,\mathrm{fg}} = \bigoplus 0_E^{*\,\mathrm{fg}}.$$

Because $c \in \operatorname{ann}_R 0_E^{*\,\mathrm{fg}}$, we see that $cm = 0$ as well. The proposition is proved. ☐

Remark 2.8. The ideal $\tau_{\mathrm{tc}}^{\mathrm{fg}}(R)$ is sometimes called the **finitistic test ideal**. We also have a *non-finitistic test ideal*, which is *a priori* possibly *smaller:*

Definition 2.9. Let R be a Noetherian ring of prime characteristic. The **non-finitistic tight closure test ideal** is the ideal

$$\tau_{\mathrm{tc}}(R) = \operatorname{ann}_R 0^*_E$$

where $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ is the direct sum of the injective hulls of the residue fields of R over all maximal ideals of R.

By definition, clearly

$$\tau_{\mathrm{tc}}(R) \subseteq \tau^{\mathrm{fg}}_{\mathrm{tc}}(R). \tag{2.9.1}$$

This leads naturally to the following important open question about test ideals, equivalent, via Matlis Duality, to our question about (1.14.2):

Conjecture 2.10. *Let R be a reduced Noetherian F-finite ring of prime characteristic. Then the non-finitistic test ideal is the same as the tight closure test ideal. That is:*

$$\tau_{\mathrm{tc}}(R) = \tau^{\mathrm{fg}}_{\mathrm{tc}}(R).$$

Conjecture 2.10 holds in the quasi-Gorenstein case by Proposition 1.13 (and the remark following it).

Remark 2.11. Conjecture 2.10 is known to be true for $\mathbb{Q}$-Gorenstein rings [**AM99**] or even in rings with isolated non-$\mathbb{Q}$-Gorenstein points [**Mac96, LS01**], as well as for standard graded rings over fields [**LS99**], and combinations thereof [**Stu08**]. It is also known to be true if the anti-canonical symbolic Rees algebra

$$R \oplus \omega_R^{(-1)} \oplus \omega_R^{(-2)} \oplus \cdots = R \oplus R(-K_R) \oplus R(-2K_R) \oplus \dots$$

is finitely generated [**CEMS18**] (this finite generation condition is a weakening of the $\mathbb{Q}$-Gorenstein condition that appears naturally in the minimal model program). If one adds a few more hypotheses, then one only needs to assume that the anti-canonical symbolic Rees algebra is finitely generated on the punctured spectrum, see [**AHP23**, Theorem 4.10] (*cf.* [**AP21**]). We will prove this, and more, in the $\mathbb{Q}$-Gorenstein case in Theorem 5.10 below.

Remark 2.12. The elements of the non-finitistic test ideal are "better" test elements: every non-zerodivisor in $\tau_{\mathrm{tc}}(R)$ is not only a tight closure test element for R, but also a tight closure test element for $\widehat{R}$ (Exercise 2.11). This is useful in proofs when we wish to reduce statements about tight closure to the complete case. Hochster and Huneke used the term **completely stable test elements** for test elements in R that remain test elements in all complete local rings of R.

Furthermore, at least in reasonable rings (for instance, in F-finite rings) elements of the non-finitistic tight closure test ideal can be used for *any* "tight closure test" in the following sense: An element $c \in \tau_{\mathrm{tc}}(R)$ if and only if whenever $m \in N_M^*$ for an arbitrary (not-necessarily finitely generated) pair of R-modules $N \subseteq M$, we have

$$cm^{p^e} \in N_M^{[p^e]}$$

for all $e \in \mathbb{N}$; see Exercise 2.5. The elements of $\tau_{\mathrm{tc}}^{\mathrm{fg}}(R)$, by contrast, are only guaranteed to "test for tight closure" when N and M are finitely generated modules.

Remark 2.13. The *non-finitistic test ideal* $\tau_{\mathrm{tc}}(R)$ is also called the **big test ideal** in the tight closure literature—its elements annihilate tight closure relations even for non-finitely generated (or "big") modules; see [**Hoc07**] where it was denoted by $\tau_b(R)$ although earlier literature denoted it by $\widetilde{\tau}(R)$. We will avoid this *big* terminology since the "big" test ideal is *a priori smaller* than the usual (tight closure) test ideal. Note that the tight closure literature usually uses the term "test ideal" for what we call the "tight closure test ideal."

Remark 2.14. Of course, it is also of interest to ask Conjecture 2.10 in more general settings, dropping the reduced or the F-finite assumptions, perhaps imposing completeness or excellence. We will see another notion of the test ideal for non-reduced (but still usually F-finite rings) in Chapter 8 where we explore a theory developed by Blickle in [**Bli13**]. If one goes beyond the excellent case however, many basic properties of tight closure seem to break down; see [**LR01, Hei10**], which includes some ideas for potential fixes.

2.2. Connection with the test ideal.

The test ideal as defined (Definition 5.18 in Chapter 1) is closely related to the tight closure test ideal:

Theorem 2.15. *Let $(R, \mathfrak{m})$ be a reduced F-finite Noetherian ring. Then the non-finitistic tight closure test ideal is equal to the test ideal for R. That is:*

$$\tau_{\mathrm{tc}}(R) = \tau(R).$$

To prove Theorem 2.15, we need the following two lemmas:

Lemma 2.16. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let E denote an injective hull of its residue field. Fix any finite R-algebra S and element $c \in S$. The Matlis dual of the R-module map*

$$\mathrm{Hom}_R(S, R) \xrightarrow{\textit{eval at } c} R \tag{2.16.1}$$

is the natural map

$$E \xrightarrow{\xi \mapsto c \otimes \xi} S \otimes_R E. \tag{2.16.2}$$

Conversely, the Matlis dual of (2.16.2) is

$$\mathrm{Hom}_{\widehat{R}}(\widehat{R} \otimes_R S, \widehat{R}) \xrightarrow{\textit{eval at } c} \widehat{R}. \tag{2.16.3}$$

Proof. Because S is a finitely generated R-module and E is injective, there is a natural isomorphism $\operatorname{Hom}_R(\operatorname{Hom}_R(S,R),E) \cong S \otimes_R E$ sending each map $[\operatorname{Hom}_R(S,R) \xrightarrow{\psi} E]$ to the element $1_S \otimes \psi(1_S) \in S \otimes_R E$. The first statement then follows easily by applying the functor $\operatorname{Hom}_R(-,E)$. For the second statement, we use the fact that Matlis dualizing Noetherian modules *twice* is the same as completing, so Matlis dualizing (2.16.1) twice is the same as applying the faithfully flat functor $\widehat{R} \otimes_R -$. $\square$

Lemma 2.17. *Let $(R, \mathfrak{m})$ be a complete local Noetherian ring, and let E denote an injective hull of its residue field. Given any exact sequence of R-modules*

$$0 \longrightarrow Z \longrightarrow E \xrightarrow{\Psi} M, \tag{2.17.1}$$

we have

$$\operatorname{ann}_R Z = \operatorname{Image}(\Psi^\vee) = (\operatorname{Image}\Psi)^\vee \cap R$$

where $\Psi^\vee$ denotes the Matlis dual of the map Ψ.

Proof. This is essentially proved in [**Smi94**, Lem 3.1]. The point is that the Matlis dual of an exact sequence

$$0 \longrightarrow Z \longrightarrow E \longrightarrow N \longrightarrow 0, \tag{2.17.2}$$

is an exact sequence

$$0 \leftarrow Z^\vee \leftarrow R \leftarrow N^\vee \leftarrow 0, \tag{2.17.3}$$

and also, by exactness,

$$(\operatorname{Image}\Psi)^\vee = \operatorname{Image}(\Psi^\vee).$$

$\square$

Proof of Theorem 2.15. Because it is reduced and F-finite, the ring R admits a strong test element c that is a non-zerodivisor (Corollary 5.22 in Chapter 1). Therefore, the test ideal $\tau(R)$ can be described as the image of the map

$$\bigoplus_{e \in \mathbb{N}} \operatorname{Hom}_R(F^e_* R, R) \to R$$

defined by sending $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ to $\phi(F^e_* c)$, by Corollary 6.16 in Chapter 1. Tensoring with the completion $\widehat{R}$, which is faithfully flat, we see that also

$$\bigoplus_{e \in \mathbb{N}} \operatorname{Hom}_{\widehat{R}}(F^e_* \widehat{R}, \widehat{R}) \xrightarrow{\Psi} \widehat{R} \tag{2.17.4}$$

has image $\tau(R)\widehat{R}$.

Applying the Matlis Dual Functor to (2.17.4) and using Lemma 2.16, we get a map

$$E \xrightarrow{\Psi^\vee} \prod_{e\in\mathbb{N}} F^e_*\widehat{R} \otimes_{\widehat{R}} E = \prod_{e\in\mathbb{N}} F^e_* R \otimes_R E \tag{2.17.5}$$

which sends each element $\xi \in E$ to $(\cdots, F^e_* c \otimes \xi, \dots)$. Because c is a strong test element for R, Proposition 2.3 implies that the kernel of the map (2.17.5) is precisely 0^*_E. On the other hand, the annihilator, in $\widehat{R}$, of the kernel of $\Psi^\vee$ is equal to $\operatorname{Image}(\Psi^{\vee\vee})$, or equivalently $\operatorname{Image}(\Psi)$, by Lemma 2.17. So finally,

$$\tau_{\mathrm{tc}}(R) = \operatorname{ann}_R 0^*_E = \operatorname{ann}_{\widehat{R}} 0^*_E \cap R = \operatorname{Image}(\Psi) \cap R = \tau(R)\widehat{R} \cap R = \tau(R).$$

□

Cycling back to the open questions (1.14.2) and 2.10, we now rephrase them as asking whether or not the strong and the tight closure test ideals are *the same* for any reduced Noetherian F-finite ring. That is, whether:

$$\tau^{\mathrm{fg}}_{\mathrm{tc}}(R) = \tau(R)? \tag{2.17.6}$$

Again, see Remark 2.11 for what is known.

2.3. The test module and tight closure. We defined, in Chapter 2, a variant of the test ideal living inside the canonical module called the *test module* (or *parameter test module*). For a Noetherian F-finite reduced ring R with canonical module[4] ω_R, the test module $\tau(\omega_R)$ is the smallest non-zero uniformly compatible submodule $\tau(\omega)$ of ω_R (Definition 5.7 in Chapter 2). Just as the test ideal for a local ring can be described in terms of tight closure in the injective hull of its residue field, the test module can be described in terms of tight closure in local cohomology:

Proposition 2.18. *Let $(R, \mathfrak{m})$ be a reduced Noetherian F-finite ring of dimension d, and suppose that R admits a canonical module, ω_R. If $\tau(\omega_R)$ denotes the test module of R, then*

$$0^*_{H^d_{\mathfrak{m}}(R)} = \left(\frac{\omega_R}{\tau(\omega_R)}\right)^\vee$$

where $(-)^\vee$ denotes the Matlis dual functor $\operatorname{Hom}_R(-, E)$.

Proof. First note that R has a strong test element $c \in R$ that is a non-zero divisor (Corollary 5.22 in Chapter 1). Therefore, the test module

[4] Note, for us, the existence of a canonical module implies that R is locally equidimensional.

can be described as the image of the map

$$\bigoplus_{e>0} F^e_*\omega_R \xrightarrow{\mathcal{T}_c} \omega_R$$

defined by sending each $F^e_* w \in F^e_*\omega_R$ to $T^e(F^e_* cw)$, where $T^e : F^e_*\omega_R \longrightarrow \omega_R$ is dual-to-Frobenius $R \xrightarrow{F^e} F^e_* R$ under the functor $\operatorname{Hom}_R(-, \omega_R)$; see Definition 5.7 and Proposition 5.6 in Chapter 2. Applying Matlis duality, the dual map

$$H^d_{\mathfrak{m}}(R) \xrightarrow{\mathcal{T}_c^{\vee}} \prod_{e>0} H^d_{\mathfrak{m}}(F^e_* R) = \prod_{e>0} F^e_* H^d_{\mathfrak{m}}(R)$$

sends each $\eta \in H^d_{\mathfrak{m}}(R)$ to $(c\eta^p, c\eta^{p^2}, c\eta^{p^3}, \dots)$. The kernel of $\mathcal{T}_c^{\vee}$ is precisely $0^*_{H^d_{\mathfrak{m}}(R)}$, because c is a tight closure test element. Thus

$$0^*_{H^d_{\mathfrak{m}}(R)} = \ker \mathcal{T}_c^{\vee} \cong (\omega_R / \operatorname{Image} \mathcal{T}_c)^{\vee} = (\omega_R/\tau(\omega_R))^{\vee},$$

as desired, with the middle isomorphism following similarly to Lemma 2.17; see Exercise 2.12. $\square$

Similar to the test ideals, we can interpret the test module $\tau(\omega_R)$ as an annihilator of a single tight closure module, suitably interpreted. Indeed, let $(R, \mathfrak{m})$ be a Noetherian local ring with canonical module ω_R, and let E be an injective hull of its residue field. Matlis duality gives an R-bilinear pairing

$$\omega_R \times H^d_{\mathfrak{m}}(R) \longrightarrow E \tag{2.18.1}$$

that allows us to view ω_R as acting on $H^d_{\mathfrak{m}}(R)$, and also view $H^d_{\mathfrak{m}}(R)$ as acting on ω_R. For a submodule $M \subseteq H^d_{\mathfrak{m}}(R)$, we can define

$$\operatorname{ann}_{\omega_R} M = \{w \in \omega_R \mid w\, M = 0\},$$

and for a submodule $J \subseteq \omega_R$,

$$\operatorname{ann}_{H^d_{\mathfrak{m}}(R)} J = \{\eta \in H^d_{\mathfrak{m}}(R) \mid J\eta = 0\}.$$

The pairing (2.18.1) is *perfect* when R is complete. In this case, one checks easily using Matlis duality that taking annihilators as above defines mutually inverse bijections between submodules of $H^d_{\mathfrak{m}}(R)$ and submodules of ω_R.

Corollary 2.19. *Let $(R, \mathfrak{m})$ be a reduced Noetherian local F-finite ring with canonical module ω_R. Then*

$$\tau(\omega_R) = \operatorname{ann}_{\omega_R} 0^*_{H^d_{\mathfrak{m}}(R)}.$$

Proof. When R is complete, this follows immediately from the perfect pairing (2.18.1) and Proposition 2.18. Now because R has a strong test element $c \in R$ that is a non-zero divisor (Corollary 5.22 in Chapter 1), one easily verifies that the tight closure is the same, whether we compute as an R-module or as an $\widehat{R}$-module. We also know that

$$\widehat{R} \otimes_R \tau(\omega_R) = \tau(\omega_{\widehat{R}}) \tag{2.19.1}$$

by Corollary 5.8 in Chapter 2. Using these two facts, we will reduce Corollary 2.19 to the complete case.

Assume Corollary 2.19 holds for $\widehat{R}$. To see that $\tau(\omega_R) \subseteq \operatorname{ann}_{\omega_R}(0^*_{H^d_\mathfrak{m}(R)})$, take any $w \in \tau(\omega_R)$. Then by (2.19.1), we have

$$1 \otimes w \in \widehat{R} \otimes_R \tau(\omega_R) = \tau(\omega_{\widehat{R}}) = \operatorname{ann}_{\omega_{\widehat{R}}} 0^*_{H^d_\mathfrak{m}}(R).$$

Since the action of $1 \otimes w$ is the same as the action of w on $H^d_\mathfrak{m}(R)$, it follows that $w\eta = 0$ for every $\eta \in 0^*_{H^d_\mathfrak{m}(R)}$. In other words, $w \in \operatorname{ann}_{\omega_R}(0^*_{H^d_\mathfrak{m}(R)})$, as desired. Now to see that $\operatorname{ann}_{\omega_R}(0^*_{H^d_\mathfrak{m}(R)}) \subseteq \tau(\omega_R)$, take any $w \in \operatorname{ann}_{\omega_R}(0^*_{H^d_\mathfrak{m}(R)})$. Then $1 \otimes_R w \in \widehat{R} \otimes_R \omega_R$ also annihilates $0^*_{H^d_\mathfrak{m}(R)}$. That is, $1 \otimes w \in \operatorname{ann}_{\omega_{\widehat{R}}}(0^*_{H^d_\mathfrak{m}(R)}) = \tau(\omega_{\widehat{R}}) = \widehat{R} \otimes \tau(\omega_R)$. Finally, because $\widehat{R}$ is faithfully flat over R, we conclude that $w \in \tau(\omega_R)$, as needed. $\square$

Remark 2.20. In the complete case (still in the world of Noetherian local reduced F-finite rings), the mutually inverse order-reserving bijections between submodules of $H^d_\mathfrak{m}(R)$ and submodules of ω_R given by taking annihilators define bijections between

(i) *compatible* submodules of ω_R in the sense of Chapter 2 Subsection 5.1; and
(ii) submodules of $H^d_\mathfrak{m}(R)$ stable under the natural Frobenius action.

This point of view was developed in [**Smi95**], where annihilators, in ω_R, of F-stable submodules of $H^d_\mathfrak{m}(R)$ were called F**-modules**.[5] See Exercise 2.14. Note that the *smallest* uniformly compatible submodule of ω_R supported on all of $\operatorname{Spec} R$ corresponds to the *largest* F-stable submodule of $H^d_\mathfrak{m}(R)$ annihilated by some non-zero-divisor, namely $0^*_{H^d_\mathfrak{m}(R)}$. See Definition 5.7 of Chapter 2.

Remark 2.21. See Exercise 2.13 for a global formula for $\tau(\omega_R)$ as an annihilator of a certain tight closure module.

Remark 2.22. Because $0^*_{H^d_\mathfrak{m}(R)} = 0^{*\,\mathrm{fg}}_{H^d_\mathfrak{m}(R)}$ for reduced F-finite Noetherian local rings (see Proposition 1.13), there is no need to introduce "weak" and "strong" variants of the parameter test module.

[5] A warning: several different notions have been called *F-modules* over the years, *cf.* [**Lyu97, EK04, BL19**].

2.4. Colon capturing. Perhaps one of best known features of tight closure is its ability to "capture" the failure of a ring to be Cohen-Macaulay. In its simplest form, the famous **Colon Capturing** property of tight closure states the following:

Theorem 2.23. *If $x_1, \ldots, x_d$ is a system of parameters for an excellent Noetherian local domain,*[6] *then*

$$(x_1, \ldots, x_i)^* :_R x_{i+1} \subseteq (x_1, \ldots, x_i)^*. \tag{2.23.1}$$

Before proving Theorem 2.23, we point out an immediate consequence: rings in which all ideals—or even all ideals generated by parameters—are tightly closed are always Cohen-Macaulay:

Corollary 2.24. *Let $(R, \mathfrak{m})$ be an excellent local Noetherian domain (or reduced equidimensional ring) in which all ideals—or even just all ideals generated by subsets of a fixed system of parameters—are tightly closed. Then R is Cohen-Macaulay.*

Proof of Theorem 2.23. We will black box the fact that R has test elements that remain test elements in the completion, although we haven't proven that (see [**HH94a**, Seciton 6]). Or the reader is invited to simply assume that R is F-finite, whence this follows from Remark 2.12.

We first reduce to the case where R is complete. A system of parameters $x_1, \ldots, x_d$ for R is also a system of parameters for $\widehat{R}$, under the natural inclusion $R \hookrightarrow \widehat{R}$. Suppose that $z \in (x_1, \ldots, x_i)^* :_R x_{i+1}$. Then $x_{i+1}z \in (x_1, \ldots, x_i)^*$ in the ring R. Since non-zerodivisors in R remain non-zero-divisors in $\widehat{R}$, also $x_{i+1}z \in (x_1, \ldots, x_i)\widehat{R}^*$. Assuming the conclusion of Theorem 2.23 holds for $\widehat{R}$, we have

$$z \in ((x_1, \ldots, x_i)\widehat{R})^*.$$

Now let $c \in R$ be strong test element for R, so that c is a tight closure test element for both R and $\widehat{R}$. In particular

$$cz^{p^e} \in (x_1^{p^e}, \ldots, x_i^{p^e})\widehat{R} \cap R = (x_1^{p^e}, \ldots, x_i^{p^e})R.$$

for all $e \geq 0$. This says that $z \in (x_1, \ldots, x_i)^*$ in R, as needed. Thus to prove Theorem 2.23, we may assume without loss of generality that R is complete.

Although $\widehat{R}$ may no longer be a domain, it is reduced and equidimensional since R is excellent.[7] So the system of parameters $x_1, \ldots, x_d$ remains a system of parameters modulo any minimal prime P, and any $z \in$

[6] The ring need not be a domain; our proof of Theorem 2.23 holds for any equidimensional local ring that admits a completely stable test element, for instance excellent equidimensional reduced local rings.

[7] See [**Sta19**, Tag 0C21] and [**Gro65**, Scholie (7.8.3)(x)]; *cf.* [**Rat71**, Theorem 3.9].

$(x_1, \ldots, x_i)^* :_R x_{i+1}$ will remain in $(x_1, \ldots, x_i)^* :_R x_{i+1}$ modulo P. Since we can check that $z \in (x_1, \ldots, x_i)^*$ modulo each minimal prime, we have now reduced to the case where R is a complete local domain. In this case, we use the Cohen Structure theorem to write R as a module finite extension of the power series subring $A = k[\![x_1, \ldots, x_d]\!]$. Now Theorem 2.23 follows immediately from Lemma 2.25 below. $\square$

Lemma 2.25. [**HH90**, Thm 4.8] *Let R be Noetherian domain*[8] *of characteristic $p > 0$, finite over some regular subring $A \subseteq R$. Then for any ideals $I, J \subseteq A$, we have*

$$(2.25.1)\qquad \begin{aligned} (IR)^* :_R JR &\subseteq ((I :_A J)R)^* \quad \textit{and} \\ (IR)^* \cap (JR)^* &\subseteq ((I \cap J)R)^*. \end{aligned}$$

Proof of Lemma 2.25. Fix a finitely generated free A-submodule $M \subseteq R$ of maximal rank, in which case R/M is a torsion A-module. So there exists non-zero $c \in A$ such that $cR \subseteq M$. Now suppose $z \in (IR)^* :_R JR$. Then for each generator j of J, $zj \in (IR)^*$. So there exists nonzero $d \in R$ such that $d(zj)^{p^e} \in I^{[p^e]}R$ for $e \gg 0$. Multiplying by c, then $cdz^{p^e} j^{p^e} \in I^{[p^e]}M$, and since this holds for all generators $j \in J \subseteq A$

$$cdz^{p^e} \in I^{[p^e]}M :_M J^{[p^e]}.$$

Now, since M is a free A-module and Frobenius is flat in A, we have (Appendix A Lemma 1.1) that $I^{[p^e]}M :_M J^{[p^e]} = (I :_M J)^{[p^e]}$. Expanding to R we see

$$cdz^{p^e} \in ((IR) :_R (JR))^{[p^e]},$$

showing that $z \in (IR :_R JR)^*$, as claimed. The proof for the intersection is similar. $\square$

2.5. Exercises.

Exercise 2.1. Show that an element $c \in R$, not in any minimal prime, is a tight closure test element for R if and only if $cN^*_M \subseteq N$ for all pairs of finitely generated modules $N \subseteq M$.

Hint: Show that if $m \in N^*_M$, then $m^{p^e} \in (N^{[p^e]})^*_M$ for all $e \geq 0$.

Exercise 2.2 ([**HH90**, 6.1(c)])**.** Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and $c \in R$ is a tight closure test element that remains a tight closure test element in $\widehat{R}$ (a completely stable test element). Suppose that $J \subseteq R$ is an ideal. Prove that $z \in J^*$ if and only if $z \in (J\widehat{R})^*$. That, is

$$J^* = (J\widehat{R})^* \cap R.$$

[8] R need not be a domain—the proof works provided that R is locally equidimensional, as in this case R will be torsion free over the regular subring A.

Exercise 2.3. Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring and $c \in R$ is a tight closure test element that remains a tight closure test element in $\widehat{R}$. For any $\mathfrak{m}$-primary ideal J, prove that

$$J^* \widehat{R} = (J\widehat{R})^*.$$

Hint: Use the previous exercise.

Exercise 2.4. Let R be a reduced F-finite Noetherian ring of prime characteristic p. Prove that an element $c \in R$, not in any minimal prime of R, is a **tight closure test element** for R if and only if for all finitely generated modules M, whenever $m \in 0^*_M$, it follows that $cm^{p^e} = 0$, or equivalently, that

$$F^e_* c \otimes m \in F^e_* R \otimes_R M$$

is the zero element.

Exercise 2.5. Let R be a reduced F-finite Noetherian ring. Let $N \subseteq M$ be an arbitrary pair of R-modules, not-necessarily finitely generated. Show that if $c \in \tau_{\mathrm{tc}}(R)$ and $m \in N^*_M$, then

$$cm^{p^e} \in N^{[p^e]}_M$$

for all $e \geq \mathbb{N}$.

Hint: Use Proposition 2.3.

Exercise 2.6. Let R be a reduced F-finite Noetherian ring. Prove that the tight closure test ideal is uniformly compatible (Definition 6.10 in Chapter 1). Deduce that the test ideal is contained in the tight closure test ideal. Of course, this also follows by combining (2.9.1) and Theorem 2.15.

Hint: Use Theorem 6.15 in Chapter 1.

Exercise 2.7. Let M be a finitely generated module over a Noetherian ring R, and suppose $x \in 0^*_M$ but $cx \neq 0$ for some $c \in R$. Then if N is chosen to be a maximal submodule of M with respect to the property that $cx \notin M$, then show that $c\, 0^*_{M/N} \neq 0$ and $\operatorname{Supp} M/N$ consists of one maximal ideal of R.

Exercise 2.8. Let $(R, \mathfrak{m})$ be an F-finite reduced Noetherian local ring of dimension d with system of parameters $x_1, \dots, x_d$. Show that $z \in (x_1^t, \dots, x_d^t)^*$ if and only if the element η in $H^d_{\mathfrak{m}}(R)$ represented by the class of z modulo $(x_1^t, \dots, x_d^t)$ in the limit (1.14.1) is in the tight closure of the zero module.

Hint: In the Cohen-Macaulay case, this is Exercise 1.11, but in general, one needs a test element and colon capturing to deal with non-injectivity of the limit (1.14.1).

Exercise 2.9. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Use Colon Capturing to prove Proposition 1.13 in the case R is reduced F-finite but not Cohen-Macaulay.

Hint: Use Exercise 2.8.

Exercise 2.10. Let R be a Noetherian ring of prime characteristic. Prove that the tight closure test ideal is contained in *every* associated prime of R that is not minimal.

Exercise 2.11. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Suppose $c \in R$ is not in any minimal prime of R. Show that if $c \in \operatorname{ann}_R 0^*_E$, then c is a tight closure test element for both R and $\widehat{R}$.

Exercise 2.12. With hypotheses and notation as in the proof of Proposition 2.18, show that $\ker \mathcal{T}_c^\vee \cong (\omega_R / \operatorname{Image} \mathcal{T}_c)^\vee$.

Hint: Use the same technique as in Lemma 2.17.

Exercise 2.13. Let R be a Noetherian F-finite locally reduced ring of prime characteristic with canonical module ω_R. Show that

$$\tau(\omega_R) = \operatorname{ann}_{\omega_R} 0^*_H.$$

where H is the module $\oplus H^{\dim R_\mathfrak{m}}_\mathfrak{m}(R)$, where we range over all maximal ideals of R.

Exercise 2.14.

3. Weak F-regularity

Many applications of tight closure stem primarily from situations in which all ideals are tightly closed.

Definition 3.1 ([**HH90**]). A Noetherian commutative ring of prime characteristic is **weakly F-regular** if every ideal is tightly closed.

Regular rings of prime characteristic are weakly F-regular, as is easily checked using Kunz's theorem on the flatness of Frobenius; see Exercise 3.1.

Weak F-regularity can be checked locally at maximal ideals:

Proposition 3.2. [**HH90**, Cor 4.15] *A Noetherian ring is weakly F-regular if and only if $R_\mathfrak{m}$ is weakly F-regular for all maximal ideals $\mathfrak{m}$ of R.*

Proof. The point is that R is weakly F-regular if and only if every ideal primary to a maximal ideal is tightly closed; see Exercise 3.2. The details are left as an exercise using following two facts:

(a) Every ideal in a Noetherian ring is an intersection of ideals primary to maximal ideals.[9]
(b) For every ideal $I \subseteq R$ primary to some maximal ideal $\mathfrak{m}$, $I^*R_\mathfrak{m} = (IR_\mathfrak{m})^*$ (Exercise 1.10).

□

We caution the reader, however, that it is *not known* whether the localization of a weakly F-regular ring at a *non-maximal* prime ideal is weakly F-regular. See Subsection 3.2.

Theorem 3.3 ([**HH90**, Corollary 5.11], [**HH94a**, Proposition 6.27])**.** *A weakly F-regular ring is normal; furthermore, an excellent weakly F-regular ring is Cohen-Macaulay.*

Proof. Let R be a weakly F-regular ring. Because the zero ideal is tightly closed, the nilradical of R is trivial, so that R is reduced. For normality, observe that the proof of Theorem 4.30 in Chapter 1 reveals that a Noetherian ring is normal if all ideals generated by a non-zerodivisor are tightly closed: Equation (4.31.1) in that proof shows that $x \in (y)^*$, so that $x \in (y)$ if R is weakly F-regular, without using the strong F-regularity hypothesis. For Cohen-Macaulayness, we can check this after localization at maximal ideals, in which case it follows immediately from Corollary 2.24. □

When all ideals are tightly closed, the same is true for all finitely generated modules:

Proposition 3.4 ([**HH90**, Prop 8.7])**.** *Let R be a weakly F-regular ring. Then every submodule N of a finitely generated module M is tightly closed.*

Proof. Take $x \in M \setminus N$. We must show that $x \notin N^*_M$. Replace N by a submodule of M maximal with respect to not containing x, and then replace M, N, and x by their images modulo N in M/N. Then x is in every non-zero submodule of M, so that $Rx \hookrightarrow M$ is an essential extension and $Rx \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. In particular M is annihilated by a power of $\mathfrak{m}$. There exists an irreducible $\mathfrak{m}$-primary ideal $Q \subseteq R$ that annihilates M and such that the Artin ring R/Q is Gorenstein (as $R_\mathfrak{m}$ is normal and normal local rings are approximately Gorenstein [**Hoc77**, Thm 1.6], or see Appendix A Lemma 11.3). So the extension $R/\mathfrak{m} \cong Rx \hookrightarrow M$ is an essential extension of R/Q-modules. It follows that M can be embedded in R/Q. It will then suffice to show that 0 is tightly closed in R/Q, i.e., that Q is tightly closed in R. □

[9]For example, if $I \subseteq \mathfrak{m}$, then $I = \bigcap_{n\in\mathbb{N}}(I + m^n)$.

Corollary 3.5. *A Noetherian ring R is weakly F-regular if and only if 1_R is a tight closure test element.*

Proof. Suppose 1 is a tight closure test element. Then whenever $z \in I^*$, we have $z = 1z \in I$. So $I^* = I$ for all I. Conversely, if R is weakly F-regular, then $N_M^{*\,\mathrm{fg}} = N$ for all finitely generated R-modules $N \subseteq M$ (Proposition 3.4). Thus $1 \in \bigcap_{N \subseteq M} N :_R N_M^{*\,\mathrm{fg}}$ and so by Definition 2.1, 1 is a tight closure test element. $\square$

In particular, in the spirit of Theorem 1.10, we have the following characterization of weak F-regularity:

Corollary 3.6. *A Noetherian ring R is weakly F-regular if and only if*

$$0_E^{*\,\mathrm{fg}} = 0$$

where $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ is the direct sum of the injective hulls of the residue fields of R over all maximal ideals of R.

Proof. The tight closure test ideal is $\mathrm{ann}_R\, 0_E^{*\,\mathrm{fg}}$ (Proposition 2.7). This contains 1 if and only if $0_E^{*\,\mathrm{fg}} = 0$. $\square$

3.1. Weak versus strong F-regularity. As the name suggests, weak F-regularity is closely related to strong F-regularity. One implication is easy:

Proposition 3.7. *Every strongly F-regular ring is weakly F-regular.*

Proof. Suppose that R is strongly F-regular, and $z \in I^*$. By definition of tight closure, there exists $c \in R$, not in any minimal prime, such that $zF_*^e c \in IF_*^e R$ for all $e \gg 0$. As strongly F-regular rings are *reduced*, c is a non-zerodivisor on R, so for all $e \gg 0$, there exists an R-linear map $\phi_e : F_*^e R \to R$ sending $F_*^e c$ to 1. Applying such ϕ_e to some inclusion $zF_*^e c \in IF_*^e R$, we have

$$z = \phi_e(F_*^e c) \in I\phi_e(F_*^e R) \subseteq I,$$

showing that I is tightly closed. $\square$

In fact, Hochster and Huneke introduced strong F-regularity in the hope it would be equivalent to weak F-regularity, as it was easier to verify many good properties (such as commutation with localization) for strong F-regularity [**HH89**]. This is perhaps the most well known conjecture in characteristic $p > 0$ commutative algebra.

Conjecture 3.8. *Let R be an F-finite Noetherian ring of prime characteristic. Then R is weakly F-regular if and only if R is strongly F-regular.*

This is of course, simply a special case of Conjecture 2.10 *cf.* (2.17.6). In other words, if $\tau_{\mathrm{tc}}^{\mathrm{fg}}(R) = R$, then is $\tau(R) = R$ as well?

Remark 3.9. Beyond the cases we have already seen in Remark 2.11, Conjecture 3.8 is also known in dimension 3 by [**Wil95**]. Furthermore, it is known also that if the anti-canonical symbolic Rees algebra $R \oplus \omega_R^{(-1)} \oplus \omega_R^{(-2)} \oplus \dots$ is finitely generated on the punctured spectrum, then if R is weakly F-regular it is also strongly F-regular [**AHP23**].

For a strongly F-regular ring, we expect that the associated anti-canonical symbolic Rees algebra is *always* finitely generated, since the same holds for Kawamata log-terminal singularities in characteristic zero as a consequence of the minimal model program [**Kol08**]. Indeed, because the minimal model program holds in dimension 3 in characteristic $p > 0$ ([**HX15, Bir16, DW19**]) and strongly F-regular singularities are KLT, Aberbach-Polstra showed that weakly F-regular rings are strongly F-regular (Conjecture 3.8 holds) in dimension 4 if $p > 5$ in [**AP21**], *cf.* [**AHP23**].

Conjecture 3.10. *If R is a Noetherian F-finite strongly F-regular domain then the anti-canonical symbolic Rees algebra*

$$R \oplus \omega_R^{(-1)} \oplus \omega_R^{(-2)} \oplus \cdots = R \oplus R(-K_R) \oplus R(-2K_R) \oplus \dots$$

is Noetherian (that is, finitely generated as an R-algebra).

By [**AHP23**], this implies Conjecture 3.8.

We expect that the tight closure test ideal defines the non-weakly F-regular locus just as the test ideal defines the non-strongly F-regular locus (Exercise 5.7 in Chapter 1), but this remains open. We do not even know whether the weakly F-regular locus is open, as we discuss below in Subsection 3.2.

3.2. Tight closure and localization.

A vexing open problem in tight closure theory is whether or not weak F-regularity is preserved under localization:

Question 3.11. Let R be an excellent weakly F-regular ring. Is it true that $W^{-1}R$ is also weakly F-regular for any multiplicatively closed set $W \subseteq R$?

Indeed, unable to settle this problem, Hochster and Huneke introduced the term F**-regular** to mean a ring all of whose localizations have the property that all ideals are tightly closed; See [**HH90**]. To avoid confusion, we will not use this term.

Remark 3.12. To settle Question 3.11, Proposition 3.2 ensures that it is enough to check that any local ring of a weakly F-regular ring is weakly F-regular.

A related open question is whether or not weak F-regularity behaves well under completion:

Question 3.13. Let $(R, \mathfrak{m})$ be an excellent weakly F-regular local ring. Is the completion $\widehat{R}$ also weakly F-regular?

Without the excellent hypotheses, this is false, see [**LR01**], *cf.* [**Hei10**].

The converse is easier: $\widehat{R}$ weakly F-regular implies that R is weakly F-regular. See Exercise 3.3.

Remark 3.14. We have seen in Chapter 1 that strong F-regularity *does* behave well under both localization (Proposition 4.23) and completion (Proposition 6.17), where we assumed the rings were F-finite. Thus affirmative resolutions to Question 3.11 and Question 3.13 are immediate for any class of rings for which we can establish Conjecture 3.8, including the classes of rings mentione din Remarks 2.11 and 3.9.

Remark 3.15. Weak F-regularity is known to pass to localizations for any finitely generated ring over a field k of infinite transcendence degree over its prime field; see [**HH94a**, Thm 8.1], where this fact is attributed to Murthy.

Importantly, tight closure *itself* does not commute with location: it is not necessarily the case that $I^*U^{-1}R = (IU^{-1}R)^*$ for all ideals:

Example 3.16. (Brenner-Monsky [**BM10**]) Let $k = \overline{\mathbb{F}_2}$ be the algebraic closure of the finite field of two elements, and set

$$R = k[x, y, z, t]/(z^4 + xyz^2 + x^3z + y^3z + tx^2y^2).$$

Then for $I = (x^4, y^4, z^4)$ and the multiplicative system $U = k[t] \setminus \{0\}$, Brenner and Monsky show that $I^*U^{-1} \neq (IU^{-1})^*$. Similar examples are constructed in [**BNS$^+$22**].

3.3. More on F-rationality. Fedder and Watanabe defined a local ring to be ***F*-rational** ([**FW89**]) if all *parameter ideals*—meaning ideals generated by subsets of systems of parameters—are tightly closed. We now relate this to our definition of F-rationality in Definition 7.9 of Chapter 1:

Proposition 3.17 ([**Smi94**, Prop 2.5])**.** *Let $(R, \mathfrak{m})$ be an equidimensional F-finite local ring of prime characteristic. Then R is F-rational if and only if every ideal generated by any subset of any system of parameters for R is tightly closed.*

Proof. Assume all ideals generated by any subset of a system of parameters are tightly closed. Then by colon capturing (Theorem 2.23), we can immediately conclude that R is Cohen-Macaulay and hence also equidimensional. Now, to show that R is F-rational, we can show that $0^*_{H^d_{\mathfrak{m}}(R)} = 0$ (Theorem 1.10), where $d = \dim R$.

Fix a system of parameters $x_1, \ldots, x_d$ for R. Then $H^d_{\mathfrak{m}}(R)$ is the increasing union of submodules of the form

$$\frac{R}{(x_1^t, \ldots, x_d^t)},$$

where $t \in \mathbb{N}$ (see the proof of Proposition 1.13). An element $\eta \in H^d_{\mathfrak{m}}(R)$ is represented by some class z modulo $(x_1^t, \ldots, x_d^t)$, and if $c\eta^{p^e} = 0$, then the class of cz^{p^e} in $\frac{R}{(x_1^{tp^e}, \ldots, x_d^{tp^e})}$ is zero. So one easily checks that $c\eta^{p^e} = 0$ for all $e \gg 0$ if and only $z \in (x_1^t, \ldots, x_d^t)^*$; see Exercise 1.11. So if by assumption, $(x_1^t, \ldots, x_d^t)^* = (x_1^t, \ldots, x_d^t)$ for all t, we can conclude that $0^*_{H^d_{\mathfrak{m}}(R)} = 0$. That is, R is F-rational.

Conversely, suppose $(R, \mathfrak{m})$ is F-rational in the sense of Definition 7.9 in Chapter 1. Then R is Cohen-Macaulay and $0^{*\,\mathrm{fg}}_{H^d_{\mathfrak{m}}(R)} = 0$ by Theorem 1.10 and Proposition 1.13. Suppose that $x_1, \ldots, x_d$ is a system of parameters for R and that there is some $z \in (x_1, \ldots, x_i)^*$. In this case,

$$z \in (x_1, \ldots, x_i, x_{i+1}^t, \ldots, x_d^t)^*$$

for all $t \in \mathbb{N}$. In particular, arguing as in the previous paragraph, we see that the the class of z in $R/(x_1, \ldots, x_i, x_{i+1}^t, \ldots, x_d^t)$ is in the tight closure of 0 in the finitely generated submodule $R/(x_1, \ldots, x_i, x_{i+1}^t, \ldots, x_d^t)$ of $H^d_{\mathfrak{m}}(R)$. But now because $0^{*\,\mathrm{fg}}_{H^d_{\mathfrak{m}}(R)} = 0$, we can conclude that it must be the zero class. That is, $z \in (x_1, \ldots, x_i, x_{i+1}^t, \ldots, x_d^t)$ for all t. This means that

$$z \in \bigcap_{t \in \mathbb{N}} (x_1, \ldots, x_i, x_{i+1}^t, \ldots, x_d^t) = (x_1, \ldots, x_i),$$

as needed. $\square$

Just as (strong or weak) F-regularity is equivalent to the triviality of the (strong or tight closure) test ideals, F-rationality can be characterized as the triviality of the *test module* (Definition 5.7 in Chapter 2). This follows immediately from Corollary 2.19 and the fact that test modules behave well under localization (Corollary 5.8 in Chapter 2). Tight closure gives another perspective on Corollary 5.14.

The following facts about F-rationality follow easily from statements about tight closure already proved in this chapter. We leave proving them as an exercise; they have all essentially been proved in the F-finite case in other parts of the text without tight closure:

Theorem 3.18. *Let $(R, \mathfrak{m})$ be an excellent Noetherian local ring in which all parameter ideals are tightly closed. Then*

(a) *R is normal;*[10]
(b) *Then R is Cohen-Macaulay;*
(c) *If R is Gorenstein, then R is weakly F-regular;*
(d) *If R is Gorenstein and F-finite, then R is strongly F-regular.*

3.4. Exercises.

Exercise 3.1. Use Kunz's theorem (Theorem 2.1 in Chapter 1) to prove that all ideals in a regular ring of prime characteristic are tightly closed.

Hint: Remember that flat maps commute with colons, *cf.* Appendix A Lemma 1.1.

Exercise 3.2. Let R be a Noetherian ring. Show that R is weakly F-regular if and only if every ideal primary to a maximal ideal is tightly closed.

Hint: Given $I \subseteq R$, let $\mathfrak{m}$ be a maximal ideal containing I. Use the fact that $I = \bigcap_{n \in \mathbb{N}}(I + \mathfrak{m}^n)$.

Exercise 3.3. Let $(R, \mathfrak{m})$ be Noetherian local ring whose completion $\widehat{R}$ is weakly F-regular. Prove that R is weakly F-regular.

Exercise 3.4. Let $R \hookrightarrow S$ be an inclusion of Noetherian rings, split as a map of R-modules. Prove that if S is weakly F-regular, then R is weakly F-regular.

Exercise 3.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring, with system of parameters $x_1, \dots, x_d$. Prove that the ideal $(x_1, \dots, x_i)$ is tightly closed if and only the ideals $(x_1, \dots, x_i, x_{i+1}^t, \dots, x_d^t)$ are tightly closed for all integers $t \geq 1$.

Exercise 3.6. Prove Theorem 3.18 (a) and (b) by examining the proofs of the corresponding statements for weakly F-regular rings.

Exercise 3.7. Prove Theorem 3.18 (c).

Hint: Remember that $E = H^d_{\mathfrak{m}}(R)$ for a Gorenstein local ring.

Exercise 3.8. **[Hoc77]** Let $(R, \mathfrak{m})$ be an approximately Gorenstein local ring; by definition, we can fix a sequence of ideals I_t, cofinal with the powers of the maximal ideal, such that each R/I_t is Gorenstein. For each t, and

[10]The proof is the same as the proof the weakly F-regular rings are normal Theorem 3.3.

$s \gg t$ so that $I_s \subseteq I_t$, let u_{ts} be an R-module generator for $I_s : I_t$ modulo I_t. Show that in this case the directed limit system

$$\frac{R}{I_t} \xrightarrow{\text{Multiplication by } u_{ts}} \frac{R}{I_s}$$

for $s \gg t$ has limit isomorphic to an injective hull E of the residue field of R. What goes wrong if we mimic the proof of Proposition 1.13 to try to prove that $0^*_E = 0^{*\mathrm{fg}}_E$ for reduced F-finite local rings?

4. Tight closure and integral extensions

An important feature of tight closure is its interaction with integral extensions:

Theorem 4.1. *Suppose that $R \subseteq S$ is a finite extension of domains of prime characteristic.*[11] *Then for every ideal $I \subseteq R$, we have $IS \cap R \subseteq I^*$.*

Proof. Tensoring the inclusion $R \hookrightarrow S$ with the fraction field of R, we easily see, after clearing denominators, that there is a R-linear map $\phi \in \mathrm{Hom}_R(S, R)$ such that $\phi(1_S) = c$, a non-zero element of R. Let $\{z_1, \dots, z_n\} \subseteq R$ be generators for the ideal I. Now if $z \in IS \cap R$, we can write $z = a_1z_1 + \cdots + a_nz_n$ for some $a_i \in S$. So also $z^{p^e} = a_1^{p^e} z_1^{p^e} + \cdots + a_n^{p^e} z_n^{p^e}$ for all $e \in \mathbb{N}$. Applying ϕ, we have

$$cz^{p^e} = \phi(a_1^{p^e})z_1^{p^e} + \cdots + \phi(a_n^{p^e})z_n^{p^e} \in I^{[p^e]} \subseteq R$$

for all e, so that $z \in I^*$. □

It is natural to ask whether there exists some integral extension S of R such that $I^* = IS \cap R$. To discuss this idea, it is helpful to consider all integral extensions at once:

Definition 4.2. [**Art71**] For an arbitrary domain R, an **absolute integral closure** of R is the integral closure of R in some fixed algebraic closure of its fraction field. We denote any such absolute integral closure by R^+.

Strictly speaking, the ring R^+ depends on the choice of an algebraic closure, but all choices are isomorphic as R-algebras.

Definition 4.3. Let R be a domain. The **plus closure** of an ideal $I \subseteq R$ is the ideal $IR^+ \cap R$ where R^+ is any[12] absolute integral closure of R. We denote the plus closure of I by I^+.

[11] more generally, R and S need not be domains if S is torsion free over R

[12] The contracted ideal $IR^+ \cap R$ is independent of the choice of R^+.

Theorem 4.1 says that $I^+ \subseteq I^*$ for all ideals in a domain of prime characteristic. We have the following partial converse due to Smith:

Theorem 4.4 ([**Smi94**])**.** *Let R be an F-finite local domain of prime characteristic p. Then $I^* = I^+$ for any ideal I generated by h elements, where h is the height of I.*

Equivalently, Theorem 4.4 says that tight closure and plus closure are the same for *parameter ideals.*

Theorem 4.4 follows from Theorem 4.12 (see Exercise 4.7), whose proof is postponed to the next section.

Remark 4.5. It is *not* the case that $I^* = IR^+ \cap R$ for all ideals. Indeed, plus closure commutes with localization (Proposition 4.9 below), but unfortunately, as we have mentioned, tight closure *does not*, see Example 3.16.

Remark 4.6. Plus closures are most interesting in rings of positive residue characteristic. Indeed, $I = I^+$ for all ideals in any normal domain containing $\mathbb{Q}$; see Exercise 4.1.

Remark 4.7. Theorem 4.4 generalizes to ideals generated by monomials in any system of parameters or indeed, to any ideal I expanded from regular subring A over which R is finite. More generally, Ian Aberbach showed that $I^* = I^+$ for any ideal I in an excellent local domain R which admits a *phantom free resolution*—meaning a complex of free modules abutting to I with the property that the boundaries are contained in the tight closure of the cycles in the free module at each spot, and that the same holds after base changing with $F^e_* R$ for every $e > 0$, [**Abe94**], *cf.* [**Abe94**]; see **??**.

4.1. Plus closure for modules.

Similar to tight closure, we define plus closure of modules as follows:

Definition 4.8. Let $N \subseteq M$ be a pair of R-modules. The **plus closure** of N in M is the R submodule N^+_M of M of elements m such that $1 \otimes m$ is in the image of the natural map

$$R^+ \otimes_R N \longrightarrow R^+ \otimes_R M.$$

Equivalently,[13]

$$N^+_M = \ker\left(M \xrightarrow{m \mapsto 1 \otimes \overline{m}} R^+ \otimes_R M/N \right),$$

where $\overline{m}$ denotes the class of m modulo N.

In some ways, plus closures are better behaved than tight closures. For example, there is no subtlety involved for infinitely generated modules (Exercise 4.2), and plus closure always commutes with localization:

[13] See Exercise 4.3

Proposition 4.9. *Let R be an arbitrary domain, and let $N \subseteq M$ be a pair of R-modules. Then for any multiplicatively closed set W,*

$$W^{-1}N_M^+ = (W^{-1}N)^+_{W^{-1}M}$$

as subsets of $W^{-1}M$.

Proof. See Exercise 4.5. $\square$

Theorem 4.1 extends to arbitrary pairs of R-modules:

Theorem 4.10. *Let R be a domain of prime characteristic. For every pair of R-modules $N \subseteq M$, $N_M^+ \subseteq N_M^{*\mathrm{fg}}$.*

Proof. Take $m \in N_M^+$. Then $m \in (N \cap M')^+_{M'}$ where M' is finitely generated (by Exercise 4.2), so it suffices to show that $N_M^+ \subseteq N_M^*$ for finitely generated modules M. Furthermore, we may assume with out loss of generality $N = 0$ (by Exercises 1.7 and 4.3). Map a finitely generated free R-module F onto M to produce an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0.$$

If $m \in 0_M^+$, then any preimage of $m \in F$ is in

$$K_F^+ = \{f \in F \mid 1 \otimes f \in \mathrm{Image}(R^+ \otimes K \longrightarrow R^+ \otimes F)\}$$

by Exercise 4.4. Fixing one $f \in K_F^+$, write

$$1 \otimes f = s_1 \otimes f_1 + \cdots + s_n \otimes f_n \in S \otimes_R F \tag{4.10.1}$$

where the $f_i \in K \subseteq F$ and S is a module finite extension of R containing the elements $s_1, \ldots, s_n$. Remembering that F is a free R-module, note that $S \otimes_R F$ is a free S-module, so we can apply Frobenius in each coordinate to get

$$1 \otimes f^{p^e} = s_1^{p^e} \otimes f_1^{p^e} + \cdots + s_n^{p^e} \otimes f_n^{p^e} \in S \otimes_R F. \tag{4.10.2}$$

As in the proof of Theorem 4.1, now there is some $\phi \in \mathrm{Hom}_R(S, R)$ such that $\phi(1_S) = c \neq 0$ (tensor with the fraction field of R, then clear denominators). The map ϕ induces a coordinate-wise map

$$S \otimes_R F \xrightarrow{\Phi} F$$

by applying ϕ to the first factor. Applying Φ to the element $1 \otimes f$ in (4.10.2), we see that

$$cf^{p^e} = K_F^{[p^e]}$$

for all $e \in \mathbb{N}$, so $f \in K_F^*$, which means that its image m is in 0_M^* by Exercise 1.8. $\square$

Theorem 4.10 raises the question:

Question 4.11. Let R be a Noetherian domain of prime characteristic. For what pairs of R-modules $N \subseteq M$ is $N_M^+ = N_M^*$?

It is *not* the case that $N_M^+ = N_M^*$ for all module pairs, or even that $I^+ = I^*$ for all ideals Remark 4.5. But one important case is essentially equivalent to Theorem 4.4:

Theorem 4.12 ([**Smi94, BST11b**])**.** *Let $(R, \mathfrak{m})$ be an F-finite*[14] *Noetherian local domain of dimension d. Then*

$$0^+_{H^d_\mathfrak{m}(R)} = 0^*_{H^d_\mathfrak{m}(R)}$$

which then necessarily equals $0^{*\,\mathrm{fg}}_{H^d_\mathfrak{m}(R)}$.

In fact, there exists a single finite extension $R \subseteq T$ such that

$$0^*_{H^d_\mathfrak{m}(R)} = \ker\left(H^d_\mathfrak{m}(R) \longrightarrow H^d_\mathfrak{m}(T)\right).$$

The proof of Theorem 4.12 is postponed until the Section 5.

4.2. Splinters.

Definition 4.13. An integral domain R is said to be a **splinter** (or **+-regular**) if it is a direct summand, as an R-module, of every module-finite extension ring.

Example 4.14. A domain containing $\mathbb{Q}$ is a splinter if and only if it is normal; see Exercise 4.1.

Example 4.15. The direct summand theorem (due to Hochster in characteristic $p > 0$ and André in mixed characteristic) says that all regular rings are splinters [**Hoc73c, And18, Bha18**].

Remark 4.16. One easily checks that splinters are preserved by localization (Exercise 4.16) and whether or not a Noetherian ring R is a splinter can be checked locally at maximal ideals. A number of basic properties of splinters are proved in [**DT23a**]: for example, splinters are preserved by étale extensions [**DT23a**, Thm A], a Noetherian local domain is a splinter if and only if its Henselization is a splinter [**DT23a**, Thm B], and in the excellent setting, also if and only if its completion is a splinter [**DT23a**, Thm C].

Proposition 4.17. *Let R be an excellent*[15] *(e.g. F-finite) Noetherian domain. The following are equivalent*

[14] F-finite is not needed if R has a tight closure test element that is a tight closure test element for $\widehat{R}$–for example, if R is excellent.

[15] Excellence is not necessary; what we need is that every localization of R at a maximal ideal is approximately Gorenstein, and this is used only to get that (c) implies (a) and (b). Since normal rings are approximately Gorenstein [**Hoc77**, Thm 1.6] or Appendix A Lemma 11.3, the excellence hypothesis can be replaced by normality.

(a) *R is a splinter;*
(b) $0^+_E = 0$ *where E is the direct sum, over all maximal ideals $\mathfrak{m}$ of R, of the injective hulls of the residue fields of the local rings $R_{\mathfrak{m}}$;*
(c) $I^+ = I$ *for all ideals.*

Proof. The proof reduces easily to the local case (see Exercise 4.8), so we assume that $(R, \mathfrak{m})$ is local and E is an injective hull of its residue field.

Recall that a finite map $R \hookrightarrow S$ splits if and only if, after tensoring with E, the map

$$E \longrightarrow S \otimes_R E$$

is injective (see Appendix A Lemma 2.4 or Lemma 7.22 in Chapter 1). Since R^+ is a direct limit of all module finite extensions of R contained in a fixed algebraic closure of its total ring of fractions, we see that splitting for *all* module finite S is the same as requiring that

$$\ker\left(E \xrightarrow{\xi \mapsto 1\otimes\xi} R^+ \otimes_R E\right) = 0.$$

So (a) and (b) are equivalent.

Clearly, (a) implies (c) for all I, since a split map

$$R \to S$$

remains split after tensoring with R/I, so $IS \cap R = I$ for any ideal of R. For the reverse implication, we observe that R is approximately Gorenstein, and so cyclic purity of the extension $R \hookrightarrow R^+$ is equivalent to purity [**Hoc77**, Thm 2.6], or see Appendix A Corollary 11.5. $\square$

Corollary 4.18 ([**HH94b**, Theorem 6.7])**.** *An excellent (e.g. F-finite) weakly F-regular ring is a splinter.*

Proof. This follows from the inclusion $0^+_E \subseteq 0^{*\,\mathrm{fg}}_E$. $\square$

Question 4.19. We now have established, for F-finite Noetherian local domains $(R, \mathfrak{m})$ of prime characteristic, that

$$0^+_E \subseteq 0^{*\,\mathrm{fg}}_E \subseteq 0^*_E \tag{4.19.1}$$

where E is an injective hull of the residue field of R. Does equality hold in (4.19.1)? In other words, is strong (or weak?) F-regularity equivalent to being a splinter for an F-finite Noetherian domain of prime characteristic?

4.3. Trace maps and trace ideals. Recall that if $R \longrightarrow S$ is any map of rings, the **trace** of S/R is the natural R-module map

$$\operatorname{Hom}_R(S, R) \xrightarrow{\phi \mapsto \phi(1_S)} R \tag{4.19.2}$$

given by evaluation at 1_S. Clearly, the trace of S/R is surjective if and only if $R \longrightarrow S$ splits as a map of R-modules. This motivates the following definition:

Definition 4.20. The **trace ideal** of a map of rings $R \longrightarrow S$ is the image $\tau_{S/R}$ of the trace map (4.19.2).

Clearly $R \longrightarrow S$ splits if and only if $\tau_{S/R} = R$. Thinking about all module finite extensions, then, we make the following definition:

Definition 4.21. Let R be a domain. The **absolute trace ideal** for R is

$$\operatorname{tr}(R) := \bigcap_S \tau_{S/R} = \bigcap_S \operatorname{Image}\left(\operatorname{Hom}_R(S,R) \xrightarrow{\text{eval at } 1_S} R\right) \tag{4.21.1}$$

where the intersection is taken over all module finite extensions S of R contained in some fixed absolute integral closure.

It follows immediately that

Proposition 4.22. *The absolute trace ideal* $\operatorname{tr}(R)$ *of a domain R is trivial if and only if R is a splinter.*

We expect that the splinter locus is open in general. For domains containing $\mathbb{Q}$, the splinter locus is the same as the normal locus, so is open in geometric settings; see Exercise 4.12. In prime characteristic, this is also relatively easy in the F-finite case:

Corollary 4.23 (**[DT23b]**)**.** *Let R be an F-finite Noetherian domain of prime characteristic. Then*

$$\{P \in \operatorname{Spec} R \mid R_P \ \text{ is a splinter}\}$$

is open in $\operatorname{Spec} R$.

Corollary 4.23 follows immediately from next theorem, because the splinter locus is a subset of the Frobenius split locus (Exercise 4.15) and the Frobenius split locus is open (Proposition 3.17 in Chapter 1):

Theorem 4.24 (**[DT23b]**)**.** *Let R be an F-finite Noetherian domain of prime characteristic. Then the locus of splinter points*

$$\{P \in \operatorname{Spec} R \mid R_P \ \text{ is a splinter}\}$$

is open. If R is Frobenius split, its complement is defined by the trace ideal $\operatorname{tr}(R)$*. Furthermore, there exists a module finite extension T of R such that* $\operatorname{tr}(R) = \tau_{T/R}$.

The point of the proof is that trace ideals are *uniformly compatible* (Definition 6.10 in Chapter 1) in prime characteristic:

Proposition 4.25. [**DMS20**, Prop 8.5.1] *Let R be a ring of prime characteristic. Then for any R-algebra S, the trace ideal $\tau_{S/R}$ is uniformly compatible.*

Proof of Proposition 4.25. You may have already shown this in Chapter 1 Exercise 6.20 but we prove it here. Take arbitrary $\psi \in \operatorname{Hom}_R(F^e_* R, R)$. We need to show that $\psi(F^e_* \tau_{S/R}) \subseteq \tau_{S/R}$. For this, take arbitrary $y \in \tau_{S/R}$ and fix some $\phi \in \operatorname{Hom}_R(S, R)$ such that $\phi(1_S) = y$. To see that $\psi(F^e_* y) \in \tau_{S/R}$, we observe that $\psi(F^e_* y)$ is the image of the element $1_S \in S$ under the composition of R-linear maps

$$S \xrightarrow{F^e} F^e_* S \xrightarrow{F^e_* \phi} F^e_* R \xrightarrow{\phi} R.$$

We conclude that $\psi(F^e_* y) \in \operatorname{Image}\left(\operatorname{Hom}_R(S, R) \xrightarrow{\text{eval at } 1_S} R\right) = \tau_{S/R}$. $\square$

Proof of Theorem 4.24. Since the splinter locus is a subset of the Frobenius split locus which we know is open from Chapter 1, it suffices to assume for the entire problem that R is Frobenius split.

Furthermore, because $\operatorname{tr}(R) = R$ if and only if R is a splinter, it suffices to show that $\operatorname{tr}(R_P) = \operatorname{tr}(R) R_P$. Note that the trace ideal of a module finite extension $R \hookrightarrow S$ satisfies

$$\begin{aligned}
\tau_{S/R} R_P &= R_P \otimes_R \operatorname{Image}\left(\operatorname{Hom}_R(S, R) \xrightarrow{\phi \mapsto \phi(1_S)} R\right) \\
&= \operatorname{Image}\left(R_P \otimes_R \operatorname{Hom}_R(S, R) \xrightarrow{\phi \mapsto \phi(1_S)} R_P\right) \\
&= \operatorname{Image}\left(\operatorname{Hom}_{R_P}(R_P \otimes_R S, R_P) \xrightarrow{\phi \mapsto \phi(1)} R_P\right) = \tau_{S_P/R_P}.
\end{aligned}$$

For a Noetherian F-finite Frobenius split ring, there are only *finitely many* uniformly compatible ideals (Theorem 6.8 in Chapter 5). So by Proposition 4.25, the intersection (4.30.1) defining $\operatorname{tr}(R)$ is actually a *finite intersection.* Of course, $\operatorname{tr}(R_P)$ is also a finite intersection of ideals of the form τ_{T/R_P} where T is a module finite extension of R_P. Since every module finite extension of R_P is the localization of some module finite extension of R, and since localization commutes with *finite* intersection, we have

$$\operatorname{tr}(R) R_P = \bigcap_S \tau_{S/R} R_P = \bigcap_{S_P} \tau_{S_P/R_P} = \operatorname{tr}(R_P) \tag{4.25.1}$$

where S ranges over a *finite set* of finite integral extensions of R contained in some fixed R^+. For the final sentence, let S be any module finite extension of R containing each of the finitely many extensions appearing in (4.25.1). Then $tr_R = \tau_{S/R}$ by Exercise 4.17. $\square$

In Theorem 5.10 in the next section, for R $\mathbb{Q}$-Gorenstein, we will also prove that $\operatorname{tr}(R) = \tau_{S/R}$ for a single $R \subseteq S$ (in fact, we will show also that $\tau(R) = \tau_{S/R}$ for some S).

Remark 4.26. Without loss of generality, we may assume the extension T in Theorem 4.24 is generically étale—meaning the corresponding extension of fraction fields is separable— by [**Sin99c**].

Remark 4.27. Theorem 4.24 is proved more generally in [**DT23b**, Thm 1.0.1], for schemes locally essentially of finite type over a Noetherian local ring of prime characteristic with geometrically regular formal fibers.

4.4. The plus test ideal. Naturally, it is worth defining the *plus closure test ideal* in analogy with the tight closure ideal:

Definition 4.28. Let R be a Noetherian domain. The **plus closure test ideal**, denoted $\tau_+(R)$, is $\operatorname{ann}_R 0^+_E$, where E is the direct sum, over all maximal ideals $\mathfrak{m}$ of R, of the injective hulls of the residue fields of R at each maximal ideal $\mathfrak{m}$.

Equivalently, the plus closure test ideal looks very much like Definition 2.6 of tight closure test ideals:

Proposition 4.29. *Let R be an Noetherian domain. Then*

$$\tau_+(R) = \bigcap_{N \subseteq M} N :_R N^+_M$$

where the intersection is taken over all pairs $N \subseteq M$ of R-modules.

Proof. Clearly $\bigcap_{N \subseteq M} N :_R N^+_M \subseteq \operatorname{ann}_R 0^+_E$, as $\{0\} \subseteq E$ is one pair of R-modules. For the reverse inclusion, fix $c \in \operatorname{ann}_R 0^+_E$. If there is a pair $N \subseteq M$ of R-modules such that $c\, N^+_M \backslash N$, then without loss of generality, we may assume N and M are finitely generated R-modules. Take $x \in N^+_M \setminus N$. Replacing N be a submodule N' containing N and maximal with respect to not containing x, we may assume that M/N is supported at one maximal ideal $\mathfrak{m}$ as in the proof of Proposition 3.4, and that the class $\overline{x}$ of x is in the plus closure of zero in M/N. Now M/N embeds in a finite direct sum of copies of an injective hull of the residue field at $\mathfrak{m}$. So in fact $c\overline{x} = 0$, and $c \in N^+_M :_R N$. $\square$

Because the trace map for a finite extension $R \hookrightarrow S$ is Matlis dual to the natural map $E \longrightarrow S \otimes_R E$, the plus closure test ideal is simply the absolute trace ideal (Definition 4.21):

Theorem 4.30. *Let R be a Noetherian domain. Then the plus test ideal is equal to the absolute trace ideal—that is,*

$$\tau_+(R) = \operatorname{tr}(R). \tag{4.30.1}$$

Theorem 4.30 follows easily from the following simple consequence of Lemma 2.16:

Proposition 4.31. *Suppose $(R, \mathfrak{m})$ is a Noetherian local domain, and let E be an injective hull of its residue field. If $R \hookrightarrow S$ is a finite extension, then*

$$\tau_{S/R} = \operatorname{Ann}_R\left(\ker\left(E \longrightarrow S \otimes_R E\right)\right) = \left(\operatorname{Image}\left(E \longrightarrow S \otimes_R E\right)\right)^{\vee} \cap R$$

Proof. Consider the exact sequence

$$\operatorname{Hom}_R(S, R) \xrightarrow{\text{eval at } 1_S} R \longrightarrow R/\tau_{S/R} \longrightarrow 0. \tag{4.31.1}$$

Let $\widehat{R}$ be the completion of R at its maximal ideal, and note that the map $\widehat{R} \longrightarrow \widehat{R} \otimes_R S := \widehat{S}$ is finite. Now applying the functor $- \otimes_R \widehat{R}$, we have an exact sequence

$$\operatorname{Hom}_{\widehat{R}}(\widehat{S}, \widehat{R}) \xrightarrow{\text{eval at } 1_{\widehat{S}}} \widehat{R} \longrightarrow \widehat{R}/\tau_{S/R}\widehat{R} \longrightarrow 0. \tag{4.31.2}$$

In particular, we have an equality of ideals $\tau_{S/R}\widehat{R} = \tau_{\widehat{S}/\widehat{R}}$.

Applying Matlis duality—either over R to (4.31.1) or over $\widehat{R}$ to (4.31.2)—produces the same exact sequence

$$0 \longrightarrow (R/\tau_{S/R})^{\vee} \longrightarrow E \xrightarrow{\xi \mapsto 1_S \otimes_R \xi} S \otimes_R E,$$

by Lemma 2.16. Denoting the kernel of the natural map $E \longrightarrow S \otimes_R E$ by 0_E^S, it follows that

$$0_E^S = \left(\frac{\widehat{R}}{\tau_{S/R}\widehat{R}}\right)^{\vee}.$$

So $\operatorname{Ann}_{\widehat{R}}(0_E^S) = \tau_{S/R}\widehat{R}$ by Lemma 2.17. Finally,

$$\operatorname{Ann}_R(0_E^S) = \left(\operatorname{Ann}_{\widehat{R}}(0_E^S)\right) \cap R = \tau_{S/R}\widehat{R} \cap R = \tau_{S/R},$$

as desired. □

Proof of Theorem 4.30. We may assume that R is local and E is the injective hull of the residue field; see Exercise 4.9. To simplify notation, let 0_E^S denote the kernel of the natural map $E \longrightarrow S \otimes_R E$ obtained by tensoring the extension $R \hookrightarrow S$ with E.

For any finite extension $R \subseteq S \subseteq R^+$ we have $0_E^S \subseteq 0_E^+$, so the reverse containment holds for their annihilators. Thus Proposition 4.31 implies that

$$\tau_+(R) \subseteq \bigcap_{S \supseteq R} \tau_{S/R} = \operatorname{tr}(R).$$

For the reverse containment, note that by definition

$$0_E^+ = \bigcup_{S \supseteq R} 0_E^S$$

where the union is taken over all finite extensions S contained in R^+. Hence

$$\tau_+(R) = \operatorname{Ann}_R 0_E^+ = \operatorname{Ann}_R \Big(\bigcup_{S \supseteq R} 0_E^S \Big) = \bigcap_{S \supseteq R} \operatorname{Ann}_R 0_E^S = \bigcap_{S \supseteq R} \tau_{S/R} = \operatorname{tr}(R),$$

as desired. $\square$

Question 4.19 can be now rephrased in terms of test ideals:

Question 4.32. For a F-finite Noetherian domain, we have

$$\tau(R) \subseteq \tau_{\mathrm{tc}}^{\mathrm{fg}}(R) \subseteq \tau_+(R) = \operatorname{tr}(R).$$

Does equality hold? In particular, is every splinter strongly F-regular?

Remark 4.33. These are all equal in the (quasi-)Gorenstein case: in that case $E = H^d_{\mathfrak{m}}(R)$; see Exercise 4.19. In Theorem 5.10 we will show equality when R is $\mathbb{Q}$-Gorenstein, *cf.* [**Sin99a**]. In general Question 4.32 is known to hold when the anti-canonical symbolic Rees algebra

$$R \oplus \omega_R^{(-1)} \oplus \omega_R^{(-2)} \oplus \cdots = R \oplus R(-K_R) \oplus R(-2K_R) \oplus \ldots$$

is Noetherian by [**CEMS18**]. However, we do not even know such finite generation for splinters in dimension ≥ 3 (in dimension 2, it follows from [**Lip69**] once one observes that splinters are F-rational and hence pseudo-rational).

If one knew that for splinters, there existed a $\mathbb{Q}$-divisor $\Delta \geq 0$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier and (R, Δ) is Kawamata log terminal (in analogy with Chapter 5 Corollary 3.21), then one could apply the minimal model program ([**HX15, Bir16, DW19**]) to deduce the finite generation at least for dimension 3 and $p > 5$.

4.5. Exercises.

Exercise 4.1. Let R be a domain containing a field of characteristic zero. Prove that $I^+ = IS \cap R$ where S is the normalization of R.

Hint: Use the field trace to show that if S is a normal domain and T is any module finite extension of S, then the inclusion $S \hookrightarrow T$ splits as an S-module map.

Exercise 4.2. Let $N \subseteq M$ be modules over an arbitrary domain R. Show that $N_M^+ = N_M^{+fg}$ where N_M^{+fg} is defined as

$$\bigcup_{M'} (N \cap M')^+_{M'}$$

where the intersection is taken over all finitely generated submodules M' of M.

Exercise 4.3. Fix a domain R and a pair of R-modules $N \subseteq M$. Prove that N_M^+ is the preimage, under the natural quotient map $M \to M/N$, of the R-module $0^+_{M/N}$.

Exercise 4.4. Let M be a module over an arbitrary domain R. Write M as a quotient of a free module, with module of relations K—that is, suppose that

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

is an exact sequence of R modules, with F a free. Prove that

$$0^+_M = \pi\left(K^+_F\right).$$

Exercise 4.5. Let R be a domain, and let $W \subseteq R$ be any multiplicatively closed set. Show that for any ideal I,

$$W^{-1}I^+ = (W^{-1}I)^+,$$

or more generally, for R-modules $N \subseteq M$, $N_M^+ \otimes_R W^{-1}R = (W^{-1}N)^+_{W^{-1}M}$

Exercise 4.6. Let R be a Noetherian F-finite domain. Suppose that all ideals generated by systems of parameters are plus closed. Prove that all ideals generated by subsets of systems of parameters are plus closed.

Exercise 4.7. Let R be an F-finite local domain of prime characteristic p with system of parameters $x_1, \dots, x_d$. Show that an element $\eta = [z + (x_1, \dots, x_d)] \in H^d_{\mathfrak{m}}(R)$ is in the plus closure of zero if and only if $z \in (x_1, \dots, x_d)^+$. Deduce that $0^+_{H^d_{\mathfrak{m}}(R)} = 0^*_{H^d_{\mathfrak{m}}(R)}$ if and only if $I^* = I^+$ for all parameter ideals.

Hint: See Exercise 2.8.

Exercise 4.8. Let R be a Noetherian ring, and let $E = \oplus_{\mathfrak{m} \in \text{m-Spec} R} E_R(R/\mathfrak{m})$, where $E_R(R/\mathfrak{m})$ is the injective hull of the R-module $R/\mathfrak{m}$ and the sum is taken over all maximal ideals of R. Show that $R_{\mathfrak{m}} \otimes_R E \cong E_{R_{\mathfrak{m}}}(R/\mathfrak{m}) = E_R(R/\mathfrak{m})$.

Exercise 4.9. Let R be a Noetherian ring and let $R \hookrightarrow S$ be a finite extension. For any prime ideal $P \in \operatorname{Spec} R$, prove that $\tau_{S/R} R_P = \tau_{S_P/R_P}$. Use this and Exercise 4.8 to show that the proof of Proposition 4.17 reduces immediately to the local case.

Exercise 4.10. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Assume that $c \in R$ is a tight closure test element, and its image in the $\mathfrak{m}$-adic completion $\widehat{R}$ is a tight closure test element for $\widehat{R}$. Prove that colon capturing hold for R. That is, prove Theorem 2.23 holds for R.

Exercise 4.11. Prove that a splinter is normal.

Exercise 4.12. An integral domain containing $\mathbb{Q}$ is a splinter if and only if it is normal.

Hint: See Exercise 4.1 or use the trace map $\mathrm{Tr} : S \longrightarrow R$ coming from the field extension $\mathcal{K}(R) \subseteq \mathcal{K}(S)$.

Exercise 4.13. Let R be any domain. Show that R is a splinter if and only if all module finite extensions $R \hookrightarrow S$ with S a *domain* are split.

Exercise 4.14. Let $R \hookrightarrow S$ be an inclusion of domains, split as a map of R-modules. Prove that if S is a splinter, then R is a splinter. More generally, prove that any domain of positive characteristic which is pure in a splinter is a splinter.

Exercise 4.15. Let R be an F-finite domain of prime characteristic. Prove that if R is a splinter, then R is Frobenius split.

Exercise 4.16. Prove that any localization of a splinter is a splinter.

Exercise 4.17. Suppose that $R \hookrightarrow S \hookrightarrow T$ are module finite extensions. Prove that $\tau_{T/R} \subseteq \tau_{S/R}$.

Exercise 4.18. Let $(R, \mathfrak{m})$ be a normal local domain, and suppose Δ is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Fix $n > 0$ the smallest positive integer such that $R(n\Delta) \cong R$, with $n \in \mathbb{N}$ minimal. Let $R \hookrightarrow S$ be the extension associated to the associated finite extension of R (see Definition 9.1 in Appendix B). Prove that if R is a splinter, then so is S.

Hint: Let $\pi \in \mathrm{Hom}_R(S, R)$ be the splitting given by projection onto the degree zero part of S. Show that π is an S-module generator for $\mathrm{Hom}_R(S, R)$, and that, by adjointness of tensor and Hom, for any finite integral extension T of S, all $\Psi \in \mathrm{Hom}_R(T, R)$ must factor through π.

Exercise 4.19. Show that for a quasi-Gorenstein F-finite domain, $\tau_{\mathrm{tc}}^{\mathrm{fg}} \subseteq \tau_+(R)$.

Hint: For a quasi-Gorenstein local ring, $\mathrm{tr}(R)$ is equal to $\tau_{S/R}$ for one fixed finite extension S of R.

5. Cohen-Macaulay properties of R^+

The colon capturing property of tight closure, together with the fact that $I^* = I^+$ for parameter ideals, raises the question as to whether all relations on systems of parameters are trivial in R^+. In fact, Hochster and Huneke proved that they are:

Theorem 5.1 ([**HH92**])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local domain of prime characteristic that admits a dualizing complex.*[16] *Then*

$$H^i_{\mathfrak{m}}(R^+) = 0$$

for all $i < \dim R$. As a consequence, any system of parameters for R is a regular sequence on R^+—that is, R^+ is a big Cohen-Macaulay algebra for R.

The "big" refers to the fact that R^+ is *not* a finitely generated R-module (or algebra). It is not known whether there exists a finitely generated R-module that is Cohen-Macaulay.

The relationship between the vanishing of $H^i_{\mathfrak{m}}(R^+)$ and relations on parameters for R will be discussed in Subsection 5.2.

Remark 5.2. Theorem 5.1 is false for rings containing a field of characteristic zero; see Exercise 4.1. However, in *mixed characteristic,* Bhargav Bhatt recently proved that if $(R, \mathfrak{m})$ is a *mixed characteristic* domain with residue field of prime characteristic, then the p-adic completion of R^+ is a big Cohen-Macaulay algebra [**Bha20**].

Corollary 5.3. *Let $(R, \mathfrak{m})$ be a Noetherian local domain of prime characteristic that admits a dualizing complex*[17]*. If R is a splinter, then R is Cohen-Macaulay.*

The same result also holds in mixed characteristic by [**Bha20**].

With the benefit of hindsight, Theorem 5.1 is now reasonably straightforward to prove, using the following:

Lemma 5.4 (The equational lemma, [**HL07**, Lem 2.2], [**Smi94**, Lem 5.2])**.** *Let $J \subseteq R$ be an ideal in a Noetherian domain R of prime characteristic. Suppose that for some $i \in \mathbb{N}$ and some $\eta \in H^i_J(R)$, the R-submodule generated by*

$$\eta, \eta^p, \eta^{p^2}, \eta^{p^3}, \ldots$$

---

[16]Recall that every F-finite local domain admits a dualizing complex by [**Gab04**]; see Chapter 2 Section 4. Even "admits a dualizing complex" is not needed if R is excellent [**HH92**].

[17]Or is excellent

is finitely generated. Then there exists a finite domain extension S of R such that η maps to zero under the natural map $H^i_J(R) \longrightarrow H^i_J(S)$ induced by the inclusion $R \hookrightarrow S$.

As a consequence, if $M \subseteq H^i_J(R)$ is an R-submodule of finite length such that $F(N) \subseteq N$, then there exists a finite integral extension S of R such that $H^i_J(R) \longrightarrow H^i_J(S)$ sends M to zero.

Theorem Theorem 5.1 follows from the following stronger result:

Theorem 5.5 (*cf.* **[HL07]**). *Suppose R is a Noetherian domain with a dualizing complex*[18] *$\omega^{\bullet}_R$ so that $\mathcal{H}^i(\omega^{\bullet}_R) = 0$ for all $i < -d$ where $d = \dim R$. Then there exists a finite extension $R \subseteq T$ such that*

$$\mathcal{H}^i\omega^{\bullet}_T \longrightarrow \mathcal{H}^i\omega^{\bullet}_R$$

is the zero map for all $i > -d$. Specializing to the case that $(R, \mathfrak{m})$ is local, we have that

$$H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(T)$$

is the zero map for all $i < d$.

This result can be generalize to the non-domain case, see Exercise 5.3.

Proof of Theorem 5.5 assuming Lemma 5.4. It suffices to do this for a single i as, given any *finite* set of finite extensions of R contained on R^+, there is one finite extension S containing them all.

Suppose we have a finite extension $R \subseteq S$ of domains such that

$$\mathcal{H}^{-i}\omega^{\bullet}_S \twoheadrightarrow N_S \hookrightarrow \mathcal{H}^{-i}\omega^{\bullet}_R$$

Notice that N_S is a finitely generated R-module since it is an image of a finitely generated R-module. We proceed by Noetherian induction on the closed set $\mathrm{Supp}(N_S)$. If $\mathrm{Supp}(N_S) = \emptyset$, we are done.

Let η denote a generic point of $\mathrm{Supp}(N_S)$. Localizing at η, the module $(N_S)_\eta$ over $R_{\mathfrak{m}_\eta}$ has support at the maximal ideal $\mathfrak{m}_\eta := \eta R_\eta$. Shifting the dualizing complex to make it normalized and applying Matlis duality, we have for some integer $j < \dim R_\eta$:

$$H^j_{\mathfrak{m}_\eta}(R_\eta) \twoheadrightarrow ((N_S)_\eta)^\vee \hookrightarrow H^j_{\mathfrak{m}_\eta}(S_\eta).$$

Set $M_S := ((N_S)_\eta)^\vee$. Notice that M_S is finite length (as it is the Matlis-dual of a finite length module). Furthermore, it is Frobenius stable as a submodule of $H^j_{\mathfrak{m}_\eta}(S_\eta)$ since M_S is simply the image of $H^j_{\mathfrak{m}_\eta}(R_\eta) \longrightarrow H^j_{\mathfrak{m}_\eta}(S_\eta)$ and

[18] For example, recall that F-finite domains always admit a dualizing complex; see Chapter 2.

Frobenius acts compatibly on the source and image of that map. Applying Lemma 5.4, there exists a finite extension domain S' of S_η, and hence of R_η, such that $H^j_{\mathfrak{m}_\eta}(S_\eta) \longrightarrow H^j_{\mathfrak{m}_\eta}(S'_\eta)$ sends M_S to zero. Thus

$$H^j_{\mathfrak{m}_\eta}(R_\eta) \longrightarrow H^j_{\mathfrak{m}_\eta}(S_\eta) \longrightarrow H^j_{\mathfrak{m}_\eta}(S'_\eta) \tag{5.5.1}$$

is the zero map.

Now, S'_η is obtained from S_η by formally adjoining finitely many variables, modding about one monic polynomials in each of those variables, and finally modding out by a minimal prime if necessary. By clearing denominators, we may assume those monic polynomials have coefficients in S and hence we can assume that S'_η is the localization of some finite domain extension $S' \supseteq S$.

Now consider the map

$$\mathcal{H}^{-i}\omega_{S'_\eta} \longrightarrow \mathcal{H}^{-i}\omega^{\bullet}_{S_\eta} \longrightarrow \mathcal{H}^{-i}\omega^{\bullet}_{R}. \tag{5.5.2}$$

After localizing at η, this composition is zero since it is Matlis dual to (5.5.1). Hence (5.5.2) has image $N_{S'}$ such that

$$\operatorname{Supp}(N_{S'}) \subsetneq \operatorname{Supp}(N_S).$$

The result then follows by Noetherian induction. $\square$

Remark 5.6. In [**SS12**], the Sannai and Singh showed a version of the above equational lemma even when one restricts to extensions $R \subseteq S$ which are generically Galois extensions with solvable $\operatorname{Gal}(\mathcal{K}(S)/\mathcal{K}(R))$.

Proof of Theorem 5.1. The fact that $H^i_{\mathfrak{m}}(R^+) = 0$ follows immediately from Theorem 5.5 and the fact that local cohomology commutes with direct limits. That is, limiting over finite extension S of R with $R \subseteq S \subseteq R^+$ we have:

$$H^i_{\mathfrak{m}}(R^+) = H^i_{\mathfrak{m}}(\varinjlim_{S \supseteq R} S) = \varinjlim_{S \supseteq R} H^i_{\mathfrak{m}}(S).$$

The fact that the transition maps are eventually zero implies $H^i_{\mathfrak{m}}(R^+) = 0$ for $i < \dim R$.

For the statement about R^+ being a big Cohen-Macaulay algebra, we use fact that the formation of R^+ commutes with localization and apply Lemma 5.21 below. $\square$

Proof sketch for Lemma 5.4. Our finite generation assumption implies that there is some $n \in \mathbb{N}$ such that η^{p^n} is in the R-submodule spanned by $\eta, \eta^p, \ldots \eta^{p^{n-1}}$. Write

$$\eta^{p^n} = r_0\eta + r_1\eta^p + \cdots r_{n-1}\eta^{p^{n-1}}$$

where $r_0, r_1, \ldots, r_{n-1} \in R$, and then consider the corresponding monic polynomial

$$g(t) = t^{p^n} - r_{n-1}t^{p^{n-1}} \cdots - r_1 t^p - r_0 t$$

in $R[t]$. Note that $g(\eta) = 0$ in $H^i_J(R)$. Let β be a cocycle representing η in the Čech complex on a set of generators for J. Because $g(\eta) = 0$, there is some coboundary $d\gamma$ such that $g(\beta) = d\gamma$. Thinking explicitly in the Čech complex, the components of this boundary produce a finite set of monic polynomials $\{g_i\}$ in $R[t]$; see Exercise 5.1.

Let S be the module finite extension of R obtained by adding all roots of all the g_i. We claim that η maps to zero under the natural map $H^i_J(R) \longrightarrow H^i_J(S)$. We leave the verification of this fact as an exercise; or see [**HL07**, Lem 2.2]. □

5.1. Tight closure and plus closure in $H^d_{\mathfrak{m}}(R)$. In the special case where R is a local ring, $J = \mathfrak{m}$ and $i = \dim R$, the Equational Lemma, Lemma 5.4, is also a key step in the proof of Theorem 4.12 that

$$0^*_{H^d_{\mathfrak{m}}(R)} = 0^+_{H^d_{\mathfrak{m}}(R)}$$

in the top local cohomology module $H^d_{\mathfrak{m}}(R)$ of an excellent local domain. We will actually prove a more general version when R is not necessarily local (but now F-finite).

Theorem 5.7. *Suppose R is an F-finite domain. Then there exists a finite domain extension $R \subseteq T$ such that the induced map $\omega_T \longrightarrow \omega_R$ has image $\tau(\omega_R)$*

$$\tau(\omega_R) = \mathrm{Image}\left(\omega_T \longrightarrow \omega_R\right)$$

and hence all sufficiently large T have this same property.

Specializing to the case that R is local, by Matlis duality, we see that $0^*_{H^d_{\mathfrak{m}}(R)} = \ker\left(H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(T)\right) = 0^+_{H^d_{\mathfrak{m}}(R)}$.

Proof. The proof is closely related to that of Theorem 5.5. We begin by considering the composition map:

$$\omega_S \longrightarrow \omega_R \longrightarrow \omega_R/\tau(\omega_R)$$

If we can find S large enough so that map is the zero map, we are done. Consider the image

$$\omega_S \twoheadrightarrow N_S \hookrightarrow \omega_R/\tau(\omega_R).$$

Our goal again is to show that we can choose S such that $\mathrm{Supp}(N_S) = \emptyset$. By Noetherian induction as before, the key computation is when R is local and N_S has finite length, so this time, for brevity, we assume we are in that case.

By local duality, our composition above Matlis dualizes to

$$\big(\omega_R/\tau(\omega_R)\big)^\vee = 0^*_{H^d_{\mathfrak{m}}(R)} \twoheadrightarrow (N_S)^\vee \hookrightarrow H^d_{\mathfrak{m}}(S)$$

We see that $(N_S)^\vee$ then has finite length and is sent to itself by F (since $0^*_{H^d_{\mathfrak{m}}(R)}$ is). Thus there exists $S \longrightarrow T$ such that the image of $(N_S)^\vee$ in $H^d_{\mathfrak{m}}(T) = 0$. The result follows. □

Remark 5.8. The hypothesis that R is a domain can also be weakened to R is reduced, locally equidimensional and with connected Spec. The details are left to the reader in Exercise 5.4

The assumption that R is F-finite in Theorem 5.7 can be weakened. Indeed, one can instead assume that there exists a submodule $\tau(\omega_R) \subseteq \omega_R$ whose formation commutes with localization and completion and such that for each $Q \in \operatorname{Spec} R$, we have that

$$\tau(\omega_R) = \operatorname{Ann}_{\omega_R} 0^*_{H_Q^{\dim R_Q}(R_Q)}.$$

This holds for instance when $(R, \mathfrak{m})$ is excellent and local and has a dualizing complex, see [**Smi95**]. The proof of Theorem 5.7 is then unchanged.

We immediately obtain the following corollary (in view of Remark 5.8, we can weaken the F-finite hypothesis to excellent if R is local).

Corollary 5.9. *Suppose $(R, \mathfrak{m})$ is an F-finite Noetherian domain. If R is quasi-Gorenstein, then since $\omega_R \cong R$ or dually, in the local case $E = H^d_{\mathfrak{m}}(R)$, we have that*

$$\tau_+(R) = \tau(R) \quad \textit{or dually, in the local case } 0^+_E = 0^*_E.$$

In fact, there exists a finite extension $R \subseteq S$ such that $\tau(R) = \tau_{S/R}$. In particular, every F-finite quasi-Gorenstein splinter is strongly F-regular.

We now generalize this to $\mathbb{Q}$-Gorenstein rings.

Theorem 5.10 (*cf.* [**Sin99a**])**.** *Suppose $(R, \mathfrak{m})$ is an F-finite Noetherian normal local domain of dimension d. If R is $\mathbb{Q}$-Gorenstein, then we have that*

$$\tau_+(R) = \tau(R) \quad \textit{or dually, in the local case} \quad 0^+_E = 0^*_E.$$

In fact, there exists a finite extension $R \subseteq S$ such that $\tau(R) = \tau_{S/R}$. In particular, every F-finite $\mathbb{Q}$-Gorenstein splinter is strongly F-regular.

Proof. This immediately reduces to the local case, and so we may assume that R is local. By Appendix B Lemma 9.5, there exists finite extension $R \subseteq S$ of normal domains such that, if $\pi_S : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is the induced map on Spec, that f^*K_R is Cartier on $\operatorname{Spec} S$. Thus, fixing a choice of $K_R \geq 0$, we can write $\pi_S^* K_R = \operatorname{div} g$ for some $g \in S$ since S is semi-local.

By Theorem 6.14 and Corollary 7.22 in Chapter 5, applied with $\Gamma = K_R$, we see that

$$\tau(R) = \tau(\omega_R, K_R) = T(\tau(\omega_S, \pi_S^* K_R)) = T(\tau(\omega_S, \operatorname{div} g)) = T(g\tau(\omega_S))$$

where $T : \omega_S \longrightarrow \omega_R$ is the Grothendieck dual of $R \longrightarrow S$. Note $R \subseteq R(K_R) = \omega_R$ since $K_R \geq 0$. If we can prove $\tau_+(R) = T(g\tau(\omega_S))$ then we are done.

Now, using Grothendieck duality for a finite map and the fact that the Hom-sets below are S_2 R-modules, we have

$$\begin{aligned} & \operatorname{Hom}_R(S, R) \\ \cong\ & \operatorname{Hom}_R(S \otimes \omega_R, \omega_R) \\ \cong\ & \operatorname{Hom}_S(S(\pi_S^* K_R), \omega_S) \\ =\ & \operatorname{Hom}_S(\tfrac{1}{g} S, \omega_S) \\ =\ & g\,\omega_S. \end{aligned}$$

Furthermore, we have the following commutative diagram:

$$\begin{tikzcd} g\,\omega_S = \operatorname{Hom}_R(S, R) \arrow[r, "T"] \arrow[d, hook] & R \arrow[d, hook] \\ \omega_S = \operatorname{Hom}_R(S, \omega_R) \arrow[r, "T"'] & \omega_R \end{tikzcd}$$

where T can be interpreted as evaluation-at-1.

By Theorem 5.7, we can choose $S' \supseteq S$ a finite extension of domains such that $\tau(\omega_S) = \operatorname{Image}\big(\omega_{S'} = \operatorname{Hom}_S(S', \omega_S) \longrightarrow \omega_S\big)$. Since we may work in R^+, for any $R \subseteq T'$, a finite extension of domains, we may find a further finite extension $T \supseteq T', S'$ and so we may compute

$$\tau_+(R) = \bigcap_{T \supseteq S'} \operatorname{Image}\Big(\operatorname{Hom}_R(T, R) \longrightarrow R\Big).$$

where the intersection runs over normal domains $T \subseteq R^+$.

As before for S, since $g \in S \subseteq T$ so that $\pi_T^* K_R = \operatorname{div}_{\operatorname{Spec} T} g$ we see that $\operatorname{Hom}_R(T, R) = g\,\omega_T$. Thus the composition

$$\operatorname{Hom}_R(T, R) \longrightarrow \operatorname{Hom}_R(S, R) \longrightarrow R$$

may be identified with

$$g\,\omega_T \longrightarrow g\,\omega_S \longrightarrow R.$$

However, $\tau(\omega_S) = \operatorname{Image}\big(\omega_T \longrightarrow \omega_S\big)$ since $T \supseteq S'$, and so we see that

$$\operatorname{Image}\Big(\operatorname{Hom}_R(T, R) \longrightarrow R\Big) = \operatorname{Image}(g\,\tau(\omega_S) \xrightarrow{T} \omega_R) = T(g\,\tau(\omega_S)).$$

Thus the intersection defining $\tau_+(R)$ is constant, and so

$$\tau_+(R) = T(g\,\tau(\omega_S)) = \tau(R)$$

as desired. $\square$

As a consequence of Theorem 5.10, since the formation of $\tau(R)$ commutes with completion ($\tau(R)\widehat{R} = \tau(\widehat{R})$) in F-finite local domains (Chapter 1 Proposition 6.17) we obtain the following.

Corollary 5.11. *If $(R, \mathfrak{m})$ is a F-finite normal $\mathbb{Q}$-Gorenstein domain then*

$$\tau_+(R)\widehat{R} = \widehat{\tau_+(R)} = \tau_+(\widehat{R}).$$

5.1.1. *An application to deformation of F-rationality.*

Theorem 5.12. *Suppose $(R, \mathfrak{m})$ is a Noetherian F-finite local ring and $f \in \mathfrak{m}$ is a non-zerodivisor such that $R/(f)$ is F-rational. Then R is also F-rational.*

Proof. Since $R/(f)$ is Cohen-Macaulay and normal, we see that R is as well. It thus suffices to show that $0 = 0^* = 0^+ \in H^d_\mathfrak{m}(R)$ where $d = \dim R$, using Theorem 4.12.

Since a normal local ring is a domain, both R and $R/(f)$ are domains. Let $(f)^+$ denote a prime ideal of R^+ lying over the prime ideal (f). It follows from Exercise 5.2 that we have the following map of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & (f) \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/(f) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (f)^+ \arrow[r] & R^+ \arrow[r] & (R/(f))^+ \arrow[r] & 0 \end{tikzcd}$$

Applying local cohomology and using that R and R^+ are (big) Cohen-Macaulay, the fact that $R/(f)$ is F-rational implies that the left vertical arrow ι below is injective

$$\begin{tikzcd} 0 \arrow[r] & H^{d-1}_\mathfrak{m}(R/(f)) \arrow[r] \arrow[d, "\iota"] & H^d_\mathfrak{m}((f)) \arrow[r, "\theta"] \arrow[d, "\kappa"] & H^d_\mathfrak{m}(R) \arrow[d] \\ 0 \arrow[r] & H^{d-1}_\mathfrak{m}((R/(f))^+) \arrow[r] & H^d_\mathfrak{m}((f)^+) \arrow[r] & H^d_\mathfrak{m}(R) \end{tikzcd}$$

Claim 5.13. $\kappa : H^d_\mathfrak{m}((f)) \longrightarrow H^d_\mathfrak{m}((f)^+)$ *injects.*

Proof of claim. Take $\eta \in H^d_\mathfrak{m}((f))$ in the socle. Since $H^d_\mathfrak{m}((f)) \cong H^d_\mathfrak{m}(R)$, we may view θ as multiplication by f and so, we see that $\theta(\eta) = 0$. In particular, we may identify η with an element of $H^{d-1}_\mathfrak{m}(R/(f))$ which we also call η. Since ι injects, by chasing the diagram we see that $\kappa(\eta) \neq 0$ and the claim is proven. $\square$

Finally, note that we have a factorization $(f) \longrightarrow fR^+ \longrightarrow (f)^+$ and so $H^d_{\mathfrak{m}}((f)) \longrightarrow H^d_{\mathfrak{m}}(fR^+)$ injects as well. Again, since f is a non-zerodivisor, we immediately see that

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(R^+)$$

injects as well, which is what we wanted to show. □

5.2. Big Cohen-Macaulay algebras and almost mathematics. In this section we present a brief sketch of the theory of Hochster's big Cohen-Macaulay modules and algebras [**Hoc75b**]. Other good (more thorough) sources for background on big Cohen-Macaulay algebras include [**Hoc07**], [**BH93**, Chapter 8], [**HH95**], and [**Die07**].

Definition 5.14 (Hochster, [**Hoc75b**])**.** Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d. We say that an R-module B (respectively R-algebra B) is a **balanced big Cohen-Macaulay module** (respectively **algebra**) if B satisfies the following two conditions:

(a) $\mathfrak{m}B \neq B$.
(b) Every system of parameters $x_1, \dots, x_d \in \mathfrak{m}$ acts as a weakly regular sequence on B. (if only *some* system of parameters acts as a regular sequence, the modifier **balanced** is removed)

We say that B is **weakly balanced big Cohen-Macaulay** if only if the second condition holds.

In the literature, most Cohen-Macaulay modules (or algebras) or required to be Noetherian. The modifier *big* above is meant to emphasize that B need not be Noetherian.

Remark 5.15. The condition (a) is called the **non-triviality condition**. Note, things like the 0-module and, if R is a domain, the fraction field $\mathcal{K}(R)$ of R, are *weakly* big Cohen-Macaulay.

Warning 5.16. The *non-triviality* condition does not necessarily localize well. Indeed, If $R = k[\![x, y]\!]/(xy)$ then the normalization $k[\![x]\!] \times k[\![y]\!] = R/(y) \times R/(x)$ is balanced big Cohen-Macaulay, but so is $S = R/(y) = k[\![x]\!]$. However, for localization at $Q = (x)$ we have that $S_Q = 0$. Hence S is a balanced big Cohen-Macaulay R-algebra but S_Q is only a *weakly* balanced big Cohen-Macaulay R_Q-algebra.

Remark 5.17. As mentioned above, B is called **(weakly) big Cohen-Macaulay** (with the *balanced* condition removed) if condition (a) is weakened to the condition that one system of parameters acts as a weakly regular sequence on B. However, if B is big Cohen-Macaulay, then $\widehat{B}$ is always *balanced* big Cohen-Macaulay by [**BS83**, Théorèm 1.7] or [**BH93**, Exercise 8.1.7 & Theorem 8.5.1].

Remark 5.18 (Existence of big Cohen-Macaulay algebras)**.** Given a Noetherian local ring $(R, \mathfrak{m})$ ring, it is not clear that a big Cohen-Macaulay module or algebra exists. It was shown by Hochster that big Cohen-Macaulay modules exists in characteristic $p > 0$ in [**Hoc75b**], also see [**Hoc73b, Hoc75a, Sha81**]. Big Cohen-Macaulay algebras were shown to exist in characteristic zero in $p > 0$ in [**HH92**]. In fact, in characteristic $p > 0$, Hochster and Huneke showed that R^+ is big Cohen-Macaulay, also see [**HH95**]. In mixed characteristic domains[19] dimension 3, building upon work of Heitmann [**Hei02**], Hochster proved that big Cohen-Macaulay algebras exist [**Hoc02**]. For the case of arbitrary dimension, using Scholze's theory of perfectoid spaces, André proved the existence of big Cohen-Macaulay algebras in [**And20**] (also see [**HM18**] and unpublished work of Gabber [**Gab18**]). Finally, in [**Bha20**], Bhatt showed in mixed characteristic that the p-adic completion (or $\mathfrak{m}$-adic completion) $\widehat{R^+}$ of R^+ is big Cohen-Macaulay.

An important strengthening of existence is the existence of **weakly functorial big Cohen-Macaulay algebra**. Meaning if $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{m})$ is a local map of Noetherian rings, does there exist big Cohen-Macaulay algebras B_R and B_S over R and S respectively, with a map between them $B_R \longrightarrow B_S$, such that the following diagram commutes.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ B_R \arrow[r] & B_S \end{tikzcd}$$

The existence of such big Cohen-Macaulay modules follows from the above in positive (or mixed) characteristic since R^+ (respectively $\widehat{R^+}$) or its completion is big Cohen-Macaulay, and they exist in characteristic 0 as well. However, this is not the first way the existence of *weakly functorial* big Cohen-Macaulay algebras was proven.

Example 5.19. If R is Cohen-Macaulay, it is a big Cohen-Macaulay algebra over itself. Since a direct limit of big Cohen-Macaulay algebras is easily seen to still be big Cohen-Macaulay, we immediately see that if R is Cohen-Macaulay, so is

$$R_{\mathrm{perf}} = \bigcup_{e \geq 0} R^{1/p^e}$$

even though R_{perf} is not Noetherian.

If B is an $(R, \mathfrak{m})$-module that is weakly (even not necessarily balanced) big Cohen-Macaulay with respect to a system of parameters $x_1, \ldots, x_d$, then

[19] In our case, this means that the ring has characteristic 0 but the residue field has characteristic $p > 0$

it is not difficult to see that

$$H^i_{\mathfrak{m}}(B) = 0 \text{ for all } i < d = \dim R, \tag{5.19.1}$$

the usual proof works. Furthermore, if B satisfies the nontriviality condition, then $B/(x_1, \ldots, x_d)B \neq 0$ and so using properties of regular sequences we see that

$$B/(x_1^t, \ldots, x_d^t) \xrightarrow{x_1 \cdots x_d} B/(x_1^{t+1}, \ldots, x_d^{t+1})$$

is injective for every t, see Appendix A Subsection 4.1. Thus the direct limit, which is isomorphic to $H^d_{\mathfrak{m}}(B)$, is nonzero. In particular, if B is big Cohen-Macaulay we see that in the derived category

$$\mathbf{R}\Gamma_{\mathfrak{m}}(B) \cong H^d_{\mathfrak{m}}(B)[d] \neq 0. \tag{5.19.2}$$

Conversely however, it is not obvious whether (5.19.1) implies that B is weakly (balanced) big Cohen-Macaulay. Fortunately, the $\mathfrak{m}$-adic completion of a B with $\mathfrak{m}B \neq B$ satisfying (5.19.1) is balanced big Cohen-Macaulay by [**BH93**, Exercise 8.1.7 & Theorem 8.5.1] (a fact we won't use).

Inspired by the local-cohomological criterion for Cohen-Macaulayness (5.19.1), Bhatt has made the following definition.

Definition 5.20 ([**Bha20**])**.** Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d. We say that an R-module (or algebra) B is **weakly cohomologically big Cohen-Macaulay** if

- for every $Q \in \operatorname{Spec} R$ and we have that
$$H^i_Q(B_Q) = 0$$
for $i < \dim R_Q$.

Lemma 5.21 ([**HL07**, Corollary 2.3(b)], [**Bha20**, Corollary 2.8])**.** *Suppose $(R, \mathfrak{m})$ is an excellent equidimensional local ring of dimension d, and B is an R-algebra. Then B is weakly balanced big Cohen-Macaulay if and only if B is weakly cohomologically big Cohen-Macaulay. As a consequence if B is weakly balanced big Cohen-Macaulay over R, then so is any localization B_Q over R_Q.*

Proof. We first show that if B is weakly balanced big Cohen-Macaulay, then in any localization, there is a system of parameters that is a regular sequence on B (in other words, B is (potentially unbalanced) big Cohen-Macaulay as well). This will immediately imply that B is weakly cohomologically big Cohen-Macaulay. Suppose $Q \in \operatorname{Spec} R$ has height h. Take $x_1, \ldots, x_h \in R$ a system of parameters for R_Q so that $\dim R/(x_1, \ldots, x_h) = \dim R - h$, which we can do using the equidimensional and excellent (and hence catenary) hypothesis and prime avoidance. Thus we can extend this sequence to $x_1, \ldots, x_h, x_{h+1}, \ldots, x_d$, a full system of parameters of R. Note

$x_1, \ldots, x_h$ are still a regular sequence on B and thus on B_Q, which proves our first statement.

For the converse, we follow the proof of [**HL07**, Corollary 2.3(b)] or [**Bha20**, Lemma 2.7] and proceed by induction on $x_1, \ldots, x_i$ part of a system of parameters $x_1, \ldots, x_d$ (the base case of $i = 0$ is clear). We want to show that x_i is not a zerodivisor on $B/(x_1, \ldots, x_{i-1})B$. Let Q be a minimal associated prime of

$$K = \ker\left(B/(x_1, \ldots, x_{i-1})B \xrightarrow{\cdot x_i} B/(x_1, \ldots, x_{i-1})B\right)$$

and note that $0 \neq H^0_Q(K_Q) \subseteq H^0_Q(B_Q/(x_1, \ldots, x_{i-1})B_Q)$. But then chasing exact sequences implies that

$$H^j_Q(B_Q/(x_1, \ldots, x_{i-2})B_Q) \neq 0 \text{ for some } j = 0, 1$$

and in general we see that

$$H^j_Q(B_Q) \neq 0 \text{ for some } j = 0, \ldots, i-1.$$

Now, since every element of K is annihilated by $(x_1, \ldots, x_i)$, see that $Q \in V(x_1, \ldots, x_i)$ and so Q has height at least i. Thus by hypothesis $H^j_Q(B_Q) = 0$ for $j = 0, \ldots, i-1$. A contradiction.

For the final statement, notice that if B is a weakly cohomologically big Cohen-Macaulay then so is B_Q as an R_Q-module for any Q. Hence B_Q is also weakly balanced big Cohen-Macaulay. $\square$

If $(A, \mathfrak{m})$ is regular and M is a *finitely generated* module. If M is faithfully flat over the regular and hence Cohen-Macaulay A, then it is Cohen-Macaulay since the regularity of a sequence will be preserved by base change to M. Conversely, it can be shown that any (maximal) Cohen-Macaulay module over A is flat, then M is faithfully flat. This follows, for instance, from the Auslander-Buchsbaum formula for modules of finite projective dimension:

$$\mathrm{projdim} M + \mathrm{depth} M = \mathrm{depth}\, R$$

which forces M to be projective or equivalently flat. We generalize this to balanced big Cohen-Macaulay modules.

Proposition 5.22 ([**HH92**, Page 77], [**Bha20**, Lemma 2.9])**.** *Suppose $(A, \mathfrak{m})$ is a regular local d-dimensional ring and B is an R-module. Then B is a balanced big Cohen-Macaulay module if and only if it is faithfully flat over A.*

Proof. Arguing as above, if B is faithfully flat over A, then it is balanced big Cohen-Macaulay (non-triviality follows since $0 \neq B \otimes A/\mathfrak{m} \cong B/\mathfrak{m}B$).

Thus we assume that B is a balanced and hence weakly cohomologically big Cohen-Macaulay, and we essentially follow the proof of [**Bha20**, Lemma 2.9] to show it is flat. By induction on dimension, we may assume that B_Q is flat over R_Q for every non-maximal $Q \in \operatorname{Spec} A$. By hypothesis, and since A is regular, there exists an i such that $\operatorname{Tor}_j(B, N) = 0$ for all $j > i$ but such that $\operatorname{Tor}_i(B, N) \neq 0$ for some finitely generated N. Hence $\operatorname{Tor}_i(B, -)$ is a left-exact functor.

Claim 5.23. *If* $k = R/\mathfrak{m}$ *then* $\operatorname{Tor}_i(B, k) \neq 0$.

Proof of claim. Choose $f \in \mathfrak{m}$. Since $\operatorname{Tor}_i(B, N) \neq 0$ is supported at $\mathfrak{m}$ (since B is flat away from $\mathfrak{m}$), we see that f is not a regular element on $\operatorname{Tor}_i(B, N)$. Factoring the multiplication-by-f-map, $N \twoheadrightarrow fN \hookrightarrow N$, since

$$\operatorname{Tor}_i(B, N) \longrightarrow \operatorname{Tor}_i(B, fN) \hookrightarrow \operatorname{Tor}_i(B, N)$$

is not injective, we see that $\operatorname{Tor}_i(B, N) \longrightarrow \operatorname{Tor}_i(B, fN)$ is also not injective. From the short exact sequence $0 \longrightarrow K \longrightarrow N \longrightarrow fN \longrightarrow 0$ we see $\operatorname{Tor}_i(B, K) \neq 0$ as well. But $K = 0 :_N f$ is the f-torsion part of N and so repeating this process for a set of generators of $\mathfrak{m}$ produces a finitely generated $\mathfrak{m}$-torsion module N with desired non-vanishing. Since N is a direct sum of copies of k claim follows. $\square$

We now have that $k \cong \mathbf{R}\Gamma_{\mathfrak{m}}(k)$ and so

$$B \otimes^{\mathbf{L}} k \cong B \otimes^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{m}}(k) \cong \mathbf{R}\Gamma_{\mathfrak{m}}(B \otimes^{\mathbf{L}} k) \cong \mathbf{R}\Gamma_{\mathfrak{m}}(B) \otimes^{\mathbf{L}} k$$

where second two isomorphisms follow from [**Sta19**, Tag 0ALZ]. However, we assumed the complex $B \otimes^{\mathbf{L}} k$ has cohomology in degree $-i < 0$ ($\operatorname{Tor}_i(B, k)$). On the other hand $\mathbf{R}\Gamma_{\mathfrak{m}}(B) \cong H^d_{\mathfrak{m}}(B)[d]$. Thus since, A is regular, N has a free resolution of length $\leq d$ and so $\mathbf{R}\Gamma_{\mathfrak{m}}(B \otimes^{\mathbf{L}} k) \cong H^d_{\mathfrak{m}}(B)[d] \otimes^{\mathbf{L}} k$ has cohomology *only* in degree ≥ 0. We have obtained our desired contradiction. $\square$

One of the consequences of the existence of big Cohen-Macaulay algebras is a proof of the direct summand theorem (due to Hochster in characteristic $p > 0$ [**Hoc73c**] and André in mixed characteristic [**And18**]). The proof is a simple application of Proposition 5.22.

Corollary 5.24. *Suppose* $(A, \mathfrak{m}) \longrightarrow (R, \mathfrak{n})$ *is a finite extension of Noetherian local rings with* A *regular. If* R *maps to a balanced big Cohen-Macaulay algebra* B*, then* $A \longrightarrow R$ *splits.*

Proof. Consider the composition $A \longrightarrow R \longrightarrow B$. It is straightforward to see that B is also a balanced big Cohen-Macaulay A-algebra and so $A \longrightarrow B$ is faithfully flat, and hence pure. Thus $A \longrightarrow R$ is also pure, and hence split, by Appendix A Lemma 2.7 and Proposition 2.3. $\square$

We began this section by proving that in characteristic $p > 0$, R^+ is cohomologically big Cohen-Macaulay (note that $\mathfrak{m}R^+ \neq R^+$ since $1 \notin \mathfrak{m}R^+$, since $1 \notin \mathfrak{m}S$ for any finite $R \subseteq S$ and so any *weakly* modifier can be removed). In the previous section, we considered a closure operation on on ideals $I \subseteq R$ by defining $I^+ = IR^+ \cap R$. Of course, for any R-algebra B also leads to a closure in the same way.

Definition 5.25. Suppose R is a ring and $N \subseteq M$ are R-modules. For any algebra B, we define the B-closure of N in M to be

$$N_M^{\mathrm{cl}_B} := \ker\Big(M \xrightarrow{m \mapsto 1 \otimes \overline{m}} B \otimes_R M/N\Big).$$

which in the case of ideals specializes to the extension-contraction: $I^{\mathrm{cl}_B} := IB \cap R$.

Note, $+$-closure is exactly $B = R^+$-closure. Closure with respect to a sufficiently large big Cohen-Macaulay algebra shares many common properties with tight closure (and in fact agrees with tight closure in characteristic $p > 0$, under moderate hypotheses [**Hoc94**, Theorem 11.1]). We highlight two facts about closures with respect to bing Cohen-Macaulay algebras.

Proposition 5.26 (*cf.* [**Hoc94**, Corollary 2.4], [**Die05**, Lemma 7.2.2], [**RG18**, Theorem 8.1])**.** *Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring and B is a balanced big Cohen-Macaulay algebra.*

(a) *If $x_1, dots, x_d$ is a system of parameters of R, then $(x_1, \ldots, x_{i-1})^{\mathrm{cl}_B} :_R x_i \subseteq (x_1, \ldots, x_i)^{\mathrm{cl}_B}$.*
(b) *Suppose that the $\mathfrak{m}$-adic completion $\widehat{R}$ is an integral domain (which is automatic if R is normal and excellent). Then for any ideal $J \subseteq R$ we have that $J^{\mathrm{cl}_B} \subseteq \overline{J}$, where $\overline{J}$ is the integral closure of J.*

Proof. For (a), notice that $(x_1, \ldots, x_{i-1})B :_B x_i = (x_1, \ldots, x_{i-1})B$ since B is balanced big Cohen-Macaulay. Hence we have that

$$(x_1, \ldots, x_{i-1})^{\mathrm{cl}_B} :_R x_i \subseteq (x_1, \ldots, x_{i-1})B :_B x_i \subseteq (x_1, \ldots, x_i)^{\mathrm{cl}_B}.$$

Intersecting back with R proves the desired result.

Now we consider (b), we *only* use the non-triviality condition of B. Let $\widehat{R}$ denote the completion of R (which we assumed is a domain). Consider the algebra $B' = B \otimes_R \widehat{R}$. It is certainly still weakly big Cohen-Macaulay (for instance, with respect to the systems of parameters of R, since $R \longrightarrow \widehat{R}$ is flat) and satisfies the non-triviality condition since $0 \neq B/\mathfrak{m}B \cong B'/\mathfrak{m}B'$ since $R \longrightarrow \widehat{R}$ is faithful.

Claim 5.27 ([**Hoc94**, Corollary 2.4, Proposition 10.5])**.** *Since B' is big Cohen-Macaulay over $\widehat{R}$, there is a non-zero $\widehat{R}$-linear map $f : B' \longrightarrow \widehat{R}$ (that is, B' is* **solid** *as an $\widehat{R}$-module, see* [**Hoc94**]*).*

Proof of claim. Note $H^d_{\mathfrak{m}}(B') \neq 0$ by (5.19.2).

By the Cohen Structure Theorem there exists a regular local ring $(A, \mathfrak{n}) \subseteq (\widehat{R}, \mathfrak{m}\widehat{R})$. Thus $0 \neq H^d_{\mathfrak{m}}(B') = H^d_{\mathfrak{n}}(B') = H^d_{\mathfrak{n}}(A) \otimes_A B'$. Since any map $B' \longrightarrow A$ factors as $B \longrightarrow \mathrm{Hom}_A(R', A) \longrightarrow A$ by Lemma 5.1 in Appendix A, and there exists a nonzero map $\mathrm{Hom}_A(R', A) \longrightarrow R'$, it suffices to show that there is a nonzero map $B' \longrightarrow A$.

Let $E = H^d_{\mathfrak{n}}(A)$ denote the injective hull of $A = A/\mathfrak{n}$ so that $E \otimes_A B' \neq 0$. Since $\mathrm{Hom}_A(-, E)$ is faithfully exact (Appendix C Remark 1.7) we see that $\mathrm{Hom}_A(B' \otimes_A E, E) \neq 0$. But using Appendix C Example 1.6, we see that

$$\mathrm{Hom}_A(B' \otimes_A E, E) \cong \mathrm{Hom}_A(B', \mathrm{Hom}_A(E, E)) \cong \mathrm{Hom}_A(B', A)$$

and so there exists a non-zero map as desired. $\square$

Now, we observe that $J^{\mathrm{cl}_B} \subseteq J^{\mathrm{cl}_{B'}} \subseteq (J\widehat{R})^{\mathrm{cl}_{B'}}$. On the other hand, $\overline{J} = \overline{J\widehat{R}} \cap R$ by [**SH06**, Proposition 1.6.2] and so it suffices to show that $(J\widehat{R})^{\mathrm{cl}_{B'}} \subseteq \overline{J\widehat{R}}$. In particular, we may assume $R = \widehat{R}$ is complete and replace B with B'.

Since $JB = J^{\mathrm{cl}_B} B$, we see that $J^n B = (J^{\mathrm{cl}_B})^n B$ for every n. Thus we have that we have that

$$J^n f(B) = f(J^n B) = f((J^{\mathrm{cl}_B})^n B) = (J^{\mathrm{cl}_B})^n f(B).$$

Since $f(B)$ is nonzero, we see that J^{cl_B} and J have the same integral closure $\overline{J}$ by [**SH06**, Corollary 6.8.12]. In particular, $J^{\mathrm{cl}_B} \subseteq \overline{J^{\mathrm{cl}_B}} = \overline{J}$. $\square$

For a domain R, one might ask if there are any intermediate algebra $R \subseteq B \subseteq R^+$ that is also big Cohen-Macaulay in characteristic $p > 0$, and the obvious thing to consider might be the perfection $R_{\mathrm{perf}} = \bigcup_{e>0} R^{1/p^e}$. While the perfection is not in general Cohen-Macaulay, it is *very close* to being Cohen-Macaulay.

Example 5.28. Suppose $(R, \mathfrak{m})$ is an Noetherian F-finite local domain of characteristic $p > 0$ and $0 \neq c \in R$ is such that R_c is Cohen-Macaulay (for instance, if c is a strong test element).

Now, $R_{\mathrm{perf}} = \bigcup_{e>0} R^{1/p^e}$ need not be big Cohen-Macaulay. For instance if R is F-injective but not Cohen-Macaulay with $H^i_{\mathfrak{m}}(R) \neq 0$ for some $i <$

$d = \dim R$, then $H^i_{\mathfrak{m}}(R_{\mathrm{perf}}) = \varinjlim H^i_{\mathfrak{m}}(R^{1/p^e}) \neq 0$ as well. See for instance Chapter 3 Subsection 5.3 for the construction of such examples.

However, R_{perf} is *close to being big Cohen-Macaulay* in several senses.

We first show that $(c)^{1/p^e} H^i_{\mathfrak{m}}(R_{\mathrm{perf}}) = 0$ for every e. Because R_c is Cohen-Macaulay, there exists an integer $n > 0$ such that $c^{p^n} H^i_{\mathfrak{m}}(R) = 0$ by Appendix C Corollary 6.5. Hence

$$c^{1/p^{e-n}} H^i_{\mathfrak{m}}(R^{1/p^e}) = 0$$

for $e \geq n$. Thus taking direct limits, the ideal

$$(c^{1/p^\infty}) = \bigcup_{e>0} (c^{1/p^e}) R_{\mathrm{perf}} = \bigcup_{e \geq n} (c^{1/p^{e-n}}) R_{\mathrm{perf}}$$

annihilates $H^i_{\mathfrak{m}}(R_{\mathrm{perf}})$.

Similarly, if c is a tight closure test element, and $x_1, \ldots, x_d$ is a regular sequence, by colon capturing we have that

$$c\big((x_1^{p^e}, \ldots, x_{i-1}^{p^e}) :_R x_i^{p^e}\big) \subseteq c(x_1^{p^e}, \ldots, x_{i-1}^{p^e})^* \subseteq (x_1^{p^e}, \ldots, x_{i-1}^{p^e})$$

for all $e > 0$. In other words, since $R \cong R^{1/p^e}$ as rings, we have that

$$c^{1/p^e}\big((x_1, \ldots, x_{i-1})R^{1/p^e} :_{R^{1/p^e}} x_i\big) \subseteq (x_1, \ldots, x_{i-1})R^{1/p^e}$$

Hence, by taking direct limits, we claim that

$$c^{1/p^e}\big((x_1, \ldots, x_{i-1})R_{\mathrm{perf}} :_{R_{\mathrm{perf}}} x_i\big) \subseteq (x_1, \ldots, x_{i-1})R_{\mathrm{perf}} \tag{5.28.1}$$

for every $e > 0$. To see this, observe any element in colon in (5.28.1) comes from a finite level and hence c^{1/p^b} "captures" it for all $b \gg 0$. Furthermore, notice that if $b > e$ and c^{1/p^b} captures the colon in the sense of (5.28.1) above, then so does c^{1/p^e}, since $c^{1/p^b} \,\big|\, c^{1/p^e}$. The claim follows.

This example leads us to the notion of almost mathematics and almost balanced big Cohen-Macaulay algebras.

Definition 5.29 (**[Fal88, Fal02, GR03, And18]**, *cf.* **[Rob10, RSS07, Shi11]**)**.** Suppose T is a ring and $J \subseteq T$ is an ideal such that $J = J^2$ and in fact the stronger condition[20] that $J \otimes_T J$ is a flat R-module.

If $N \subseteq M$ are T-modules, we say that $x \in M$ is **J-almost in** N if $Jx \subseteq N$.

We say that a T-module M is **J-almost zero** if $JM = 0$.

[20]By [**GR03**, Proposition 2.1.7(ii)] $J \otimes_T J$ being flat implies $J = J^2$

We say that an element $x \in T$ is **J-almost weakly regular** on a module M, if $\ker(M \xrightarrow{\cdot x} M)$ is J-almost zero.

If $(R, \mathfrak{m})$ is a Noetherian local ring and T is a R-algebra, we say that T is **J-almost weakly balanced big Cohen-Macaulay** if for every system of parameters $x_1, \dots, x_d$ for R, we have that x_i is J-almost weakly regular on $T/(x_1, \dots, x_{i-1})T$. If additionally we have that $T/\mathfrak{m}T$ is not J-almost zero, then we say that T is **J-almost balanced big Cohen-Macaulay.**

Note that the non-triviality condition in Definition 5.29 is stronger than the one in Definition 5.14 (something can be almost zero while not being zero) while the condition on systems of parameters is weaker (one can be almost weakly regular without being regular).

Example 5.30. Suppose $(R, \mathfrak{m})$ is an excellent Noetherian local domain of characteristic $p > 0$ and $c \in \mathfrak{m} \subseteq R$ a tight closure test element. Set $J = (c^{1/p^\infty}) \subseteq R_{\mathrm{perf}}$. It straightforward to see that $J^2 = J$ and even more that $J \otimes_T J$ is flat (since it is a direct limit of flat modules).

Example 5.28 shows that the perfection R_{perf} is an (c^{1/p^∞})-*almost weakly balanced big Cohen-Macaulay.* Let us quickly observe that R_{perf} also satisfies the almost variant of the non-triviality condition. If $c^{1/p^e}(R_{\mathrm{perf}}/\mathfrak{m}R_{\mathrm{perf}}) = 0$ for all $e > 0$, then

$$c^{1/p^e} 1 \in \mathfrak{m}R_{\mathrm{perf}}$$

for all $e > 0$. We prove this as follows. Take $v : \mathcal{K}(R) \setminus 0 \longrightarrow \mathbb{Z}$ a discrete valuation such that $v(R) \geq 0$ and $R \cap v_{\geq 0} = \mathfrak{m}$. One way to construct such a v is to blow up $\mathfrak{m}$ and normalize, and on some chart, localize at a height one prime lying over $\mathfrak{m}$ to find such a valuation ring with the desired property.

We can extend v uniquely to a valuation $v : \mathcal{K}(R)_{\mathrm{perf}} \setminus 0 \longrightarrow \mathbb{Q}$. Since $c \in \mathfrak{m}$ is not a unit, we have that $v(c) > 0$, and so the sequence of positive numbers $v(c^{1/p^e})$ limits to zero. On the other hand, $v(\mathfrak{m}R_{\mathrm{perf}}) \geq 1$ since $v(x) \geq 1$ for every $x \in \mathfrak{m}$. This is a contradiction.

Finally we note that, thanks to Gabber, there is a straightforward way to map an *almost* big-Cohen-Macaulay algebra to an honest big-Cohen-Macaulay algebra (also see [**Hoc02**]).

Proposition 5.31 (Gabber, [**Gab18**, Page 2])**.** *Suppose $(R, \mathfrak{m})$ is a local Noetherian ring and T is an R-algebra and $c \in T$ has a compatible system of p-power roots $c^{1/p^e} \in T$. Suppose that T is $J = (c^{1/p^\infty})$-almost (weakly) balanced big Cohen-Macaulay. Let $T' = \prod_{\mathbb{N}} T$ (with the diagonal map $T \longrightarrow T'$) and let $W \subseteq T'$ be the multiplicative set generated by $\mathbf{c} := (c, c^{1/p}, c^{1/p^2}, \dots)$. Then $B := W^{-1}T'$ is (weakly)balanced big Cohen-Macaulay.*

PROOF. Before beginning, observe that because $(x_1, \ldots, x_{i-1})$ is finitely generated, $(x_1, \ldots, x_{i-1})T' = \prod_{\mathbb{N}}(x_1, \ldots, x_{i-1})$.

We first handle the case where T is only weakly balanced big Cohen-Macaulay. Consider $(x_1, \ldots, x_d)$ a system of parameters for R. Suppose $\bar{t} \in \ker(T/(x_1, \ldots, x_{i-1}) \xrightarrow{\cdot x_i} T/(x_1, \ldots, x_{i-1}))$ lifting to some $t \in T$. We know that $(c^{1/p^\infty})x_i \in (x_1, \ldots, x_{i-1})$. Consider the diagonal map $d : T \longrightarrow T'$, identifying an element with its image, we see that $\mathbf{c}x_i = \mathbf{c}d(x_i) \in (x_1, \ldots, x_{i-1})T'$. Hence, if we invert $\mathbf{c}$ we see that $x_i \in (x_1, \ldots, x_{i-1})B$ as desired. This proves that B is weakly balanced big Cohen-Macaulay.

We now prove the non-triviality condition (a) from Definition 5.14. Suppose $1 \in \mathfrak{m}B = W^{-1}\mathfrak{m}T'$. But by properties of localization this means that $\mathbf{c}^{p^l} \in \mathfrak{m}B'$ for some $p^l > 0$. This implies that $c^{p^l/p^e} \in \mathfrak{m}T$ for all $e \gg 0$. In particular, we have that $c^{1/p^e} \in \mathfrak{m}T$ for all $e \gg 0$ and so $T/\mathfrak{m}T$ is (c^{1/p^e})-almost zero, a contradiction. $\square$

Remark 5.32. The same construction can also be used for the following purposes (which we will use, again see [**Gab18**]).

(a) $x \in R$, $J \subseteq R$ has the property that $(c^{1/p^\infty})x \in JT$ (that is x is almost in JT), then $x \in JT'$. We verified this in the special case that $J = (x_1, \ldots, x_{i-1})$ above.
(b) If $\eta \in H^i_{\mathfrak{m}}(T)$ is (c^{1/p^∞})-almost zero (that is, $c^{1/p^e}\eta = 0$ for all $e > 0$) then the image of η, $\overline{\eta'} \in H^i_{\mathfrak{m}}(T')$ is zero.

We finally point out one important property of big Cohen-Macaulay algebras over *regular* ring. Note that if a domain R is essentially of finite type type over a field, or complete, there exists a Noether(-Cohen) normalization $A \subseteq R$ (a finite extension) and so a balanced big Cohen-Macaulay algebra for R also becomes a balanced big Cohen-Macaulay algebra for A.

Theorem 5.33. *Suppose $(A, \mathfrak{m})$ is a regular local ring and B is an R-algebra. Then the following are equivalent.*

(a) B is R-flat.
(b) B is weakly cohomologically big Cohen-Macaulay.
(c) B is weakly balanced big Cohen-Macaulay.

PROOF. First suppose that B is R-flat. Then for any $Q \in \operatorname{Spec} R$ we have that $H^i_Q(B_Q) = H^i_Q(R_Q) \otimes_{R_Q} B_Q$ since B_Q is a flat R_Q-module[21]. Now,

[21] essentially by flat base change for cohomology on $\operatorname{Spec} R_Q$ or by [**BS98**, Lemma 4.3.1] which asserts that the target is Noetherian – this is not needed in the proof.

if $i < \dim R_Q$, then we immediately see that $H^i_Q(R_Q) = 0$, since regular ring are Cohen-Macaulay, and $H^i_Q(B_Q) = 0$ as well. This shows that (a) $\Rightarrow$ (b).

Since regular local rings are catenary (we did not need the full power of excellence), we see that (b) and (c) are equivalent by Lemma 5.21. $\square$

5.3. Characterizations of tight closure via almost mathematics and big Cohen-Macaulay algebras. We can characterize tight closure of ideals via big Cohen-Macaulay algebras.

Theorem 5.34 (*cf.* [**Hoc94**])**.** *Suppose that* $(R, \mathfrak{m})$ *is a Noetherian complete local domain of characteristic* $p > 0$ *and* $I \subseteq R$ *is an ideal. Then* $x \in I^*$ *if and only if* $x \in I^{\mathrm{cl}_B}$ *(that is,* $x \in IB$*) for some balanced big Cohen-Macaulay* R*-algebra* B*.*

Proof. First, suppose $x \in I^*$. Then for some test element $c \in R$ we have that $cx^{p^e} \in I^{[p^e]}$ so that $c^{1/p^e}x \in IR^{1/p^e} \subseteq IR_{\mathrm{perf}}$. Set $T = R_{\mathrm{perf}}$ and apply the construction of Proposition 5.31 to construct B. Using the notation from that result, $\mathbf{c}x \in IT'$ and so $x \in IB \cap R$.

Next suppose that $x \in IB$ for some balanced big Cohen-Macaulay R-algebra B. We must prove that $x \in I^*$. We follow the argument of [**HH94b**]. Our hypothesis implies that $x^{p^e} \in (IB)^{[p^e]} = I^{[p^e]}B$ for all $e > 0$. We note by Claim 5.27 that there exists a nonzero map $\phi : B \longrightarrow R$ with $c = \phi(b) \neq 0$, hence since $bx^{p^e} \in I^{[p^e]}B$, applying the R-linear map ϕ we see that

$$cx^{p^e} \in I^{[p^e]}\phi(B) \subseteq I^{[p^e]}$$

for all $e > 0$. Hence $x \in I^*$ as desired. $\square$

Remark 5.35. Again, the hypothesis that R is complete can be weakened in various ways, see for instance [**Hoc94**, Theorem 11.1]. However, since containment in tight closure may be checked after completion Exercise 2.2, and a balanced big Cohen-Macaulay algebra over R maps to one over $\widehat{R}$, the above tends to be enough for applications.

Almost mathematics can be used to characterize tight closure.

Theorem 5.36 ([**HH90**, Theorem 6.9], *cf.* [**HH91**])**.** *Suppose that* $(R, \mathfrak{m})$ *is a Noetherian complete local domain ring of characteristic* $p > 0$ *and* $c \in R$ *is a tight closure test element. Suppose* $I \subseteq R$ *is an ideal. Then*

$$x \in I^* \text{ if and only if } (c^{1/p^\infty})x \subseteq IR_{\mathrm{perf}}.$$

The condition on the right side of the displayed equation could be phrased as saying that x **is** (c^{1/p^∞})**-almost in** IR_{perf}.

Proof. If $x \in I^*$, then $c^{1/p^e} x \in (I^{[p^e]})^{1/p^e} = IR^{1/p^e}$ for all $e > 0$. Hence taking a union we see that $(c^{1/p^\infty})x \subseteq IR_{\mathrm{perf}}$. We needed no completeness hypothesis.

For the reverse, suppose that $(c^{1/p^\infty})x \subseteq IR_{\mathrm{perf}}$. By the hypothesis that $(c^{1/p^\infty})x \subseteq IR_{\mathrm{perf}}$, we know that for each $e > 0$, there exists a e' (which we may assume satisfies $e' \geq e$) such that $c^{1/p^e} x \in IR^{1/p^{e'}}$, however there is no clear (for instance even linear) relation between e and e'. Instead, we simply apply Theorem 5.34.

Consider $T := R_{\mathrm{perf}}$, which we know is an almost balanced big Cohen-Macaulay algebra by Example 5.30. Now apply the construction of Proposition 5.31 to form a big Cohen-Macaulay algebra B. By Remark 5.32 we see that $x \in IB$ and hence $x \in I^*$ by Theorem 5.34. $\square$

The proof we gave above is not the original proof, which instead relied upon the ideas around Chapter 6 Theorem 7.4. We sketch the original proof in Exercise 5.9

5.4. Mixed characteristic. One of the applications of the existence of big Cohen-Macaulay algebras in mixed characteristic (*cf.* [**And20, HM18**]) is that it allows one to develop a theory of singularities in mixed characteristic. From the perspective of closure operations $(-)^{\mathrm{cl}_B}$, a theory of test ideals was introduced in [**PRG21**]. Another approach closer in spirit with the earlier chapters of this book can be found in the work of Ma-Schwede (*cf.* [**MS18, MS21**]). Compare also with [**INS23**]. These ideas have been developed further by numerous authors, see for instance [**ST21, TY23, BMP$^+$23, Rob22, HLS22, Mur22b, CLM$^+$23**].

On the other hand, a number of people have explored closure-theoretic definitions which mimic tight closure, see for instance [**Hoc94, Hei01, BS12, EH18, HM21, Jia21**]. Also see Definition 5.42 below for additional discussion.

We will not delve into any of this mixed-characteristic theory too deeply but we will briefly highlight some definitions and results which give a flavor of some of this work and which closely connect to earlier chapters of this book

Definition 5.37. Suppose $(R, \mathfrak{m})$ is a complete Noetherian local ring and B is a balanced big Cohen-Macaulay algebra. We can define the **test module of R along B** to be:

$$\tau_B(\omega_R) := \Big(\mathrm{Image}\Big(H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(B)\Big)\Big)^\vee = \mathrm{Ann}_{\omega_R} \ker\Big(H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(B)\Big).$$

If R is in characteristic $p > 0$, or in mixed characteristic and one restricts to *perfectoid*[22] big Cohen-Macaulay algebras B, then it turns out there there is a "big enough" $\mathbb{B}$ so that

$$\tau_{\mathbb{B}}(\omega_R) = \sum_B \tau_B(\omega_R),$$

see **[Die07, MS21]** (again, in mixed characteristic we sum over perfectoid B). The point is that, in those cases, two big Cohen-Macaulay algebras R can map to a third.

Regardless, this gives a notion of rational singularities defined from big Cohen-Macaulay algebras.

Definition 5.38. A Noetherian local ring $(R, \mathfrak{m})$ is (**perfectoid**[23]) **big Cohen-Macaulay rational** (or simply **(p)BCM-rational**) if

(a) R is Cohen-Macaulay and
(b) $H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(B)$ injects for every (perfectoid) balanced big Cohen-Macaulay R-algebra B (this is equivalent to $\tau_{\widehat{B}}(\omega_{\widehat{R}}) = \omega_{\widehat{R}}$).

BCM-rational singularities agree with F-rational singularities (by arguments essentially the same as those in the section above) and are always pseudo-rational, see **[MS21]** and *cf.* **[Smi97a]** as well as Definition 2.28 and [**?**] in Chapter 6.

Conjecture 5.39 (Ma-Schwede)**.** *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of characteristic 0 that has rational singularities. Then R is BCM-rational.*

To conclude this section, we mention that big Cohen-Macaulay rational singularities deform.

Theorem 5.40. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and $f \in \mathfrak{m}$ is a non-zerodivisor. If $R/(f)$ is BCM-rational, then so is R.*

Proof. Left to the reader in Exercise 5.11. $\square$

Remark 5.41. As an application, it can be shown that if $(R, \mathfrak{m})$ is a ring finite type over $\mathbb{Q}$, and after reduction to characteristic $p > 0$, we have that $(R_p, \mathfrak{m}_p)$ is F-rational *for a single* p (that does not have to be sufficiently large), then R has rational singularities in characteristic zero. This means one can effectively confirm rationality of singularities in characteristic 0 using a computer to check in characteristic $p > 0$. (Also compare with **[Zhu17]** from which similar results can also be obtained).

[22] The definition is not important here, it is enough to know this is a mixed characteristic analog of perfect.

[23] if R is mixed characteristic

The rough idea is that if $(R_{\mathbb{Z}}, \mathfrak{m}_{\mathbb{Z}})$ is the family of models of R over $\mathbb{Z}$ (note $\mathfrak{m}_{\mathbb{Z}}$ is *not* maximal), then $R_{(\mathfrak{m}_{\mathbb{Z}}+(p))}/(p)$ is F-rational, and hence BCM-rational. Thus $R_{(\mathfrak{m}_{\mathbb{Z}}+(p))}$ is BCM-rational and hence pseudo-rational. If follows that the localization $R = (R_{\mathbb{Z}})_{(\mathfrak{m}_{\mathbb{Z}}+(p))} \otimes_{\mathbb{Z}} \mathbb{Q}$ is also pseudo-rational and hence has rational singularities. See [**MS21, MST$^+$22, ST21**] for details and generalizations.

We conclude this section with some alternate closure constructions in mixed characteristic, related to the above, and inspired by tight closure.

Definition 5.42 ([**Hei01**])**.** Suppose $(R, \mathfrak{m})$ is a Noetherian local domain with $p \in \mathfrak{m}$. If $I \subseteq R$ is an ideal then the **extended plus closure** of I, is the set

$$I^{\mathrm{epf}} := \{x \in R \mid \exists c \neq 0, \forall N \in \mathbb{N}, \forall \epsilon > 0, c^{\epsilon} x \in (I, p^N)R^+\}.$$

By the sorts of arguments above, it is easy to see that extended plus closure agrees with tight closure in characteristic $p > 0$. It is closely related to closures induced from perfectoid BCM algebras, as in Theorem 5.34, see [**CLM$^+$23**, Proposition 5.2.5]. Other properties of extended plus closure have been developed here: [**HM21**]. A variant of the above, **weak epf closure** is defined in [**Jia21**]:

$$I^{\mathrm{wepf}} := \bigcap_N (I, p^N)^{\mathrm{epf}}$$

These closure operations, and the theory developed around them in the above references, was used in [**Mur22a**] to give new proofs of certain symbolic power containments in mixed characteristic (more closely related to the proofs of Hochster-Huneke via tight closure, [**HH02**]).

Remark 5.43. Closure operations in general have been a topic of substantial study in commutative algebra. For instance, having a sufficiently good closure operation implies the existence of big Cohen-Macaulay modules and algebras. To learn more, see for instance: [**Die10, RG18, DRG19, ERGV23**]. Some other related closure operations not mentioned here can be found in [**Eps12, EH18, BS12, HV04**].

5.5. Exercises.

Exercise 5.1.

Exercise 5.2. Suppose R is an integral domain and $Q \subseteq R$ is a prime ideal. Show that for any prime $Q^+ \subseteq R^+$ lying over Q (which necessarily exist since $R \subseteq R^+$ is integral) we have that

$$R^+/Q^+ \cong (R/Q)^+.$$

Exercise 5.3. Prove that a version Theorem 5.5 holds when R is not necessarily a domain but is instead reduced, locally equidimensional, and has connected Spec.

Hint: Reduce to the domain case by letting $Q_1, \dots, Q_t$ be the minimal primes of R and considering the finite extension $R \longrightarrow \prod_i R/Q_i$.

Exercise 5.4. Prove the conclusion of Theorem 5.7 still holds if instead of assuming that R is an F-finite domain, one assume that R is F-finite, reduced, is locally equidimensional and has connected Spec.

Exercise 5.5. The equational lemma can be generalized to schemes:

Exercise 5.6 (*cf.* [**Bha12**, Proposition 4.2])**.** Suppose that X is an integral Noetherian scheme and with $A = H^0(X, \mathcal{O}_X)$ is a Noetherian ring. Suppose further that $J \subseteq A$ is an ideal with $Z = V(J\mathcal{O}_X) \subseteq X$ (note, an interesting case is when $Z = X$). Suppose $M \subseteq H^i_Z(X, \mathcal{O}_X)$ is a Frobenius stable submodule[24] that is finitely generated as an A-module. Then there exists a finite surjective map $\pi : Y \longrightarrow X$ such that M is sent to zero via

$$H^i_Z(X, \mathcal{O}_X) \longrightarrow H^i_{\pi^{-1}Z}(Y, \mathcal{O}_Y).$$

Exercise 5.7 ([**Bha12**])**.** Using Exercise 5.6, prove the following globalization of Theorem 5.5.

Suppose that $X \longrightarrow \operatorname{Spec} R$ is a proper birational map between integral Noetherian schemes. Show that for every $i > 0$ there exists a finite surjective map $\pi : Y \longrightarrow X$ of Noetherian integral schemes such that:

(a) $H^i(X, \mathcal{O}_X) \longrightarrow H^i(Y, \mathcal{O}_Y)$ is the zero map for $i > 0$.
(b) $H^i(Y, \omega_Y) \longrightarrow H^i(X, \omega_X)$, induced by $\pi_*\omega_Y \longrightarrow \omega_X$ (dual to $\mathcal{O}_X \longrightarrow \mathcal{O}_Y$) is the zero map for $i > 0$.

Hint: For the first statement, take $Z = X$. For the second work locally to assume $(R, \mathfrak{m})$ is local and set $J = \mathfrak{m}$. The second statement becomes easier if you assume that X is Cohen-Macaulay (and since Macaulayfications exist by [**Kaw00**]), you are invited to do so.

Exercise 5.8. A domain R is called a **derived splinter** if for every proper surjective map $\pi : X \longrightarrow \operatorname{Spec} R$ we have that $\mathcal{O}_{\operatorname{Spec} R} \longrightarrow \mathbf{R}\pi_*\mathcal{O}_X$ splits (in the derived category). Prove that a strongly F-regular ring is a derived splinter.

[24]Meaning that $F(M) \subseteq M$ where F is the Frobenius action on $H^i_Z(X, \mathcal{O}_X)$.

Exercise 5.9. Use the following steps to give another proof of Theorem 5.36 which doesn't mention big Cohen-Macaulay algebras – following the strategy of [**HH90**, Theorem 6.9] (which proves a more general result, see Exercise 5.10). In this proof you will give another proof that

$$\text{if } (c^{1/p^\infty})x \in JR_{\text{perf}} \text{ then } x \in J.$$

First note that by Gabber's generalization of the Cohen-Structure theorem, [**Ill14**, Théroème VI.2.1.1] (*cf.* [**KS18, Ska16**]), there exists $A \subseteq R$ a Noether normalization which is generically étale ($\mathcal{K}(A) \subseteq \mathcal{K}(R)$ is separable).

(a) Prove that we may assume, without loss of generality, that $c \in A \cap \mathcal{J}(R/A)$ (see Definition 7.1 in Chapter 6).
(b) Prove that $A_{\text{perf}}[R] \cong A_{\text{perf}} \otimes_A R$, and hence the left side is a flat R-module (*cf.* Chapter 6 Lemma 7.9)
(c) Use (7.10.1) in Chapter 6 to show that
$$c^{1/p^e} R_{\text{perf}} \subseteq A_{\text{perf}}[R^{1/p^e}]$$
for every $e > 0$
(d) Use the previous to parts to prove that $c^{2/p^e} x \in IR^{1/p^e}$ for all $e > 0$ and to then conclude that $x \in I^*$.

Exercise 5.10 ([**HH90**, Theorem 6.9])**.** Consider the following generalization of Theorem 5.36.

Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian complete local domain of characteristic $p > 0$ with generically étale[25] Noether-Cohen-Gabber normalization $k[\![x_1, \dots, x_d]\!] = A \subseteq R$. Consider the $\mathfrak{m}_A$-adic discrete valuation v on A (geometrically, the one coming from the blowup of $\mathfrak{m}_A$). This extends uniquely to a $\mathbb{Q}$-valuation on R_{perf}.

Fix $x \in R$ and $I \subseteq R$ is an ideal. Show that

$$x \in I^*$$

if and only if there exist a sequence of non-unit elements $c_1, c_2, \dots \in A_{\text{perf}}$ with

$$\lim_{e \longrightarrow \infty} v(c_e) = 0$$

and such that $c_e x \in IR_{\text{perf}}$ for all $e > 0$.

For further generalizations, see [**HH91**].

Hint: You can either generalize Proposition 5.31 and the proof of Theorem 5.36, or you can mimic the argument given above in Exercise 5.9.

[25] Such Noether normalizations exists by what is usually called the Cohen-Gabber theorem, [**Ill14**, Théroème VI.2.1.1] also see [**KS18, Ska16**].

Exercise 5.11. Prove Theorem 5.40. That is, show that if $R/(f)$ is BCM-rational and $f \in \mathfrak{m}$ is a nonzero divisor in $(R, \mathfrak{m})$, then R is also BCM-rational.

Hint: R is certainly Cohen-Macaulay. Next observe that for any balanced big Cohen-Macaulay R-algebra B, we have that $R/(f)$ is balanced big Cohen-Macaulay over $R/(f)$. Use a strategy similar to that of Theorem 5.12.

CHAPTER 8

Cartier modules and modules with Frobenius action

Warning, this chapter is likely to undergo very substantial revision.

1. p^{-1}-linear maps and Cartier modules

Suppose R is a ring of characteristic $p > 0$ and that M, N are R-modules.

Definition 1.1 (p-linear maps)**.** A p^e**-linear map** is a map $\phi : M \longrightarrow N$ such that:

(a) ϕ is additive (meaning $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in M$)
(b) for any $x \in M$ and $r \in R$, we have $\phi(rx) = r^{p^e}\phi(x)$.

More generally, for any integer $e \geq 0$ a p^e**-linear map** is an additive map $\phi : M \longrightarrow N$ satisfying the property that $\phi(rx) = r^{p^e}x$.

We see that a p^e-linear map is nothing more than a R-module homomorphism $\psi : M \longrightarrow F^e_* N$ with different notation. The advantage of using this language is it really shrinks the notation. The disadvantage of course is it can be hard to keep track of which R-module action is being used where.

Now we move onto p^{-1}-linear maps which are defined by taking roots instead of raising to powers.

Definition 1.2 (p^{-e}-linear maps)**.** Fix an integer $e \geq 0$, a p^{-e}**-linear map** is a map $\phi : M \longrightarrow N$ such that

(a) ϕ is additive,
(b) for any $x \in M$ and $r \in R$, we have $\phi(r^{p^e}x) = rx$.

As above, a p^{-e}-linear map is simply an R-module homomorphism $\psi : F^e_* M \longrightarrow N$.

Example 1.3. The most common example of a p-linear map is Frobenius itself $F : R \longrightarrow R$. The Frobenius splittings studied since Chapter 1 give excellent examples of p^{-1}-linear maps.

Given a p^{-d}-linear map $\phi : L \longrightarrow M$ and a p^{-e}-linear map $\psi : M \longrightarrow N$, the composition

$$\phi \circ \psi : L \xrightarrow{\psi} M \xrightarrow{\phi} N$$

is a p^{-e-d}-linear map since

$$\phi\big(\psi\big(r^{p^{e+d}}x\big)\big) = \phi\big(\psi\big((r^{p^d})^{p^e}x\big)\big) = \phi\big((r^{p^d})\psi(x)\big) = r\phi\big(\psi(x)\big).$$

Throughout the rest of this section, we largely consider p-linear and p^{-1}-linear maps from a module to itself. One advantage of this assumption is that it allows us to self-compose these maps without the notation complications of (2.0.1). In particular, if $\phi : L \longrightarrow L$ is a p^{-e}-linear map, then by self-composition, $\phi^n : L \longrightarrow L$ is a p^{-ne}-linear map.

Definition 1.4 (Cartier modules, [**BB11**])**.** Suppose R is a ring of characteristic $p > 0$.

(a) A **Cartier module (of degree** e**)** is an R-module with a given p^{-e}-linear map $\phi : M \longrightarrow M$ called the **structural morphism (of the Cartier module)**. It is usually denoted by the pair (M, ϕ) or if the ϕ is obvious, simply by the module M.
(b) A **morphism of Cartier modules** $h : (M, \phi) \longrightarrow (N, \psi)$ is an R-module map, also denoted by $h : M \longrightarrow N$, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ N & \xrightarrow[\psi]{} & N. \end{array}$$

Implicitly, both ϕ and ψ are p^{-e}-linear maps for the same *same* e.
(c) A **Cartier submodule** of a module (M, ϕ) is a sub-R-module $N \subseteq M$ such that $\phi|_N(N) \subseteq N$.
(d) A Cartier module is called **finite** (respectively **finitely presented**) if M is a finite R-module (respectively a finitely presented R-module).
(e) A Cartier module (M, ϕ) is called F**-pure** if $\phi(M) = M$ (*i.e.* , if ϕ is surjective).
(f) A Cartier module (M, ϕ) is called **nilpotent** if $\phi^n(M) = 0$ for some $n \gg 0$. The minimal n with this property is called **order of nilpotence**.

Lemma 1.5 ([**BB11**])**.** *If* (M, ϕ) *is an* F*-pure Cartier module and* $z^2M = 0$*, then* $zM = 0$*. In particular,* $\operatorname{Ann}_R M$ *is a radical ideal.*

Proof. If $z^2M = 0$, then $z^{p^e}M = 0$ so that $0 = \phi(z^{p^e}M) = z\phi(M) = zM$ completing the proof. $\square$

Example 1.6. Suppose that k is a perfect field and that $S = k[x_1, \ldots, x_n]$. Then it is easy to see that $\mathrm{Hom}_S(F^e_* S, S) \cong F^e_* S$ see Chapter 1 **??** and more generally Chapter 2 Corollary 3.16. Hence the evaluation-at-1 map $\mathrm{Hom}_S(F^e_* S, S) \longrightarrow S$ can be identified with some map $\phi : F^e_* S \longrightarrow S$, which turns out to be a generating map, and hence a Cartier map which we also label $\phi : S \longrightarrow S$.

More generally, if S is any ring such that $\mathrm{Hom}_S(F^e_* S, S) \cong F^e_* S$, then the same set of identifications yield a Cartier map $\phi : S \longrightarrow S$ whose corresponding element in $\mathrm{Hom}_S(F^e_* S, S)$ generates that set as a $F^e_* S$-module, see Exercise 2.1.

The terminology *F-pure* comes from the definition of an F-pure ring, see Chapter 1 Subsection 7.6, but there is no pure map of modules necessarily involved.

Example 1.7. Consider an F-injective F-finite ring R. Then the map $F_* \omega_R \longrightarrow \omega_R$ is surjective and so we may view ω_R as an F-pure Cartier module. However, the Frobenius map on R is not always pure since there are F-injective singularities that are not F-pure, see Chapter 4 Example 2.16.

Definition 1.8 (Modules with a Frobenius action)**.** Dually, a module M with a p^e-linear map $\phi : M \longrightarrow M$ is called a **module with a(n e-iterated) Frobenius action.** A morphism of modules with Frobenius action is defined analogously as above.

Lemma 1.9. *Suppose that $(M, \phi) \xrightarrow{h} (N, \psi)$ is a map of Cartier modules. Then the following inherit Cartier-module structure.*

(a) *The R-module kernel, $K = \ker h$ with structural map $\phi|_K$.*
(b) *The R-module cokernel, $C = N/h(M)$ with structural map ψ_C defined by $\psi_c(\overline{x})) = \overline{\psi}(\overline{x})$.*
(c) *The R-module image $I = \phi(M)$ with structural map $\psi|_I$.*

Proof. This follows immediately from the displayed diagram in Definition 1.4. $\square$

Cartier modules behave well with respect to completion and localization.

Lemma 1.10 (Localization and completion of Cartier modules)**.** *Suppose (M, ϕ) is a Cartier module.*

(a) *Suppose that $W \subseteq R$ is a multiplicative system, then $W^{-1}M$ inherits the structure of a Cartier-module with $(W^{-1}\phi)(x/w^{p^e}) = \phi(x)/w$.*

(b) *Suppose that $\mathfrak{b} \subseteq R$ is an ideal and let $\widehat{\ }$ denote completion with respect to this ideal. Then $\widehat{M}$ inherits the structure of a Cartier-module.*

PROOF. Part (a) is easy, notice that any $x/w \in W^{-1}M$ can be rewritten as xw^{p^e-1}/w^{p^e}. Alternately, one can view ϕ as $\phi : F^e_* M \longrightarrow M$ and then simply apply the functor $\otimes_R W^{-1}R$.

For part (b), tensoring by $\widehat{R}$ may not work because $F_* M$ is not necessarily a finite R-module, note we are not even assuming that R is F-finite. However, consider the completion (and compare with Lemma 1.14)

$$\begin{array}{rl} & \widehat{F^e_* M} \\ = & \varprojlim F^e_* M/(\mathfrak{b}^n F^e_* M) \\ = & \varprojlim F^e_* M/(F^e_*(\mathfrak{b}^{[p^e]})^n M) \\ \cong & F^e_* \varprojlim M/((\mathfrak{b}^{[p^e]})^n M) \\ \cong & F^e_* \varprojlim M/(\mathfrak{b}^n M) \\ = & F^e_* \widehat{M}. \end{array}$$

Thus applying the completion to $F^e_* M \longrightarrow M$ yields the desired Cartier structure on the completion $\widehat{M}$. $\square$

We also observe the following important fact closely related to localization.

Lemma 1.11 (Lifting Cartier submodules from localizations)**.** *Suppose (M, ϕ) is a module, $W \subseteq R$ is a multiplicative set, and $N' \subseteq W^{-1}M$ is a Cartier submodule. Then there exists a Cartier submodule $N \subseteq M$ such that $W^{-1}N = N'$.*

PROOF. First let $N_1 \subseteq M$ be any submodule such that $W^{-1}N_1 = N'$. Of course N_1 is probably not a Cartier submodule since we may have $\phi(N_1) \not\subseteq N1$, so set $N_2 = N_1 + \phi(N_1)$. This still may not be Cartier so set $N_3 = N_2 + \phi(N_2) = N_1 + \phi(N_1) + \phi^2(N_1)$. This process yields an ascending chain of submodules so let $N = \bigcup_e N_e$. By construction, this is a Cartier module. Furthermore, since direct limits commute with localization $W^{-1}N = \bigcup_e W^{-1}N_e$. But each $W^{-1}N_e = W^{-1}N_1 + W^{-1}\phi(N_1) + \cdots + W^{-1}\phi^{n-1}(N_1) = N'$ and hence $W^{-1}N = N'$. $\square$

Inspired by the previous proof, we make the following definition.

Definition 1.12. Fix a Cartier module (M, ϕ) and a submodule $N_1 \subseteq M$. Then $N = \bigcup_{e \geq 0} \phi^e(N_1)$ is the **Cartier submodule generated by** N_1. In the case that M is Noetherian, we can take $N = \bigcup_{e=0}^n \phi^e(N_1)$ for some n.

1.1. Matlis duals of Cartier modules – modules with Frobenius action. Suppose $(R, \mathfrak{m}, k)$ is an F-finite complete local ring of characteristic $p > 0$. We fix E to be an injective hull of the residue field k. Recall Matlis duality (apply $\operatorname{Hom}(-, E)$) from Appendix C Section 1.

By Appendix C Lemma 1.8 $\operatorname{Hom}_R(F_* R, E)$ is an injective hull of the residue field of $F_* R$, hence

$$\operatorname{Hom}_R(F_* R, E) \cong F_* E. \tag{1.12.1}$$

We fix this isomorphism once and for all. Based on this choice we have:

Lemma 1.13 (Matlis duality and Frobenius pushforward)**.** *For any Noetherian or Artinian module M over $(R, \mathfrak{m}, k)$ an F-finite complete local ring of characteristic $p > 0$ with E the injective hull of k, we have a functorial isomorphism for every $e > 0$*

$$(F^e_* M)^\vee = \operatorname{Hom}_R(F^e_* M, E) \cong F^e_*(M^\vee) = F^e_* \operatorname{Hom}_R(M, E) = (F^e_* M)^\vee.$$

This depends on the choice of isomorphism (1.12.1) (for $e = 1$) but on no other choices.

Proof. We begin with the special case that $M = R$ and induct on e. The base case is given. Then by Hom-$\otimes$ adjointness and induction:

$$\begin{array}{rl} & \operatorname{Hom}_R(F^e_* R, E) \\ \cong & \operatorname{Hom}_R((F^e_* R) \otimes_{F^{e-1}_* R} F^{e-1}_* R, E) \\ \cong & \operatorname{Hom}_{F^{e-1}_* R}(F^e_* R, \operatorname{Hom}_R(F^{e-1}_* R, E)) \\ \cong & \operatorname{Hom}_{F^{e-1}_* R}(F^e_* R, F^{e-1}_* E) \\ \cong & F^{e-1}_* \operatorname{Hom}_R(F_* R, E) \\ \cong & F^e_* E. \end{array}$$

Now for a general case of M we likewise have

$$\begin{array}{rl} & \operatorname{Hom}_R(F^e_* M, E) \\ \cong & \operatorname{Hom}_R(F^e_* M \otimes_{F^e_* R} F^e_* R, E) \\ \cong & \operatorname{Hom}_{F^e_* R}(F^e_* M, \operatorname{Hom}_R(F^e_* R, E)) \\ \cong & \operatorname{Hom}_{F^e_* R}(F^e_* M, F^e_* E) \\ \cong & F^e_* \operatorname{Hom}_R(M, E) \end{array}$$

which completes the proof. $\square$

Suppose now that $\phi : F^e_* M \longrightarrow M$ is an R-module map with M a Noetherian module (which of course makes M into a Cartier module). We apply the functor $\operatorname{Hom}_R(\_\_, E)$ and obtain a map using Lemma 1.13

$$\phi^\vee : M^\vee = \operatorname{Hom}_R(M, E) \longrightarrow \operatorname{Hom}_R(F^e_* M, E) \cong F^e_*(M^\vee).$$

On the other hand, given an Artinian module N with a p^e-linear map, which we represent by $\phi : N \longrightarrow F^e_* N$, we apply Matlis duality again and obtain

$$F^e_*(N^\vee) \cong \operatorname{Hom}_R(F^e_* N, E) \longrightarrow N^\vee.$$

Hence, we obtain:

Proposition 1.14. *Over an F-finite complete Noetherian local ring $(R, \mathfrak{m}, k)$, Matlis duality induces an equivalence of categories between finite Cartier modules and Artinian modules with a Frobenius action.*

Example 1.15. Consider the e-iterated Frobenius map $R \longrightarrow F_*^e R$. Set $d = \dim R$ and apply the functor $H^d_{\mathfrak{m}}(\_\_)$ and obtain

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(F_*^e R).$$

Using Čech cohomology (or any number of other arguments), for instance, it is easy easy to see that $H^d_{\mathfrak{m}}(F_*^e R) = F_*^e H^d_{\mathfrak{m}}(R)$. In particular, $H^d_{\mathfrak{m}}(R)$ is a module with Frobenius action.

The Matlis dual of $H^d_{\mathfrak{m}}(R)$ is ω_R, a canonical module of R. Hence dual to the Frobenius action on $H^d_{\mathfrak{m}}(R)$ we have a canonical p^{-e}-linear map $\omega_R \longrightarrow \omega_R$.

We should check that the induced p^{-e}-linear map on $\omega_R = (H^d_{\mathfrak{m}}(R))^\vee$ is the same as the usual trace map $F_*^e \omega_R \longrightarrow \omega_R$. Note that the *usual* map

$$F_*^e \omega_R \longrightarrow \omega_R \tag{1.15.1}$$

is defined by choosing an isomorphism $F_*^e \omega_R \cong \operatorname{Hom}_R(F_*^e R, \omega_R)$ and then applying evaluation-at-1. First we need the following.

Lemma 1.16. *For an F-finite Noetherian complete local ring $(R, \mathfrak{m}, k)$, the choice of isomorphism $\operatorname{Hom}_R(F_* R, E) \cong F_* E$ induces an isomorphism $\operatorname{Hom}_R(F_* R, \omega_R) \cong F_* \omega_R$.*

Proof. Given an isomorphism $\operatorname{Hom}_R(F_* R, E) \cong F_* E$, we see that

$$\begin{array}{rl} & \operatorname{Hom}_R(F_* R, \omega_R) \\ \cong & \operatorname{Hom}_R(F_* R, \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), E)) \\ \cong & \operatorname{Hom}_R((F_* R) \otimes_R H^d_{\mathfrak{m}}(R), E) \\ \cong & \operatorname{Hom}_R(H^d_{\mathfrak{m}}(F_* R), E) \\ \cong & F_* \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), E) \end{array}$$

where the last isomorphism is Lemma 1.13 and the second to last isomorphism is Lemma 7.2. But $\operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), E) = (H^d_{\mathfrak{m}}(R))^\vee$ is ω_R. Thus the choice of an isomorphism $\operatorname{Hom}_R(F_* R, E) \cong F_* E$ determines an isomorphism $\operatorname{Hom}_R(F_* R, \omega_R) \cong F_* \omega_R$. ☐

Proposition 1.17. *Suppose that $(R, \mathfrak{m}, k)$ is an F-finite complete local ring with a fixed isomorphism $F_*^e E \cong \operatorname{Hom}_R(F_*^e R, E)$ inducing an isomorphism $F_*^e \omega_R \cong \operatorname{Hom}_R(F_*^e R, \omega_R)$ as in Lemma 1.16. Then the map $F_*^e \omega_R \longrightarrow \omega_R$ obtained in Example 1.15 coincides with the usual map from (1.15.1).*

PROOF. It is sufficient to show that the Matlis dual of the map

$$F^e_*\omega_R \cong F_* \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), E) \cong \operatorname{Hom}_R(F_*R, \omega_R) \xrightarrow{\text{eval@1}} \omega_R$$

coincides with the map $H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(F^e_*R)$ induced by Frobenius.

Notice that the evaluation at 1 map $\operatorname{Hom}_R(F_*R, \omega_R) \longrightarrow \omega_R$ is identified with $\operatorname{Hom}_R(F_*R, \omega_R) \longrightarrow \operatorname{Hom}_R(R, \omega_R)$ induced by the Frobenius map $R \longrightarrow F^e_*R$. Hence the Matlis dual of the map labeled eval@1 is just $\operatorname{Hom}_R(\operatorname{Hom}_R(R, \omega_R), E) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(F_*R, \omega_R), E)$ which is naturally identified with $H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(F^e_*R)$ by local duality. $\square$

Remark 1.18 (Virtues of Matlis duality)**.** The notion of Cartier module seems best well behaved for F-finite schemes. However Artinian modules over a local ring with a Frobenius action are pleasant to deal with even outside of the F-finite case. Thus commutative algebraists especially have developed a great deal of theory of F-singularities in the local (excellent) not necessarily F-finite case.

1.2. Anti-nilpotence. One notion which has shown to be quite important for Artinian modules with a Frobenius action is anti-nilpotence.

Definition 1.19 (Anti-nilpotent, [**EH08**])**.** Suppose (N, F) is an Artinian module with a Frobenius action (*i.e.* $F : N \longrightarrow N$ is a p^e-linear map). Then (N, F) is called **anti-nilpotent** if for every F-compatible submodule $L \subseteq N$ (that is, $F(L) \subseteq L$) one has that the induced action $\overline{F} : N/L \longrightarrow N/L$ is injective.

Dually a Cartier module (M, ϕ) is called **Cartier anti-nilpotent** if every Cartier submodule $(N, \phi|_N)$ of (M, ϕ) is F-pure (that is, $\phi|_N$ is surjective).

We first observe that Cartier anti-nilpotence behaves well for Cartier modules over F-finite rings.

Lemma 1.20. *Suppose (M, ϕ) is a Cartier module. Then the following are equivalent.*

(a) (M, ϕ) is Cartier anti-nilpotent.
(b) For every multiplicative set $W \subseteq R$, $(W^{-1}M, W^{-1}\phi)$ is also Cartier anti-nilpotent.
(c) For every $Q \in \operatorname{Spec} R$, (M_Q, ϕ_Q) is Cartier anti-nilpotent.
(d) For every maximal $\mathfrak{m} \in \text{m-}\operatorname{Spec} R$, $(M_{\mathfrak{m}}, \phi_{\mathfrak{m}})$ is Cartier anti-nilpotent.

PROOF. Suppose (a) holds and $W \subseteq R$ is a multiplicative set. Further suppose that $N' \subseteq W^{-1}M$ is a Cartier submodule such that the induced

Cartier action on N' is not surjective. By Lemma 1.11 there exists a Cartier submodule $N \subseteq M$ such that $N' = W^{-1}N$. Hence the Cartier action on N is not surjective. This proves that (a) implies (b).

Certainly (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and so it remains to show that (d) implies (a). Thus suppose that $N \subseteq M$ is a Cartier submodule and $\phi(N) \neq N$. But then $N/\phi(N)$ is non-zero and hence non-zero at some maximal ideal $\mathfrak{m}$. □

We have the following connection between anti-nilpotence of modules with Frobenius action and Cartier modules.

Lemma 1.21. *Suppose $(R, \mathfrak{m})$ is a Noetherian F-finite local ring and (M, ϕ) is a finite Cartier module. Let $(M^\vee, F)$ denote the dual module with Frobenius action. If $(M^\vee, F)$ is anti-nilpotent, then (M, ϕ) is Cartier anti-nilpotent. The converse also holds if R is complete.*

PROOF. A Cartier submodule $N \subseteq M$ is F-pure if and only if the Matlis dual quotient $M^\vee/N^\vee$ has injective F-action. This proves the first statement. The second statement follows from Proposition 1.14. □

Note that for a non-complete ring, the completion of a Cartier (or Matlis dual) of a Cartier module can create submodules inaccessible over R. See Exercise 1.4.

If the Cartier module is $(M = R, \phi)$ is F-pure, then it is Cartier anti-nilpotent.

Lemma 1.22. *Suppose that R is a ring and (R, ϕ) is an F-pure Cartier module. Then (R, ϕ) is anti-nilpotent.*

PROOF. Suppose that $J \subseteq R$ is a Cartier submodule. Choose $x \in R$ such that $\phi(x) = 1$. Next choose $z \in J$. Notice that $z^{p^e}x \in J$ and so $\phi(z^{p^e}x) = z\phi(x) = z$ and so $\phi|_J$ is surjective. This proves that (R, ϕ) is anti-nilpotent. □

However, F-pure Cartier modules are generally not anti-nilpotent. Indeed, even in the case that of the Cartier module ω_R, there is an example in [**EH08**, Example 2.16] where $F_*\omega_R \longrightarrow \omega_R$ is surjective but not anti-nilpotent.

1.3. Anti-nilpotence as a measure of singularities. It turns out that if R is F-split, the canonical Frobenius action on the $H^i_\mathfrak{m}(R)$ is anti-nilpotent (and hence so is the Frobenius action on ω_R). First we first need a Lemma.

For motivation, suppose that $f : R \longrightarrow S$ is a map of rings and $\phi : S \longrightarrow R$ is an R-linear map. Then for any ideal $I \subseteq R$, consider the extension to S, $IS = f(I) \cdot S$. It is easy to see that $\phi(IS) \subseteq I$ by the R-linearity of ϕ and the fact that I is an ideal. Below, we generalize this simple observation to local cohomology.

Lemma 1.23. *Suppose that $f : R \longrightarrow S$ is a morphism of Noetherian rings, $I \subseteq R$ is an ideal and $N \subseteq H^i_I(R)$ is a submodule. Let $(f(N))_S \subseteq H^i_I(S)$ denote the S-submodule of $H^i_I(S)$ generated by $(H^i_I(f))(N)$. Suppose that $\phi : S \longrightarrow R$ is an R-linear map, then we have that*

$$(H^i_I(\phi))((f(N))_S) \subseteq N,$$

in other words, the map on local cohomology induced by ϕ sends $(f(N))_S$ back into N.

Proof. By tensoring the sequence $R \xrightarrow{f} S \xrightarrow{\phi} R$ with L and $H^i_I(R)$, we obtain the following diagram where at this point we know the top rectangle is commutative:

$$\begin{tikzcd}
N \otimes_R S \arrow[rr, "N \otimes \phi"] \arrow[d, "\rho"'] & & N \arrow[d, hook] \\
H^i_I(R) \otimes_R S \arrow[rr, "\phi' = H^i_I(R) \otimes \psi"] \arrow[dr, "\mu"'] & & H^i_I(R) \\
& H^i_I(S) \arrow[ur, "H^i_{\mathfrak{m}}(\phi)"'] &
\end{tikzcd}$$

We next need to explain what the map μ is and why the diagram commutes. The existence of μ follows from tensoring the Čech descriptions of local cohomology $H^i_I(R)$ with a ring extension (in this case $R \longrightarrow S$). Indeed, if $x_1, \dots, x_n$ generate I, then explicitly for Čech classes and $b \in S$ we define μ to be:

$$\left[\bigoplus_{j_1,\dots,j_i} \frac{a_{j_1,\dots,j_i}}{x_{j_1}^{n_1} \dots x_{j_i}^{n_i}}\right] \otimes b \overset{\mu}{\longmapsto} \left[\bigoplus_{j_1,\dots,j_i} \frac{f(a_{j_1,\dots,j_i})b}{x_{j_1}^{n_1} \dots x_{j_i}^{n_i}}\right] \tag{1.23.1}$$

Tensoring the Čech complex for $H^i_I(R)$ with S we see that cycles get mapped to cycles and boundaries to boundaries, hence the map μ exists.

The map ϕ' sends the left side of (1.23.1) to $\left[\bigoplus_{j_1,\dots,j_i} \frac{a_{j_1,\dots,j_i}}{x_{j_1}^{n_1} \dots x_{j_i}^{n_i}}\right] \cdot \phi(b)$ which equals $\left[\bigoplus_{j_1,\dots,j_i} \frac{a_{j_1,\dots,j_i}\phi(b)}{x_{j_1}^{n_1} \dots x_{j_i}^{n_i}}\right]$. The right side of image of (1.23.1) also agrees with $\left[\bigoplus_{j_1,\dots,j_i} \frac{(a_{j_1,\dots,j_i}).b}{x_{j_1}^{n_1} \dots x_{j_i}^{n_i}}\right]$ where the $(a_{j_1,\dots,j_i}).\bullet$ denotes action by an

R-module element. Hence $H^i_I(\phi)$ sends this to $\left[\bigoplus_{j_1,\dots,j_i} \frac{a_{j_1,\dots,j_i}\phi(b)}{x_{j_1}^{n_1}\dots x_{j_i}^{n_i}}\right]$ and so the diagram commutes.

Now, we claim that the image of $\mu \circ \rho : N \otimes_R S \longrightarrow H^i_I(S)$ is $(f(N))_S$. To see that, notice that the diagram below commutes and the composition in the bottom row is $H^i_I(f)$.

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & N \otimes S \arrow[d, "\rho"'] \arrow[dr, "\mu\circ\rho"] & \\
H^i_I(R) \arrow[r] \arrow[rr, bend right, "H^i_I(f)"'] & H^i_I(R) \otimes_R S \arrow[r, "\mu"'] & H^i_I(S)
\end{tikzcd}$$

It follows immediately that $(\mu \circ \rho)(N \otimes S)$ is the S-module generated by the image of N as claimed.

Finally, the commutativity of the first diagram then implies the lemma. $\square$

Theorem 1.24 (**[Ma12]**). *Suppose that R is F-pure and $\mathfrak{m} \subseteq R$ is a maximal ideal. Then for every i, $H^i_{\mathfrak{m}}(R)$ is anti-nilpotent with respect to the canonical Frobenius action F. Dually, if R is F-finite and F-split, then $(\omega_R, \Phi : F^e_* \omega_R \longrightarrow \omega_R)$ is an anti-nilpotent Cartier module.*

Proof. The local cohomology along $\mathfrak{m}$ of R and its completion $\widehat{R}$ are the same, hence we may assume R is complete. But since R is F-pure, and hence by Appendix A Corollary 2.5 we have that R is F-split.

If $H^i_{\mathfrak{m}}(S)$ is not anti-nilpotent, then there exists $N \subseteq H^i_{\mathfrak{m}}(R)$ which is F-compatible and some $z \in H^i_{\mathfrak{m}}(R)$ such that $z \notin N$ but $F(z) \in N$. Consider the descending chain of F-compatible submodules:

$$(F(N))_R \supseteq (F^2(N))_R = (F((F(N))_R))_R \supseteq (F^3(N))_R \supseteq \dots$$

Since $H^i_{\mathfrak{m}}(R)$ is Artinian, this chain stabilizes at some $N' = (F^e(N))_R$ and so $N' = (F(N'))_R$. Note $z \notin N \supseteq N'$ but $F^e(z) \in N'$, and hence replacing z by an image under some iterated Frobenius if necessary, we may assume that $z \notin N'$ but $F(z) \in N'$. Now, since R is F-split, we fix a splitting $R \xrightarrow{F} R \xrightarrow{\phi} R$. We apply Lemma 1.23 and see that $H^i_{\mathfrak{m}}(\phi)((F(N'))_R) \subseteq N$. Since $F(z) \in (F(N'))_R$, we see that $z = \phi(F(z)) \in N'$, a contradiction. This proves the local statement.

Now, notice if all a Cartier module's localizations are anti-nilpotent, the Cartier module itself is anti-nilpotent. $\square$

We have the following (admittedly confusing) definition.

Definition 1.25. A ring of characteristic $p > 0$ is called ***F*-anti-nilpotent** if for every maximal ideal $\mathfrak{m} \subseteq R$, we have that the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is anti-nilpotent for every i. If R is F-finite, we say that R is **Cartier anti-nilpotent** if each $\mathcal{H}^{-i}(\omega_R^{\bullet})$ is Cartier anti-nilpotent.

Certainly if R is F-anti-nilpotent (or F-finite and Cartier anti-nilpotent) then R is F-injective (0 is always a Cartier/F-submodule) and so we have:
(1.25.1)
(F-split) $\Rightarrow$ (F-pure) $\Rightarrow$ (F-anti-nilpotent) $\Rightarrow$ (Cartier anti-nilpotent) $\Rightarrow$ (F-injective)

Lemma 1.26. *A Noetherian local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ is F-anti-nilpotent if and only if the completion $\widehat{R}$ is F-anti-nilpotent. Furthermore, if R is F-finite and F-anti-nilpotent then it is Cartier anti-nilpotent.*

PROOF. The first statement follows since $H^i_{\mathfrak{m}}(R) = H^i_{\mathfrak{m}}(\widehat{R})$. The second follows from Lemma 1.21. □

However, we do not know the answer to the following questions

Question 1.27.
(a) If (M, ϕ) is a finite anti-nilpotent Cartier module over an F-finite local ring $(R, \mathfrak{m})$ is the completion of M anti-nilpotent as well?
(b) If R is F-finite and Cartier anti-nilpotent, is it F-anti-nilpotent?
(c) If R is F-anti-nilpotent, are its localizations also? Is the F-anti-nilpotent locus open?

Throughout this book, we have been considering questions like the following, if $(R, \mathfrak{m})$ is local, $f \in \mathfrak{m}$ is a regular element and $R/(f)$ is F-split (respectively F-injective), then is R F-split (respectively F-injective)? See for instance Chapter 1 Corollary 7.24 and Chapter 1 Theorem 7.14. The following is among the best results so far towards Chapter 1 Conjecture 7.16.

Theorem 1.28 ([**MQ18**], *cf.* [**HMS14**])**.** *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and $f \in \mathfrak{m}$ is a regular element such that $R/(f)$ is F-anti-nilpotent (for example, if it is F-pure) then R is F-anti-nilpotent and hence F-injective.*

The following proof closely follows those in [**HMS14**] and [**MQ18**].

Proof. Consider the diagram of local cohomology:

$$\begin{tikzcd}
\dots \arrow[r] & H^i_{\mathfrak{m}}(R) \arrow[r, "\alpha"] \arrow[d, "F^e"'] & H^i_{\mathfrak{m}}(R/(f)) \arrow[r, "\beta"] \arrow[d, "F^e_{R/(x)}"'] & H^{i+1}_{\mathfrak{m}}(R) \arrow[r, "\cdot f"] \arrow[d, "(\cdot f^{p-1})\circ F^e"'] & H^{i+1}_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & \dots \\
\dots \arrow[r] & H^i_{\mathfrak{m}}(R) \arrow[r, "\alpha"'] & H^i_{\mathfrak{m}}(R/(f)) \arrow[r, "\beta"'] & H^{i+1}_{\mathfrak{m}}(R) \arrow[r, "\cdot f"'] & H^{i+1}_{\mathfrak{m}}(R) \arrow[r] & \dots
\end{tikzcd}$$

Claim 1.29. *In the diagram above, the following equivalent statements hold for every i:*

(a) α is zero.
(b) β is injective.
(c) $\cdot f$ is surjective.

Proof of claim. We have the following factorization of α for any a.

$$\alpha : H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(R/(f^a)) \xrightarrow{\alpha_a} H^i_{\mathfrak{m}}(R/(f))$$

Now, we may factor $F_{R/(x)}$ as
(1.29.1)

$$F^e_{R/(f)} : H^i_{\mathfrak{m}}(R/(f)) \xrightarrow{F^e_R} H^i_{\mathfrak{m}}(R/(f^{p^e})) \longrightarrow H^i_{\mathfrak{m}}(R/(f^a)) \xrightarrow{\alpha_a} H^i_{\mathfrak{m}}(R/(f))$$

for some $1 \leq a \leq p^e$ where the later are maps are just induced by the surjections $R/(f^{p^e}) \longrightarrow R/(f^a) \longrightarrow R/(f)$.

For each a, the image N_a of $H^i_{\mathfrak{m}}(R/(f^a)) \longrightarrow H^i_{\mathfrak{m}}(R/(f))$ is a F-submodule (and hence an F^e-submodule) of $H^i_{\mathfrak{m}}(R/(f))$. Choose $z \in H^i_{\mathfrak{m}}(R/(f))$ and image $\overline{z} \in H^i_{\mathfrak{m}}((f))/N_a$. If we choose $e \geq a$, we see that the image of z in $H^i_{\mathfrak{m}}(R/(f^a))$ via (1.29.1), maps to $F^e(z)$ in $H^i_{\mathfrak{m}}(R/(f))$. Hence $F^e(z) \in N_a$.

$$\begin{tikzcd}
H^i_{\mathfrak{m}}(R/(f^a)) \arrow[r] \arrow[d] & H^i_{\mathfrak{m}}(R/(f)) \arrow[d, "F^e"] \arrow[dl, dotted, "(1.29.1)"'] \\
H^i_{\mathfrak{m}}(R/(f^a)) \arrow[r] & H^i_{\mathfrak{m}}(R/(f))
\end{tikzcd}$$

Thus $z \in N_a$ as well since Frobenius acts anti-nilpotently on $H^i_{\mathfrak{m}}(R/(f))$. This shows that each

$$H^i_{\mathfrak{m}}(R/(f^a)) \longrightarrow H^i_{\mathfrak{m}}(R/(f))$$

is the zero map. Hence we have just shown that $H^i_{\mathfrak{m}}(R/(f^a)) \xrightarrow{\cdot f} H^i_{\mathfrak{m}}(R/(f^a))$ surjects for each a.

Now, consider the direct system $\ldots \longrightarrow R/(f^a) \xrightarrow{1 \mapsto f} R/(f^{a+1}) \longrightarrow \ldots$, the direct limit of this system is $H^1_{(f)}(R)$ and so

$$\lim H^i_{\mathfrak{m}}(R/(f^a)) \cong H^i_{\mathfrak{m}}(H^1_{(f)}(R)) \cong H^{i+1}_{\mathfrak{m}}(R)$$

where the last equality comes from a spectral sequence or composition of derived functors argument and the fact that $H^0_{(f)}(R) = 0$ since R is a regular element (that is, $\mathbf{R}\Gamma_{(f)}(R) = H^1_{(f)}(R)[-1]$). Applying this limit to the surjective maps

$$H^i_{\mathfrak{m}}(R/(f^a)) \xrightarrow{\cdot f} H^i_{\mathfrak{m}}(R/(f^a))$$

we obtain that $H^{i+1}_{\mathfrak{m}}(R) \xrightarrow{\cdot f} H^{i+1}_{\mathfrak{m}}(R)$ surjects for every i. This proves Claim 1.29. $\square$

We now have short exact sequence for every $i > 0$:

$$0 \longrightarrow H^i_{\mathfrak{m}}(R/(f)) \xrightarrow{\beta} H^{i+1}_{\mathfrak{m}}(R) \xrightarrow{\cdot f} H^{i+1}_{\mathfrak{m}}(R) \longrightarrow 0$$

In fact, if we mimic the proof of Chapter 1 Theorem 7.14, we obtain directly that R is F-injective. Instead, we generalize that proof to prove that R itself is anti-nilpotent.

Choose $N \subseteq H^{i+1}_{\mathfrak{m}}(R)$ an F-stable submodule. Fix $L = \bigcap_a f^a \cdot N$ and observe that $f \cdot L = L$, because $H^{i+1}_{\mathfrak{m}}(R)$ is finite, this intersection is actually a finite intersection $L = f^a \cdot N$ for some $a \gg 0$ and in fact, in a little bit, we shall see that $N = L$ (but it is a convenient device for now). Notice that since $F(f^a N) = f^{ap^e} F(N) \subseteq f^{ap^e} N$, we have that L is also F-stable. Let $L' = \beta^{-1}(L) = L \cap H^i_{\mathfrak{m}}(R/(f)) \subseteq H^i_{\mathfrak{m}}(R/(f))$ denote its inverse image. L' is an F-stable submodule as well.

Claim 1.30. $0 \longrightarrow H^i_{\mathfrak{m}}(R)\big/L' \xrightarrow{\overline{\beta}} H^{i+1}_{\mathfrak{m}}(R)\big/L \xrightarrow{\cdot f} H^{i+1}_{\mathfrak{m}}(R)\big/L \longrightarrow 0$ *is exact.*

Proof of claim. The only place where this might not be exact is in the middle. Suppose then that $\overline{z} \in H^{i+1}_{\mathfrak{m}}(R)\big/L$ has the property that $f \cdot \overline{z} = 0$, that is $f \cdot z \in L$. But then because $f \cdot L = L$, there exists $y \in L$ so that $f \cdot y = f \cdot z$ and thus $f \cdot (z - y) = 0$. Hence there is some element mapping to $z - y$, let us call it $w \in \ker(H^{i+1}_{\mathfrak{m}}(R) \xrightarrow{\cdot f} H^{i+1}_{\mathfrak{m}}(R)) = H^i_{\mathfrak{m}}(R/(f))$. Since

$$\frac{H^i_{\mathfrak{m}}(R/(f))}{L'} = \frac{H^i_{\mathfrak{m}}(R/(f))}{L \cap H^i_{\mathfrak{m}}(R/(f))} \cong \frac{H^i_{\mathfrak{m}}(R/(f)) + L}{L}$$

we see that $\overline{w + y} \in \frac{H^i_{\mathfrak{m}}(R/(f)) + L}{L}$ maps to $\overline{z}$. This proves Claim 1.30. $\square$

Suppose now that $z \in H^{i+1}_{\mathfrak{m}}(R)\big/L$ and that $f^{p^e - 1} F^e(z) = 0$. Without loss of generality we can assume that $\mathfrak{m} z = 0$ (otherwise replace z with multiples until this is true). Hence $f \cdot z = 0$ and z has a preimage in

$H^i_{\mathfrak{m}}(R)/L' \subseteq H^i_{\mathfrak{m}}(R)/L$. But Frobenius acts injectively there and so $z = 0$. This proves that $f^{p^e-1} \circ F$ is injective on $H^{i+1}_{\mathfrak{m}}(R)/L$.

We finally show that $N = L$ which will complete the proof. Notice $L = f^{p^e-1} \cdot N$ for some $e \gg 0$. The map $N/L \xrightarrow{(\cdot f^{p^e-1})\circ F} H^{i+1}_{\mathfrak{m}}(R)/L$ is injective because it is the restriction of injective map to a smaller domain. But $f^{p^e-1}F(N) \subseteq f^{p^e-1}N = L$ and so our injective map is zero. Thus $N/L = 0$ and so $N = L$. $\square$

Remark 1.31. An element f such that $H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot f} H^i_{\mathfrak{m}}(R)$ is surjective for every i is called a **surjective element**. The key part of the above proof was showing that if $R/(f)$ is anti-nilpotent, then f is a surjective element for R. For a more detailed illumination of the theory of such elements, see [**MQ18**].

1.4. Exercises.

Exercise 1.1. Suppose that (M, ϕ) is a Cartier module and $Q \subseteq R$ is a finitely generated ideal. Show that

$$\Gamma_Q(M) = \{x \in M \mid Q^m x = 0 \text{ for some } m > 0\}$$

is Cartier submodule of M.

Exercise 1.2. Suppose (N, ϕ) is a module with an e-iterated Frobenius action ϕ. Show that (N, ϕ) is anti-nilpotent if and only if for each $z \in N$, we have that $z \in (\phi(z), \phi^2(z), \dots)_R$.

Exercise 1.3. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and E is the injective hull of the residue field $R/\mathfrak{m}$, show that any injective p^e-linear map F on E is anti-nilpotent.

Hint: Reduce to the case that R is complete. Suppose $N \subseteq E$ is an F-submodule and $z \in E$ is such that $F(z) \in N$. For any $a \in \operatorname{Ann}_R N$ and notice that $aF(z) = 0$. Use the injectivity to deduce that $az = 0$ as well.

Remark 1.32. Exercise 1.3, which is a dual version of Lemma 1.22 was pointed out to us by Neil Epstein.

Exercise 1.4. Consider the ring $R = \mathbb{F}_p[x, y]/(y^2 - x^2(x-1))$ defining a nodal singularity, and observe the completion at (x, y) is isomorphic to $\mathbb{F}_p[\![u, v]\!]/(uv) =: \widehat{R}$, see [**Har77**, Chapter I, Section 4].

Consider the $\Phi \in \operatorname{Hom}_R(F_*R, R)$ a generating homomorphism. Viewing Φ as a p^{-1}-linear map, show that (R, Φ) is an F-pure Cartier module and that $(\mathfrak{m}, \Phi|_{\mathfrak{m}})$ is a "simple" Cartier module. However, show that the same module under completion is not simple. Likewise show that the Matlis dual $(\mathfrak{m}^\vee, F)$ is not simple as a module with Frobenius action.

2. Finiteness properties of Cartier modules

Cartier modules have remarkably strong finiteness properties. The first fundamental finiteness result for Cartier modules is as follows, and observe it is a generalization of Chapter 4 Theorem 6.18 from rings to modules.

As in the previous section, we omit the F_*s frequently working with p^{-e}-linear maps instead of R-linear maps. We will try to avoid confusion by referring to $\phi : M \longrightarrow M$ explicitly as a p^{-e}-linear map in such cases.

Suppose (M, ϕ) is a Cartier module where ϕ is a p^{-e}-linear map. Notice that for every integer $n > 0$, we obtain a p^{-ne}-linear map

$$\phi^n : \underbrace{M \xrightarrow{\phi} M \xrightarrow{\phi} \cdots \xrightarrow{\phi} M}_{n-\text{times}}. \tag{2.0.1}$$

Certainly we have the chain of containments

$$\cdots \subseteq \phi^{n+1}(M) \subseteq \phi^n(M) \subseteq \cdots \subseteq \phi(M) \subseteq M.$$

Of course these containments might all be equality if $M = R$ and ϕ is a Frobenius splitting, but in general this descending chain of submodules always stabilizes if M is Noetherian. We actually proved this a slightly different context in Chapter 4 Theorem 6.18 so we simply recall the result.

Theorem 2.1. *Suppose (M, ϕ) is a finite Cartier module, with respect to a p^{-e}-linear map $\phi : M \longrightarrow M$, over a (not necessarily F-finite) Noetherian ring R. Then the descending chain of submodules*

$$M \supseteq \phi(M) \supseteq \phi^2(M) \supseteq \dots$$

eventually stabilizes. In other words there exists an n_0 such that $\phi^n(M) = \phi^{n_0}(M)$ for all $n \geq n_0$.

This stabilization is one of the foundational theorems in the theory of F-singularities. Versions of it were due to Hartshorne-Speiser, Lyubeznik and Gabber it is sometimes called *HSLG stabilization* [**HS77, Lyu97, Gab04**].

Definition 2.2. The smallest n such that $\phi^{n+1}(M) = \phi^n(M)$ is called the **HSLG-number**. Dually, if (N, F) is a module with a Frobenius action, then smallest n such that $\ker(F^n) = \ker(F^{n+1})$ is called the **HSL**-number.

Notice that if $\phi^{n+1}(M) = \phi^n(M)$, then by applying ϕ to both sides we have $\phi^{n+d}(M) = \phi^n(M)$ for all $d > 0$.

Definition 2.3. For a Cartier module (M, ϕ), with p^{-e}-linear ϕ, the stable image $\phi^n(M)$ for $n \gg 0$ is denoted by $\underline{M} = \sigma(M) = \sigma(M, \phi)$.

Example 2.4. Suppose $R = \mathbb{Z}/3\mathbb{Z}[x, y, z]$, let $M = R$ and $\Phi : M \longrightarrow M$ be the p^{-1}-linear map which sends the monomial $x^2y^2z^2 \mapsto 1$ and sends the other monomials $x^iy^jz^k \mapsto 0$, $0 \leq i, j, k \leq 2$. This is exactly the generating map Chapter 2 Remark 1.4 (although we are suppressing the F_* notation in this example).

Next let $f = x^4+y^4+z^4$ and consider the map $\phi(\_\_) = \Phi(f^{3-1} \cdot \_\_)$. This is corresponds to the generating map of $\mathrm{Hom}(F_*R/(f), R/(f))$ via Chapter 4 Theorem 2.1. We want to understand the image of ϕ. First we expand

$$f^2 = x^8 + 2x^4y^4 + 2x^4z^4 + y^8 + 2y^4z^4 + z^8.$$

We need to understand what Φ does to f^2 and multiples of f^2. As an R-module $F_*(f^2)$ is generated by $x^iy^jz^kf^2$ for $0 \leq i, j, k \leq 2 = p - 1$. Hence we see that

$$\mathrm{Image}(\phi) = \sum_{0 \leq i,j,k \leq 2} (\Phi(x^iy^jz^kf^2)).$$

This computation is summarized in the following table, only the products with non-zero image are included.

Monomial	Product	Image
y^2z^2	$\mathbf{x^8y^2z^2} + 2x^4y^6z^2 + 2x^4y^2z^6 + y^{10}z^2 + 2y^6z^6 + y^2z^{10}$	x^2
xyz^2	$x^9yz^2 + \mathbf{2x^5y^5z^2} + 2x^5yz^6 + xy^9z^2 + 2xy^5z^6 + xyz^{10}$	$2xy$
xy^2z	$x^9y^2z + 2x^5y^6z + \mathbf{2x^5y^2z^5} + xy^{10}z + 2xy^6z^5 + xy^2z^9$	$2xz$
x^2z^2	$x^{10}z^2 + 2x^6y^4z^2 + 2x^6z^6 + \mathbf{x^2y^8z^2} + 2x^2y^4z^6 + x^2z^{10}$	y^2
x^2yz	$x^{10}yz + 2x^6y^5z + 2x^6yz^5 + x^2y^9z + \mathbf{2x^2y^5z^5} + x^2yz^9$	$2yz$
x^2y^2	$x^{10}y^2 + 2x^6y^6 + 2x^6y^2z^4 + x^2y^{10} + 2x^2y^6z^4 + \mathbf{x^2y^2z^8}$	z^2

and so $\phi(M) = (x^2, 2xy, 2xz, y^2, 2yz, z^2) \cdot M$. Let us explain the computation done in the table in slightly more detail. In the first row, when we multiply f^2 by y^2z^2. The first (bold) monomial term is $x^8y^2z^2 = (x^2)^3(x^2y^2z^2)$ and then Φ of that monomial is $x^2 \cdot 1 = x^2$. The other monomials in the first row are sent to zero. The other rows are similar.

But now we must also compute $\phi^2(M)$. However, each entry in the left "Monomial" column is already an element of $(x^2, 2xy, 2xz, y^2, 2yz, z^2)$ and so we see that $\phi^2(M) = \phi((x^2, 2xy, 2xz, y^2, 2yz, z^2) \cdot M) = (x^2, 2xy, 2xz, y^2, 2yz, z^2) \cdot M$.

Definition 2.5 (**[BB11]**)**.** A Cartier module (M, ϕ) is called **nilpotent** if $\phi^i(M) = 0$ for some i. It is called **locally nilpotent** if it is a union of its nilpotent Cartier submodules.

We have the following important observation whose proof is immediate from the definition.

Proposition 2.6 ([**BB11**])**.** *If M is a finite Cartier module over a Noetherian ring R, then the module $M/\underline{M}$ is nilpotent, furthermore $\underline{M}$ is the smallest Cartier submodule with this property.*

In other words, $M/\underline{M}$ is the *maximal nilpotent quotient of M.*

Of course, we can also consider nilpotent *submodules.*

Proposition 2.7. *Suppose R is Noetherian and (M, ϕ) is a finite Cartier module. Then there exists $M_{\mathrm{nil}} \subseteq M$ the maximum nilpotent Cartier submodule. Furthermore, we have that M/M_{nil} has no nilpotent Cartier submodules.*

PROOF. Since a sum of nilpotent Cartier submodules is nilpotent, the existence of M_{nil} is clear. If $N \subseteq M$ is such that $N/(M_{\mathrm{nil}} \cap N)$ is nilpotent, then $\phi^n(N) \subseteq M_{\mathrm{nil}}$ for some n. Hence N is itself nilpotent and so $N \subseteq M_{\mathrm{nil}}$ and we have proven the proposition. $\square$

2.1. Adjoint Cartier module structures. Blickle and Böckle proved several important finiteness results for Cartier modules which we now attack. First however we need to describe an adjoint approach to Cartier modules.

Given any Cartier module (M, ϕ), we can view $\phi \in \mathrm{Hom}_R(F^e_* M, M)$. But we have natural isomorphisms

$$\begin{array}{ll} & \mathrm{Hom}_R(F^e_* M, M) \\ \cong & \mathrm{Hom}_R(F^e_* M \otimes_{F^e_* R} F^e_* R, M) \\ \cong & \mathrm{Hom}_{F^e_* R}(F^e_* M, \mathrm{Hom}_R(F^e_* R, M)) \\ = & F^e_* \mathrm{Hom}_R(M, (F^e)^\flat M). \end{array}$$

Recall that for any map of rings $f : R \longrightarrow S$ and R-module, we have $f^\flat M = \mathrm{Hom}_R(S, M)$ as an the S-module . In our case, $S = F^e_* R$, which is abstractly isomorphic to R. Hence we view $(F^e)^\flat M$ as an R-module.

Thus the choice of a ϕ is the same data as a choice of $\kappa = \kappa_\phi \in \mathrm{Hom}_R(M, (F^e)^\flat M)$. Notice that since $(F^e)^\flat \circ (F^d)^\flat = (F^{e+d})^\flat$ one can compose

$$M \xrightarrow{\kappa} (F^e)^\flat M \xrightarrow{(F^e)^\flat \kappa} (F^{2e})^\flat M$$

and obtain the map we call κ^2. More generally, we define $\kappa^i = ((F^e)^\flat \kappa^{i-1}) \circ \kappa$. If we are keeping track for F^e_* structures, then we can identify κ^i with an element of $\mathrm{Hom}_{F^{ie}_* R}\left(F^{ie}_* M, \mathrm{Hom}_R(F^{ie}_* R, M)\right)$.

Lemma 2.8. *If a p^{-e}-linear map $\phi : M \longrightarrow M$ is adjoint to $\kappa_\phi : M \longrightarrow (F^e)^\flat M$ (in the sense above) then ϕ^i is adjoint to κ^i, in other words $\kappa^i_\phi = \kappa_{\phi^i}$ (also defined as above).*

PROOF. This is left as an exercise to the reader in Exercise 2.5. □

Example 2.9 (The Cartier module structure of $F^\flat M$)**.** Applying $F^\flat$ to κ corresponding to ϕ, we obtain the following.

$$F^\flat M \xrightarrow{F^\flat \kappa} F^\flat (F^e)^\flat M \cong (F^{e+1})^\flat M \cong (F^e)^\flat F^\flat M$$

which implies immediately that $F^\flat M$ is a Cartier module.

Explicitly, from the non-adjoint perspective, we also can see this structure as follows. Apply $\mathrm{Hom}_R(F_* R, \bullet)$ to the map $\phi : F^e_* M \longrightarrow M$ to obtain

$$\mathrm{Hom}_R(F_* R, F^e_* M) \longrightarrow \mathrm{Hom}_R(F_* R, M).$$

Now, we have the map of rings $F_* R \longrightarrow F^{e+1}_* R$ and so we obtain the composition:

$$\mathrm{Hom}_{F^e_* R}(F^{e+1}_* R, F^e_* M) \subseteq \mathrm{Hom}_R(F^{e+1}_* R, F^e_* M) \longrightarrow \mathrm{Hom}_R(F_* R, F^e_* M) \longrightarrow \mathrm{Hom}_R(F_* R, M)$$

where the first inclusion exists because every $F^e_* R$-module homomorphism is an R-module homomorphism. Now, $\mathrm{Hom}_{F^e_* R}(F^{e+1}_* R, F^e_* M) \cong F^e_* \mathrm{Hom}_R(F_* R, M)$ and hence we obtain our Cartier module structure on $\mathrm{Hom}_R(F_* R, M)$, which we identify with $F^\flat M$.

We have the following simple but crucial observation.

Lemma 2.10. *Suppose that (L, ϕ) is a Cartier module. Then $\phi^i = 0$ if and only if the adjoint map $\kappa_{\phi^i} = \kappa^i_\phi$ is zero.*

PROOF. This follows from the isomorphism:

$$\mathrm{Hom}_R(F^{ie}_* L, L) \cong F^{ei}_* \mathrm{Hom}_R(L, (F^{ie})^\flat L).$$

□

Besides yielding a straightforward way to see the Cartier-module structure on $F^\flat M$ as in Example 2.9, the adjoint approach has another large advantage.

Say that we would like to identify $M_{\mathrm{nil}} \subseteq M$. The kernel of $F^e_* M \longrightarrow M$ is not an $F^e_* R$-submodule, but only an R-submodule, and so it is unlikely to be a Cartier submodule. However, the kernel of the adjoint κ is already an R-module. Furthermore, if $K_i = \ker \kappa^i$, then since we can factor $\kappa^{i+1} = (F^{ei})^\flat \kappa \circ \kappa^i$, we see that

$$K_i = \ker \kappa^i \subseteq K_{i+1} = \ker \kappa^{i+1}. \tag{2.10.1}$$

On the other hand, from the diagram:

$$\begin{tikzcd} K_{i+1} \arrow[r] \arrow[d, hook] & \kappa(K_{i+1}) \arrow[r] \arrow[d, hook] & \kappa^{i+1}(K_{i+1}) = 0 \arrow[d, hook] \\ M \arrow[r, "\kappa"'] & (F^e)^\flat M \arrow[r, "(F^e)^\flat \kappa^i"'] & (F^{ie})^\flat M \end{tikzcd}$$

We see that $\kappa(K_{i+1}) \subseteq \ker\big((F^e)^\flat \kappa^i\big)$. Since $(F^e)^\flat$ is left exact, we have that $\ker\big((F^e)^\flat \kappa^i\big) = (F^e)^\flat(\ker \kappa^i) = (F^e)^\flat K_i$. Thus combining this with (2.10.1), we see that:

$$\kappa(K_{i+1}) \subseteq (F^e)^\flat K_i \subseteq (F^e)^\flat K_{i+1}. \tag{2.10.2}$$

Thus we can view $\kappa|_{K_{i+1}} \in \operatorname{Hom}_R(K_{i+1}, (F^e)^\flat K_{i+1})$. In other words, we have just shown the following.

Theorem 2.11. *If (M, ϕ) is a Cartier module with adjoint structural map $\kappa : M \longrightarrow (F^e)^\flat M$, then $K_i = \ker \kappa^i \subseteq M$ is a Cartier submodule of M.*

We thus also obtain the following.

Proposition 2.12. *Suppose that R is Noetherian and (M, ϕ) is a finite Cartier module with adjoint action κ as above. Then $M_{\mathrm{nil}} = \bigcup_i \ker \kappa^i$ is the maximal nilpotent Cartier submodule of M.*

Proof. We already know each $\ker \kappa^i$ is a Cartier submodule, and by Lemma 2.10, we see it is nilpotent. On the other hand, if $N \subseteq M$ is a Cartier submodule and $\phi^i(N) = 0$, then $\kappa^i(N) = 0$ as well and so $N \subseteq K_i$. This completes the proof. □

We also obtain the following way to detect if $M_{\mathrm{nil}} = 0$.

Corollary 2.13. *Suppose R is Noetherian and (M, ϕ) is a finite Cartier module and κ is the adjoint map as above. Then $M_{\mathrm{nil}} = 0$ if and only if κ is injective.*

Proof. If $M_{\mathrm{nil}} = 0$ then certainly κ is injective. Conversely, if κ is injective, then so is $(F^e)^\flat \kappa$ (since $(F^e)^\flat$ is left exact), and hence so is κ^2 as it is a composition of injective maps. Continuing in this way shows that κ^i injects for all i and hence $M_{\mathrm{nil}} = 0$ by Proposition 2.12. □

2.2. Descending chains of Cartier submodules. Our next goal is to show that finite Cartier modules satisfy the DCC property up to nilpotence. First we need the following which should be compared to the locus a test ideal cuts out in the Spec of a ring.

Proposition 2.14. *Let R be an F-finite ring and (M, C) a finite F-pure Cartier module. Then there exists a closed set $Y = V(I) \subseteq X = \operatorname{Spec} R$ such that:*

(a) *$X \setminus Y$ is dense.*
(b) *For all Cartier-module quotients M/N whose support also does not contain an irreducible component of X we have that $\operatorname{Supp}(M/N) \subseteq Y$ and that $IM \subseteq N$.*

PROOF. We first show that we may assume that X is irreducible. Indeed, if the proposition is true for irreducible X, then let $\{X_j\}$ denote the irreducible components of X and let U_j be an open dense affine[1] subset of $\{z \in X_j \mid z \notin X_i, i \neq j\}$. Obviously $\bigcup_j U_j = \coprod_j U_j$ is a dense affine open set of X and so we may replace X by $\bigcup_j U_j$. But then X is the disjoint union of its irreducible components and the result follows.

We now assume that X is irreducible and in fact we can assume that X is reduced since M has radical annihilator by Lemma 1.5. Next we claim that we may assume that $M_{\mathrm{nil}} = 0$.

To show this claim, suppose we can prove the proposition for $\underline{M} = M/M_{\mathrm{nil}}$, a Cartier submodule which has no nilpotent submodules. Choose Y a subset for $\underline{M}$. Suppose that $N \subseteq M$ is a Cartier submodule such that $\operatorname{Supp}(M/N)$ does not contain all of X and is not contained in Y. Choose a point $x \in X \setminus Y$ a generic point of the support of M/N and replace X by $\operatorname{Spec} \mathcal{O}_{X,x}$, so that $Y = \emptyset$ and replace M and N by M_x and N_x respectively. It follows that $M/(M_{\mathrm{nil}} + N) \cong \underline{M}/(N/(N \cap M_{\mathrm{nil}}))$ must be zero since $\underline{M}$ has no non-trivial quotients on Y. In other words $M_{\mathrm{nil}} + N = M$. It immediately follows that M/N is nilpotent, which is impossible since M/N is F-pure (since M is). Hence we can replace M by $\underline{M}$.

We let $U = \operatorname{Spec} R[f^{-1}] \subseteq X$ be an open affine set where

(1) $U \subseteq X_{\mathrm{reg}}$ and $F^e_* R[f^{-1}]$ is a free $R[f^{-1}]$-module.
(2) $M|_U$ is a free $R[f^{-1}]$-module.
(3) $\kappa|_U : M|_U \longrightarrow (F^e)^\flat M|_U$ is surjective.

and set $Y = X \setminus U$. Only (3) might be non-obvious, indeed, κ is already injective since $M_{\mathrm{nil}} = 0$. But $(F^e)^\flat M|_U$ is also locally free, of the same rank as $M|_U$, and so since $\kappa|_U$ is generically injective, it is also generically surjective and so we can satisfy (3) shrinking U further if necessary.

[1]We only assume affine because so far we are talking about Cartier modules over a ring, and not on schemes.

Suppose now that $N \subseteq M$ is a Cartier submodule such that $\mathrm{Supp}(M/N)$ does not contain X and is not contained in Y. Localizing at the generic point η of $\mathrm{Supp}(M/N)$ and so replacing X by $\mathrm{Spec}\,\mathcal{O}_{X,\eta} = \mathrm{Spec}\,R$ allows to assume that $U = X$, $Y = \emptyset$ and that M/N is a finite length F-pure Cartier module. Consider the diagram

$$\begin{tikzcd} M \arrow[r, "\kappa_M"] \arrow[d, "\alpha"'] & (F^e)^\flat M \arrow[d, "\beta"] \\ M/N \arrow[r, "\kappa_{M/N}"'] & (F^e)^\flat(M/N) \end{tikzcd}$$

Note β is surjective since $F^e_* R$ is free. Thus $\kappa_{M/N}$ is also surjective since κ_M is. Hence $\ell_R(M/N) \geq \ell_R\big((F^e)^\flat M/N\big)$ where again $\ell_R(\underline{\ \ })$ is the R-module length. We claim that this is impossible for finite length modules. Indeed since $(F^e)^\flat$ is exact (since $F^e_* R$ is free) by taking a Jordan-Hölder resolution of M, it suffices to show that $\ell_R\big((F^e)^\flat(R/\mathfrak{m})\big) > 1$. Note that

$$\begin{array}{rl} & \ell_R\big((F^e)^\flat R/\mathfrak{m}\big) \\ = & \ell_{F^e_* R}(\mathrm{Hom}_R(F^e_* R, R/\mathfrak{m})) \\ = & \ell_{F^e_* R}\big(\mathrm{Hom}_{R/\mathfrak{m}}(F^e_*(R/\mathfrak{m}^{[p^e]}), R/\mathfrak{m})\big) \\ = & \ell_{F^e_* R}\big(F^e_*(R/\mathfrak{m}^{[p^e]})\big) \\ > & 1 \end{array}$$

where the final inequality holds because R has dimension ≥ 1. The fact that $IM \subseteq N$ follows immediately from the fact that $\mathrm{Ann}_R(M/N)$ is radical (since M/N is F-pure, since M is). ☐

We come to our promised theorem.

Theorem 2.15 ([**BB11**, Proposition 4.6])**.** *Suppose that M is a finite Cartier module over an F-finite Noetherian ring R. Then any descending chain of Cartier submodules*

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots \tag{2.15.1}$$

stabilizes up to nilpotence, meaning that for all $i \gg 0$ the quotients M_i/M_{i+1} are nilpotent.

Proof. First consider the chain

$$\underline{M} = \underline{M_0} \supseteq \underline{M_1} \supseteq \underline{M_2} \supseteq \ldots \tag{2.15.2}$$

and notice that M_i/M_{i+1} is nilpotent if and only if $\phi^n(M_i) \subseteq M_{i+1}$ for all $n \gg 0$ but then $\phi^n(M_i) = \underline{M_{i+1}}$ for all $n \gg 0$ as well (for even larger n). Hence M_i/M_{i+1} is nilpotent if and only if $\underline{M_i} = \underline{M_{i+1}}$. In particular (2.15.1) stabilizes up to nilpotence if and only if (2.15.2) honestly stabilizes. Hence we replace (2.15.1) by (2.15.2) and assume that each M_i is F-pure. Thus we can also assume that R is reduced.

We now proceed on the induction of of the dimension of R (the base case of $\dim R = 0$ being clear since then M is Artinian). Note that each M_i has finite rank on each irreducible component of $\operatorname{Spec} R$ and so these finite ranks must eventually stabilize, say for all $i \geq i_0$. Let $Y = V(J) \subseteq \operatorname{Spec} R$ denote the closed set for M_{i_0} coming from Proposition 2.14 so that for any $i \geq i_0$ we have $JM_{i_0} \subseteq M_i$. By truncating (2.15.1) before i_0 we may henceforth assume that $JM \subseteq M_i$ for all i.

Consider $M' = \sum_{i\geq 0} \phi^i(JM)$, the unique smallest Cartier module containing JM and observe $M' \subseteq M_i$ for all $i \geq 0$. But now (2.15.1) stabilizes if and only if

$$M/M' = M_0/M' \supseteq M_1/M' \supseteq M_2/M' \supseteq \dots \tag{2.15.3}$$

stabilizes. Hence we can replace R by a quotient R/J since $\operatorname{Supp}(M/M') \subseteq V(I)$. Our induction hypothesis then completes the proof. $\square$

As a consequence, we obtain the following Jordan-Hölder theory.

Corollary 2.16 (Jordan-Hölder for Cartier modules)**.** *Suppose R is an F-finite Noetherian ring and M is a finite Cartier module. Then there exists a chain of Cartier submodules*

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_t = \underline{M} = \sigma(M)$$

where each submodule M_i is F-pure, and each $Q_{i+1} = M_{i+1}/M_i$ is simple up to nilpotence (that is, every proper Cartier submodule of Q_{i+1} is contained in $(Q_{i+1})_{\mathrm{nil}}$). Furthermore, the length of any such chain is constant and the factors

$$\frac{Q_{i+1}}{(Q_{i+1})_{\mathrm{nil}}}$$

are unique up to ordering and isomorphism in any such chain.

Proof. The proof is essentially the same as Jordan-Hölder decomposition for finite length modules, and so follows from Theorem 2.15. We leave it as an exercise to the reader in Exercise 2.3. $\square$

Definition 2.17. We say the **length of a Cartier module** (M, ϕ) is the number t appearing in a Jordan-Hölder decomposition of Cartier modules as in Corollary 2.16.

2.3. Finiteness properties of morphisms between Cartier modules and consequences.

The main goal of this section is to show that the set of Cartier-module homomorphisms between two Cartier modules is a finite set. Once this is achieved, we obtain that there are finitely many Cartier submodules up to nilpotence in Theorem 2.23. We begin with an example.

Example 2.18. Suppose R is an F-finite integral domain. Suppose M is a torsion free rank-1 R-module with p^{-e}-linear maps $\phi : M \longrightarrow M$ making it into a Cartier module. Then we claim that $\mathrm{Hom}_{\mathrm{Cartier}}(M, M)$ is a finite set.

We observe that an R-module map $\alpha : M \longrightarrow M$ is multiplication by some element $\lambda \in R$. If α is also map of Cartier modules we then have the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[d, "\phi"] \\ M \arrow[r, "\alpha"'] & M. \end{tikzcd}$$

Hence $\phi(\alpha(m)) = \alpha(\phi(m))$ for all $m \in M$. In other words $\phi(\lambda m) = \lambda\phi(m) = \phi(\lambda^{p^e} m)$ for all $m \in M$. Now consider $\mathrm{Hom}_{p^{-e}}(M, M)$ as an R-module by pre-multiplication (in other words, consider $\mathrm{Hom}_R(F^e_* M, M)$ as an $F^e_* R$-module). We see that $\mathrm{Hom}_R(M, M)$ with this R-action is still a rank-1 torsion free R-module and so if the two elements $\phi(\lambda \cdot \_\_) = \phi(\lambda^{p^e} \cdot \_\_)$ are equal, then $\lambda = \lambda^{p^e}$ and so $\lambda \in \mathbb{F}_{p^e}$. This proves that $\mathrm{Hom}_{\mathrm{Cartier}}(M, M) \subseteq \mathbb{F}_{p^e}$ and is finite as claimed.

We next consider what happens when $R = k$ is a field.

Theorem 2.19. *Suppose k is an F-finite field and (M, ϕ) and (N, ψ) are finite Cartier modules over k (in particular M and N are finite dimensional k-vector spaces). Then* $\mathrm{Hom}_{\mathrm{Cartier}}(M, N)$ *is a finite dimensional $\mathbb{F}_p$-vector space.*

PROOF. Suppose $m = \mathrm{rank}_k M$ and $n = \mathrm{rank}_k N$. This is a linear algebra problem.

□

Now our goal is to generalize these results by reducing to the case where R is a field. We start by considering what occurs when (M, ϕ) is a *simple Cartier module*; meaning that M has exactly one proper Cartier submodule, 0.

Lemma 2.20. *Suppose R is Noetherian and (M, ϕ) is a simple Cartier module, then M has exactly one associated prime.*

PROOF. Suppose Q is an associated prime. There exists some $x \in M$ with $\mathrm{Ann}_R\, x = Q$ and so the submodule $\Gamma_Q(M) \subseteq M$ is nonzero. It is also a Cartier submodule by Exercise 1.1 and thus $\Gamma_Q(M) = M$ since M is simple. If $Q' = \mathrm{Ann}_R\, y$ is another associated prime then we see that $Q^m y = 0$ for some m and thus $Q^m \subseteq Q'$ and so taking radicals $Q \subseteq Q'$. Reversing roles shows that $Q' \subseteq Q$ and completes the proof. $\square$

Lemma 2.21. *Suppose R is Noetherian and (M, ϕ) is a finite simple nilpotent Cartier module. Then $\phi = 0$ and M is a simple R-module (in particular, $M = R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \in \operatorname{m-Spec} R$).*

PROOF. Let $\kappa : M \longrightarrow (F^e)^\flat(R)$ denote the adjoint map. Each $K = \ker \kappa$ is a Cartier submodule by Theorem 2.11. Since ϕ is nilpotent, we have that κ cannot be injective by Corollary 2.13 and hence $K = M$. Thus $\phi = 0$, and the Cartier submodules of M are simply the ordinary submodules and so M must be simple as an R-module as well. $\square$

Using the above, we can obtain the following.

Theorem 2.22. *If (M, ϕ) and (N, ψ) are Cartier modules with respect to p^{-e}-linear maps ϕ, ψ with M F-pure and $N_{\mathrm{nil}} = 0$, then $\operatorname{Hom}_{\mathrm{Cartier}}(M, N)$ is a finite dimensional vector space over $\mathbb{F}_{p^e}$ and in particular a finite set.*

PROOF. Since $\phi(M) = M$, for any $f : M \longrightarrow N$ a map of Cartier modules, we have that $\psi(f(M)) = f(M)$. Hence $f(M) \subseteq \underline{N}$ and so we may replace N by $\underline{N}$ and so assume that N is F-pure. Likewise since $f(M_{\mathrm{nil}}) \subseteq N_{\mathrm{nil}} = 0$, we may replace M by M/M_{nil} and so assume that $M_{\mathrm{nil}} = 0$. In other words, we may assume that both M, N are F-pure and both $M_{\mathrm{nil}} = 0$ and $N_{\mathrm{nil}} = 0$.

Let s, t be the Cartier module lengths of M and N respectively. We proceed by induction on the sum $s + t$. If M or N are zero, there is nothing to do and so we assume $s, t > 0$. We begin with the base case where $s, t = 1$, and so M, N are simple. In this case any non-zero map $f : M \longrightarrow N$ is an isomorphism (since M and N are simple) and so M, N have the same unique associated prime Q by Lemma 2.20. We may replace R by R/Q. Furthermore, since M and N are then torsion free, any map $M \longrightarrow N$ induces a unique map on the localizations $M_Q \longrightarrow N_Q$. But now R_Q is a field, and so we can apply Theorem 2.19. This proves the base case.

Suppose first that $s > 1$ and take a decomposition $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_s = \underline{M} = M$ of Cartier modules with each M_i F-pure and such that the quotients M_{i+1}/M_i are simple up to nilpotence. We have the following diagram with exact bottom row:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathrm{Cartier}}(M/M_1, N) \arrow[r] \arrow[d, hook] & \operatorname{Hom}_{\mathrm{Cartier}}(M, N) \arrow[r] \arrow[d, hook] & \operatorname{Hom}_{\mathrm{Cartier}}(M_1, N) \arrow[d, hook] \\
0 \arrow[r] & \operatorname{Hom}(M/M_1, N) \arrow[r] & \operatorname{Hom}(M, N) \arrow[r] & \operatorname{Hom}(M_1, N).
\end{tikzcd}$$

We also claim that the top row is exact. Indeed, if $M \longrightarrow N$ is a map of Cartier modules that restricts to the zero map $M_1 \longrightarrow N$, then it factors through some $M/M_1 \longrightarrow N$.

Notice that $\mathrm{Hom}_{\mathrm{Cartier}}(M_1, N)$ is finite by induction. As above, since $N_{\mathrm{nil}} = 0$, if $P = M/M_1$ then $\mathrm{Hom}(P, N) = \mathrm{Hom}(P/P_{\mathrm{nil}}, N)$ and so $\mathrm{Hom}(M/M_1, N)$ is also finite by induction. Hence the finiteness of $\mathrm{Hom}_{\mathrm{Cartier}}(M, N)$ follows.

Suppose $t > 1$ and a decomposition $0 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_t = \underline{N} = N$, of F-pure Cartier modules with simple quotients up to nilpotence. Let $Q = N/N_1$ and fix N' to be the preimage in N of Q_{nil}. Then $N' \supseteq N_1$, $\underline{N'} = N_1$, and N/N' is no nilpotent submodules. Form the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Cartier}}(M, N') \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{\mathrm{Cartier}}(M, N) \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{\mathrm{Cartier}}(M, N/N') \arrow[d, hook] \\
0 \arrow[r] & \mathrm{Hom}(M, N') \arrow[r] & \mathrm{Hom}(M, N) \arrow[r] & \mathrm{Hom}(M, N/N')
\end{tikzcd}$$

As before, it is straightforward to see the top row is exact. We see that $\mathrm{Hom}_{\mathrm{Cartier}}(M, N')$ is finite by induction. Furthermore, since $(N/N')_{\mathrm{nil}} = 0$, we see that $\mathrm{Hom}_{\mathrm{Cartier}}(M, N/N')$ is also finite. Hence $\mathrm{Hom}_{\mathrm{Cartier}}(M, N)$ is finite as well. □

Theorem 2.23. *If (M, ϕ) is a finite Cartier module, then there are finitely many F-pure Cartier submodules of (M, ϕ).*

Proof. We consider a Jordan-Hölder decomposition $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_t = \underline{M} = \sigma(M)$ of $(\underline{M}, \phi)$ via F-pure Cartier submodules as in Corollary 2.16. The quotients $\{Q_j/(Q_j)_{\mathrm{nil}}\}$ which are simple Cartier modules, are independent of the decomposition. Of course every F-pure simple Cartier submodule N of M is one of these quotients Q_j and since $\mathrm{Hom}_{\mathrm{Cart}}(N, M)$ is a finite set, there are only finitely many submodules of M isomorphic to N. Thus there are finitely many simple F-pure Cartier submodules of M.

We thus proceed by induction on the length of $\underline{M} = \sigma(M)$ in the category of F-pure Cartier modules, and $N_1, \ldots, N_t$ the finitely many simple F-pure Cartier submodules of M. By induction, there are finitely many F-pure Cartier modules in each of the M/N_j. These correspond bijectively to the F-pure Cartier submodules of M containing N_j. Since every F-pure Cartier submodule of M certainly contains one of the N_j, we are done. □

2.4. Exercises.

Exercise 2.1. Suppose R is a Noetherian F-finite ring such that $\mathrm{Hom}_R(F^e_* R, R) \cong F^e_* R$. Show that the map $\phi : F^e_* R \cong \mathrm{Hom}_R(F^e_* R, R) \longrightarrow R$ induced by evaluation-at-1 generates $\mathrm{Hom}_R(F^e_* R, R)$ as an R-module.

Exercise 2.2. Suppose R is Noetherian and (M, ϕ) is a finite Cartier module. Show that M is nilpotent if and only if it is locally nilpotent.

Exercise 2.3. Prove Jordan-Hölder for Cartier modules as in Corollary 2.16.

Exercise 2.4. Suppose R is an Noetherian ring and (M, ϕ) is a Cartier module. Suppose that $R \longrightarrow S$ is a ring homomorphism. Give $\mathrm{Hom}(S, M)$ a structure of a Cartier module over S.

Hint: We did this for the Frobenius map in section Example 2.9.

Exercise 2.5. Prove Lemma 2.8.

Exercise 2.6. Consider $R = \mathbb{F}_{5^2}[x, y, z]$, let $\mathfrak{m} = (x, y, z)$, $f = x^4 + y^4 + z^4$ and set $\phi = f^{5-1} \cdot \Phi$ similar to Example 2.4 where Φ corresponds to a generating homomorphism of $\mathrm{Hom}_R(F_*R, R)$. Let $\lambda \in \mathbb{F}_{5^2} \setminus \mathbb{F}_5$ and show that

$$\mathfrak{m}^2 + (x + \lambda y)$$

is an (R, ϕ^2) Cartier-submodule but that it is *not* an (R, ϕ)-Cartier submodule. More generally, for any $\lambda \in \mathbb{F}_{5^e} \setminus \mathbb{F}_5$, we have that $\mathfrak{m}^2 + (x + \lambda y)$ is an (M, ϕ^e) Cartier submodule but not a (M, ϕ) Cartier submodule.

3. Cartier algebras and test modules

We have seen various notions of singularities of pairs (R, Δ) and $(R, \mathfrak{a}^t)$. We defined these pairs by only considering special maps $\phi : F^e_* R \longrightarrow R$ (or equivalently considering a p^{-e}-linear map). If one is considering a set of such maps $\mathscr{C}_e \subseteq \mathrm{Hom}_{p^{-e}}(R, R)$ for various e, there is one property of such maps has been implicitly used more than any other.

(Composition) If $\phi \in \mathscr{C}_e$ and $\psi \in \mathscr{C}_d$, then $\phi \circ \psi \in \mathscr{C}_{e+f}$.

For example if $R \longrightarrow F_*R$ is split with a fixed splitting ϕ, then $\phi^2 = \phi \circ (F_*\phi)$ splits $R \longrightarrow F^2_*R$. Blickle pointed out the following general framework for thinking about such actions on modules in general.

Definition 3.1 ([**Bli13, Sch11**])**.** Suppose R is a ring of characteristic $p > 0$. A **Cartier algebra (over** R**)** is a (typically non-commutative) graded ring $\mathscr{C} = \bigoplus_{e \geq 0} \mathscr{C}_e$ such that $r \cdot \phi_e = \phi_e \cdot r^{p^e}$ for all $r \in R$ and where $\mathscr{C}_0 = R$. Note that a Cartier algebra is *not an R-algebra* as R is typically not central, although it is an $\mathbb{F}_p$-algebra.

A **principal Cartier algebra** is a Cartier algebra that is generated as a ring by $\mathscr{C}_0$ and some fixed $\phi \in \mathscr{C}_e$. In this case we write $\mathscr{C} = \mathscr{C}_0(\phi)$.

Typically, we have that $\mathscr{C}_e \subseteq \bigoplus_{e \geq 0} \operatorname{Hom}_{p^{-e}}(R, R)$ (where the ring multiplication $\phi \cdot \psi = \phi \circ \psi$ is composition as in Section 3), but this is not always the case. Indeed, the choice of a graded ring homomorphism $\mathscr{C} \longrightarrow \bigoplus_{e \geq 0} \operatorname{Hom}_{p^{-e}}(R, R)$ is the same as the choice of $\mathscr{C}$-module structure on R.

Example 3.2 (Cartier algebras we have already seen)**.** If R is an F-finite ring, then $\mathscr{C} = \bigoplus_{e \geq 0} \operatorname{Hom}_{p^{-e}}(R, R)$ is a Cartier algebra, called the **full Cartier algebra**. This is written as $\mathscr{C}^R = \bigoplus \mathscr{C}^R_e$.

Ideal pairs. Next suppose that $\mathfrak{a}$ is an ideal and $t \geq 0$ is a real number. For each e consider

$$(F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil}) \cdot \operatorname{Hom}_R(F^e_* R, R) \subseteq \operatorname{Hom}_R(F^e_* R, R).$$

Viewing this as p^{-e}-linear maps, we are considering sums of $\phi \in \operatorname{Hom}_{p^{-e}}(R, R)$ that are pre-composed by multiplication an element of $\mathfrak{a}^{\lceil t(p^e-1) \rceil}$. Hence we have the corresponding

$$\mathscr{C}^{\mathfrak{a}^t}_e \subseteq \mathscr{C}^R_e = \operatorname{Hom}_{p^{-e}}(R, R).$$

We see that

$$\mathscr{C}^{\mathfrak{a}^t} := \bigoplus_{e \geq 0} \mathscr{C}^{\mathfrak{a}^t}_e$$

is a Cartier algebra. To see it is closed under composition one must observe that

$$p^f \lceil t(p^e - 1) \rceil + \lceil t(p^f - 1) \rceil \geq \lceil t(p^{e+f} - 1) \rceil$$

which implies that

$$(\mathfrak{a}^{\lceil t(p^e-1) \rceil})^{[p^f]} \mathfrak{a}^{\lceil t(p^f-1) \rceil} \subseteq \mathfrak{a}^{\lceil t(p^{e+f}-1) \rceil}$$

and hence that

$$\mathscr{C}^{\mathfrak{a}^t}_e \cdot \mathscr{C}^{\mathfrak{a}^t}_f \subseteq \mathscr{C}^{\mathfrak{a}^t}_{e+f}.$$

Divisor pairs. Finally, suppose that R is F-finite and normal and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on $X = \operatorname{Spec} R$. We set

$$\mathscr{C}^\Delta_e := \operatorname{Hom}_{p^{-e}}(R(\lceil (p^e - 1)\Delta \rceil)) \subseteq \mathscr{C}^R_e.$$

In other words, we are considering maps $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\Delta_\phi \geq \Delta$ just as in Chapter 5 Setting 5.17 (notice that in this chapter, we use p^{-e}-linear maps instead of R-linear maps $F^e_* R \longrightarrow R$, but as we have seen this is only a notation distinction).

It follows from Chapter 5 Lemma 2.3 (or a computation similar to the ideal case above) that $\mathscr{C}^\Delta_e \cdot \mathscr{C}^\Delta_f \subseteq \mathscr{C}^\Delta_{e+f}$ and hence we have a Cartier algebra:

$$\mathscr{C}^\Delta = \bigoplus_{e \geq 0} \mathscr{C}^\Delta_e.$$

Thus another natural property one might desire for maps on a ring, we have also sometimes restricted them. For instance if $R \longrightarrow F^2_* R$ splits, then we restrict that splitting to $F_* R$ to see that $R \longrightarrow F_* R$ splits as well. This is only reasonable when the module we are working with has a canonical map $M \longrightarrow F_* M$, but that does happen often enough.

> (Restriction) If $\phi \in \mathscr{C}_e$ and $f \leq e$, then $\phi|_{F^f_* R} : F^f_* R \longrightarrow F^e_* R \xrightarrow{\phi} R$ is in $\mathscr{C}_d$ (here $F^f_* R \longrightarrow F^e_* R$ is identified with the $e - f$-iterated Frobenius).

Unfortunately, when dealing with Cartier algebras defined by pairs, restriction and composition are not always compatible, as the next example shows.

Example 3.3 (Restriction vs composition)**.** Suppose $R = \mathbb{F}_p[x]$ is an F-finite ring, $x \in R$ and $t \geq 0$. We consider the Cartier algebra $\mathscr{C}^{x^t}_e$. Let us explore what it means for this collection of p^{-e}-linear maps to satisfy the restriction condition Section 3. Let $\Phi^e \in \operatorname{Hom}_{p^{-e}}(R, R)$ denote a generating homomorphism, choose $\phi \in \mathscr{C}^{x^t}_e$ and write

$$\phi(-) = \Phi^e(x^{\lceil t(p^e-1)\rceil} v \cdot -)$$

for some $v \in R$. For any $f \leq e$, we have the composition (the restriction map)

$$\begin{tikzcd} \psi : R \arrow[r, "F^{e-f}"] & R \arrow[r, "\cdot x^{\lceil t(p^e-1)\rceil} v"] \arrow[rr, bend right, "\phi"'] & R \arrow[r, "\Phi"] & R \end{tikzcd}$$

and we are asking whether or not $\psi \in \mathscr{C}^{x^t}_f$. We can write this composition ψ as $\Phi^f(w \cdot -)$. We see that w is a multiple of $x^{\lceil t(p^f-1)\rceil}$ if and only if $\psi \in \mathscr{C}^{x^t}_f$. This is equivalent to requiring that $w^{p^{e-f}}$ is a multiple of $(x^{\lceil t(p^f-1)\rceil})^{p^{e-f}} = x^{p^{e-f}\lceil t(p^f-1)\rceil}$. Now, we already know that $w^{p^{e-f}}$ is a multiple of $x^{\lceil t(p^e-1)\rceil}$. Thus we are reduced to asking whether being a multiple of $x^{\lceil t(p^e-1)\rceil}$ implies you are a multiple of $x^{p^{e-f}\lceil t(p^f-1)\rceil}$, that is, whether or not

$$\lceil t(p^e - 1)\rceil \geq p^{e-f}\lceil t(p^f - 1)\rceil.$$

Indeed, this does not always happen, if $t(p^f - 1)$ is an integer, this follows. But in general, it is not true. For instance if $f = 1, e = 2$ and $t = 1/(p^e - 1)$, then $\lceil t(p^e - 1)\rceil = 1$ but $p^{e-f}\lceil t(p^f - 1)\rceil = p \cdot 1 = p$.

Perhaps though there is a different choice of rounding. Indeed, if one sets $\mathscr{D}^{x^t,e} \subseteq \operatorname{Hom}_{p^{-e}}(R, R)$ to be the maps obtained by pre-composition with multiplication by $x^{\lfloor tp^e\rfloor}$, then we see that

$$\lfloor tp^e\rfloor \geq p^{e-f}\lfloor tp^f\rfloor$$

for any $e > f$. Hence $\phi \in \mathscr{D}^{x^t,e}$ restricts to $\psi \in \mathscr{D}^{x^t,f}$.

It is worth noticing that if $0 < t < 1$ and $(p^e - 1)t$ is an integer, then $\lfloor tp^e \rfloor = \lceil t(p^e - 1) \rceil$ and so $\mathscr{D}^{x^t,e} = \mathscr{C}_e^{x^t}$.

Indeed, the same composition works for divisors. For R a normal domain and $\Delta \geq 0$ a $\mathbb{Q}$-divisor, set

$$\mathscr{D}^{\Delta,e} = \text{Image}\Big(\operatorname{Hom}_{p^{-e}}(R(\lfloor tp^e\Delta \rfloor), R) \longrightarrow \operatorname{Hom}_{p^{-e}}(R, R)\Big).$$

We obtain that elements of $\mathscr{D}^{\Delta,e}$ restrict to $\mathscr{D}^{\Delta,f}$, indeed, this may be checked in codimension 1 at DVRs where the computation is the same as the above.

In general, $\mathscr{D}^{\Delta,e}$ and $\mathscr{C}^{\Delta,e}$ are incomparable and $\mathscr{D}^{\Delta}$ is setup to be closed under restriction while $\mathscr{C}^{\Delta}$ is closed under composition.

In summary:

- The choice of rounding $\lceil t(p^e - 1) \rceil$ is the natural choice for composition of maps.
- The choice of rounding $\lfloor tp^e \rfloor$ is the natural choice for restriction of maps.

3.1. Modules of a Cartier algebra.

Definition 3.4 ([**Bli13**])**.** Suppose that R is a ring and $\mathscr{C}$ is a Cartier algebra. We say that M is a **Cartier module (of $\mathscr{C}$)** or simply a **$\mathscr{C}$-module** if M is a left $\mathscr{C}$-module. An R-submodule $N \subseteq M$ is called a **Cartier submodule** if the restricted action makes N into left $\mathscr{C}$-module.

Example 3.5 (Test ideals as Cartier submodules)**.** If R is a ring, and $\mathscr{C}$ is a Cartier-module, then R is a $\mathscr{C}$-module. Indeed for each $\phi \in \mathscr{C}_e$ and $x \in R$, then $\phi \cdot x = \phi(x) \in R$.

Next suppose that R is F-finite and reduced. Then the test ideal $\tau(R)$ is an $\mathscr{C}$-module, for any $\mathscr{C}$. Indeed, it suffices to show that $\tau(R)$ is an $\mathscr{C}^R$-module where $\mathscr{C}^R$ is the full Cartier algebra. But in fact, by definition, $\tau(R)$ is the smallest ideal $J \subseteq R$, not contained in any minimal prime of R, such that $\phi(J) \subseteq J$ for all homogeneous $\phi \in \mathscr{C}^R$. Hence $\tau(R)$ is the smallest ideal not contained in any minimal prime of R that is also a $\mathscr{C}^R$-submodule of R.

If R is F-finite normal domain and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor, then we likewise have that $\tau(R, \Delta)$ is the smallest non-zero ideal that is a $\mathscr{C}^{\Delta}$-submodule of R.

Likewise if R is a regular domain, then using the notation from Chapter 4 Section 5, it is not difficult to see that $\tau(R, \mathfrak{a}^t)$ is a $\mathscr{C}^{\mathfrak{a}^t}$-submodule of R (indeed, the smallest non-zero $\mathscr{C}^{\mathfrak{a}^t}$-module).

Finally, assuming R is F-finite and reduced, suppose that $\phi \in \mathscr{C}^R$ is a homogeneous element (of degree e). Let $R(\phi) \subseteq \mathscr{C}$ denote the principal Cartier algebra generated by ϕ. Then we immediately see that $\tau(R, \phi) = \tau(R, \Delta_\phi)$ is the smallest non-zero $R(\phi)$-submodule of R.

Observation 3.6. Suppose R is a ring of characteristic $p > 0$ and $\mathscr{C}$ is a Cartier algebra. If $W \subseteq R$ is a multiplicative set and M is a $\mathscr{C}$-module, then $W^{-1}M$ is an $\mathscr{C}$-module as well. Indeed, if $\phi \in \mathscr{C}_e$ and $x/w^n \in W^{-1}M$, then we may define

$$\phi \cdot (x/w) = \phi \cdot (xw^{p^e-1}/w^{p^e}) = (\phi(xw^{p^e-1}))/w.$$

Finally, we consider what happens when a $\mathscr{C}$-module M satisfies $I \cdot M = 0$ for some ideal of R.

Lemma 3.7. *Suppose R is a Noetherian ring and $\mathscr{C}$ is a Cartier algebra. If $I \subseteq R$ is an ideal, then $\mathscr{C} \cdot I$ is a two-sided ideal of $\mathscr{C}$ and in particular, $\mathscr{C}/(\mathscr{C} \cdot I)$ is an R/I-Cartier algebra. Finally, if M is an $\mathscr{C}$-module and $I \cdot M = 0$, then we obtain that M/I is also canonically a $\mathscr{C}/(\mathscr{C} \cdot I)$-module.*

PROOF. We have that $I \cdot \mathscr{C}_e = \mathscr{C}_e \cdot I^{[p^e]} \subseteq \mathscr{C}_e \cdot I$ which proves it is a two-sided ideal. The other statements follow. $\square$

3.2. Nilpotence, F-pure submodules and HSLGB-stabilization.

Suppose that R is a ring and $\mathscr{C}$ is a Cartier algebra. We write $\mathscr{C}_+ := \bigoplus_{e \geq 1} \mathscr{C}_e$. Notice that if M is a $\mathscr{C}$-module, then $\mathscr{C}_+ \cdot M \subseteq M$ is a $\mathscr{C}$-module as well, as is $\mathscr{C}_+^2 \cdot M = \mathscr{C}_+ \cdot \mathscr{C}_+ \cdot M$, etc.

Definition 3.8 (**[Bli13]**)**.** Suppose R is a Noetherian ring, $\mathscr{C}$ is a Cartier algebra and M is a Cartier module of $\mathscr{C}$. We say that M is **nilpotent** if $\mathscr{C}_+^i \cdot M = 0$ for some $i \geq 0$. A map of $\mathscr{C}$-modules $g : M \longrightarrow N$ is called a **nil-isomorphism** if the kernel and cokernel of g are nilpotent.

Lemma 3.9. *Suppose R is Noetherian, $\mathscr{C}$ is a Cartier algebra, and M is a Cartier module that is finitely generated as an R-module. Then there exists $M_{\mathrm{nil}} \subseteq M$, the maximal nilpotent Cartier submodule of R. Furthermore, M/M_{nil} has no nilpotent submodules.*

PROOF. Indeed, if M_1, M_2 are nilpotent $\mathscr{C}$-submodules of M, then we see that $M_1 + M_2$ is also nilpotent (choose i so that both $\mathscr{C}_+^i M_1 = 0$ and $\mathscr{C}_+^i M_2 = 0$). Since M is Noetherian, the sum of the nilpotent submodules stabilizes, and so stays nilpotent. This completes the proof of the first part.

For the second part, suppose that $N \subseteq M/M_{\mathrm{nil}}$ is nilpotent. Then $\mathscr{C}^j N = 0$. Let $N' \supseteq M_{\mathrm{nil}}$ be the corresponding $\mathscr{C}$-submodule of M. Then we see that $\mathscr{C}^j N' \subseteq M_{\mathrm{nil}}$ and hence N' is nilpotent, and so $N' \subseteq M_{\mathrm{nil}}$. Thus $N = 0$. $\square$

We have the notion of F-pure Cartier module.

Definition 3.10 ([**Bli13**])**.** Suppose $\mathscr{C}$ is a Cartier algebra and M is a $\mathscr{C}$-module. We say that $(M, \mathscr{C})$ is ***F*-pure** if $\mathscr{C}_+ \cdot M = M$.

Lemma 3.11. *Suppose that R is a ring of characteristic $p > 0$, $\mathscr{C}$ is a Cartier algebra and M an $\mathscr{C}$-module. Then for any multiplicative subset $W \subseteq R$ we have that $W^{-1}(\mathscr{C}_+ \cdot M) = \mathscr{C}_+ \cdot (W^{-1}M) \subseteq W^{-1}M$.*

PROOF. For $\phi \in \mathscr{C}_e$, notice we may write $(\phi.x)/w = (\phi.(x/w^{p^e}))$ for $x \in M$ and $w \in W$. The result follows. $\square$

Lemma 3.12. *Suppose that R is a ring of characteristic $p > 0$, $\mathscr{C}$ is a Cartier algebra and M an $\mathscr{C}$-module. Then for any finitely generated ideal $I \subseteq R$, we have that $H^0_I(M)$ is a Cartier submodule of M.*

PROOF. Suppose $x \in H^0_I(M)$. Then since $I^n x = 0$, we see that $I^n \mathscr{C}_e \cdot x = \mathscr{C}_e \cdot ((I^n)^{[p^e]} x) = \mathscr{C}_e \cdot 0 = 0$. Hence $H^0_I(M)$ is a Cartier submodule as desired. $\square$

We then have the following generalization of Theorem 2.1 which we call HSLGB-stabilization (the added"B" stands for M. Blickle, who first proved it).

Theorem 3.13 (HSLGB stabilization)**.** *Suppose that R is a Noetherian ring of characteristic $p > 0$, $\mathscr{C}$ is a Cartier algebra, and M is a Cartier module that is finitely generated as an R-module. Then the descending chain*

$$M \supseteq \mathscr{C}_+ \cdot M \supseteq \mathscr{C}_+^2 \cdot M \supseteq \mathscr{C}_+^3 \cdot M \supseteq \dots$$

eventually stabilizes.

PROOF. The proof mimics Theorem 2.1 and we repeatedly use Lemma 3.11 among the first steps. If $(\mathscr{C}_+^i \cdot M)_Q = (\mathscr{C}_+^{i+1} \cdot M)))_Q$ for some $Q \in \operatorname{Spec} R$, then that equality holds on a neighborhood U of Q since M is Noetherian. Hence applying $\mathscr{C}_+$ to both sides repeatedly, we see that

$$(\mathscr{C}_+^j \cdot M)_{Q'} = (\mathscr{C}_+^{j+1} \cdot M)))_{Q'}$$

for all $Q' \in U$ and all $j \geq i$. Thus the locus where where $\mathscr{C}_+^i \cdot M$ is not equal to $\mathscr{C}^i \cdot M$ is a closed subset of $\operatorname{Spec} R$ that decreases as i increases. By Noetherian induction this stabilizes, and so we have a closed set $Z \subseteq \operatorname{Spec} R$

such that $\operatorname{Supp}\left(\frac{\mathscr{C}_+^i \cdot M}{\mathscr{C}_+^{i+1} \cdot M}\right) = Z$ for all $i \gg 0$. Replacing M by $\mathscr{C}^i M$ for some large i, we may assume that $\operatorname{Supp}\left(\frac{\mathscr{C}_+^i \cdot M}{\mathscr{C}_+^{i+1} \cdot M}\right) = Z$ for all i. We let $\eta \in Z$ be a generic point of the now constant Z_i and now we replace R by R_η.

We have reduced to the case that $(R, \mathfrak{m})$ is local and $\operatorname{Supp} \frac{\mathscr{C}_+^i \cdot M}{\mathscr{C}_+^{i+1} \cdot M}\bigg) = \{\mathfrak{m}\}$ for all $i > 0$. As in Theorem 2.1, we write $\mathfrak{m} = (x_1, \dots x_t)$ and choose N so that $x_j M \subseteq \mathscr{C}_+ \cdot M$ for every j. We shall show by induction that $x_j^{2N} M \subseteq \mathscr{C}_+^n M$ for all n (the base case $n = 0$ is clear). Assuming the case for n we have that

$$x_j^{2N} M \subseteq x_j^N (\mathscr{C}_+ \cdot M) \subseteq \mathscr{C}_+ \cdot (x_j^{pN} M) \subseteq \mathscr{C}_+ \cdot (\mathscr{C}_+^n \cdot M) = \mathscr{C}_+^{n+1}.$$

In particular, we see that $M/(x_1^{2N}, \dots, x_t^{2N})M$ has finite length and hence $\mathscr{C}_+^i \cdot M/(x_1^{2N}, \dots, x_t^{2N})M$ must eventually stabilize. The result follows. $\square$

Definition 3.14 (**[Bli13]**)**.** With notation as in Theorem 3.13, we define $\underline{M} = \sigma(M, \mathscr{C})$ to be $\mathscr{C}_+^i \cdot M$ for $i \gg 0$. We call it the **maximal F-pure submodule of** $(M, \mathscr{C})$.

It is worth noticing that, in the notation above $M/\underline{M}$, is nilpotent, indeed it is the maximal nilpotent quotient. Indeed $\underline{M} \longrightarrow M$ is a nil-isomorphism as is $M \longrightarrow M/M_{\mathrm{nil}}$.

Remark 3.15. While we regularly use the $\underline{M}$ notation, when working with more than one Cartier algebra simultaneously, we will switch to the $\sigma(M, \mathscr{C})$ notation.

Example 3.16 ($\underline{R}$ in a non-reduced ring)**.** Let $S = \mathbb{F}_p[x]$ and consider the ring $R = \mathbb{F}_p[x]/(x^n)$ and let $\mathscr{C}$ be the full Cartier algebra on R. Consider the p^{-1}-linear map $\phi(-) = \Phi_S(x^{n(p-1)}-)$ where Φ_S generates $\operatorname{Hom}_{p^{-1}}(S, S)$. By Fedder's Lemma Chapter 4 Theorem 2.1, this induces $\overline{\phi} \in \operatorname{Hom}_{p^{-1}}(R, R)$ which generates that Hom-set and so $\mathscr{C}$ is a principal Cartier algebra generated by $\overline{\phi}$.

We consider the image of $\overline{\phi}^m$ which we compute by taking the image of ϕ^m, which is $\Phi^m((x^{n(p^m-1)}))$. We may assume that $p^m \geq n$ and so write $n(p^m - 1) = (n-1)p^m + (p^m - n)$. It follows from Chapter 1 **??** that

$$\Phi^m((x^{n(p^m-1)})) = (x^{n-1}).$$

In particular, we see that $\underline{R}$ is the ideal generated by x^{n-1}. This perhaps should not be surprising in view of Lemma 1.5.

3.3. Test modules.

Definition 3.17 (Test modules, [**Bli13**])**.** Suppose R is an F-finite Noetherian ring, $\mathscr{C}$ is a Cartier algebra and M is a $\mathscr{C}$-module that is finitely generated as an R-module. We define the **test module**

$$\tau(M, \mathscr{C})$$

to be the smallest $\mathscr{C}$-submodule (if it exists) N of M so that $H^0_Q(N_Q) \longrightarrow H^0_Q(M_Q)$ is a nil-isomorphism for all Q an associated prime of M.

Furthermore, under the same hypotheses, additionally let $\mathscr{P}$ be a collection of prime ideals of R, so that each $Q \in \mathscr{P}$ is an associated prime of M/N for some $\mathscr{C}$-submodule N. We define the **adjoint submodule**

$$\mathrm{adj}_{\mathscr{P}}(M, \mathscr{C})$$

to be the smallest $\mathscr{C}$-submodule (if it exists) N of M so that $N_Q = \underline{M}_Q$ for all $Q \in \mathscr{P}$.

Remark 3.18 (Alternate definitions)**.** In case that $\mathscr{P}$ is the set of minimal associated primes of M, our definition of adjoint submodule agrees with the original definition of test submodule in [**Bli13**]. Our definition of test submodule is taken from [**BS19**].

It is not clear that test modules and adjoint modules exist in this generality even if R is F-finite, and in fact, it is an open question, see Conjecture 3.23 below.

Example 3.19. If R is F-finite and reduced, we showed that $\tau(R, \mathscr{C})$ exists when $\mathscr{C}$ is a principal Cartier algebra in Chapter 5 Section 5, and of the full Cartier algebra in Chapter 5 **??**. When R is additionally normal and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor, we also showed that the test ideal $\tau(R, \mathscr{C}^{\Delta})$ exists in the same section.

Next assume that R is normal and that that $\Delta = D + \Gamma$ where D is prime, $\Gamma \geq 0$ and the D-coefficient of Γ is zero. Suppose that Q is the prime ideal defining D and set $\mathscr{P} = \{Q\}$. Then the ideal defined in of Chapter 5 Exercise 5.13 is $\mathrm{adj}_{\mathscr{P}}(R, \mathscr{C}^{\Delta})$, see also Chapter 5 Section 7.

Definition 3.20 (F-regularity, [**Bli13**])**.** Given a Noetherian ring R, a Cartier algebra $\mathscr{C}$ and an $\mathscr{C}$-module M that is finitely generated as an R-module, we say that $(M, \mathscr{C})$ is ***F*-regular** if $\tau(M, \mathscr{C}) = M$.

Example 3.21. Suppose R is an F-finite Noetherian domain, $\mathscr{C}$ is the full Cartier algebra, and set $M = R$. Then since there is only a single associated prime of R, we see that $(M, \mathscr{C})$ is F-regular if and only if R is strongly F-regular in the usual sense.

In general however, $\tau(R) \subseteq R$ is an $\mathscr{C}$-submodule and we always have that $(\tau(R), \mathscr{C})$ is F-regular.

Corollary 3.22. *We have that $\tau(M,\mathscr{C}) \subseteq \underline{M}$ if it exists and so if $\tau(M,\mathscr{C}) = M$ then $(M,\mathscr{C})$ is F-pure.*

PROOF. Certainly we have that $\underline{M} \subseteq M$ and this inclusion is a nil-isomorphism and hence stays that way after applying $H^0_Q(-_Q)$ for any Q an associated prime M, proving the first statement. Hence if we have $\tau(M,\mathscr{C}) = M$ then $M = \tau(M,\mathscr{C}) \subseteq \underline{M} \subseteq M$ proving the corollary. □

Conjecture 3.23 (Blickle)**.** *If R is an F-finite Noetherian ring, $\mathscr{C}$ is a Cartier algebra and M is an $\mathscr{C}$-module, then $\tau(M,\mathscr{C})$ exists.*

There are three main cases where test modules of $(M,\mathscr{C})$ are known to exist.

(a) If every associated prime Q of $\underline{M}$ is minimal and the R/Q-rank of $\underline{M}_Q$ equals 1 for all such Q. [**Bli13**]
(b) If every associated prime Q of $\underline{M}$ is minimal and $\mathscr{C}$ is a principal Cartier algebra. [**Bli13**]
(c) If R is finite type over an F-finite field. [**BS19**]

We shall only prove the first case. The final case uses ideas similar to [**BMS08**].

We now begin to tackle the existence of test modules.

Proposition 3.24 ([**Bli13**, Theorem 3.9])**.** *With notation as in Definition 3.17, the test module $\tau(M,\mathscr{C})$ exists if there is some $c \notin Q$ for any Q an associated prime of M such that for the localizations $R_c, \mathscr{C}_c$ and M_c we have that $\tau(M_c,\mathscr{C}_c)$ exists and agrees with $\underline{M}_c$. In this case:*

$$\tau(M,\mathscr{C}) = \sum_{i\geq 0} \mathscr{C}^i_+(c\cdot \underline{M}).$$

PROOF. We may assume that $M = \underline{M}$ by Corollary 3.22. Suppose $N \subseteq M$ is an $\mathscr{C}$-submodule such that $H^0_Q(N_Q) \longrightarrow H^0_Q(M_Q)$ is a nil-isomorphism for all Q an associated prime of M. We then have that $N_c = M_c$ since this property is preserved after localizing at c. Hence since M is finitely generated, there is a t such that $c^t M \subseteq N$, or in other words $c^t(M/N) = 0$. But now since $M = \underline{M}$, we have that $M/N = \underline{M/N}$ and so $c(M/N) = 0$ by Exercise 3.3. Thus $cM \subseteq N$. Now, the sum

$$L = \sum_{i\geq 0} \mathscr{C}^i_+(c\cdot M)$$

is a Cartier submodule of M and is contained in N. Furthermore, we have that $H^0_Q(L_Q) = H^0_Q(M_Q)$ by construction again, proving that $\tau(M,\mathscr{C}) = L$. □

Remark 3.25. The above is not quite the most general statement, as explained in [**BS19**]. Indeed, it might happen that R is a domain and $M = M_1 \oplus M_2$ where $M_1 = R$ needs a test element $c_1 \in Q$ for some prime Q (perhaps Q defines the singular locus), but that $\mathrm{Supp}(M_2) \subseteq V(Q)$. Hence c_1 can't be used as a test element for the M_2 side of the direct sum. This issue is addressed by using a multiple test elements (one for each associated prime). Instead, we choose to work in a simpler setting where instead we frequently assume that every associated prime of M is minimal. For more general results see [**BS19**].

Example 3.26. Suppose for instance that $M = R$ has a unique associated prime but is non-reduced. Then the c we must select in Proposition 3.24 will not be in the minimal prime. But then it (or its powers) will not be in $\underline{R}$ since we expect that $\underline{R}$ will be contained in the minimal prime, see Example 3.16.

Proposition 3.24 above roughly says that test modules exist if we have a theory of test elements. A partial converse can be found in Exercise 3.5.

Lemma 3.27. *With notation as in Definition 3.17, suppose $M = \underline{M}$ has a single associated prime Q and that $\mathrm{rank}_{R/Q} M = 1$. Then $\tau(M, \mathscr{C})$ exists.*

Proof. We may replace R by R/Q and $\mathscr{C}$ by $\mathscr{C}/(\mathscr{C} \cdot Q)$ via Lemma 3.7. Since M is a single associated prime and is rank 1, we may assume that $M \subseteq R$ is a non-zero ideal. Hence, if we choose $0 \neq c_1 \in M \subseteq R$, we have that $M_{c_1} = R_{c_1}$.

Next suppose that $\mathscr{C}' = R(\phi) \subseteq \mathscr{C}$ is a principal Cartier algebra generated by $R = \mathscr{C}_0$ and some non-zero $\phi \in \mathscr{C}_e$. The action of ϕ_c on $M_c = R_c$ may be identified with an element of $\mathrm{Hom}_{p^{-e}}(R_c, R_c)$. Thus, by Chapter 5 Corollary 5.9 we see that $\tau(M_{c_1}, \mathscr{C}'_{c_2}) \subseteq M_{c_1} = R_{c_1}$ exists. Choose $0 \neq c_2 \in \tau(M_{c_1}, \mathscr{C}'_{c_1})$ set $c = c_1 c_2$ we see that $\tau(M_c, \mathscr{C}'_c) = M_c$. It follows since any $\mathscr{C}$-module is also a $\mathscr{C}'$-module, that $\tau(M_c, \mathscr{C}_c) = M_c$. Now apply Proposition 3.24. $\square$

Theorem 3.28 ([**Bli13**])**.** *Suppose R is an F-finite Noetherian ring, $\mathscr{C}$ is a Cartier algebra and M is an $\mathscr{C}$-module. Suppose that every associated prime Q of $\underline{M}$ is minimal and $\mathrm{rank}_{R/Q} \underline{M}_Q \leq 1$ for each such Q. Then $\tau(M, \mathscr{C})$ exists.*

Proof. We may assume that $M = \underline{M}$ by Corollary 3.22 (notice the set of associated primes might change, but if Q is an associated prime of M by not of $\underline{M}$, then $H^0_Q(\underline{M}) = 0$). For each associated (hence minimal) prime Q of M, we have that $H^0_Q(M)$ is a $\mathscr{C}$submodule of M with a single associated prime. Hence $\tau(H^0_Q(M), \mathscr{C})$ exists. It follows that

$$L = \sum_Q \tau(H^0_Q(M), \mathscr{C})$$

is a $\mathscr{C}$-submodule of M. Note that $H^0(L_Q) = H^0(\tau(H^0_Q(M), \mathscr{C})_Q) = H^0(M_Q)$ completing the proof. □

3.4. Test modules as sums of test modules of principal Cartier algebras. Our goal in this section is to show that the test ideal $\tau(R, \mathscr{C})$ is the sum of the test modules generated by principal sub-Cartier algebras of $\mathscr{C}$, at least when R is reduced (which implies that every associated prime of R, or any submodule $I \subseteq R$, is minimal).

Proposition 3.29. *Suppose that R is an F-finite reduced Noetherian ring, $\mathscr{C}$ is a Cartier algebra and we have that $I \subseteq R$ is an $\mathscr{C}$-module. Then there exists a finitely generated Cartier $\mathscr{C}' \subseteq \mathscr{C}$ so that*

$$\tau(I, \mathscr{C}') = \tau(I, \mathscr{C}).$$

Proof. We have a map $\mathscr{C} \longrightarrow \mathscr{C}^R$ (the full Cartier algebra), and so replacing $\mathscr{C}$ by its image, we may assume that $\mathscr{C} \subseteq \mathscr{C}^R$. Since the test modules exist by Theorem 3.28, by Exercise 3.1, for each finitely generated $\mathscr{C}'' \subseteq \mathscr{C}' \subseteq \mathscr{C}$ we have that $\tau(I, \mathscr{C}'') \subseteq \tau(I, \mathscr{C}')$. On the other hand, suppose $x \in \tau(I, \mathscr{C})$. Then choosing c in R not in any minimal prime of *the module* I so that $I_c = R_c$ and also that $(R_c, \mathscr{C}_c)$ and $(R_c, \mathscr{K}_c)$ are both F-regular by Exercise 3.5, we have that

$$x \in \sum_{i \geq 0} \mathscr{C}^i_+(cI) = \tau(I, \mathscr{C}).$$

It follows that $x \in \sum_{i=1}^t \phi_i cI$ for some $\phi_i \in \mathscr{C}_{e_i}$. Now then, if $\mathscr{L}$ is the Cartier algebra generated by the ϕ_i's, we see that $x \in \tau(I, \mathscr{L})$. Hence $\tau(I, \mathscr{C})$ is the filtered union of the $\tau(I, \mathscr{C}')$ for $\mathscr{C}'$ finitely generated. This proves the result by the Noetherian property. □

Now suppose that $\phi_1, \cdots, \phi_m \in \mathscr{C}_+$ are homogeneous elements generating the $\mathscr{C}'$ we constructed in Proposition 3.29. If $\mathscr{C}_i$ is the Cartier algebra generated by ϕ_i, we can find c_i not in any minimal prime of I so that $\tau(R_{c_i}, (\mathscr{C}_i)_{c_i}) = \sigma(R_{c_i}, (\mathscr{C}_i)_{c_i})$. The product $c = c_1 \cdots c_m$ will not be in any associated prime of I and so $\tau(R_c, (\mathscr{C}_i)_c) = \sigma(R_c, (\mathscr{C}_i)_c)$ as well. What now if you took $\phi_1 \circ \phi_2$, or more generally an arbitrary monomial ϕ in the ϕ_i? We will need to find a c so that $\tau(R_c, (\mathscr{C}_i)_c) = \sigma(R_c, (\mathscr{C}_i)_c)$ for all c.

Lemma 3.30. *With notation as in Proposition 3.29, suppose $\mathscr{C}$ is a finitely generated Cartier algebra generated by homogeneous $\phi_1, \ldots, \phi_m$. Then there exists a c not in any minimal prime of the* module I *such that for any*

$\phi = \phi_{i_1} \circ \cdots \circ \phi_{i_s}$ *a monomial in the* ϕ_i*, we have that* $\tau(I_c, (R(\phi))_c) = \sigma(I_c, (R(\phi))_c)$.

PROOF. We may assume that $\mathscr{C} \subseteq \mathscr{C}^R$, the full Cartier algebra. Choose c_0 not in any minimal prime of the module I such that

$$I_{c_0} = R_{c_0} = \prod R_j$$

is a finite product of regular domains (on for each minimal prime Q_j of the module I). Let $R' = R_{c_0}$, $\mathscr{C}' = \mathscr{C}_{c_0}$, and ϕ'_i the image of ϕ_i in $\mathscr{C}'$. Notice that each ϕ'_i induces a Cartier action on each $R_j = H^0_{Q_j}(R_{c_0})$.

For each minimal prime Q_j of I and each ϕ_i, we have two cases:

(a) ϕ_i is zero on R_j, in which case set $c_{i,j} = 1$.
(b) ϕ_i non-zero on R_j, in which case choose $c_{i,j} \in R$ not in any minimal prime of I so that if $\widetilde{R} = (R_j)_{c_{i,Q}}$, then the image of ϕ_i generates $\mathrm{Hom}_{p^{-e_i}}(\widetilde{R}, \widetilde{R})$.

Let $c = c_0 \cdot \prod_{i,j} c_{i,j}$. Then $I_c = R_c$ is a product of regular domains $R_{c,j}$.

Consider a monomial ϕ as in the statement of the lemma, and its induced Cartier action $\phi_{c,j}$ on $R_{c,j}$. Either we have that $\phi_{c,j}$ is a composition of generators of Hom-sets and so $\phi_{c,j}$ also generates a Hom-set, or we have that $\phi_{c,j} = 0$. In either case we have that

$$\tau(R_{c,j}, R(\phi_{c,j})) = \sigma(R_{c,j}, R(\phi_{c,j})).$$

The result follows. $\square$

Theorem 3.31. *Suppose that R is an F-finite reduced Noetherian ring and that $\mathscr{C}$ is a Cartier algebra with an action on an ideal $I \subseteq$. Then*

$$\tau(I, \mathscr{C}) = \sum_{e>0} \sum_{\phi \in \mathscr{C}_e} \tau(I, R(\phi)).$$

In particular, since R is Noetherian, $\tau(I, \mathscr{C})$ is a finite *sum of test modules of principal Cartier algebras.*

PROOF. The containment $\supseteq$ is clear. We may assume that $\mathscr{C}$ is finitely generated by $\phi_i \in \mathscr{C}_{e_i}$ by Proposition 3.29.

Choose $c \in R$ not in any minimal prime of the module I, as in Lemma 3.30 and so that $\tau(I_c, \mathscr{C}_c) = \sigma(I_c, \mathscr{C}_c)$ using Exercise 3.5. Then

$$\tau(I, \mathscr{C}) = \sum_{i \geq 0} \mathscr{C}^i_+(cI)$$

by Proposition 3.24. In particular, we thus obtain that

$$\tau(I, \mathscr{C}) = \sum_{\psi} \psi(cI)$$

where ψ runs over monomials in the ϕ_i. But on the other hand, each $\psi(cI) \subseteq \tau(I, R(\psi))$ again by Proposition 3.24. Thus

$$\tau(I, \mathscr{C}) \subseteq \sum_{\psi} \tau(I, R(\psi)) \subseteq \sum_{e>0} \sum_{\phi \in \mathscr{C}_e} \tau(I, R(\phi)).$$

where the inner sum runs over ψ monomials in the ϕ_i. This completes the proof. □

We immediately obtain the following corollary.

Corollary 3.32. *Suppose R is an F-finite normal domain and $\mathscr{C}$ is a Cartier algebra with an action on R. Then*

$$\tau(R, \mathscr{C}) = \sum_{e \geq 0} \sum_{\phi \in \mathscr{C}_e} \tau(R, \Delta_\phi).$$

Proof. If $\phi \in \mathscr{C}_e$, then $\tau(R, \Delta_\phi) = \tau(R, \phi) = \tau(R, R(\phi))$ where the first equality is Chapter 5 Theorem 5.24 and the second is essentially by definition. The result now follows from Theorem 3.31. □

Conjecture 3.33. *With notation as in Theorem 3.31, there exists a single principal Cartier algebra $R(\psi)$ so that $\tau(I, \mathscr{C}) = \tau(I, R(\psi))$.*

3.5. Exercises.

Exercise 3.1. Suppose that R is F-finite and Noetherian, $\mathscr{C}$ is a Cartier algebra, and M is a $\mathscr{C}$-module. If $\mathscr{C}' \subseteq \mathscr{C}$ is a sub-Cartier-algebra, show that $\sigma(M, \mathscr{C}') \subseteq \sigma(M, \mathscr{C})$ and show that $\tau(M, \mathscr{C}') \subseteq \tau(M, \mathscr{C})$ assuming the test modules exist.

Exercise 3.2. Prove the analog of Lemma 1.11 for Cartier algebras. Explicitly, suppose R is an F-finite Noetherian ring, $\mathscr{C}$ is a Cartier algebra, and M is an $\mathscr{C}$-module. Suppose further that $W \subseteq R$ is multiplicative set and $N' \subseteq W^{-1}M$ is an $W^{-1}\mathscr{C}$-submodule. Show that there exists an $\mathscr{C}$-module $N \subseteq M$ such that $W^{-1}N = N'$.

Exercise 3.3. Suppose that R is an Noetherian ring, $\mathscr{C}$ is a Cartier algebra and M is an F-pure $\mathscr{C}$-module. Show that if $c \in R$ is such that $c^t M = 0$, then $cM = 0$, generalizing Lemma 1.5.

Exercise 3.4. Suppose R is an F-finite Noetherian ring with a unique associated prime Q, but that R is non-reduced. Suppose that $\mathscr{C}$ is a Cartier algebra and R is an $\mathscr{C}$-module. Prove that $\underline{R} \subseteq Q$.

Exercise 3.5. Suppose that R is F-finite and Noetherian, $\mathscr{C}$ is a Cartier algebra and M is an $\mathscr{C}$-module that is finite as an R-module. Suppose further that every associated prime of M is minimal and that $\tau(M, \mathscr{C})$ exists. Prove that there exists a $c \in R$ such that $c \notin Q$ for every associated prime of M, and such that $(M_c, \mathscr{C}_c)$ is F-regular. This can be viewed as a partial converse of Proposition 3.24.

Exercise 3.6. Suppose that R is F-finite and Noetherian, $\mathscr{C}$ is a Cartier algebra and M is an $\mathscr{C}$-module that is finite as an R-module and such that every associated prime of R is minimal. If $\mathscr{C}' \subseteq \mathscr{C}$ is a Cartier subalgebra and $\tau(M, \mathscr{C}')$ exists, show that $\tau(M, \mathscr{C})$ exists as well.

4. Lyubeznik's F-modules

Warning, this section is not yet written.

CHAPTER 9

Hilbert-Kunz multiplicity and F-signature

In Chapter 1 we saw that an F-finite local ring $(R, \mathfrak{m})$ is regular if and only if $F^e_* R$ is a free R-module for some, or equivalently every, $e > 0$. F-signature and Hilbert-Kunz multiplicity are measurements of how free $F^e_* R$ is, asymptotically.

Suppose for the moment that $(R, \mathfrak{m}, k = k^p)$ is an F-finite local domain of dimension d with perfect residue field k, then

Hilbert-Kunz mulitiplicity: measures $\mu_R(F^e_* R)$, the number generators it takes to generate $F^e_* R$ as an R-module, compared to a regular ring of the same dimension. Explicitly:

$$e_{\mathrm{HK}}(R) = \lim_{e \longrightarrow \infty} \frac{\mu_R(F^e_* R)}{p^{ed}}.$$

Slogan: More singular rings have larger Hilbert-Kunz multiplicity.

See Definition 1.4 and Example 1.7.

F-signature: measures how many simultaneous free R-module summands $F^e_* R$ has (denoted $\mathrm{frk}(F^e_* R)$), compared to a regular ring of the same dimension. In other words, it measures the *percentage* of $F^e_* R$ that is a free R-module (asymptotically).

$$s(R) = \lim_{e \longrightarrow \infty} \frac{\mathrm{frk}(F^e_* R)}{p^{ed}}$$

Slogan: More singular rings have smaller F-signature.

See Definition 1.9

Note if R is regular and k is perfect, $F^e_* R$ is free of rank p^{ed} – reduce to the complete case and then see Example 1.19 in Chapter 1.

1. Definitions of Hilbert-Kunz multiplicity and F-signature

Both Hilbert-Kunz multiplicity and F-signature are defined as limits, whose existence is not obvious. Indeed, Kunz studied the ratio defining Hilbert-Kunz multiplicity in [**Kun76**] but its existence as a limit was shown by [**Mon83**]. Likewise Smith-Van Den Bergh [**SVdB97**] implicitly studied the limit defining F-signature, Huneke-Leuschke [**HL02**] explicitly defined it as an object of interest, but it was not shown to exist until work of Tucker [**Tuc12**] (although several special cases were known before [**HL02, Yao06, AE06, Abe08**], for instance of the ring was $\mathbb{Q}$-Gorenstein outside a set of codimension ≤ 1). We will not cover all we might about these important topics: for additional reading see the surveys [**Hun13, Cha21**].

We follow the approach of Polstra-Tucker [**PT18**] to show that F-signature and Hilbert-Kunz multiplicity both exist.

Before embarking on that, we need some preliminaries related to lengths of modules.

1.1. Preliminaries on lengths of modules. We begin with some simple observations about fields K of characteristic $p > 0$. In what follows, note that we can identify $K^{p^e} \subseteq K$ with $K \subseteq F^e_* K$, or with $K \subseteq K^{1/p^e}$.

Proposition 1.1. *Suppose that K is an F-finite field.*

(a) For any $e > 0$,
$$[F^e_* K : K] = [F_* K : K]^e \quad \textit{or equivalently} \quad [K : K^{p^e}] = [K : K^p]^e.$$

(b) For any finite extension $L \supseteq K$ we have that $[F_* L : L] = [F_* K : K]$.

(c) If $L \supseteq K$ is a finite separable extension of K, then, working in a fixed algebraic closure, $L \otimes_K K^{1/p^e} \cong LK^{1/p^e} = L^{1/p^e}$.

Proof. For (a), notice that $[K : F^2_* K] = [K : F_* K][F_* K : F^2_* K] = [K : F_* K]^2$, now repeat.

For (b), the commutative diagram

$$\begin{tikzcd} L & K \arrow[l, hook'] \\ L^p \arrow[u, hook] & K^p, \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

ensures that $[L : K][K : K^p] = [L : K^p] = [L : L^p][L^p : K^p]$. But $[L^p : K^p] = [L : K]$, now cancel.

Part (c) follows from linear disjointness (*cf.* Chapter 1 Proposition 5.9). $\square$

We will need the following observations about lengths of modules over Noetherian rings. We did a special case of this in Proposition 1.10 in Chapter 1.

Proposition 1.2. *Suppose $(R, \mathfrak{m}, k)$ is a Noetherian local ring and M is a finitely generated R-module (frequently $M = R$ or a quotient thereof).*

(a) If $(R, \mathfrak{m}, k) \longrightarrow (S, \mathfrak{n}, l)$ is a faithfully flat extension of Noetherian local rings with $\mathfrak{m}S = \mathfrak{n}$ then for every $e > 0$,

$$\ell_R\left(\frac{M}{\mathfrak{m}^{[p^e]}M}\right) = \ell_S\left(\frac{(M \otimes_R S)}{\mathfrak{n}^{[p^e]}(M \otimes_R S)}\right).$$

Frequently we take $S = \widehat{R} \otimes_k \overline{k}$ (which has algebraically closed residue field and hence is F-finite).

(b) If M has finite length and k is F-finite, then

$$\ell_R(F^e_* M) = [k : k^p]^e\, \ell_R(M) \text{ and so} \tag{1.2.1}$$

$$\mu_R(F^e_* R) = [k : k^p]^e\, \ell_R\Big(R/\mathfrak{m}^{[p^e]}\Big). \tag{1.2.2}$$

(c) If k is perfect, then $\ell_R(F^e_* M) = \ell_R(M)$ and $\mu_R(F^e_* R) = \ell_R\big(R/\mathfrak{m}^{[p^e]}\big)$

Proof. For Proposition 1.2 (a), since $\mathfrak{n}^{[p^e]} = \mathfrak{m}^{[p^e]}S$ we have that

$$\frac{M}{\mathfrak{m}^{[p^e]}M} \otimes_R S \cong \frac{M \otimes_R S}{\mathfrak{n}^{[p^e]}(M \otimes_R S)}.$$

The result follows by [**Sta19**, Tag 02M1].

For (b), since $R \to F^e_* R$ is a module finite map of rings, we have, using Proposition 1.1 (a), that

$$\ell_R(F^e_* M) = [F^e_* k : k]\ell_{F^e_* R}(F^e_* M) = [k : k^p]^e \ell_R(M),$$

so (1.2.1) follows. For the second statement, we use Nakayama's Lemma to see that

$$\mu_R(F^e_* R) = \ell_R\left(\frac{F^e_* R}{\mathfrak{m} F^e_* R}\right) = \ell_R\left(F^e_* \frac{R}{\mathfrak{m}^{[p^e]}}\right)$$

So the statement follows from (1.2.1) applied to the finite length module $M = R/\mathfrak{m}^{[p^e]}$. (c) is just a special case of (b). $\square$

The following simple inequality will become quite important to us.

Lemma 1.3. *Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian local ring of dimension d. Then for every $e > 0$,*

$$\ell_R\Big(R/\mathfrak{m}^{[p^e]}\Big) \geq (p^e)^d.$$

If R is regular, this is equality.

PROOF. By Proposition 1.2 (a), we may assume that $(R, \mathfrak{m}, k)$ is complete with algebraically closed residue field $k = \overline{k}$. Furthermore, modding out a by a minimal prime will not increase length and so we may assume that R is an integral domain.

Now, $\ell_R\big(R/\mathfrak{m}^{[p^e]}\big) = \mu_R(F^e_* R)$ (Proposition 1.2 (c)) and $\mu_R(F^e_* R) \geq [F^e_* L : L]$, where L is the fraction field of R (module generators remain generators after localization). By the Noether-Cohen normalization $R \supseteq A = k[\![x_1, \dots, x_d]\!]$, and that $[F^e_* L : L] = [F^e_* K : K]$ where K is the fraction field of A by Proposition 1.1 (b). However, $[F^e_* K : K] = p^{ed}$ (Chapter 1 Example 1.19), so that $\ell_R\big(R/m^{[p^e]}\big) \leq [F^e_* L : L] = [F^e_* K : K] = p^{ed}$, completing the proof. $\square$

We will see below in Corollary 1.12 (for domains, and then more generally in Exercise 2.2) that

$$e_{\mathrm{HK}}(R) \geq 1.$$

1.2. Hilbert-Kunz multiplicity. We are now ready to define Hilbert-Kunz multiplicity.

Definition 1.4. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d. We define the **Hilbert-Kunz multiplicity of** R to be the limit:

$$e_{\mathrm{HK}}(R) := \lim_{e \longrightarrow \infty} \frac{\ell_R\big(R/\mathfrak{m}^{[p^e]}\big)}{p^{ed}}$$

if it exists.[1]

We have already seen in Lemma 1.3 that $\ell_R\big(R/\mathfrak{m}^{[p^e]}\big) \geq p^{ed}$ and so $e_{\mathrm{HK}}(R) \geq 1$. It is immediate that $R/\mathfrak{m}^{[p^e]} \cong \widehat{R}/\widehat{\mathfrak{m}}^{[p^e]}$ where $\widehat{-}$ denotes the $\mathfrak{m}$-adic completion. The following lemma is an immediate consequence.

Lemma 1.5. *With notation as in Definition 1.4, we have that* $e_{\mathrm{HK}}(R) = e_{\mathrm{HK}}(\widehat{R})$.

Before moving on to F-signature, we give an example and then reinterpret Hilbert-Kunz multiplicity (as promised the introduction to the chapter) when R is F-finite.

Example 1.6 (Regular rings)**.** Suppose that $(R, \mathfrak{m})$ is a regular local ring. By Lemma 1.5 we may assume R is complete and so $R \cong k[\![x_1, \dots, x_d]\!]$ by the Cohen-Structure Theorem. In this case $\ell_R\big(R/\mathfrak{m}^{[p^e]}\big) = p^{ed}$ since $R/\mathfrak{m}^{[p^e]}$ is generated by the monomials $x_1^{a_1} \cdots x_d^{a_d}$ for $0 \leq a_i < p^e$. In particular, we immediately see that

$$e_{\mathrm{HK}}(R) = 1.$$

[1] It does as we shall see.

Example 1.7. (Perfect and imperfect residue fields) Suppose that $(R, \mathfrak{m}, k)$ is an F-finite Noetherian local ring with k a perfect field. Then for any finitely generated module M, the minimal number of generators of M is equal to $\ell_R(M/\mathfrak{m}M)$ by Nakayama's lemma. By Proposition 1.2 (b) and using that k is perfect, we have that the number of generators of $F^e_* M$ as an R-module is equal to

$$\ell_R((F^e_* M)/\mathfrak{m}M) = \ell_R\Big(F^e_*(M/\mathfrak{m}^{[p^e]}M)\Big) = \ell_R\Big(M/\mathfrak{m}^{[p^e]}M\Big).$$

In other words, the numerator of the fraction defining the Hilbert-Kunz multiplicity is exactly the minimal number of generators of $F^e_* R$ as an R-module. Hence, in view of Example 1.6, we see that

$$e_{\mathrm{HK}}(R) = \lim_{e \longrightarrow \infty} \frac{\text{Number of generators of } F^e_* R}{\text{Expected number of generators of } F^e_* R \text{ if } R \text{ was regular}}.$$

Written out more tersely, in the case that k is *perfect* and R is dimension d, we have that

$$e_{\mathrm{HK}}(R) = \lim_{e \longrightarrow \infty} \frac{\mu(F^e_* R)}{p^{ed}}.$$

If the residue field in *imperfect*, then $\mu(F^e_* R)$ is not the length of $R/\mathfrak{m}^{[p^e]}$. If k is F-finite, set $p^\gamma = [k : k^p]$. We have that

$$\mu(F^e_* R) = \ell_R((F^e_* R)/\mathfrak{m}) = \ell_R\Big(F^e_* R/\mathfrak{m}^{[p^e]}\Big) = p^{\gamma e}\, \ell_R\Big(R/\mathfrak{m}^{[p^e]}\Big)$$

by Proposition 1.2 (b). Hence, even if k is imperfect but F-finite we have that

$$e_{\mathrm{HK}}(R) = \lim_{e \longrightarrow \infty} \frac{\mu(F^e_* R)}{p^{e(d+\gamma)}} \tag{1.7.1}$$

We now move on to the definition of F-signature.

1.3. F-signature. Now we measure how free the module $F^e_* R$ is in a different way. Suppose $(R, \mathfrak{m}, k)$ is a local ring and M is a finite R-module (we will shortly take $M = F^e_* R$).

Definition 1.8. We define the **free rank of** M, denoted $\mathrm{frk}(M)$, to be the maximal number a such that there exists a surjection of R-modules $M \twoheadrightarrow R^{\oplus a}$. Notice that this surjection must split since $R^{\oplus a}$ is free and hence projective.

In particular, if $\mathrm{frk}(M) = a$, then we can write $M \cong R^{\oplus a} \oplus N$ where N has no free-module summands (this decomposition is not unique). Conversely, if $M \cong R^{\oplus b} \oplus L$ where L has no R-module summand, then by Lemma 1.13 below, we will see below that $\mathrm{frk}(M) = b$.

We can now define F-signature.

Definition 1.9 (F-signature)**.** Suppose $(R, \mathfrak{m})$ is an F-finite local domain of characteristic $p > 0$ and dimension d. The **F-signature of** R, is defined to be (if the limit exists[2]):

$$s(R) := \lim_{e \longrightarrow \infty} \frac{\operatorname{frk}(F^e_* R)}{\operatorname{rank}(F^e_* R)}$$

where $\operatorname{rank}(F^e_* R) = [F^e_* \mathcal{K}(R) : \mathcal{K}(R)]$ is the generic rank.

Intuitively, $s(R)$ is the *fraction of* $F^e_* R$ that is a free R-module, at least as $e \longrightarrow \infty$. In other words:

$$0 \leq s(R) \leq 1.$$

Remark 1.10. We point out that if instead of assuming R is a domain we assume it is reduced and equidimensional, then $\operatorname{rank} F^e_* R$ still makes sense (the number $[F^e_* \mathcal{K}(R/Q) : \mathcal{K}(R/Q)]$ is independent of the minimal prime $Q \subseteq R$ by Proposition 1.11 below). Hence one can still *define* F-signature via the formula above (and the limit still exists). However, it turns out that if $s(R) > 0$, even with this definition, then R is normal and hence R is a domain.

Before diving into some alternate ways to measure *free rank*, we spend a little time understanding (generic) *rank for F-finite rings.*

Proposition 1.11. *Let R be a Noetherian F-finite ring. Suppose that $Q, \mathfrak{q} \in$* $\operatorname{Spec} R$ *with $Q \subseteq \mathfrak{q}$. Then for any $e \geq 0$*

$$\begin{aligned} [F^e_* \mathcal{K}(R/Q) : \mathcal{K}(R/Q)] &= [F^e_* \mathcal{K}(R/\mathfrak{q}) : \mathcal{K}(R/\mathfrak{q})]\, p^{e \dim(R_\mathfrak{q}/QR_\mathfrak{q})} \\ &= [F_* \mathcal{K}(R/\mathfrak{q}) : \mathcal{K}(R/\mathfrak{q})]^e\, p^{e \dim(R_\mathfrak{q}/QR_\mathfrak{q})} \end{aligned}$$

Proof. By modding out by Q we may assume that $Q = 0$ and by localizing at $\mathfrak{q}$ we may assume that $\mathfrak{q}$ is maximal; hence we may assume that $(R, \mathfrak{m} = \mathfrak{q}, k = \mathcal{K}(R/\mathfrak{q}))$ is local. By Exercise 1.11 we may assume that $(R, \mathfrak{m}, k)$ is a complete domain. By taking a Noether normalization $A = k[\![x_1, \ldots, x_n]\!] \subseteq R$ and using Proposition 1.1 (b) as well as a direct computation (Chapter 1 Example 1.19), the first equality follows. The second equality is simply Proposition 1.1 (a). $\square$

The argument above does not use the excellence of R. In fact, this is the key observation needed to show that F-finite rings are *catenary*[3] (**[Kun76]**), a part of the definition of excellence, see Exercise 1.12.

[2]It does, as we shall see.

[3]For any two fixed primes, $\mathfrak{q}, Q$, the length of a maximal chain of intermediate primes is constant.

Regardless, it follows that if $(R, \mathfrak{m}, k)$ is an F-finite domain of dimension d, then by Proposition 1.11 (setting $Q = (0)$ and $\mathfrak{q} = \mathfrak{m}$) we see that

$$\operatorname{rank}_R(F^e_* R) = p^{ed}[F^e_* k : k] = p^{ed}[F_* k : k]^e. \tag{1.11.1}$$

Before continuing, we point out a corollary for Hilbert-Kunz multiplicity.

Corollary 1.12. *Suppose $(R, \mathfrak{m})$ is a Noetherian F-finite local domain. Then*

$$e_{\mathrm{HK}}(R) = \lim_{e \longrightarrow \infty} \frac{\mu(F^e_* R)}{\operatorname{rank} F^e_* R}.$$

In particular, $e_{\mathrm{HK}}(R) \geq 1$.

Proof. We simply apply (1.11.1) to (1.7.1). The final statement follows since the rank cannot exceed the minimal number of generators. $\square$

1.4. Observations on free rank. We point out some basic facts about free rank.

Lemma 1.13 ([**PT18**, Lemma 2.1])**.** *Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian local domain and M, M' are finite R-modules and $N \subseteq M$.*

(a) If $M \cong R^{\oplus b} \oplus L$ where $\operatorname{frk}(L) = 0$, then $\operatorname{frk}(M) = b$.
(b) We have that $\operatorname{frk}(M \oplus M') = \operatorname{frk}(M) + \operatorname{frk}(M')$.
(c) We have $\operatorname{frk}(M/N) \leq \operatorname{frk}(M)$
(d) We have $\operatorname{frk}(M) \leq \operatorname{frk}(N) + \mu(M/N)$ where $\mu(-)$ is the minimal number of generators of $(-)$.

We closely follow the proof of [**PT18**, Lemma 2.1].

Proof. We first show (a). Indeed, suppose that there is a surjection

$$\phi : R^{\oplus b} \oplus L \twoheadrightarrow R^{\oplus a}$$

where $a > b$ and L has no free R-module summands. We view both $R^{\oplus b}$ and L as submodules of the source of ϕ and so we see that $\operatorname{Image}(\phi) = \phi(R^{\oplus b}) + \phi(L)$. Since L has no free R-module summands, we claim that $\phi(L) \subseteq \mathfrak{m} R^{\oplus a}$. This will complete the proof since then the image of ϕ modulo $\mathfrak{m}$ will be the same as $\phi(L)$ modulo $\mathfrak{m}$, and $R^{\oplus b} \longrightarrow R^{\oplus a}$ cannot be surjective. Thus to see the claim, observe that if an element of $\phi(L) \subseteq R^{\oplus a}$ has a unit in some entry, then projecting on to that entry provides a surjection from $\phi(L)$ to R, and hence a surjection from L to R. This is impossible since L has no free R-module summands. This proves (a).

For (b), notice if we set $a = \operatorname{frk}(M)$ and $a' = \operatorname{frk}(M)$, then we can write

$$M \oplus M' = R^{\oplus a} \oplus L \oplus R^{\oplus b} \oplus L'$$

where L, L' have no free summands. It thus suffices to show that $L \oplus L'$ has no free R-module summands. Indeed, suppose that there was a surjective map $\phi : L \oplus L' \longrightarrow R$. By hypothesis, $\phi(L) \subseteq \mathfrak{m}$ and $\phi(L') \subseteq \mathfrak{m}$, but then $\phi(L \oplus L') = \phi(L) + \phi(L') \subseteq \mathfrak{m} \subsetneq R$, a contradiction.

For (b), let $c = \operatorname{frk}(M/N)$. Then we have a surjective map $M/N \twoheadrightarrow \oplus R^{\oplus c}$. Hence we have a surjective map $M \twoheadrightarrow M/N \twoheadrightarrow \oplus R^{\oplus c}$ and so $\operatorname{frk}(M) \geq c$ as desired.

Finally, for (d), write $a = \operatorname{frk}(M)$, $b = \operatorname{frk}(N)$. Let d be the maximal number such that there exists a surjective:

$$\phi : M \twoheadrightarrow R^{\oplus d}$$

and such that $\phi|_N$ is also surjective (that is, $\phi(N) = R^{\oplus d}$). It may happen that $d = 0$. Set $K = \ker \phi$ and $L = \ker \phi|_N$ so that $L \subseteq K$ and so that both $M \cong R^{\oplus d} \oplus K$ and $N \cong R^{\oplus d} \oplus L$. Now, if $\psi : K \twoheadrightarrow R^{\oplus \operatorname{frk}(K)}$ is surjective, then notice that $\psi(L) \subseteq \mathfrak{m}\, R^{\oplus \operatorname{frk}(K)}$ as if not, we could have made d bigger. In particular, we have a surjection induced by ψ:

$$\overline{\psi} : K/L \longrightarrow (R/\mathfrak{m})^{\oplus \operatorname{frk}(K)} = k^{\oplus \operatorname{frk}(K)}.$$

But $K/L \cong M/N$. Thus we have a surjection

$$\frac{M/N}{\mathfrak{m}\,(M/N)} \twoheadrightarrow k^{\oplus \operatorname{frk}(K)}.$$

It follows that $\mu(M/N) \geq \operatorname{frk}(K)$ and so:

$$\operatorname{frk}(M) = d + \operatorname{frk}(K) \leq \operatorname{frk}(N) + \mu(M/N)$$

since $d \leq \operatorname{frk}(N)$. □

Another useful way to compute the free rank is the following construction.

Definition 1.14. Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring and M is a finite R-module. Denote the set:

$$I(M) := \big\{ x \in M \mid \phi(x) \in \mathfrak{m} \text{ for all } \phi \in \operatorname{Hom}_R(M, R) \big\}$$

It is not difficult to verify that $I(M)$ is a submodule, see Exercise 1.1.

Lemma 1.15. *With notation as in Definition 1.14, we have that*

$$\operatorname{frk}(M) = \ell_R(M/I(M)).$$

In fact, if we have an R-module decomposition $M = R^{\oplus \operatorname{frk}(M)} \oplus L$, then $I(M) = \mathfrak{m} R^{\oplus \operatorname{frk}(M)} \oplus L$.

Proof. The first statement is a consequence of the second, and so we prove the second statement. Certainly $L \subseteq I(M)$ as L has no free R-summands. We also see that $\mathfrak{m}R^{\oplus \operatorname{frk}(M)}$ has no free R-summands since $\mathfrak{m}R$ has none. Thus $\mathfrak{m}R^{\oplus \operatorname{frk}(M)} \oplus L \subseteq I(M)$. Now suppose that $x = f \oplus l \in I(M)$ with $f \in R^{\oplus \operatorname{frk}(M)}$ and $l \in L$. Hence $f = x - l \in I(M)$ as well since $I(M)$ is a submodule of M. Thus $f \in I(R^{\oplus \operatorname{frk}(M)}) = \mathfrak{m}R^{\oplus \operatorname{frk}(M)}$. This is what we wanted to show. $\square$

In the case that $M = F^e_* R$, also have that $I(F^e_* R)$ is an $F^e_* R$-submodule of $F^e_* R$ by Exercise 1.1. In other words, it corresponds to an ideal of R. We make the following definition, originally appearing in [**AE05**].

Definition 1.16. Suppose that R is an F-finite local ring of characteristic $p > 0$. We define the ideal

$$I_e(R) = \{x \in R \mid F^e_* x \in I(F^e_* R)\}.$$

It follows from Lemma 1.15 that $\operatorname{frk}(F^e_* R) = \ell_R(R/I_e(R))$.

By Lemma 1.15 and Proposition 1.2 (b) we know that

$$\operatorname{frk}(F^e_* R) = \ell_R(F^e_* R/I_e(R)) = [F^e_* k : k]\ell_R(R/I_e(R)).$$

Hence, combining this with (1.11.1), we have that:

Lemma 1.17. *Suppose $(R, \mathfrak{m})$ is a Noetherian F-finite local domain.*

$$s(R) = \lim_{e \longrightarrow \infty} \frac{\ell_R(R/I_e(R))}{p^{ed}}$$

The formula in Lemma 1.17 looks very similar to our definition of Hilbert-Kunz multiplicity. We are just modding out by the $\mathfrak{m}$-primary[4] ideal $I_e(R)$ instead of the $\mathfrak{m}$-primary ideal $\mathfrak{m}^{[p^e]}$, see Exercise 1.2. We saw that Hilbert-Kunz multiplicity is bounded below by 1 whereas F-signature is bounded above by 1. The formula Lemma 1.17 for the F-signature in terms of these ideals $I_e(R)$ was first observed by Aberbach-Enescu and Yao in [**AE05, Yao06**].

We need additional facts about the ideals $I_e(R)$ which will be used in the proof of the existence of F-signature.

Lemma 1.18. *Suppose that $(R, \mathfrak{m})$ is an F-finite local domain of characteristic $p > 0$. Then we have that*

$$I_e(R)^{[p]} \subseteq I_{e+1}(R).$$

Furthermore, for any $\phi \in \operatorname{Hom}_R(F_* R, R)$, we have that

$$\phi(F_* I_{e+1}(R)) \subseteq I_e(R).$$

[4] unless $I_e(R) = R$

PROOF. Suppose $\phi \in \mathrm{Hom}_R(F^{e+1}_* R, R)$ and consider $\phi|_{F^e_* R} \in \mathrm{Hom}_R(F^e_* R, R)$. By definition
$$\phi|_{F^e_* R}(F^e_* x) \in \mathfrak{m}.$$
for each $x \in I_e(R)$. Since under the inclusion $F^e_* R \hookrightarrow F^{e+1}_* R$ we have $F^e_* x \mapsto F^{e+1}_* x^p$, we see that $x^p \in I_{e+1}(R)$. Hence $I_e(R)^{[p]} \subseteq I_{e+1}(R)$ as desired.

The second statement we leave to the reader in Exercise 1.4. $\square$

Note that the sequence ideals $\mathfrak{m}^{[p^e]}$ also satisfy the containment from Lemma 1.18.

1.5. Existence. We now follow the existence proof of Polstra-Tucker. We begin with several lemmas.

Lemma 1.19. *Suppose that R is an F-finite Noetherian domain and* $\mathrm{rank}(F_* R) = p^m$. *Then there exists a short exact sequence of R-modules:*
$$0 \longrightarrow R^{\oplus p^m} \longrightarrow F_* R \longrightarrow M \longrightarrow 0$$
where M is torsion.

PROOF. At the level of fraction fields, we have an isomorphism
$$\mathcal{K}(R)^{\oplus p^m} \longrightarrow F_* \mathcal{K}(R).$$
Restricting this to $R^{\oplus p^m} \subseteq \mathcal{K}(R)^{\oplus p^m}$ and clearing denominators (of the images of a basis for $R^{\oplus p^m}$) if necessary produces an injective map $R^{\oplus p^m} \hookrightarrow F_* R$. The cokernel is torsion since tensoring it with $\mathcal{K}(R)$ is zero. $\square$

Lemma 1.20. *Suppose $(R, \mathfrak{m})$ is an F-finite Noetherian local ring of characteristic $p > 0$ and M is a finite R-module. If $D = \dim \mathrm{Supp}\, M$, there is a constant C such that*
$$\ell_R\Big(M/(\mathfrak{m}^{[p^e]} M)\Big) \leq p^{eD} C$$
for all $e > 0$. In particular, if $d = \dim R > \dim \mathrm{Supp}\, M$, then
$$\lim_{e \longrightarrow \infty} \frac{\ell_R\big(M/(\mathfrak{m}^{[p^e]} M)\big)}{p^{ed}} = 0.$$

PROOF. For the first statement, if $J = \mathrm{Ann}_R M$ then the R-length of $M/\mathfrak{m}^{[p^e]} M$ is equal to its R/J-length, hence we can assume that $R = R/J$. Now, if $R^{\oplus m} \longrightarrow M$ is surjective, then it stays surjective after modding out by $\mathfrak{m}^{[p^e]}$ and it suffices to assume that $M = R^{\oplus m}$ and hence that $M = R$. Finally, notice that if $t = \mu(\mathfrak{m}) = \ell_R\big(\mathfrak{m}/\mathfrak{m}^2\big)$ then
$$\mathfrak{m}^{tp^e} \subseteq \mathfrak{m}^{[p^e]}.$$

Hence

$$\begin{aligned}\lim_{e \longrightarrow \infty} \frac{\ell_R\big(R/\mathfrak{m}^{[p^e]}\big)}{p^{eD}} &\leq \lim_{e \longrightarrow \infty} \frac{\ell_R\big(R/\mathfrak{m}^{tp^e}\big)}{p^{eD}} \\ &= \lim_{i \longrightarrow \infty} \frac{\ell_R\big(R/\mathfrak{m}^{ti}\big)}{i^D} \\ &= \lim_{i \longrightarrow \infty} t^D \frac{\ell_R\big(R/\mathfrak{m}^{ti}\big)}{(ti)^D} \\ &= t^D e(R)\end{aligned}$$

where $e(R)$ is the Hilbert-Samuel multiplicity of R, see for instance [**Mat89**, Chapter 13]. The second part of the lemma is a direct consequence of the first. □

We now need the following formal lemma which will imply existence.

Lemma 1.21. *Suppose p is prime, $d > 0$ is an integer, and $\{a_e\}_{e \geq 0}$ is a sequence of real numbers so that $\left\{\frac{a_e}{p^{ed}}\right\}_{e\geq 0}$ is bounded below. Suppose that*

$$\frac{a_{e+1}}{p^{(e+1)d}} \leq \frac{a_e}{p^{ed}} + \frac{C}{p^e}$$

for some constant C. Then the limit

$$\lim_{e \longrightarrow \infty} \frac{a_e}{p^{ed}} =: \eta$$

exists and is finite. Furthermore, $\eta - \frac{a_e}{p^{ed}} \leq \frac{2C}{p^e}$.

Proof. Consider what happens when we iterate our inequality:

$$\frac{a_{e+2}}{p^{(e+2)d}} \leq \frac{a_{e+1}}{p^{(e+1)d}} + \frac{C}{p^{e+1}} \leq \frac{a_e}{p^{ed}} + \frac{C}{p^e} + \frac{C}{p^{e+1}} = \frac{a_e}{p^{ed}} + \frac{C}{p^e}\big(1 + \frac{1}{p}\big).$$

Continuing in this way we obtain that

$$\frac{a_{e+f}}{p^{(e+f)d}} \leq \frac{a_e}{p^{ed}} + \frac{C}{p^e}\Big(1 + \frac{1}{p} + \cdots + \frac{1}{p^{f-1}}\Big) \leq \frac{a_e}{p^{ed}} + \frac{2C}{p^e}.$$

Thus, taking lim sup by sending $f \longrightarrow \infty$ we obtain:

$$a^+ := \limsup_{g \longrightarrow \infty} \frac{a_g}{p^{gd}} = \limsup_{f \longrightarrow \infty} \frac{a_{e+f}}{p^{(e+f)d}} \leq \frac{a_e}{p^{ed}} + \frac{2C}{p^e}$$

Now, taking lim inf and sending $e \longrightarrow \infty$ we see that

$$a^+ \leq \liminf_{e \longrightarrow \infty} \frac{a_e}{p^{ed}} + \frac{2C}{p^e} = \liminf_{e \longrightarrow \infty} \frac{a_e}{p^{ed}} =: a^-$$

Thus the limit $\eta = a^+ = a^-$ exists as desired. For the final statement, we actually proved above that $a^+ - \frac{a_e}{p^{ed}} \leq \frac{2C}{p^e}$ in the displayed equation defining a^+. □

Theorem 1.22 ([**PT18**], *cf.* [**Mon83, Tuc12**]). *Suppose $(R, \mathfrak{m})$ is an F-finite local Noetherian domain of dimension d and that J_e is a sequence of $\mathfrak{m}$-primary ideals such that*

$$J_e^{[p]} \subseteq J_{e+1}.$$

Then the limit

$$\lim_{e \longrightarrow \infty} \frac{1}{p^{ed}} \ell_R(R/J_e)$$

exists. In particular, both F-signature and Hilbert-Kunz multiplicity exist for Noetherian domains.

This *only* proves the existence of Hilbert-Kunz multiplicity for R an F-finite domain, and we relegate the general proof to the next section (the strategy is to reduce to the case of an F-finite domain), see Theorem 2.26.

PROOF. Since J_1 is $\mathfrak{m}$-primary, we have that $\mathfrak{m}^{[p^f]} \subseteq J_1$ for some f, and so $\mathfrak{m}^{[p^{f+e-1}]} \subseteq J_e$ for all $e > 0$. Hence, truncating and then shifting the numbering of our sequence J_e, which does not impact the existence of the limit, we may assume that $\mathfrak{m}^{[p^e]} \subseteq J_e$ for all $e > 0$.

Suppose $k = R/\mathfrak{m}$ is the residue field and $p^\gamma = [k : k^p]$. By Proposition 1.11 we know that $p^{\gamma+d} = \operatorname{rank}(F_* R)$. Therefore, by using Lemma 1.19, we have the following:

$$0 \longrightarrow R^{\oplus p^{\gamma+d}} \xrightarrow{\kappa} F_* R \longrightarrow M \longrightarrow 0$$

where M is torsion. This is the only place we used the property that R is a domain, *cf.* Exercise 1.9.

Because we know that $J_e F_* R = F_* J_e^{[p]} \subseteq F_* J_{e+1}$, if we take κ and mod out by $F^e_* J_e$, we obtain a map

$$\overline{\kappa} : (R/J_e)^{\oplus p^{d+\gamma}} \twoheadrightarrow F_*(R/J_{e+1}).$$

Hence, we have that:

$$\begin{aligned} p^\gamma \ell_R(R/J_{e+1}) &= \ell_R(F_*(R/J_{e+1})) \\ &\leq \ell_R\Big((R/J_e)^{\oplus p^{d+\gamma}}\Big) + \ell_R(\operatorname{coker} \kappa) \\ &= p^{d+\gamma} \ell_R(R/J_e) + \ell_R(\operatorname{coker} \kappa). \end{aligned} \tag{1.22.1}$$

where we used Proposition 1.2 (b) for the first equality.

Claim 1.23. $\ell_R(\operatorname{coker} \overline{\kappa}) \leq \ell_R\big(M/\mathfrak{m}^{[p^e]} M\big)$.

PROOF. Notice that $\operatorname{coker}\overline{\kappa}$ is a quotient of $F_*(R/J_{e+1})$ and hence also a quotient of $F_*(R/\mathfrak{m}^{[p^{e+1}]})$. Since we know that $\mathfrak{m}^{[p^e]}$ annihilates $F_*(R/\mathfrak{m}^{[p^{e+1}]})$, it also must annihilate $\operatorname{coker}\overline{\kappa}$.

Next, we observe that $\operatorname{coker}\kappa$ is also a quotient of M. Hence it is a quotient of $M/\mathfrak{m}^{[p^e]}$. This proves the claim. $\square$

We return to the main proof. By Lemma 1.20, there exists a constant C, depending on M, so that $\ell_R\big(M/\mathfrak{m}^{[p^e]}M\big) \leq Cp^{e(d-1)}$ (recall that $\dim \operatorname{Supp} M < d = \dim R$). Combining this, Claim 1.23, and (1.22.1), we see that

$$p^\gamma \ell_R(R/J_{e+1}) \leq p^{d+\gamma}\ell_R(R/J_e) + Cp^{e(d-1)}.$$

We divide both sides by $p^{\gamma+(e+1)d}$ and so obtain that

$$\frac{\ell_R(R/J_{e+1})}{p^{(e+1)d}} \leq \frac{\ell_R(R/J_e)}{p^{ed}} + \frac{Cp^{-\gamma-d}}{p^e}.$$

This completes the proof by Lemma 1.21.

The existence of the F-signature limit follows from setting $J_e := I_e(R)$, noting it is $\mathfrak{m}$-primary by Exercise 1.2, and then using the first part of Lemma 1.18. $\square$

We note one corollary of the proof for later use.

Corollary 1.24 ([**PT18**, Theorem 4.3])**.** *With notation as in Theorem 1.22, and additionally assuming that $\mathfrak{m}^{[p]} \subseteq J_1$, there exists a constant D depending on R (but* not *on the sequence of J_e) so that if $\eta = \lim_{e\longrightarrow\infty} \frac{1}{p^{ed}}\ell_R(R/J_e)$, then*

$$\eta - \frac{1}{p^{ed}}\ell_R(R/J_e) \leq \frac{D}{p^e}$$

for all $e > 0$.

PROOF. Use the final statement of Lemma 1.21 and set $D = 2Cp^{-\gamma-d}$ where C is as in the proof (and note it does not depend on the J_e). $\square$

A similar strategy can also be used to prove the following result from which the existence of F-signature and Hilbert-Kunz multiplicity also follows (in the F-finite domain case).

Theorem 1.25 ([**PT18**], *cf.* [**Mon83, Tuc12**])**.** *Suppose $(R, \mathfrak{m})$ is an F-finite local Noetherian domain of dimension d and that J_e is a sequence of ideals satisfying two properties:*

(a) *there exists a nonzero* $\phi \in \operatorname{Hom}_R(F_* R, R)$ *such that* $\phi(F_* J_{e+1}) \subseteq J_e$ *for every* $e > 0$, *and*
(b) *there exists a* $\mathfrak{m}$*-primary ideal* I *such that* $I^{[p^e]} \subseteq J_e$ *for every* $e > 0$.

Then the limit

$$\lim_{e \longrightarrow \infty} \frac{1}{p^{ed}} \ell_R(R/J_e) := \eta$$

exists. Furthermore, if $I = \mathfrak{m}$ *then there exists a constant* D, *depending on* ϕ *(but not depending on the* J_e*), such that*

$$\frac{1}{p^{ed}} \ell_R(R/J_e) - \eta \leq D/p^e.$$

Proof. We omit the proof as it is similar in flavor to that of Theorem 1.22 and Corollary 1.24. As in the start of that proof, one may reduce to the case that $I = \mathfrak{m}$. We do point out that one replaces Lemma 1.19 and Lemma 1.21 with Exercise 1.6 and Exercise 1.7 respectively. Also, the second statement in Lemma 1.18 replaces the first for the application to F-signature. See Exercise 1.8. $\square$

1.6. Localization and semi-continuity. Suppose $(R, \mathfrak{m}, k)$ is a Noetherian F-finite local domain. If $Q \in \operatorname{Spec} R$, then we would like to compare Hilbert-Kunz multiplicity and F-signature for R and for R_Q. In general, we expect the singularities of R_Q to be *no more severe* than the singularities of R, and so we should expect

$$e_{\mathrm{HK}}(R) \geq e_{\mathrm{HK}}(R_Q) \quad \text{and} \quad s(R) \leq s(R_Q). \tag{1.25.1}$$

We prove this below.

Lemma 1.26. *With notation as above, (1.25.1) holds.*

Proof. F-signature is particularly easy. The point is that $\operatorname{frk} F^e_* R \leq \operatorname{frk} F^e_* R_Q$ while generic rank stays the same. Taking the limit proves the statement about F-signature.

For Hilbert-Kunz, the strategy is similar. A set of generators of $F^e_* R$ as an R-module certainly localize to the set of generators of $F^e_* R_Q$ as an R_Q-module, so $\mu_R(F^e_* R) \geq \mu_{R_Q}(F^e_* R_Q)$. Now, by (1.2.2) in Proposition 1.2 (b) we know that

$$\begin{array}{rl} & [F_* k : k]^e \, \ell_R\big(R/\mathfrak{m}^{[p^e]}\big) \\ = & \mu_R(F^e_* R) \\ \geq & \mu_{R_Q}(F^e_* R_Q) \\ = & [F_* k(Q) : k(Q)]^e \, \ell_{R_Q}\big(R_Q/(QR_Q)^{[p^e]}\big) \end{array} \tag{1.26.1}$$

where $k(Q) = \mathcal{K}(R_Q) = R_Q/QR_Q$. By Proposition 1.11 we know that
$p^{e \dim R}\,[F_* k : k]^e = p^{e \dim R_Q} p^{e \dim R/Q}\,[F_* k : k]^e = p^{e \dim R_Q}[F_* k(Q) : k(Q)]^e$.
Dividing (1.26.1) by this term and sending e to ∞ yields the desired

$$e_{\mathrm{HK}}(R) \geq e_{\mathrm{HK}}(R_Q).$$

□

It is then natural to ask if more is true. Suppose R is an F-finite but not necessarily local domain. If $\mathfrak{q} \in \operatorname{Spec} R$, one might actually hope that if $e_{\mathrm{HK}}(R_{\mathfrak{q}}) > a \in \mathbb{R}$, then there exists a neighborhood U of $\mathfrak{q}$ such that for all $\mathfrak{p} \in U$, we have that

$$e_{\mathrm{HK}}(R_{\mathfrak{p}}) > a$$

In other words, that the assignment $\operatorname{Spec} R \xrightarrow{\mathfrak{q} \mapsto e_{\mathrm{HK}}(R_{\mathfrak{q}})} \mathbb{R}$ is *upper semi-continuous*. This, and the analogous statement for F-signature, are both true.

Theorem 1.27 (**[Smi16, Pol18]**, *cf.* **[PT18, ES05]**). *Suppose R is a Noetherian F-finite domain*[5]*. Then the assignment*

$$\operatorname{Spec} R \xrightarrow{\mathfrak{q} \mapsto e_{\mathrm{HK}}(R_{\mathfrak{q}})} \mathbb{R}$$

is upper semi-continuous and the assignment

$$\operatorname{Spec} R \xrightarrow{\mathfrak{q} \mapsto s(R_{\mathfrak{q}})} \mathbb{R}$$

is lower semi-continuous.

Proof. We omit the proof, referring to the above sources instead. However, we say a few words about the idea, following **[PT18]**. Suppose $\mathfrak{q} \in \operatorname{Spec} R$ with $d = \dim R_{\mathfrak{q}}$. By combining Corollary 1.24 and Theorem 1.25, we constructed $D(\mathfrak{q})$ so that for the limiting value η we have that

$$\left| \eta - \frac{\ell_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/J_e)}{p^{ed}} \right| = \left| \eta - \frac{[F^e_* k(\mathfrak{q}) : k(\mathfrak{q})]\; \ell_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/J_e)}{\operatorname{rank}(F^e_* R)} \right| \leq \frac{D(\mathfrak{q})}{p^e}.$$

Now we restrict to the ideals $J_e = (\mathfrak{q}R_{\mathfrak{q}})^{[p^e]}$ or $J_e = I_e(R_{\mathfrak{q}})$. It turns out one can find a constant D that makes the above inequality simultaneously for each $R_{\mathfrak{q}}$. Indeed, by looking at how we constructed D, it was based primarily on properties of torsion cokernels M of short exact sequences, which can be constructed before localization, $0 \longrightarrow R^{\oplus \operatorname{rank}(F_* R)} \longrightarrow F_* R \longrightarrow M \longrightarrow 0$ or of $0 \longrightarrow F_* R \longrightarrow R^{\oplus \operatorname{rank}(F_* R)} \longrightarrow M \longrightarrow 0$. For example, the Hilbert-Samuel multiplicity of $R_{\mathfrak{q}}/\operatorname{Ann}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$ and number of generators of $M_{\mathfrak{q}}$ show up, which have upper bounds independent of $\mathfrak{q}$.

[5]these assumptions can be weakened

With a D that produces the above displayed inequality simultaneously at each prime $\mathfrak{q}$, the result then reduces to fixing a single $e \gg 0$ and then showing things semicontinuity of

$$\frac{[F^e_* k(\mathfrak{q}) : k(\mathfrak{q})]\ \ell_{R_\mathfrak{q}}\big(R_\mathfrak{q}/\mathfrak{q}^{[p^e]}\big)}{\operatorname{rank}(F^e_* R)} = \frac{\mu_{R_\mathfrak{q}}(F^e_* R_\mathfrak{q})}{\operatorname{rank}(F^e_* R)}$$

or of

$$\frac{[F^e_* k(\mathfrak{q}) : k(\mathfrak{q})]\ \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/I_e(R_\mathfrak{q}))}{\operatorname{rank}(F^e_* R)} = \frac{\operatorname{frk} R_\mathfrak{q} F^e_* R_\mathfrak{q}}{\operatorname{rank}(F^e_* R)}.$$

$\square$

Remark 1.28. In [**DSPY19**], De Stefani, Polstra, and Yao showed that one could naturally define Hilbert-Kunz multiplicity and F-signature for F-finite domains that are *not necessarily local*. That is, they defined

$$e_{\mathrm{HK}}(R) = \lim_{e \longrightarrow \infty} \frac{\mu_R(F^e_* R)}{\operatorname{rank}(F^e_* R)}$$

and

$$s(R) = \lim_{e \longrightarrow \infty} \frac{\operatorname{frk}_R(F^e_* R)}{\operatorname{rank}(F^e_* R)}$$

and showed that these notions behave in many of the same ways that we will see that Hilbert-Kunz multiplicity and F-signature do. Furthermore, they showed that $e_{\mathrm{HK}}(R)$ is the maximum of the Hilbert-Kunz multiplicities of the localizations, while $s(R)$ is the minimum of the F-signature of the localizations.

1.7. F-signature for Cartier algebras. We now briefly describe F-signature for Cartier algebras. There is one rather annoying issue that comes up. If $\mathscr{C} = \bigoplus \mathscr{C}_e$ is a Cartier algebra, then it can happen that $\mathscr{C}_e = 0$ for certain $e > 0$. Indeed, if $\phi : R \longrightarrow R$ is an p^{-2}-linear map and $\mathscr{C}$ is the Cartier algebra generated by ϕ, then $\mathscr{C}_e = 0$ for all e odd.

To address this, we introduce the following notation.

Notation 1.29. Suppose that $(R, \mathfrak{m})$ is an F-finite local domain and that

$$\mathscr{C} = \bigoplus_{e \geq 0} \mathscr{C}_e \subseteq \bigoplus_{e \geq 0} \operatorname{Hom}_{p^{-e}}(R, R)$$

is a Cartier algebra under composition. Set $\Gamma_{\mathscr{C}} := \{e \geq 0 \mid \mathscr{C}_e \neq 0\}$.

In what follows, we interpret $\mathscr{C}_e$ as a subset of $\operatorname{Hom}_R(F^e_* R, R)$ instead of using p^{-e}-linear maps.

Now, if $F^e_* R \cong R \oplus M$, then the projection map onto the factor R may or may not live in a given Cartier algebra $\mathscr{C}$. It is possible to take this approach to describe the R-summands of R that come from $\mathscr{C}$, and use that to define F-signature for $(R, \mathscr{C})$. We leave this perspective to the reader in Exercise 1.13 and instead directly define the ideals I_e.

Definition 1.30. Suppose that $(R, \mathfrak{m})$ is an F-finite local domain and $\mathscr{C}$ is a Cartier algebra as in Notation 1.29. For each $e \in \Gamma_{\mathscr{C}}$, set

$$I_e^{\mathscr{C}}(R) = \{x \in R \mid \phi(F_*^e x) \subseteq \mathfrak{m} \text{ for all } \phi \in \mathscr{C}_e\}.$$

With this definition in places, we can define F-signature for pairs.

Definition 1.31 ([**BST11a**])**.** Suppose $(R, \mathfrak{m})$ is an F-finite Noetherian domain of dimension d and that $\mathscr{C}$ is a Cartier algebra with with $\Gamma_{\mathscr{C}}$ as above. We define the ***F*-signature of** $(R, \mathscr{C})$ as the limit (if it exists[6])

$$s(R, \mathscr{C}) := \lim_{e \in \Gamma_{\mathscr{C}}} \frac{\ell_R\big(R/I_e^{\mathscr{C}}\big)}{p^{ed}}.$$

In particular, we may define F-signature for pairs (R, Δ) or $(R, \mathfrak{a}^t)$ by defining it for the corresponding Cartier algebra.

Now, one might hope that $I_e^{\mathscr{C}}(R)^{[p]} \subseteq I_{e+1}^{\mathscr{C}}$ but that is not the case generally. However, for any $\phi \in \mathscr{C}^f$, the fact that the Cartier algebra is closed under composition (that is, it's a non-commutative ring) guarantees that

$$\phi(F_*^f I_{e+f}^{\mathscr{C}}(R)) \subseteq I_e^{\mathscr{C}}. \tag{1.31.1}$$

See Exercise 1.4. This containment can play the same role as the containment $J_e^{[p]} \subseteq J_{e+1}$ did for the purpose of proving limits exist.

Theorem 1.32 ([**BST11a, PT18**])**.** *Suppose $(R, \mathfrak{m})$ is an F-finite local domain and $\mathscr{C}$ is a Cartier algebra with $\Gamma_{\mathscr{C}}$ as in Notation 1.29. Then the F-signature limit*

$$\lim_{e \in \Gamma_{\mathscr{C}}} \frac{\ell_R\big(R/I_e^{\mathscr{C}}\big)}{p^{ed}}$$

exists.

PROOF. We do not work out the details as they are similar to Theorem 1.25 (and hence to Theorem 1.22). The key difference is that one must keep track of the degrees $\Gamma_{\mathscr{C}}$ where non-zero maps exist. ☐

Remark 1.33. There are other ways to generalize F-signature without pairs. Taking the perspective of [**SVdB97**], consider an indecomposable R-module M. One might ask how many times M occurs, in an indecomposable decomposition of $F_*^e R$ as e goes to infinity. This question is explored in [**HN15, HS17, HN21**], for instance for certain quotient singularities.

[6] it still does

Another approach is to what is the maximum value b_e such that $F^e_* R$ surjects onto $M^{\oplus b_e}$. The case that $M = \omega_R$ has been of particular interest, see [**San15**].

1.8. Exercises.

Exercise 1.1. Prove that the set $I(M) = I_R(M)$ is an R-submodule of M. Furthermore, show that $I_R(F^e_* M)$ is an $F^e_* R$-submodule of $F^e_* M$.

Exercise 1.2. Suppose that R is an F-finite local ring. Prove that $\mathfrak{m}^{[p^e]} \subseteq I_e(R)$ and conclude that $I_e(R)$ is always $\mathfrak{m}$-primary (if it is a proper ideal).

Exercise 1.3. Suppose $(R, \mathfrak{m})$ is an F-finite regular ring. Explain why $I_e(R) = \mathfrak{m}^{[p^e]}$.

Exercise 1.4. Suppose that $(R, \mathfrak{m})$ is an F-finite local ring of characteristic $p > 0$. Show that

$$\phi(F^d_* I_{e+d}(R)) \subseteq I_e(R)$$

for all $\phi \in \operatorname{Hom}_R(F^d_* R, R)$. This completes the proof of Lemma 1.18.

Exercise 1.5. Verify the assertions of Remark 1.10. Namely, suppose that $(R, \mathfrak{m}, k)$ is a Noetherian F-finite reduced and equidimensional local ring of dimension d.

(a) Show that the number $[F^e_* \mathcal{K}(R/Q) : \mathcal{K}(R/Q)]$ is independent of the choice of minimal prime, and we call this number the rank of $F^e_* R$.
(b) Verify that $\operatorname{frk} F^e_* R = [F^e_* k : k]\, \ell_R(R/I_e(R))$.
(c) Show that if $\mathfrak{c} \subseteq R$ is the conductor, that $\mathfrak{c} \subseteq I_e(R)$ for all e.
(d) Conclude that if R is non-normal, that

$$0 = \lim_{e \longrightarrow \infty} \frac{1}{p^{ed}} (R/I_e(R)) =: s(R).$$

Exercise 1.6. Suppose that R is an F-finite Noetherian domain and $\operatorname{rank}(F^e_* R) = p^m$. Show there exists a short exact sequence of R-modules:

$$0 \longrightarrow F^e_* R \xrightarrow{\psi_1 \oplus \cdots \oplus \psi_{p^m}} R^{\oplus p^m} \longrightarrow M \longrightarrow 0$$

where M is torsion. Furthermore, for any fixed nonzero $\phi \in \operatorname{Hom}_R(F^e_* R, R)$, show that we may assume that each $\psi_i = \phi \star d_i$ for some $0 \neq d_i \in R$.

Exercise 1.7 ([**PT18**, Lemma 3.5(ii)])**.** Suppose p is prime, $d > 0$ is an integer, and $\{b_e\}_{e \geq 0}$ is a sequence of real numbers so that $\left\{\frac{b_e}{p^{ed}}\right\}_{e \geq 0}$ is bounded above. Suppose that

$$\frac{b_e}{p^{ed}} \leq \frac{b_{e+1}}{p^{(e+1)d}} + \frac{C}{p^e}$$

for some constant C. Then the limit

$$\lim_{e \longrightarrow \infty} \frac{b_e}{p^{ed}}$$

exists.

Exercise 1.8. Prove Theorem 1.25.

Exercise 1.9. Suppose that R is an F-finite Noetherian *reduced* ring and $\operatorname{rank}(F_* R) = p^m$. Then show that there exists a short exact sequence of R-modules:

$$0 \longrightarrow R^{\oplus p^m} \longrightarrow F_* R \longrightarrow M \longrightarrow 0$$

where M is torsion. This generalizes Lemma 1.19 from out of the reduced case. Conclude that $e_{\mathrm{HK}}(R)$ exists for F-finite reduced local rings.

Exercise 1.10. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and $f \in \mathfrak{m}$ is a non-zerodivisor and that M is a finitely generated $R/(f)$-module. Prove that $\ell_R\big(\operatorname{Tor}_1^R(R/\mathfrak{m}^{[p^e]}, M)\big) \leq O(p^{e(d-1)})$.

Hint: Take a $R/(f)$-module free-resolution of M and use that to reduce to the case where $M = R/(f)$. Now resolve M as an R-module.

Exercise 1.11. Suppose that $(R, \mathfrak{m})$ is an F-finite local domain with $(\widehat{R}, \widehat{\mathfrak{m}})$ its completion. Choose Q a minimal prime of $\widehat{R}$. Let $K = \mathcal{K}(R)$ denote the fraction field of R and $L = \mathcal{K}(\widehat{R}/Q)$ the fraction field of $\widehat{R}/Q$. Show that, for any $e \geq 0$

$$[F^e_* K : K] = [F^e_* L : L].$$

Hint: Choose an injection $R^{\oplus \alpha} \hookrightarrow F^e_* R$ whose cokernel is torsion. Tensor this up with $\widehat{R}$.

Exercise 1.12. Use Proposition 1.11 to directly prove that F-finite rings are universally catenary. In other words, if R is F-finite, prove that $R[x_1, \dots, x_m]$ is catenary.

Exercise 1.13. With notation as in Definition 1.30, define the **free rank of $F^e_* R$ with respect to $\mathscr{C}_e$** (denoted $\operatorname{frk}^{\mathscr{C}}(F^e_* R)$) to be the largest number a so that there is a surjection

$$F^e_* R \xrightarrow{(\psi_1 \oplus \cdots \oplus \psi_a)} R^{\oplus a}$$

where each individual $\psi_i \in \mathscr{C}$.

Show that

$$\ell_R\left(\frac{R}{I_e^{\mathscr{C}}(R)}\right) = \operatorname{frk}^{\mathscr{C}}(F^e_* R).$$

2. Perspectives on, and generalizations of, Hilbert-Kunz multiplicity

In this section we explore some generalizations of Hilbert-Kunz multiplicity with respect to non-maximal ideals and define Hilbert-Kunz multiplicity for modules. Before beginning on this however, it is instructive to recall the theory of Hilbert-Samuel multiplicity. A nice treatment of this material can be found in [**SH06**, Chapter 11].

Definition 2.1. Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring of dimension$d$, I is $\mathfrak{m}$-primary, and $M \neq 0$ is a finite R-module.

The **Hilbert-Samuel function of I on M** is the function

$$n \mapsto \ell_R(M/I^n M),$$

it is eventually polynomial of degree $\dim \operatorname{Supp} M$ for $n \gg 0$ (called the **Hilbert-Samuel polynomial**).

We define the **multiplicity of I on M**, denoted $e(I; M)$, to be $d!$ times the leading coefficient of the associated Hilbert-Samuel polynomial, that is:

$$e(I; M) := \lim_{n \longrightarrow \infty} \frac{d!}{n^d} \ell_R\left(\frac{M}{I^n M}\right).$$

When $M = R$, we simply write $e(I)$ for $e(I; M)$.

The **multiplicity of R**, $e(R)$, is then defined to be $e(\mathfrak{m}, R)$.

It is easy to see that Hilbert-Kunz multiplicity

$$e_{\mathrm{HK}}(R) := \lim_{e \longrightarrow \infty} \ell_R\left(R/\mathfrak{m}^{[p^e]}\right)$$

is modeled after the more classical Hilbert-Samuel multiplicity.

Our goal in this section is to generalize much of the theory of Hilbert-Samuel multiplicity to Hilbert-Kunz multiplicity.

We begin with the following result characterizing multiplicity.

Theorem 2.2 ([**Ree61**])**.** *Let $(R, \mathfrak{m})$ be a Noetherian excellent domain*[7] *and $I \subseteq J$ are $\mathfrak{m}$-primary ideals. Then*

$$\overline{I} = \overline{J} \quad \textit{if and only if} \quad e(I) = e(J)$$

where $\overline{(-)}$ denotes the integral closure of $(-)$. In fact, the implication $\Rightarrow$ holds for any Noetherian local ring.

It is natural to ask whether there is a similar story for Hilbert-Kunz multiplicity. The answer is yes, with *tight closure* replacing integral closure!

[7] or just that it's completion is equidimensional

2.1. Hilbert-Kunz multiplicity and tight closure. We begin with a generalization of Hilbert-Kunz multiplicity to non-maximal ideals.

Definition 2.3. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d and I is $\mathfrak{m}$-primary. We define the **Hilbert-Kunz multiplicity of** R to be the limit:

$$e_{\mathrm{HK}}(I) := \lim_{e \longrightarrow \infty} \frac{\ell_R\big(R/I^{[p^e]}\big)}{p^{ed}}.$$

Note, the limit exists by Theorem 1.22.

Let us take a moment to compare Hilbert-Kunz multiplicity with Hilbert-Samuel multiplicity.

Proposition 2.4. *Suppose $(R, \mathfrak{m})$ is Noetherian local ring of dimension d and I is an $\mathfrak{m}$-primary ideal. Then*

$$e(I)/d! \leq e_{\mathrm{HK}}(I) \leq e(I).$$

Furthermore, when I is generated by d-elements (a parameter ideal) we have that $e_{\mathrm{HK}}(I) = e(I)$.

Proof. Since $I^{p^e} \supseteq I^{[p^e]}$ we have that

$$\ell_R\big(R/I^{p^e}\big) \leq \ell_R\Big(R/I^{[p^e]}\Big)$$

Dividing by $p^{ed} = (p^e)^d$ and taking a limit $e \longrightarrow \infty$ yields

$$e(I)/d! \leq e_{\mathrm{HK}}(I).$$

For the next inequality, note that both Hilbert-Samuel and Hilbert-Kunz multiplicity are agnostic to the completion of R (we are taking the length of finite length modules) and so it is harmless to assume $(R, \mathfrak{m}, k)$ is complete. It is also not difficult to see that we may replace R by $R \otimes_k \overline{k}$ and so assume that k is algebraically closed, see Lemma 2.17 below for more details. In particular, since k is infinite, by [**SH06**, Proposition 8.3.7 and Corollary 8.3.9] there exists an ideal $J \subseteq I$ with $\overline{J} = \overline{I}$ where J is generated by d elements (a *minimal reduction of* I). Since $J \subseteq I$ we see that $e_{\mathrm{HK}}(I) \leq e_{\mathrm{HK}}(J)$ and by Theorem 2.2 we have that $e(I) = e(J)$. Hence, it suffices to show $e_{\mathrm{HK}}(J) = e(J)$.

By a formula of Lech for computing multiplicity ([**Lec57**], [**Mat89**, Theorem 14.12], [**SH06**, 11.2.10]) we have that

$$e(J) = \lim_{n \longrightarrow \infty} \frac{\ell_R(R/(f_1^n, \ldots, f_d^n))}{n^d}.$$

Note there is no $d!$-term, and we are only taking powers of the generators (there are no cross-terms). Hence it immediately follows that

$$e(I) = e_{\mathrm{HK}}(I)$$

and the proof is complete. □

Theorem 2.5 ([**HH90**, Theorem 8.17], *cf.* [**Hun13**, Theorem 5.5])**.** *Suppose that $(R, \mathfrak{m})$ is a Noetherian F-finite local domain*[8] *of characteristic $p > 0$ and dimension d and that $I \subseteq J$ are $\mathfrak{m}$-primary ideals. Then*

$$I^* = J^* \quad \textit{if and only if} \quad e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(J).$$

In fact, the implication $\Rightarrow$ holds for any Noetherian local ring.

Proof. Suppose $I^* = J^*$. We follow the argument of [**Hun13**, Proposition 5.4]. Write $J = (u_1, \ldots, u_t)$. We can find c, not in any minimal prime, such that $cu_i^{p^e} \in J^{[p^e]}$ for $e \gg 0$ and $i = 1, \ldots, t$. Hence

$$c(J^{[p^e]}/I^{[p^e]}) = 0$$

for $e \gg 0$. Mapping basis elements to generators, we have a surjection:

$$\Big(\frac{(R/(c))}{I^{[p^e]}(R/(c))}\Big)^{\oplus t} \twoheadrightarrow J^{[p^e]}/I^{[p^e]}.$$

As $e \longrightarrow \infty$, the length of the left grows like $p^{e(d-1)}\, t\, e_{\mathrm{HK}}(IR/(c))$ since $\dim R/(c) = d - 1$. Thus the length of the left side is bounded above by $Cp^{e(d-1)}$ for some constant C. From the short exact sequence:

$$0 \longrightarrow J^{[p^e]}/I^{[p^e]} \longrightarrow R/I^{[p^e]} \longrightarrow R/J^{[p^e]} \longrightarrow 0$$

we see that

$$\ell_R\Big(R/I^{[p^e]}\Big) = \ell_R\Big(R/J^{[p^e]}\Big) + \ell_R\Big(J^{[p^e]}/I^{[p^e]}\Big) \leq \ell_R\Big(R/J^{[p^e]}\Big) + Cp^{e(d-1)}.$$

Dividing by p^{ed} and sending $e \longrightarrow \infty$ proves the implication ($\Rightarrow$) without any

For the implication ($\Leftarrow$), we prove it first when $(R, \mathfrak{m}, k)$ is a complete local domain and then explain how to reduce to that case. Thus suppose $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(J)$ where R is complete. Without loss of generality we may assume that $J = I + (u)$. We must prove that $u \in I^*$.

Let $k[\![x_1, \ldots, x_d]\!] = A \subseteq R$ be a generically étale Noether-Cohen-Gabber normalization (which exists by the Cohen-Gabber theorem, [**Ill14**, Théroème VI.2.1.1], *cf.* [**KS18, Ska16**]) and let $v : \mathcal{K}(A) \setminus \{0\} \longrightarrow \mathbb{Z}$ be the $\mathfrak{m}_A$-adic discrete valuation which we can extend to a $\mathbb{Q}$-valuation on R_{perf} that we also call v.

Claim 2.6 (*cf.* [**Shi07**, Proposition 2.15], [**CLM$^+$23**, Lemma 4.0.13])**.** *There exists a sequence of elements $c_1, c_2, \dots \in A$ such that $\lim v(c_e^{1/p^e}) = \lim v(c_e)/p^e \longrightarrow 0$ and such that*

$$c_e u^{p^e} \in I^{[p^e]}$$

[8] or more generally that its completion is reduced and equidimensional and R has a completely stable test element

for all $e > 0$.

Proof of claim. Suppose not. Hence we can choose $E > 0$ so that

$$\frac{1}{p^E} \leq \inf\{v(b^{1/p^e}) = v(b)/p^e \mid b \in A, bu^{p^e} \in I^{[p^e]}\}.$$

Let $N_e = \{c \in A \mid v(c^{1/p^e}) = v(c)/p^e \geq 1/p^E\}$, an ideal of A. In fact, if $e > E$ then $N_e = \mathfrak{m}^{p^{e-E}}$. We notice that there exists an injection:

$$0 \longrightarrow (A/N_e) \xrightarrow{1 \mapsto u^{p^e}} J^{[p^e]}/I^{[p^e]}$$

We thus see that

$$\lim_{e \longrightarrow \infty} \frac{\ell_A(A/N_e)}{p^{ed}} = \lim_{e \longrightarrow \infty} \frac{\ell_A\Big(A/\mathfrak{m}^{p^{e-E}}\Big)}{p^{ed}} = \frac{e(A)}{d! p^E} > 0.$$

Furthermore,

$$\ell_A\Big(J^{[p^e]}/I^{[p^e]}\Big) = \ell_R\Big(J^{[p^e]}/I^{[p^e]}\Big) = \ell_R\Big(R/I^{[p^e]}\Big) - \ell_R\Big(R/J^{[p^e]}\Big).$$

Dividing by p^{ed} and sending $e \longrightarrow 0$ contradicts our assumption that the difference $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J) = 0$ and so proves the claim. $\square$

With the claim in place, we simply apply Chapter 7 Exercise 5.10.

Now we explain how reduce to the case of a complete local domain. As e_{HK} is unaffected by completion ($R/I^{[p^e]} = \widehat{R}/(I\widehat{R})^{[p^e]}$) and because $(I\widehat{R})^* = I^*\widehat{R}$ by Exercise 2.3 in Chapter 7, we may assume that $R = \widehat{R}$ is complete. Additionally, by Chapter 7 Exercise 1.2 and Exercise 2.8 (which uses Proposition 2.22 below) we may assume that R is a domain. $\square$

Remark 2.7. It is natural to ask if one can define a meaningful Hilbert-Kunz multiplicity along non-$\mathfrak{m}$-primary ideals. This is done in [**EY17**], although we will not explore it here.

Remark 2.8. If an ideal $\mathfrak{a} \subseteq R$ is $\mathfrak{m}$-primary, Vraciu introduced a notion of Hilbert-Kunz multiplicity for pairs $(R, \mathfrak{a}^t)$ [**Vra08**]. She then showed an analog of the above result for a better-behaved version of tight closure of pairs.

2.2. F-signature as minimal relative Hilbert-Kunz multiplicity.

Under mild hypotheses, we learned that two $\mathfrak{m}$-primary ideals $I \subseteq J \subseteq R$ have the same tight closure if and only if $e(I) = e(J)$. It is thus natural to ask how close $e(I)$ and $e(J)$ can be in a weakly F-regular ring and $J \neq I$ (when $I^* = I$). These sorts of questions were formally studied in [**WY04**] and it was observed in [**Yao06**] that this notion is connected to F-signature. This has been made precise in the following result.

Theorem 2.9 ([**PT18**], *cf.* [**WY04**, Proposition 1.7, Question 1.10], [**Yao06**])**.** *Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian F-finite local ring, then*

$$s(R) = \inf_{I \subsetneq J} \left\{ \frac{e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)}{\ell_R(J/I)} \right\} = \inf_{I \subsetneq J} \Big\{ e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J) \Big\}$$

where the infimums run over pairs $\mathfrak{m}$-primary ideals $I \subsetneq J$.

We will prove this in several steps.

Lemma 2.10 ([**HL02**, Proof of Theorem 11], [**WY04**, Proposition 1.7], [**PT18**, Lemma 6.1])**.** *With notation as in Theorem 2.9, if $I \subsetneq J$ are $\mathfrak{m}$-primary ideals, then*

$$\ell_R(R/I_e(R)) \leq \frac{\ell_R\big(R/I^{[p^e]}\big) - \ell_R\big(R/J^{[p^e]}\big)}{\ell_R(J/I)}$$

and so

$$s(R) \leq \frac{e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)}{\ell_R(J/I)}.$$

Proof. Consider a sequence of $\mathfrak{m}$-primary ideals

$$I = I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_m = J$$

where each $I_{i+1}/I_i \cong k$. If we can show that $s(R) \leq \frac{e_{\mathrm{HK}}(I_i) - e_{\mathrm{HK}}(I_{i+1})}{1}$ then by summing up and solving for $s(R)$, the result will be proven. Hence, without loss of generality we may assume that $J = I + (x)$ and that $J/I \cong k$.

We claim there is a map

$$J^{[p^e]}/I^{[p^e]} \xrightarrow{\psi} R/I_e(R)$$

where $\psi(x^{p^e}) = 1$ (and so ψ is surjective). Note $J^{[p^e]}/I^{[p^e]} \cong \frac{R}{I^{[p^e]} : J^{[p^e]}}$ is cyclic (generated by x^{p^e}). Hence it suffices to show that $I^{[p^e]} : J^{[p^e]} \subseteq I_e(R)$, or in other words that if

$$zx^{p^e} \in I^{[p^e]} \tag{2.10.1}$$

then $z \in I_e(R)$. But if $z \notin I_e(R)$, then there exists $\phi : F^e_* R \longrightarrow R$ such that $\phi(F^e_* z) = 1$. Then applying ϕ to (2.10.1) we see that $x \in I$, a contradiction. Thus ψ exists.

Since ψ is surjective:

$$\ell_R\Big(R/I^{[p^e]}\Big) - \ell_R\Big(R/J^{[p^e]}\Big) = \ell_R\Big(J^{[p^e]}/I^{[p^e]}\Big) \geq \ell_R(R/I_e(R)).$$

Dividing by p^{ed} and taking a limit proves the lemma. $\square$

We need another characterization of $I_e(R)$. Recall that if a Noetherian local ring $(R, \mathfrak{m}, k)$ is *approximately Gorenstein* Appendix A Section 11, then there exists a sequence of $\mathfrak{m}$-primary ideals, $J_1 \supseteq J_2 \supseteq \dots$ such that each R/J_j is Gorenstein. By Proposition 11.6, we see that

$$E = \bigcup_j \operatorname{Ann}_E J_j \cong \bigcup_j E_{R/J_j} \cong \varinjlim R/J_j.$$

Note in each R/J_j we can find $\overline{u_j}$, a generator of the socle $\operatorname{Ann}_{R/J_j} \mathfrak{m} = 0 :_{R/J_j} \mathfrak{m}$ (the socle is 1-dimensional over k since R/J_j is Gorenstein). From the construction we may assume that u_js map to each other in the direct limit. We take $u_j \in R$, a pre-image of $\overline{u_j}$.

Lemma 2.11. *If $(R, \mathfrak{m})$ is an F-finite Noetherian domain, and $J_1 \supseteq J_2 \supseteq \dots$ and $u_1, u_2, \dots$ is as above. Then*

$$I_e(R) = (I_{j_e}^{[p^e]} : u_{j_e}^{p^e})$$

for any $j_e \gg 0$ (depending on $e > 0$). Furthermore, for any fixed e, the ideals $(I_j^{[p^e]} : u_j^{p^e})$ form an increasing sequence as j increases.

Proof. By Lemma 2.4 in Appendix A we see that

$$I_e(R) = \{x \in R \mid E \xrightarrow{\eta \mapsto \eta \otimes F_*^e x} E \otimes F_*^e R \text{ is not injective}\}$$

Note, checking whether the map above is injective is equivalent to checking if u maps to zero, since u generates the socle.

If a map in the definition above is not injective, if must fail to be injective after tensoring with some R/J_j (who direct limit together to get E). In other words, we may check if $x \in I_e(R)$ by checking if

$$R/J_j \xrightarrow{y \mapsto y \otimes F_*^e x} R/J_j \otimes F_*^e R \cong F_*^e(R/J_j^{[p^e]})$$

satisfies $u_j \mapsto F_*^e x u_j^{p^e} = 0$. That is, if $x u_j^{p^e} \in J_j^{[p^e]}$ or in other words if $x \in J_j^{[p^e]} : u_j^{p^e}$. Thus

$$I_e(R) = \bigcup_j (I_j^{[p^e]} : u_j^{p^e}).$$

Finally, to complete the proof it suffices to show that $(I_j^{[p^e]} : u_j^{p^e}) \subseteq (I_{j+1}^{[p^e]} : u_{j+1}^{p^e})$. We have a map $R/I_j \longrightarrow R/I_{j+1}$ sending $u_j \mapsto u_{j+1}$. Tensoring with $F_*^e R$ gives us a map $R/I_j^{[p^e]} \longrightarrow R/I_{j+1}^{[p^e]}$ sending $u_j^{p^e} \mapsto u_{j+1}^{p^{e+1}}$. Thus if $z u_j^{p^e} \in I_j^{p^e}$ we see that $z u_{j+1}^{p^e} \in I_{j+1}^{p^e}$, which is what we wanted. $\square$

We are now ready to prove our main result of the subsection.

Proof of Theorem 2.9. We closely follow the proof of [**PT18**, Theorem 6.4]. Using that R is approximately Gorenstein, we can choose I_j's and u_j's as in Lemma 2.11. By Lemma 2.10, it is sufficient to show that:

$$s(R) = \lim_{j\longrightarrow\infty} \big(e_{\mathrm{HK}}(I_j) - e_{\mathrm{HK}}(I_j + (u_j))\big).$$

We consider the following sequences of ideals: $I_{j,e} = (I_j^{[p^e]} : u_j^{p^e})$ noticing that $I_e(R) = I_{j,e}$ for all $e \gg 0$ by Lemma 2.11. Certainly

$$I_{j,e}^{[p]} = (I_j^{[p^e]} : u_j^{[p^e]})^{[p]} \subseteq (I_j^{[p^{e+1}]} : u_j^{[p^{e+1}]}) = I_{j,e+1}.$$

Similarly, since $\mathfrak{m}u_j \in I_j$ we see that $\mathfrak{m}^{[p]} \in I_{j,1}$.

The following claim is the crucial insight, and it uses the uniformity statement Corollary 1.24.

Claim 2.12 ([**PT18**, Theorem 6.3])**.** *We have that*

$$s(R) = \lim_{j\longrightarrow\infty} \lim_{e\longrightarrow\infty} \frac{1}{p^{ed}} \ell_R(R/I_{j,e}).$$

Proof of claim. Using Theorem 1.22 and our observations on the $I_{j,e}$ above, we can write

$$\eta_j := \lim_{e\longrightarrow\infty} \frac{1}{p^{ed}} \ell_R(R/I_{j,e})$$

and by Corollary 1.24, we have that there exists D, independent of j, such that

$$\eta_j \leq \frac{1}{p^{ed}} \ell_R(R/I_{j,e}) + \frac{D}{p^e}.$$

Since, using Lemma 2.11, $I_{j,e} \subseteq I_{j+1,e} \subseteq I_e(R)$, we have that $\ell_R(I_e(R)) \leq \ell_R(R/I_{j+1,e}) \leq \ell_R(R/I_{j,e})$ and so

$$s(R) \leq \eta_{j+1} \leq \eta_j \leq \frac{1}{p^{ed}} \ell_R(R/I_{j,e}) + \frac{D}{p^e}.$$

Note this forces $\lim\limits_{j\longrightarrow\infty} \eta_j$ to exist. Hence, sending $j \longrightarrow \infty$ and then sending $e \longrightarrow \infty$ we see that

$$s(R) \leq \lim_{j\longrightarrow\infty} \eta_j \leq s(R).$$

But by definition:

$$\lim_{j\longrightarrow\infty} \eta_j = \lim_{j\longrightarrow\infty} \lim_{e\longrightarrow\infty} \frac{1}{p^{ed}} \ell_R(R/I_{j,e})$$

and the claim is proven. ☐

With the claim in place, notice, using the short exact sequences:

$$0 \longrightarrow \frac{R}{I_{j,e}} \xrightarrow{1\mapsto u^{p^e}} \frac{R}{I_j^{[p^e]}} \longrightarrow \frac{R}{I_j^{[p^e]} + (u_j^{[p^e]})} \tag{2.12.1}$$

that

$$\begin{aligned}
s(R) &= \lim_{j\longrightarrow\infty}\ \lim_{e\longrightarrow\infty} \frac{1}{p^{ed}} \ell_R(R/I_{j,e}) && \text{(Claim 2.12)}\\
&= \lim_{j\longrightarrow\infty}\ \lim_{e\longrightarrow\infty} \frac{1}{p^{ed}} \left(\ell_R\Big(R/I_j^{[p^e]}\Big) - \ell_R\left(\frac{R}{I_j^{[p^e]} + (u_j^{[p^e]})}\right)\right) && (2.12.1)\\
&= \lim_{j\longrightarrow\infty} \big(e_{\mathrm{HK}}(I_j) - e_{\mathrm{HK}}(I_j + (u_j))\big)
\end{aligned}$$

which completes the proof. $\square$

There is an important open conjecture related to this work.

Conjecture 2.13 (Watanabe-Yoshida)**.** *There exist $\mathfrak{m}$-primary ideals $I \subsetneq J$ such that*

$$s(R) = e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J).$$

The infimum in Theorem 2.9 is a minimum.

Remark 2.14. Suppose $(R, \mathfrak{m})$ is a Noetherian F-finite local domain and that $s(R) > 0$. One may check that a ring is weakly F-regular by checking that all $\mathfrak{m}$-primary ideals are tightly closed (Chapter 7 Exercise 3.2). Therefore, if $I \subsetneq I^*$ is $\mathfrak{m}$-primary, since $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I^*)$ we see that $s(R) = 0$. Thus if $s(R) > 0$, we see that R is weakly F-regular.

In Section 4 we will prove a result of Aberbach-Leuschke ([**AL03**]) which shows that $s(R) > 0$ if and only if R is *strongly* F-regular. Hence, if R is a weakly F-regular F-finite ring and Conjecture 2.13 has a positive answer for some $I \subsetneq J$, then since $I^* = I, J^* = J$ we have $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J) \neq 0$, and so $s(R) > 0$ and so R is strongly F-regular. In other words,

$$\text{Conjecture 2.13 } (\Rightarrow) \left(\begin{array}{c}\text{weak implies strong } F\text{-regularity,}\\ \text{Chapter 7 Conjecture 3.8}\end{array}\right).$$

Note there is a direct proof of Conjecture 2.13 when R is Gorenstein, see Exercise 2.3, but we already knew that weak F-regularity and strong F-regularity coincided in the Gorenstein (or even $\mathbb{Q}$-Gorenstein) case: Chapter 7 Theorem 5.10.

2.3. Further generalizations of Hilbert-Kunz multiplicity. In the previous section, we showed that Hilbert-Kunz multiplicity existed for domains. We will now generalize this to arbitrary Noetherian rings of characteristic $p > 0$. In doing this, it will actually be convenient to generalize Hilbert-Kunz multiplicities to modules.

Definition 2.15 (Hilbert-Kunz multiplicity)**.** Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d, I is $\mathfrak{m}$-primary, and M is a finitely generated R-module. We define the **Hilbert-Kunz multiplicity of** I **on** M to be

$$e_{\mathrm{HK}}(I; M) := \limsup_{e \longrightarrow \infty} \frac{\ell_R\big(M/I^{[p^e]}M\big)}{p^{ed}}.$$

When $I = \mathfrak{m}$, this is simply denoted by $e_{\mathrm{HK}}(M)$ and called **Hilbert-Kunz multiplicity of** M. The function $p^e \mapsto \frac{\ell_R(M/I^{[p^e]}M)}{p^{ed}}$ is called the **Hilbert-Kunz function of** I **on** M.

Note if R is an F-finite local domain, then then $\ell_R\big(M/\mathfrak{m}^{[p^e]}M\big)[F_*k : k]^e = \mu_R(F^e_* M)$ and $p^{ed}[F_*k : k]^e = \operatorname{rank} F^e_* R$ (see Proposition 1.2 (b) and (1.11.1)). Hence

$$e_{\mathrm{HK}}(\mathfrak{m}; M) = \lim_{e \longrightarrow \infty} \frac{\mu_R(F^e_* M)}{\operatorname{rank} F^e_* R} \tag{2.15.1}$$

so again we are simply comparing the numbers of generators of $F^e_* M$ to the number we expect if R is regular.

Warning 2.16. The Hilbert-Kunz function is *not* typically eventually a polynomial in p^e. However, there exists a second coefficient when $(R, \mathfrak{m}, k = k^p)$ is *normal* and excellent of dimension d. That is,

$$\ell_R\Big(M/I^{[p^e]}M\Big) = e_{\mathrm{HK}}(I; M)(p^e)^d + \beta(p^e)^{d-1} + O(q^{d-2}).$$

For details, including a study of β, see [**HMM04**]. Also see [**HY09, CK16**] for some generalizations and see [**Hun13**, Section 7] for an exposition.

We will show this lim sup is in fact a limit. The problem will eventually reduce to the case when $M = R$ is an integral domain, which we have already solved. We make some simplifying observations.

Lemma 2.17. *Suppose* $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ *is a faithfully flat extension of Noetherian local rings of the same dimension and that* $\mathfrak{n} = \mathfrak{m}S$*. Suppose* $I \subseteq R$ *is* $\mathfrak{m}$*-primary. Then*

$$e_{\mathrm{HK}}(I, M) = e_{\mathrm{HK}}(IS, M \otimes_R S)$$

Furthermore, the associated Hilbert-Kunz functions are also the same.

PROOF. This is a direct consequence of Proposition 1.2 (a). $\square$

Remark 2.18 (Assuming our ring is F-finite)**.** In view of Lemma 2.17, when computing Hilbert-Kunz multiplicity of a module, or proving that the lim sup in Definition 2.15 is a limit, we may assume that R is complete and F-finite. In fact, we may assume it has an algebraically closed residue field.

Simply notice that $R \longrightarrow \widehat{R}\widehat{\otimes}_k \overline{k}$ satisfies the conditions of the lemma where $k \subseteq \widehat{R}$ is a coefficient field.

Our next goal is to create tools that let us reduce to the case that R is reduced. We first state the following lemma.

Lemma 2.19. *Suppose $(R, \mathfrak{m}, k = k^p)$ is a complete Noetherian d-dimensional local ring with perfect residue field, I is an $\mathfrak{m}$-primary ideal, and M is any finitely generated R-module. Then*

$$e_{\mathrm{HK}}(I; F^f_* M) = p^{fd} e_{\mathrm{HK}}(I; M).$$

In fact, the associated Hilbert-Kunz functions differ by multiplication by p^{fd}.

Proof. Since k has perfect residue field, the R-length of a module N is the same as the R-length of $F^f_* N$. Hence

$$\frac{\ell_R\Big((F^f_* M)/I^{[p^e]}\Big)}{p^{ed}} = \frac{\ell_R\Big(F^f_*(M/I^{[p^{e+f}]}M)\Big)}{p^{ed}} = p^{fd}\frac{\ell_R\Big(M/I^{[p^{e+f}]}M\Big)}{p^{(e+f)d}}.$$

Sending $e \longrightarrow \infty$ completes the proof. $\square$

Lemma 2.20. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring, $J \subseteq R$ is an ideal and $(\overline{R} = R/J, \overline{\mathfrak{m}} = \mathfrak{m}/J)$ is the quotient. Suppose that $\dim R = \dim \overline{R}$ and that M is a finite R-module such that*

$$J \subseteq \mathrm{Ann}_R(M).$$

Then we may view M as an $\overline{R}$-module and, for any $\mathfrak{m}$-primary ideal I (with $\overline{I} := I\overline{R}$) we have

$$e_{\mathrm{HK}}(I, M) = e_{\mathrm{HK}}(\overline{I}, M).$$

In fact, the associated Hilbert-Kunz functions are identical. In particular, if $J = \sqrt{(0)}$ is the nilradical and $\sqrt{(0)}M = 0$, then we may compute the Hilbert-Kunz multiplicity over $R_{\mathrm{red}} = \overline{R} = R/\sqrt{(0)}$.

Proof. The relevant lengths are identical. $\square$

Remark 2.21 (Assuming our ring is reduced)**.** Notice that for any finitely generated module M, we have that $\mathrm{Ann}_R(F^e_* M) \supseteq \sqrt{0}$ for $e \gg 0$, since $\sqrt{(0)}^{[p^e]} = 0$ for $e \gg 0$. Thus, by Lemma 2.19 and Lemma 2.20, if we want to compute the Hilbert-Kunz multiplicity of I on M, or to prove that the lim sup in Definition 2.15 is a limit, we may assume that R is reduced.

The next tool is also quite useful.

Proposition 2.22 ([**Mon83**, Lemma 1.3], *cf.* [**Hun13**, Lemma 3.10])**.** *Suppose $(R,\mathfrak{m})$ is a Noetherian local ring and I is an $\mathfrak{m}$-primary ideal. Further suppose that M, N are finitely generated R-modules such that $M_\mathfrak{q} \cong N_\mathfrak{q}$ for each minimal prime of R such that* $\dim(R/\mathfrak{q}) = \dim R$*. Then*

$$e_{\mathrm{HK}}(I;M) = e_{\mathrm{HK}}(I;N).$$

In fact, the associated Hilbert-Kunz functions agree up to a term of order $O(p^{e(d-1)})$.

Proof. There exists a map $\phi : M \longrightarrow N$ such that $\phi_\mathfrak{q} : M_\mathfrak{q} \longrightarrow N_\mathfrak{q}$ is an isomorphism for all the minimal primes $\mathfrak{q}$ as above, see Exercise 2.7.

We have a sequence $M \xrightarrow{\phi} N \longrightarrow C \longrightarrow 0$ where $\dim C < d$. By tensoring with $R/I^{[p^e]}$ we obtain the exact sequence

$$M/I^{[p^e]}M \longrightarrow N/I^{[p^e]}M \longrightarrow C/I^{[p^e]}M \longrightarrow 0.$$

Hence, by Lemma 1.20,

$$\ell_R\Big(N/I^{[p^e]}N\Big) \leq \ell_R\Big(M/I^{[p^e]}M\Big) + \ell_R\Big(C/I^{[p^e]}C\Big) \leq \ell_R\Big(M/I^{[p^e]}M\Big) + O(p^{e(d-1)}).$$

Reversing the roles of M and N completes the proof. $\square$

As a consequence, Hilbert-Kunz multiplicity of modules is essentially the same as Hilbert-Kunz multiplicity of rings:

Corollary 2.23. *Suppose $(R,\mathfrak{m})$ is a Noetherian local domain, I is an $\mathfrak{m}$-primary ideal, and M is an R-module of generic rank* $n = \mathrm{rank}(M)$*. Then*

$$e_{\mathrm{HK}}(I;M) = n\, e_{\mathrm{HK}}(I;R).$$

Next we prove that Hilbert-Kunz multiplicity behaves well with respect to short exact sequences.

Proposition 2.24. *Suppose $(R,\mathfrak{m})$ is a Noetherian local ring of dimension d, I is an $\mathfrak{m}$-primary ideal, and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of finite R-modules. Then

$$e_{\mathrm{HK}}(I;M) = e_{\mathrm{HK}}(I;L) + e_{\mathrm{HK}}(I;N).$$

Proof. It suffices to prove that

$$\ell_R\Big(M/I^{[p^e]}M\Big) = \ell_R\Big(L/I^{[p^e]}I\Big) + \ell_R\Big(N/I^{[p^e]}N\Big) + O(p^{e(d-1)}).$$

But M and $L \oplus N$ agree at the minimal primes of R, now use Proposition 2.22. $\square$

We can use this to obtain:

Corollary 2.25. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring, I is an $\mathfrak{m}$-primary ideal, and $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ is a filtration of a finitely generated R-module M with quotients $Q_1 = M_1/M_0, \ldots, Q_n = M_n/M_{n-1}$. Then*

$$e_{\mathrm{HK}}(I; M) = \sum_{i=1}^{n} e_{\mathrm{HK}}(I; Q_i).$$

Finally, we prove that Hilbert-Kunz multiplicity exists as a limit in general.

Theorem 2.26. *If $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring, I is an $\mathfrak{m}$-primary ideal, and M is a finitely generated R-module, then*

$$e_{\mathrm{HK}}(I; M) = \lim_{e \longrightarrow \infty} \frac{\ell_R\big(M/I^{[p^e]}M\big)}{p^{ed}}.$$

In particular the limit exists and is finite.

Proof. Without loss of generality, using Remark 2.18 and Remark 2.21, we may assume that R is complete, has perfect residue field, and is reduced.

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ denote the minimal primes of R such that $\dim(R/\mathfrak{q}_i) = d$. For each of these, may find a free $R_i := R/\mathfrak{q}_i$-module F_i, of rank r_i, so that $(F_i)_{\mathfrak{q}_i} \cong M_{\mathfrak{q}_i}$. Set

$$L = \bigoplus_{i=1}^{t} F_i.$$

Hence, by Proposition 2.22,

$$\ell_R\Big(M/I^{[p^e]}M\Big) \leq \ell_R\Big(L/I^{[p^e]}L\Big) + O(p^{e(d-1)}).$$

Next, notice that

$$\ell_R\Big(L/I^{[p^e]}L\Big) = \sum_{i=1}^{t} \ell_R\Big(F_i/I^{[p^e]}F_i\Big) = \sum_{i=1}^{t} r_i \ell_{R_i}\Big(R_i/I^{[p^e]}R_i\Big).$$

But each limit

$$\lim_{e \longrightarrow \infty} \frac{\ell_{R_i}\big(R_i/I^{[p^e]}R_i\big)}{p^{ed}} = e_{\mathrm{HK}}(IR_i; R_i)$$

exists by Theorem 1.22 since each R_i is an F-finite domain. The result follows. $\square$

2.4. Exercises.

Exercise 2.1. Suppose $(R, \mathfrak{m})$ is a regular local ring and I is a $\mathfrak{m}$-primary ideal. Prove that

$$e_{\mathrm{HK}}(I) = \ell_R(R/I).$$

Hint: Frobenius is flat.

Exercise 2.2. Prove for a general Noetherian local ring $(R, \mathfrak{m})$ that $e_{\mathrm{HK}}(R) \geq 1$.

Hint: Reduce to the F-finite domain case and use Corollary 1.12.

Exercise 2.3 ([**WY00**, Theorem 2.1])**.** Suppose $(R, \mathfrak{m})$ is a Gorenstein F-finite local domain and $I = (x_1, \ldots, x_d)$ is a system of parameters. Let $u \in R$ be an element that maps to the socle in R/I. Show that

$$s(R) = e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I + (u)).$$

Hint: Notice that $I_e := I^{[p^e]}$ forms an approximately Gorenstein sequence with u^{p^e} generating the socle in the quotients. Now use the (proof of) Theorem 2.9.

Exercise 2.4. Suppose $(R, \mathfrak{m})$ is a Noetherian F-finite local domain. Show that $s(R) = s(\widehat{R})$.

Hint: It is possible that $\widehat{R}$ is not a domain even if R is. However, if $\widehat{R}$ is not a domain, then R is not normal, and so $s(R) = 0$, see Exercise 1.5 or Remark 2.14.

Exercise 2.5. Suppose $(R, \mathfrak{m}, k)$ is a Noetherian F-finite local ring and E is the injective hull of the residue field k. Show that $I_e(R) = \{x \in R \mid E \xrightarrow{z \mapsto z \otimes F^e_* R} E \otimes_R F^e_* R\}$.

Note the set above does not need an F-finite hypothesis. Hence, in [**Yao06**], Yao defined the F-signature to be $\lim_{e \longrightarrow \infty} \frac{1}{p^{ed}} \ell_R(R/I_e(R))$ for $I_e(R)$ as above.

Exercise 2.6. Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian complete F-finite local domain and $k \subseteq l$ is a field extension with l also F-finite. Show that

$$s(R) = s(R \widehat{\otimes}_k l).$$

For a vast generalizations see [**Yao06**, Theorem 5.6].

Exercise 2.7. Suppose R is a Noetherian ring, $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ is a finite set of minimal primes, and M, N are finite R-modules such that $M_{\mathfrak{q}_i} \cong N_{\mathfrak{q}_i}$

for $i = 1, \ldots, t$. Prove that there exists a map $\phi : M \longrightarrow N$ such that $\phi_{\mathfrak{q}_i} : M_{\mathfrak{q}_i} \xrightarrow{\sim} N_{\mathfrak{q}_i}$ is an isomorphism for $i = 1, \ldots, t$.

Hint: Consider the multiplicative set $W := \bigcap_{i=1}^{t}(R \setminus \mathfrak{q}_i)$ and use the fact that $W^{-1}R = R_{\mathfrak{q}_1} \times \cdots \times R_{\mathfrak{q}_t}$.

Exercise 2.8. Suppose that $(R, \mathfrak{m})$ is a Noetherian local equidimensional ring with minimal primes $Q_1, \ldots, Q_n$ and that J is $\mathfrak{m}$-primary. Write $R_i = R/Q_i$. Prove that

$$e_{\mathrm{HK}}(J) = \sum_{i=1}^{t} e_{\mathrm{HK}}(JR_i).$$

Hint: Use Proposition 2.22.

Definition 2.27. A Noetherian local ring $(R, \mathfrak{m})$ (or a graded ring $(R, \mathfrak{m} = R_{>0})$) is said to be of **finite Frobenius representation type** (or **FFRT**) if when considering every decomposition of $F^e_* R$ into (graded)indecomposables:

$$F^e_* R \cong M_{e,1} \oplus \cdots \oplus M_{e,t_e}$$

there are only finitely many modules $M_{e,i}$ up to isomorphism (running over all $e > 0$).

Exercise 2.9.

3. Values of F-signature and Hilbert-Kunz multiplicity

Hilbert-Kunz multiplicity and F-signature are notoriously difficult to compute, and no general technique for computing Hilbert-Kunz multiplicity is known, although see for example [**HM93, Con96, Sei97, Mon98, BC97, CH98, Eto02, MT04, MT06, ?, BH06, MS13, Tri17, RS15, ES19, GKV21**] – a *far* from exhaustive list, where many interesting examples are computed.

In fact, we know that the Hilbert-Kunz multiplicity need not be rational [**Bre13**] (*cf.* [**Tri05, Bre06**]). While that example is not so explicit, Monsky [**Mon08**] has also conjectured that the Hilbert-Kunz multiplicity of $R = \mathbb{F}_2[x, y, z, u, v]/(uv + x^3 + y^3 + xyz)$ is

$$e_{\mathrm{HK}}(R) = \frac{4}{3} + \frac{5}{14\sqrt{7}},$$

In fact, Monsky has also proposed an example with transcendental Hilbert-Kunz multiplicity [**Mon09**].

It is suspected that F-signature need not be rational (or even algebraic) as well, indeed if Monsky's proposed counter example above has irrational Hilbert-Kunz multiplicity, it also has irrational F-signature by the following proposition.

Proposition 3.1 ([**HL02**, Proposition 13]). *Suppose $(R, \mathfrak{m}, k)$ is a F-finite Cohen-Macaulay local domain of dimension d. Then*

$$(e(R) - 1)(1 - s(R)) \geq e_{\mathrm{HK}}(R) - 1.$$

Furthermore, if the Hilbert-Samuel multiplicity $e(R) = 2$, then

$$s(R) + e_{\mathrm{HK}}(R) = 2.$$

If $s(R) = 0$ then the above simply says that $e(R) \geq e_{\mathrm{HK}}(R)$, which we already knew by Proposition 2.4.

Proof. Without loss of generality we may assume that R is complete (Exercise 2.4). Furthermore, since all the relevant invariants are unchanged when passing to $R \widehat{\otimes}_R \overline{k}$ (Exercise 2.6) we may assume that k is infinite.

By [**SH06**, Proposition 8.3.7 and Corollary 8.3.9], there exists a $\mathfrak{m}$-primary ideal $I = (x_1, \dots, x_d)$ whose integral closure $\overline{I} = \mathfrak{m}$ (a *minimal reduction of* $\mathfrak{m}$).

Now, $e(R) := e(\mathfrak{m}) = e(\overline{I}) = e(I) = \ell_R(R/I)$ (see [**SH06**, Proposition 11.1.10(2)] for the final equality using that R is Cohen-Macaulay). Since $e(R) - 1 = \ell_R(\mathfrak{m}/I)$, Lemma 2.10 implies that

$$s(R)(e(R) - 1) \leq e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(\mathfrak{m}) = e(R) - e_{\mathrm{HK}}(R).$$

Manipulating the inequality yields

$$e_{\mathrm{HK}}(R) - 1 \leq e(R) - 1 - s(R)(e(R) - 1) = (e(R) - 1)(1 - s(R))$$

as desired.

Finally, when $e(R) = 2$, since $\ell_R(\mathfrak{m}/I) = 1$, we see that $\mathfrak{m}$ is generated by $d+1$ elements, and so R is a quotient of a $(d+1)$-dimensional regular ring by the Cohen-Structure theorem. It follows that R is a complete intersection and hence R is Gorenstein and furthermore that $\mathfrak{m} = I + (u)$ for a socle generator. By Exercise 2.3, and using that $e_{\mathrm{HK}}(I) = e(I) = 2$, we have

$$s(R) = e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I + (u)) = 2 - e_{\mathrm{HK}}(R)$$

as desired. $\square$

3.1. Using Hilbert-Kunz multiplicity and F-signature to characterize regular rings. Of course, if R is regular, it follows from the definitions that

$$e_{\mathrm{HK}}(R) = s(R) = 1.$$

It is natural to ask about the converse. Recall that if $e(R) = 1$ then R is regular ([**Nag62**, Theorem 40.6]).

The following should be viewed as a variant of Kunz' criterion for regular rings. For instance, for F-signature, it says that if the percentage of $F^e_* R$ that is free limits to zero as $e \longrightarrow \infty$, then R is *actually regular*.

Theorem 3.2 ([**WY00, HL02**], *cf.* [**HY02**])**.** *Suppose that* $(R, \mathfrak{m}, k)$ *is a Noetherian F-finite local domain*[9] *ring. Then the following are equivalent.*

(a) *R is regular.*
(b) $e_{\mathrm{HK}}(R) = 1$.
(c) $s(R) = 1$.

PROOF. First suppose that $s(R) = 1$, we will show $e(R) = 1$. Note if $s(R) = 1 > 0$ then R is weakly F-regular by Remark 2.14, and so R is Cohen-Macaulay. Then Proposition 3.1 applies and we see that $0 = e_{\mathrm{HK}}(R) - 1$.

We omit the proof that $e_{\mathrm{HK}}(R) = 1$ implies that R is regular. There are at least two direct proofs of this fact: [**WY00, HY02**] (also see the exposition in [**Hun13**]). Both proofs deduce that $\ell_R\big(R/\mathfrak{m}^{[p]}\big) = p^d$, and from there the result is straightforward. Since it is not difficult to reduce to the case where R is complete with perfect residue field, then $\ell_R\big(R/\mathfrak{m}^{[p]}\big) = p^d$ implies that $F_* R$ is generated by $\operatorname{rank} F_* R$ elements. It follows that $F_* R$ is free and so R is regular by Kunz' theorem. See the next section for generalizations however.

A proof that $s(R) = 1$ implies that R is regular (without passing through results on Hilbert-Kunz multiplicity) can be found in [**MP20**]. $\square$

3.2. Lower bounds on Hilbert-Kunz multiplicity and upper bounds on F-signature. It is natural to ask how close Hilbert-Kunz multiplicity (or F-signature) can be to 1 without R being regular. This question has been explored by many authors, see for instance [**WY00, BE04, WY05, AE08, CDHZ12, AE13, NnBS20, JNS$^+$23**].

The rest of this section has yet to be written.

[9] these conditions can be relaxed in various ways, see the references

3.3. Toric singularities.

3.4. Various interesting Hilbert-Kunz functions.

3.5. Lech's conjecture for Hilbert-Kunz multiplicity.

3.6. Limit Hilbert-Kunz and limit F-signature. [BLM12, Tri07, Tri19, Shi18]

3.7. F-signature functions for pairs and p-fractals.

3.8. Exercises.

Exercise 3.1. Suppose that $R = k[\![x, y, z]\!]/(x, y) \cap (z)$. Prove that $e_{\mathrm{HK}}(R) = 1$ and explain why this is not a contradiction.

Exercise 3.2 (Curves)**.**

Exercise 3.3.

Exercise 3.4 ([**Tri18**])**.**

4. Positivity of F-signature and an application to the étale fundamental group

Our first goal in this section is to show that a strongly F-regular ring $(R, \mathfrak{m})$ is strongly F-regular if and only if $s(R) > 0$, a result originally due to Aberbach-Leuschke [**AL03**], see [**HL02**, Theorem 11] for the Gorenstein case.

4.1. Rings with positive F-signature are strongly F-regular. We have already seen that $s(R) > 0$ implies that R is weakly F-regular in Remark 2.14. We now show it is strongly F-regular.

Theorem 4.1. *Suppose $(R, \mathfrak{m})$ is a Noetherian F-finite local domain of dimension $d > 0$ such that $s(R) > 0$. Then R is strongly F-regular.*

Proof. Suppose R is not strongly F-regular, then $\tau(R) \subseteq \mathfrak{m}$. It follows, since $\tau(R)$ is uniformly F-compatible, that for each integer $e > 0$, we have that $\tau(R) \subseteq I_e(R)$. Now, as we have seen, $\mathfrak{m}^{[p^e]} \subseteq I_e(R)$ and so

$$\ell_R(R/I_e(R)) \leq \ell_R\Big(R/(\tau(R) + \mathfrak{m}^{[p^e]})\Big).$$

But $e \mapsto \ell_R\big(R/(\tau(R) + \mathfrak{m}^{[p^e]})\big)$ is the Hilbert-Kunz function of the ring $R/\tau(R)$, a ring of dimension $< d = \dim R$. In particular, that function is bounded above by a function $e \mapsto Cp^{e \dim R/\tau(R)}$, see Lemma 1.20. Thus, we see that

$$s(R) = \lim_{e \longrightarrow \infty} \frac{1}{p^{ed}} \ell_R(R/I_e(R)) = 0.$$

This completes the proof. □

Remark 4.2. The previous result also holds for R a non-domain if one make the definition, as we have suggested before,

$$s(R) = \lim_{e \longrightarrow \infty} \frac{1}{p^{ed}} \ell_R(R/I_e(R)).$$

The proof is the same.

4.2. Strongly F-regular rings have positive F-signature.

Theorem 4.3 ([**AL03**]). *Suppose $(R, \mathfrak{m})$ is a strongly F-regular F-finite local ring of dimension $d > 0$. Then $s(R) > 0$.*

We follow a a particularly insightful proof due Polstra-Tucker [**PT18**, Theorem 5.1]. Another proof, using particularly simple methods (and which avoids Gabber's generalization of the Cohen structure theorem) can be found in [**Pol22**], see [**MP20**] for an exposition.

Proof. Without loss of generality, we may assume R is complete (Exercise 2.4). By Gabber's generalization of the Cohen structure theorem, [**Ill14**, Théroème VI.2.1.1], we can find a Noether normalization

$$k[\![x_1, \ldots, x_d]\!] \subseteq A \subseteq R$$

which is generically étale. Fixing some nonzero element $c \in \mathcal{J}(R/A) \cap A$ where $\mathcal{J}(R/A)$ is the Jacobian ideal of R over A, we see from Chapter 6 (7.10.1) that

$$c\, F^e_* R \subset R \otimes_A F^e_* A \cong R[F^e_* A]$$

for all $e > 0$. Since R is strongly F-regular, there exists some $e_0 \geq 0$ and $\phi : F^{e_0}_* R \longrightarrow R$ such that $\phi(F^{e_0}_* c) = 1$.

Fix a basis $F^e_* a_1, \ldots, F^e_* a_m$ for $F^e_* A$ over A, with associated projection maps $\pi_i : F^e_* A \longrightarrow A$. Note $m = \operatorname{rank}_A(F^e_* A) = [F^e_* \mathcal{K}(R) : \mathcal{K}(R)]$ by Proposition 1.1 (b).

We have the following compositions:

$$\pi'_i : F^e_* R \xrightarrow{\cdot c} R[F^e_* A] \cong R \otimes_A F^e_* A \xrightarrow{R \otimes_A \pi_i} R$$

which satisfies $\pi'(F^e_* a_i) = c$ and $\pi'(F^e_* a_j) = 0$ for $j \neq i$. Now, the compositions:

$$\psi_i := \phi \star \pi'_i : F^{e+e_0}_* R \longrightarrow R$$

have the property that:

$$\psi_i(F^{e+e_0}_* a_i) = 1 \quad \text{and} \quad \psi_i(F^{e+e_0}_* a_j) = 0 \text{ for } j \neq i.$$

Thus

$$F^{e+e_0}_* R \xrightarrow{\psi_1 \oplus \cdots \oplus \psi_m} R^{\oplus m}$$

is surjective. In particular, $\operatorname{frk}(F^{e+e_0}_* R) \geq m = [F^e_* \mathcal{K}(R) : \mathcal{K}(R)]$. Hence

$$s(R) = \lim_{e \longrightarrow \infty} \frac{\operatorname{frk}(F^{e+e_0}_* R)}{\operatorname{rank}(F^{e+e_0}_* R)} \geq \lim_{e \longrightarrow \infty} \frac{[F^e_* \mathcal{K}(R) : \mathcal{K}(R)]}{[F^{e+e_0}_* \mathcal{K}(R) : \mathcal{K}(R)]} = \frac{1}{[F^{e_0}_* \mathcal{K}(R) : \mathcal{K}(R)]}$$

which is bigger than zero. $\square$

The above gives an explicit, and even computable, upper bound for $s(R)$, namely $\frac{1}{[F^{e_0}_* \mathcal{K}(R):\mathcal{K}(R)]}$.

Remark 4.4 (F-rational signature)**.** One might ask whether there is a similar way to control F-rationality. Indeed, that has been a topic of considerable study, starting with a preprint of Hochster-Yao from 2009 that became [**HY22**], and continuing in [**San15, ST23**].

The **relative F-rational signature** (introduced by Smirnov-Tucker, *cf.* Hochter-Yao's **F-rational signature**) is the infimum:

$$s_{\text{rel}} = \inf_{(\underline{x}) \subsetneq I} \frac{e_{HK}(\underline{x}) - e_{\text{HK}}(I)}{\ell_R(R/(\underline{x})) - \ell_R(R/I)}$$

where the infimum runs over all systems of parameters $\underline{x} := x_1, \ldots, x_d$ and ideals containing $(\underline{x})$. This turns out to be equal to the dual F-signature of Sannai:

$$s_{\text{dual}}(R) = \lim_{e \longrightarrow \infty} \frac{b_e}{\operatorname{rank} F^e_* R}$$

where b_e is the maximum value such that there exists a surjection $F^e_* R \twoheadrightarrow \omega_R^{\oplus b_e}$. This equality was shown [**ST23**] who also showed that the limit exists. These numbers detect F-rationality, in the same way that F-signature detects F-regularity (indeed, that's the motivation behind their introduction).

4.3. Transformation rules for F-signature under finite maps. If $R \subseteq S$ is finite, the goal of this subsection is to relate the F-signatures of R and S. Perhaps the most important case is when $R \subseteq S$ is **quasi-étale**, or in other words étale in codimension 1.

Theorem 4.5 ([**CRST18**], [**VK12**, Lemma 2.6.1], *cf.* [**HL02**, Proposition 19])**.** *Suppose that $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, l)$ is a finite split inclusion of Noetherian F-finite local normal domains that is étale-in-codimension-1. Then*

$$s(S) = \frac{[\mathcal{K}(S) : \mathcal{K}(R)]}{[l : k]}\, s(R)$$

The condition that $R \subseteq S$ is split is not very restrictive since strongly F-regular rings are splinters, see Exercise 6.20 in Chapter 1 or see Chapter 7.

Proof. This proof will use the trace map $\mathrm{Tr} : S \longrightarrow R$. There are three key facts we need about Tr coming from Chapter 5 Section 7.

(a) $\mathrm{Tr}(\mathfrak{n}) \subseteq \mathfrak{m}$.
(b) Tr generates $\mathrm{Hom}_R(S, R)$ as a S-module.
(c) $\mathrm{Tr}(S) = R$.

Property (a) is Chapter 5 Exercise 7.3. Property (b) follows since the ramification divisor is zero (since $R \subseteq S$ is étale-in-codimension-1) and Tr corresponds to the ramification divisor by Chapter 5 Lemma 7.11. The final condition (b) holds since $R \subseteq S$ splits, and hence there must be some surjective $\kappa : S \longrightarrow R$. But κ is a pre-multiple of Tr by (b), and so Tr must also be surjective.

Write $b_e = \mathrm{frk}_R(F^e_* S)$ and notice that $b_e = \ell_R\Big(\frac{F^e_* S}{I(F^e_* S)}\Big)$ by Lemma 1.15. Taking a short exact sequence (Lemma 1.19)

$$0 \longrightarrow R^{\oplus[\mathcal{K}(S):\mathcal{K}(R)]} \longrightarrow S \longrightarrow C \longrightarrow 0$$

where C is torsion, we thus see that

$$\mathrm{frk}_R(F^e_* S) \leq [\mathcal{K}(S) : \mathcal{K}(R)]\, \mathrm{frk}_R\, F^e_* R + \mu_R(F^e_* C)$$

Dividing by $\mathrm{rank}\, F^e_* R$ and sending $e \longrightarrow \infty$ yields, since C is torsion,

$$\limsup_{e \longrightarrow \infty} \frac{\mathrm{frk}_R(F^e_* S)}{\mathrm{rank}\, F^e_* R} \leq [\mathcal{K}(S) : \mathcal{K}(R)] s(R) + e_{\mathrm{HK}}(C) = [\mathcal{K}(S) : \mathcal{K}(R)] s(R).$$

On the other hand taking a short exact sequence (Exercise 1.6)

$$0 \longrightarrow S \longrightarrow R^{\oplus[\mathcal{K}(S):\mathcal{K}(R)]} \longrightarrow B \longrightarrow 0$$

and repeating the steps, proves the other inequality (with a $\liminf$), and hence:

$$[\mathcal{K}(S) : \mathcal{K}(R)] s(R) = \lim_{e \longrightarrow \infty} \frac{\mathrm{frk}_R(F^e_* S)}{\mathrm{rank}\, F^e_* R}. \tag{4.5.1}$$

Our next goal is to show that

$$\lim_{e \longrightarrow \infty} \frac{\operatorname{frk}_R(F^e_* S)}{\operatorname{rank} F^e_* R} = [l : k] s(S) = \tag{4.5.2}$$

which combined with (4.5.1) will complete the proof.

We first decompose as an S-module:

$$F^e_* S \cong S^{\oplus c_e} \oplus N_e.$$

Claim 4.6. $\operatorname{frk}_R(N_e) = 0$.

Proof of claim. Indeed, thanks to (b), any map $\phi : N_e \longrightarrow R$ must factor as

$$N_e \xrightarrow{\psi} S \xrightarrow{\mathrm{Tr}} R$$

by Appendix A Lemma 5.1. Now, $\psi(N_e) \subseteq \mathfrak{n}$ and thus using (a) we see that $\phi(N_e) = \mathrm{Tr}(\psi(N_e)) \subseteq \mathrm{Tr}(\mathfrak{n}) \subseteq \mathfrak{m}$ which proves the claim. $\square$

Similarly:

Claim 4.7. $\operatorname{frk}_R(S) = [l : k]$.

Proof of claim. Notice that

$$I(S) := \{x \in S \mid \phi(x) \in \mathfrak{m} \text{ for all } \phi \in \operatorname{Hom}_R(S, R)\}$$

contains $\mathfrak{n}$ since Tr generates $\operatorname{Hom}_R(S, R)$. On the other hand $I(S) \neq S$ since $R \subseteq S$ splits. Hence $I(S) = \mathfrak{n}$. By Lemma 1.15 we see that $\operatorname{frk}_R(S) = \ell_R(S/I(S)) = \ell_R(S/\mathfrak{n}) = [l : k]$ proving the second claim. $\square$

Combining the two claims, we see that $\operatorname{frk}_R(F^e_* S) = [l : k] c_e = [l : k] \operatorname{frk}_S(F^e_* S)$ and so dividing by $\operatorname{rank}_R(F^e_* R) = \operatorname{rank}_S(F^e_* S)$ (Proposition 1.1 (b)), and taking a limit we see that (4.5.2) holds. This completes the proof. $\square$

We didn't actually need that $R \subseteq S$ was étale-in-codimension 1, the same proof works without change if we assume properties (a), (b), (c) for some map $T : S \longrightarrow R$.

Corollary 4.8 (**[VK12, CR22]**). *Suppose $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, l)$ is a finite extension of Noetherian F-finite domains. Further suppose that there exists $T \in \operatorname{Hom}_R(S, R)$ such that*

(a) $T(\mathfrak{n}) \subseteq \mathfrak{m}$.
(b) T generates $\operatorname{Hom}_R(S, R)$ as a S-module.
(c) $T(S) = R$.

Then

$$s(S) = \frac{[\mathcal{K}(S) : \mathcal{K}(R)]}{[l : k]}\, s(R)$$

Note that if R is strongly F-regular, then the third condition $T(S) = R$ follows from the fact that T generates $\mathrm{Hom}_R(S, R)$, see Chapter 5 Exercise 7.12.

It immediately follows that if R is strongly F-regular, so that $s(R) > 0$, then so is S. A fact we already knew from

Indeed, there such are extensions. For instance, any cyclic cover of degree n associated to a divisor D of index n will satisfy the condition above. See Appendix B Section 9.

The above results let people compute the F-signature of numerous examples. For instance, we can compute the F-signature of quotient singularities, or even of quotients of rings whose F-signature we know. The following example illustrates this.

Example 4.9 ([**WY00**, Theorem 5.4], [**HL02**, Example 18])**.** Consider $T = k[\![x, y, z]\!]$ where $k = \overline{k}$ is an algebraically closed field of characteristic $p > 5$. The hypersurface quotients $R = T/(f)$ of multiplicity 2 are classified (up to change of variables) in [**Art77**], and appear on the list below (these singularities are called *Du Val singularities* or *rational double points*). Each of them has a finite extension $R \subseteq S$ that in most cases is étale in codimension 1, and in the remaining cases S is a *cyclic cover* where p divides the index as above (for A_n, this happens when $p|(n-1)$ and for D_n when $p|(n-2)$, also *cf.* [**CRMP$^+$21**]). Hence, we can write down the F-signature, and, by Proposition 3.1, we can write down the Hilbert-Kunz multiplicity too.

name	f	$[\mathcal{K}(S) : \mathcal{K}(R)]$	$s(R)$	$e_{\mathrm{HK}}(R)$
$A_n (n \geq 1)$	$x^2 + y^2 + z^{n+1}$	$n + 1$	$1/(n + 1)$	$2 - 1/(n + 1)$
$D_n (n \geq 4)$	$x^2 + y^2 z + z^{n-1}$	$4(n - 2)$	$1/(4(n - 2))$	$2 - 1/(4(n - 2))$
E_6	$x^2 + y^3 + z^4$	$1/24$	24	$2 - (1/24)$
E_7	$x^2 + y^3 + yz^3$	$1/48$	48	$2 - (1/48)$
E_8	$x^2 + y^3 + z^5$	$1/120$	120	$2 - (1/120)$

One can weaken the strong conditions on T (for instance, the étale-in-codimension-1) by instead working with pairs. We state such a result whose proof is similar, but unavoidably more technical. For $\Delta \geq 0$ a $\mathbb{Q}$-divisor we define $s(R, \Delta)$ to be $s(R, \mathscr{C}^{\Delta})$, as in Subsection 1.7.

Theorem 4.10 ([**CRS23**, Theorem C and Proposition 3.14], *cf.* [**CRST18**, Theorem 4.4])**.** *Suppose $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{m}, l)$ is an F-finite extension of normal domains with induced $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. Suppose $T \in \operatorname{Hom}_R(S, R)$ is surjective and satisfies $T(\mathfrak{n}) \subseteq \mathfrak{m}$ (many cases of interest, simply set $T = \operatorname{Tr}$). Set D_T to be the divisor on $\operatorname{Spec} S$ corresponding to the map T as in Chapter 5 Proposition 7.10 (so D_T is the ramification divisor if $T = \operatorname{Tr}$). Then if $\Delta - D_T \geq 0$, we have that*

$$s(S, f^*\Delta - D_T) = s(R, \Delta).$$

PROOF. We omit the proof referring above to [**CRS23**] who prove even stronger results including general results on transformation of F-signature of Cartier algebras. □

4.4. An application to the divisor class group. Recall that the divisor class group $\operatorname{Cl}(R)$ of a Noetherian normal domain R, is the set Weil divisors on $\operatorname{Spec} R$ modulo linear equivalence.

Theorem 4.11 ([**Pol22**, Corollary 3.3], [**CR22**, Corollary 5.1])**.** *Suppose $(R, \mathfrak{m}, k)$ is a strongly F-regular local ring. If D is a $\mathbb{Q}$-Cartier Weil divisor on $\operatorname{Spec} R$ with index n (that is, $n > 0$ is the smallest integer such that $nD \sim 0$), then*

$$n < 1/s(R).$$

Furthermore, the torsion part of the divisor class group of R is finite with

$$|\operatorname{Cl}(R)_{\text{tors}}| < 1/s(R).$$

Hence $\operatorname{Cl}(R)$ is torsion free if $s(R) > 1/2$.

Note that Polstra proved the finiteness of the divisor class group without directly using F-signature.

PROOF. The first statement simply follows since we can always take a finite cyclic cover S of degree n associated to D, since then Corollary 4.8 applies, we have

$$1 \geq s(S) = [\mathcal{K}(S) : \mathcal{K}(R)] s(R) = ns(R)$$

and so $\frac{n \leq 1}{s(R)}$. Note this already rules out any torsion in the divisor class group of $s(R) > 1/2$.

For the second statement, we follow the strategy of [**CR22**, Corollary 5.1]. If D_1 is a torsion divisor of index n_1, let $S_1 \supseteq R$ be the associated cyclic cover (of degree n_1). By [**TW92**, Corollary 2.6], then kernel of the induced

$$\operatorname{Cl}(R) \xrightarrow{f_1^*} \operatorname{Cl}(S_1)$$

is generated by the equivalence class of D, that is $|\ker(f_1^*)| = n$. As above, we have that $s(S_1) = n_1 s(R)$. Repeating this process, if D_2 is torsion another divisor on $\operatorname{Spec} R$, whose class is not in $\ker(f_1^*)$, then $f_1^* D_2$ on $\operatorname{Spec} S_1$ is still torsion, say of index n_2. Hence we can find a further local extension $S_2 \supseteq S_1$, a cyclic cover associated to $f_1^* D_2$, of degree n_2. Again, $s(S_2) = n_2 s(S_1) = n_1 n_2 s(R)$. Since F-signature is bounded above by 1, this process must stop and we thus have a sequence of maps of Abelian groups

$$\mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(S_1) \longrightarrow \mathrm{Cl}(S_2) \longrightarrow \ldots \longrightarrow \mathrm{Cl}(S_n)$$

where $\mathrm{Cl}(R)_{\mathrm{tors}}$ is in the kernel of the composition $\mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(S_n)$. But each map in that sequence has finite kernel, by construction, and the result follows. $\square$

The goal of the remainder of the chapter is to take the above arguments and apply them to the *étale fundamental group.*

4.5. An application of F-signature to the étale fundamental group. A classical way to study a singularity in complex algebraic geometry, especially an isolated one $x \in X$, is to study the fundamental group of the link of the singularity[10]. This investigation began with Mumford, who showed that for a normal surface singularity, having trivial fundamental group is equivalent to being nonsingular [**Mum61**], *cf.* [**Fle75, CS93**].

In characteristic $p > 0$, we lack such techniques, but one can use the étale fundamental group of Grothendieck (and generalizations) to measure something like the pro-finite completion of the fundamental group of the link in the complex setting. Unfortunately, regularity of a normal surface singularity is not equivalent to triviality of the étale fundamental group in characteristic $p > 0$ [**Art77**], *cf.* [**CS93**], although with more general fundamental groups which we won't define here, one can characterize smoothness for $p > 3$ [**EV10**]. Our goal is to study the étale fundamental group of strongly F-regular singularities.

We define the étale fundamental group in a very special setting relevant to us. For more general definitions, see for instance [**Gro63, Mur67**].

Definition 4.12. Suppose U is a Noetherian integral normal scheme with fraction field K. The **étale fundamental group** of U to be

$$\pi_1^{\mathrm{t}}(U) = \varprojlim \mathrm{Gal}(\mathcal{K}(V)/K)$$

where the inverse limit runs over connected V with finite étale $V \longrightarrow U$ which are generically Galois and with a compatible map from $\operatorname{Spec} K_{\mathrm{sep}}$. In

[10]If $X \subseteq \mathbb{A}^{\mathbb{C}}$, the link is a small euclidean ball B centered at x intersected with X, that is $B \cap (X \setminus \{x\})$, if x is not an isolated singularity, it is natural to instead consider $B \cap X_{\mathrm{nonsing}}$

other words, the fraction field of the scheme V comes with an embedding into K_{sep}, $\mathcal{K}(V)/K$ is Galois, and the induced $V \longrightarrow U$ is Galois. Note V is normal and hence integral in our special setup.

Example 4.13. If $R = k$ is a field then the étale fundamental group of $\operatorname{Spec} R$ is simply the absolute Galois group of k,

$$\pi_1^{\mathrm{t}}(\operatorname{Spec} k) = \operatorname{Gal}(k_{\mathrm{sep}}/k)$$

On the other hand if $(R, \mathfrak{m}, k_{\mathrm{sep}})$ is a Noetherian complete local domain with separably closed residue field, then

$$\pi_1^{\mathrm{t}}(\operatorname{Spec} R) = \{1\}$$

Indeed, a finite $R \subseteq S$ is a product of local rings by the Henselian property, (9) in [**Sta19**, Tag 04GG] and so we may assume S is local. Now suppose $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, l)$ is finite étale. We know that $\mathfrak{n} = \mathfrak{m}S$ (étale implies unramified) and so the base change $k = R/\mathfrak{m} \longrightarrow S/\mathfrak{n} = l$ is also finite étale, hence $k = l$ since k is separably closed. But then $\mu_S (=) \dim_k(S/\mathfrak{m}S) = 1$, and so S is a free R-module of rank 1. Hence $R = S$ and the étale fundamental group is trivial since the limiting set is trivial.

If k is a field of characteristic zero, then $\mathbb{A}^1_k$ is trivial (the complex plane is simply connected). However, in characteristic $p > 0$ the fundamental group of $\mathbb{A}^1_k$ is not trivial, there are étale coverings of $\mathbb{A}^1$, Artin-Schreier coverings, that are wildly ramified at infinity.

In our case, we will assume that R is a Noetherian F-finite normal domain, and $U \subseteq \operatorname{Spec} R$. Then for any finite $V \longrightarrow U$ with V integral, we can take the integral closure of R in $\mathcal{K}(V)$ and so obtain a finite extension $R \subseteq S$ of normal domains with $V = f^{-1}(U)$ if $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is the induced map. Thus studying finite étale

$$V \longrightarrow U$$

as above is the same as studying finite extensions of normal domains $R \subseteq S$ where are étale over U. In other words,

$$\pi_1^{\mathrm{t}}(U) = \varprojlim \operatorname{Gal}(\mathcal{K}(S)/\mathcal{K}(R))$$

where $S \subseteq \mathcal{K}(R)_{\mathrm{sep}}$ runs over normal finite generically Galois extensions of R which are *tale* over U.

Theorem 4.14 ([**CR22**])**.** *Suppose $(R, \mathfrak{m}, k = k_{\mathrm{sep}})$ is a strongly F-regular complete local ring of dimension ≥ 2 with separably closed residue field. Suppose $U \subseteq \operatorname{Spec} R$ is an open set whose complement has codimension ≥ 2, for instance set U to be the regular locus of $\operatorname{Spec} R$ or $U = \operatorname{Spec} R \setminus \mathfrak{m}$.*

$$\left|\pi_1^{\mathrm{t}}(U)\right| \leq \frac{1}{s(R)}.$$

In particular, the étale fundamental group is finite.

PROOF. If $S, S' \supseteq R$ are finite normal domain extensions in $\mathcal{K}(R)_{\mathrm{sep}}$ which are generically Galois and étale over U. Consider $S \otimes_R S'$. This is étale over U but it need not be irreducible. Normalizing and then modding out by minimal prime we obtain a map $S \otimes_R S' \twoheadrightarrow S'' \subseteq \mathcal{K}(R)_{\mathrm{sep}}$. Since even $S \otimes_R S'$ is étale over U (being a composition of étale maps over U), we see that $R \subseteq S''$ is finite étale[11] over U and generically Galois ($\mathcal{K}(S'') = \mathcal{K}(S')\mathcal{K}(S)$).

Because of the above, if we find an absolute bound N on $[\mathcal{K}(S) : \mathcal{K}(R)]$ for such extensions $S \supseteq R$, there is a unique maximal such extension $S \supseteq R$. It would then follow that $\pi_1^t(U)$ is finite of order bounded by N.

Finally, suppose $(R, \mathfrak{m}, k) \supseteq (S, \mathfrak{n}, l)$ is a finite extension of local domains which is étale over U. By Exercise 4.1 and the fact that $k = k_{\mathrm{sep}}$, we see that $k = l$. Therefore, by Theorem 4.5 we see that

$$1 \geq s(S) = [\mathcal{K}(S) : \mathcal{K}(R)] s(R)$$

and so $[\mathcal{K}(S) : \mathcal{K}(R)] \leq 1/s(R)$. This provides our absolute bound and completes the proof. ☐

Remark 4.15. The results above were originally inspired by a conjecture of Kollár
cite[Question 26]KollarNewExamples that for KLT singularities over $\mathbb{C}$, the étale fundamental group of the link is finite (see also [**GZ94, FKL93, KM99**] for earlier related results). This was proven by Braun [**Bra21**] (and even more that $B \cap X_{\mathrm{nonsing}}$ has finite fundamental group where B is a small ball around the point $x \in X$ of interest). Earlier work in characteristic zero includes [**Xu14**] (*cf.* [**GKP16, TX17**]) who proved finiteness of the étale fundamental group of the singularity (which equals the profinite completion of the fundamental group of the link). In fact, using normalized volume instead of F-signature, there are similar bounds on the size fundamental group in characteristic zero since,

(a) Normalized volume is positive for KLT singularities, [**Li18**, Corollary 3.4].
(b) Normalized volume is bounded above by d^d if $d = \dim X$.*cf.* [**Blu18**].
(c) Normalized volume satisfies a transformation rule analogous to Theorem 4.5, [**XZ21**].

Indeed, it is expected that normalized volume and F-signature are quite closely related [**LLX20, Zhu23**]. However, as mentioned in, we do not even know if, for a KLT singularity $(R, \mathfrak{m})$ of essentially finite type over a field of

[11] For more general versions of this, useful for computing étale fundamental groups in general, see [**Mur67**].

characteristic 0 (so that R has open strongly F-regular type by Chapter 6 Theorem 4.23), we know that, if

$$\limsup_{p} s(R_p) > 0.$$

Other related results on étale fundamental groups can be found in, for example, [**BCRG$^+$19, Sti17, BJNnB19, JS22, BGO17, CLM$^+$23**].

4.6. Exercises.

Exercise 4.1. Suppose that $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, l)$ is a finite, étale-in-codimension-1, and split extension of Noetherian domains with R normal (so that $\mathrm{Tr}(S) \subseteq R$). Show that $k \subseteq l$ is separable.

Hint: Show that there is an induced map $\overline{\mathrm{Tr}} : l \longrightarrow k$ and show that it is a multiple of the field trace for $k \subseteq l$. If you get stuck, see [**AR63**, Proposition 3.1].

Exercise 4.2. Let $R \subseteq S = k[x, \dots, x_d]$ be an nth Veronese subring of S (that is, R is the k-algebra generated by the monomials of degree n). Prove that $s(S) = 1/n$.

Exercise 4.3. Consider the subring $R = k[u^{n+1}, u^{n+1}, uv] \subseteq k[u, v] = S$. Show that S can also be obtained as a cyclic cover of index $n + 1$ from R. Conclude that $s(R) = 1/(n + 1)$.

CHAPTER 10

Geometric and global applications of Frobenius

Warning, this chapter will likely undergo substantial revision and additions (numerous references are also missing). We also hope to generalize some of the results written here.

The insight into singularities in prime characteristic documented in earlier chapters of this book ultimately helped enable progress in the minimal model program for prime characteristic threefolds and fourfolds; [**HX15, Bir16, BW17, Wal18, DW19, HW22, HW23, Cas21**].

Suppose X is a F-finite Noetherian scheme. Then on each affine chart $U_i = \operatorname{Spec} R_i$ we can define a test ideal $\tau(R_i)$.Because the formation of the test ideal commutes with localization, we then obtain a test ideal sheaf

$$\tau(X) \subseteq \mathcal{O}_X,$$

a coherent sheaf so that $\Gamma(U_i, \tau(X)) = \tau(R_i)$. Likewise if X is normal and $\Delta \geq 0$ is an $\mathbb{R}$-divisor, then we can define $\tau(X, \Delta)$ analogously. If ω_X is a canonical sheaf on X, then similar statements can be made for $\tau(\omega_X) \subseteq \omega_X$, the sheaf such that $\Gamma(U_i, \tau(\omega_X)) = \tau(\omega_{R_i})$. Analogous statements hold for other variants.

In this chapter, we are going to use this construction

1. Special Frobenius-stable global sections

Suppose that k is an F-finite field of characteristic $p > 0$ and X is a projective variety over k. Further suppose that L a Weil divisor on X. Now, the Frobenius map induces

$$F^e_* \mathcal{O}_X(K_X) = F^e_* \omega_X \longrightarrow \omega_X = \mathcal{O}_X(K_X)$$

as we have seen and so twist this map by $\mathcal{O}_X(L)$ (and S2-ify/reflexify if necessary) to obtain

$$F^e_*(\mathcal{O}_X(K_X + p^e L)) = ((F^e_* \omega_X) \otimes \mathscr{L})^{S2} \longrightarrow (\omega_X \otimes \mathscr{L})^{S2} = \mathcal{O}_X(K_X + L).$$

We are going to use this map produce a special sub-vector space of $H^0(X, \mathcal{O}_X(K_X + L))$, in other words, a special linear system in $|H^0(X, \mathcal{O}_X(K_X + L))|$.

Definition 1.1 (Frobenius stable special sections)**.** With X and L as above, we define

$$S^0(X, \mathcal{O}_X(K_X+L)) := \text{Image}\Big(H^0(X, F^e_*(\mathcal{O}_X(K_X+p^eL))) \longrightarrow H^0(X, \mathcal{O}_X(K_X+L))\Big)$$

for $e \gg 0$. This is the set of *(dual) Frobenius stable sections* of $H^0(X, \mathcal{O}_X(K_X+L))$.

Notice that $H^0(X, \omega_X \otimes \mathscr{L})$ is a finite dimensional k-vector space, and so the images that appear in Definition 1.1 are descending submodules of a finitely generated k-vector space. Hence Definition 1.1 does not depend on the choice of $e \gg 0$. We do not even need to apply any variant of the HSLG theorem Chapter 8 Theorem 2.1.

In a slightly different direction, assume now that L is a Cartier divisor (so we didn't need to S2ify). The construction above only used that ω_X had the structure of a Cartier module since we just twisted $F^e_*\omega_X \longrightarrow \omega_X$ by L, and took global sections. Indeed, suppose that $(\mathscr{M}, \phi)$ is a Cartier-module on X, that is, an $\mathcal{O}_X$-module with a given *structural map* $\phi : F^e_*\mathscr{M} \longrightarrow \mathscr{M}$, then we make the following definition.

Definition 1.2 (Frobenius stable special sections for Cartier modules)**.** If $(\mathscr{M}, \phi)$ is any Cartier-module on X and $\mathscr{L}$ is a line bundle, we define

$$S^0(X, \phi, \mathscr{M} \otimes \mathscr{L}) := \text{Image}\Big(H^0(X, F^{ne}_*(\mathscr{M} \otimes \mathscr{L}^{p^{ne}})) \longrightarrow H^0(X, \mathscr{M} \otimes \mathscr{L})\Big)$$

for $n \gg 0$.

There are some important cases when $S^0 = H^0$. First, when X is globally F-split they are the same.

Lemma 1.3. *With notation as above, if X is globally F-split, then $S^0(X, \mathcal{O}_X(K_X+L)) = H^0(X, \mathcal{O}_X(K_X + L))$ for any Weil divisor L.*

PROOF. Each map $F^e_*\mathcal{O}_X(K_X) \longrightarrow \mathcal{O}_X(K_X)$ is split surjective. Hence it stays split surjective after tensoring by $\mathcal{O}_X(L)$, S2-ifying, and after taking cohomology. $\square$

Proposition 1.4. *Suppose $\mathscr{L} = \mathcal{O}_X(L)$ is an ample line bundle and $(\mathscr{M}, \phi)$ is an F-pure Cartier module. Then $S^0(X, \phi, \mathscr{M} \otimes \mathscr{L}^m) = H^0(X, \mathscr{M} \otimes \mathscr{L}^m)$ for $m \gg 0$. In particular, if X is F-injective then $S^0(X, \mathcal{O}_X(K_X + mL)) = H^0(X, \mathcal{O}_X(K_X + mL))$ for $m \gg 0$.*

PROOF. Using that $\mathscr{L}$ is ample, we have that for $m \gg 0$, the map induced by ϕ

$$F^e_*(\mathscr{M} \otimes \mathscr{L}^{mp^e}) \cong (F^e_*\mathscr{M}) \otimes \mathscr{L}^m \longrightarrow \mathscr{M} \otimes \mathscr{L}^m$$

is surjective on global sections (it was already surjective locally by the F-purity hypothesis). Hence the map

$$F_*^{(n+1)e}(\mathscr{M} \otimes \mathscr{L}^{mp^{(n+1)e}}) \longrightarrow F_*^{ne}(\mathscr{M} \otimes \mathscr{L}^{mp^{ne}})$$

surjects on global sections as well for all n. Composing these maps yields that

$$F_*^{(n+1)e}(\mathscr{M} \otimes \mathscr{L}^{mp^{(n+1)e}}) \longrightarrow \mathscr{L}^m \longrightarrow \mathscr{M} \otimes \mathscr{L}^m$$

is surjective on global sections as desired. The result follows. $\square$

Some of the key F-pure Cartier modules we will work with include $\tau(\omega_X)$ and $\sigma(\omega_X) = \underline{\omega_X}$.

In the next subsection we will show that we can choose this $m \gg 0$ effectively if instead of requiring that $S^0 = H^0$, we simply require that S^0 globally generates $\mathscr{M} \otimes \mathscr{L}^m$. First however, we consider some examples.

Example 1.5 (Projective space)**.** Suppose that X is $\mathbb{P}^n$. Then X is Frobenius split and so $S^0(X, \omega_X \otimes \mathcal{O}_X(n)) = H^0(X, \omega_X \otimes \mathcal{O}_X(n))$.

Example 1.6 (Elliptic curves)**.** Suppose now that X is a smooth curve over an algebraically closed field of characteristic $p > 0$ (notice that $\mathcal{O}_X = \omega_X$). If X is ordinary, it is Frobenius split and so $H^0(X, \mathscr{L}) = S^0(X, \mathscr{L})$ for any line bundle $\mathscr{L}$. If X is supersingular, then $S^0(X, \mathcal{O}_X) = 0$.

Next suppose that D is any divisor of positive degree on X, then we claim that:

$$S^0(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(D)).$$

Indeed, it is sufficient to show that $H^0(X, F_*\mathcal{O}_X(pD)) \longrightarrow H^0(X, \mathcal{O}_X(D))$ is surjective, or Serre-dually that

$$H^1(X, \mathcal{O}_X(-D)) \longrightarrow H^1(X, F_*\mathcal{O}_X(-pD))$$

injects. However, it was shown in [**Oda71**, Theorem 2.17] that this is injective as long as $\deg D > 0$ (in fact, similar results for vector bundles are also shown there).

Example 1.7 (Higher genus curves)**.** Suppose that X is a smooth projective curve over an algebraically closed field of characteristic $p > 0$. Suppose that $\mathscr{L}$ is a line bundle of degree ≥ 2. Then we claim that $|S^0(X, \omega_X \otimes \mathscr{L})|$ is a base point free linear system. Fix a closed point $Q \in X$. We must show that $S^0(X, \omega_X \otimes \mathscr{L}) \subseteq H^0(X, \omega_X \otimes \mathscr{L})$ has a section that does not vanish at Q. We form the following diagram for $e \gg 0$, where $Q' := p^e Q$ is the corresponding non-reduced scheme.

$$\begin{tikzcd}
H^0(X, F^e_*(\omega_X \otimes \mathscr{L}^{p^e})) \arrow[r] \arrow[d, two heads] & H^0(Q', F^e_*(\omega_{Q'} \otimes \mathscr{L}^{p^e})) \arrow[r] \arrow[dd, two heads] & 0 \arrow[dd] \\
S^0(X, \omega_X \otimes \mathscr{L}) \arrow[d, hook] & & \\
H^0(X, \omega_X \otimes \mathscr{L}) \arrow[r] & H^0(Q, \omega_Q \otimes \mathscr{L}) \arrow[r] & H^1(X, \omega_X \otimes \mathscr{L} \otimes \mathcal{O}_X(-Q))
\end{tikzcd}$$

The top right entry $H^1(X, F^e_*(\omega_X \otimes \mathscr{L}^{p^e} \otimes \mathcal{O}_X(-p^e Q)))$ is zero by Serre vanishing since $\deg(\mathscr{L} \otimes \mathcal{O}_X(-Q)) \geq 1$ and hence $\mathscr{L} \otimes \mathcal{O}_X(-Q)$ is ample. The middle vertical arrow is surjective since Q is a point and in particular Frobenius split so that $F^e_* \omega_Q \longrightarrow F^e_* \omega_{Q'} \longrightarrow \omega_Q$ is surjective. The diagram then immediately implies that $S^0(X, \omega_X \otimes \mathscr{L})$ has a section that does not vanish at Q.

In fact, it was also shown in [**Tan72**, Theorem 15] that $H^1(X, \mathscr{L}^{-1}) \longrightarrow H^1(X, \mathscr{L}^{-p})$ injects as long as $\deg \mathscr{L} > \frac{2g-2}{p}$ where g is the genus of X. Hence by Serre duality, $S^0(X, \omega_X \otimes \mathscr{L}) = H^0(X, \omega_X \otimes \mathscr{L})$ for all $\mathscr{L}$ with $\deg \mathscr{L} > \frac{2g-2}{p}$.

We will generalize the diagram in the previous example in various ways in the next section.

1.1. Effective global generation. We first recall that a sheaf $\mathscr{F}$ is said to be 0-regular with respect to an ample globally generated line bundle $\mathscr{L}$ if $H^i(X, \mathscr{F} \otimes \mathscr{L}^{-i}) = 0$ for all $i > 0$. In which case, we have that $\mathscr{F}$ is globally generated. See Chapter 6 Definition 3.18 or [**Laz04b**] for more details.

Theorem 1.8 (*cf.* [**Kee08, Smi97b**]). *Suppose that X is projective over an F-finite field k and suppose that $(\mathscr{M}, \phi)$ is an F-pure Cartier module on X with* $\dim \operatorname{Supp} \mathscr{M} = d$. *Assume that $\mathscr{L}$ is ample and globally generated*[1] *and $\mathscr{A}$ is ample. Then*

$$\mathscr{M} \otimes \mathscr{L}^d \otimes \mathscr{A}$$

is globally generated by $S^0(X, \phi, \mathscr{M} \otimes \mathscr{L}^d \otimes \mathscr{A})$.

Proof. Consider the surjective map

$$F^{me}_*(\mathscr{M} \otimes \mathscr{L}^{dp^{me}} \otimes \mathscr{A}^{p^{me}}) \cong ((F^{me}_* \mathscr{M}) \otimes \mathscr{L}^d \otimes \mathscr{A}) \longrightarrow \mathscr{M} \otimes \mathscr{L}^d \otimes \mathscr{A}.$$

If we can show that the left side is globally generated as an $\mathcal{O}_X$-module, we see that the images of those global sections (that is, $S^0(X, \mathscr{M} \otimes \mathscr{L}^d \otimes \mathscr{A})$

[1] Meaning that $H^0(X, \mathscr{L})$ globally generates $\mathscr{L}$ as an $\mathcal{O}_X$-module

assuming $m \gg 0$) will also globally generate $\mathscr{M} \otimes \mathscr{L}^d \otimes \mathscr{A}$. We thus show that $F^{me}_*(\mathscr{M} \otimes \mathscr{L}^{dp^{me}} \otimes \mathscr{A}^{p^{me}})$ is 0-regular as a $\mathcal{O}_X$-module.

Now,

$H^i(X, \mathscr{L}^{-i} \otimes F^{me}_*(\mathscr{M} \otimes \mathscr{L}^{dp^{me}} \otimes \mathscr{A}^{p^{me}})) = H^i(X, \otimes F^{me}_*(\mathscr{M} \otimes \mathscr{L}^{(d-i)p^{me}} \otimes \mathscr{A}^{p^{me}}))$

which is zero by Serre vanishing for $i \leq d$ and for $i > d$ by dimension reasons. The result follows. $\square$

Corollary 1.9. *Suppose that X is a projective d-dimensional F-injective over an F-finite field. Suppose that $\mathscr{L}$ is globally generated ample and $\mathscr{A}$ is ample. Then $\omega_X \otimes \mathscr{L}^d \otimes \mathscr{A}$ is globally generated by $S^0(X, \omega_X \otimes \mathscr{L}^d \otimes \mathscr{A})$.*

It is a conjecture for smooth d-dimensional varieties in characteristic zero that $\omega_X \otimes \mathscr{L}^{d+1}$ is globally generated for any ample line bundle $\mathscr{L}$, including the case that $\mathscr{L}$ is not globally generated. This is called **Fujita's freeness conjecture**. Some progress on it in low dimensions can be found in [], []. The theorem of Angerhrn and Siu [**AS95**] (*cf.* [**Laz04b**, Section 10.3]) gives a quadratic (as opposed to linear, $d+1$) bound on what power of $\mathscr{L}$ is needed so that $\omega_X \otimes \mathscr{L}$ is globally generated. In characteristic $p > 0$, Fujita's freeness conjecture is known to be false even for surfaces [].

1.2. Connections with section rings. We now provide an alternate description of $S^0(X, \omega_X \otimes \mathscr{L})$ when $\mathscr{L}$ is ample. We work with graded rings, see **??** for more details.

Suppose X is a normal projective variety of dimension > 0 over an F-finite field. Set

$$S = \bigoplus_{i \geq 0} H^0(X, \mathscr{L}^i).$$

This is a graded ring. The module

$$\omega_S = \bigoplus_{i \in \mathbb{Z}} H^0(X, \omega_X \otimes \mathscr{L}^i)$$

is a graded canonical module for S. See **?? ??**. The maps

$$F^e_*(\omega_X \otimes \mathscr{L}^{p^e i}) \longrightarrow \omega_X \otimes \mathscr{L}^i$$

induce maps $F^e_* \omega_S \longrightarrow \omega_S$, which is in fact the dual to Frobenius on S, **??**.

Based on these observations, we see that $S^0(X, \omega_X \otimes \mathscr{L}^i)$ is nothing else but ith graded pieces of the F-pure Cartier module $\sigma(\omega_S) = \underline{\omega_S}$. Explicitly:

Proposition 1.10. *With notation as above, $S^0(X, \omega_X \otimes \mathscr{L}^i) = [\sigma(\omega_S)]_i$.*

It is of course possible to likewise describe $S^0(X, \phi, \mathscr{M} \otimes \mathscr{L}^i)$ for a Cartier module $(\mathscr{M}, \phi)$. See Exercise 1.5.

1.3. Special sections for pairs and alternate formulations. There is another common formulation of S^0 which hides the divisor K_X. Suppose that X is a normal projective variety over an F-finite field. Fix L to be a Weil divisor and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $(p^e-1)(L-K_X-\Delta)$ is Cartier (in particular, $(p^e-1)\Delta$ has integer coefficients).

In this case, we have maps dual to Frobenius:

$$\ldots \longrightarrow F^{2e}_* \mathcal{O}_X(K_X) \longrightarrow F^e_* \mathcal{O}_X(K_X) \longrightarrow \mathcal{O}_X(K_X).$$

Twisting by $L-K_X$, and reflexifying we have:

$$\ldots \longrightarrow F^{2e}_* \mathcal{O}_X(p^{2e}L+(1-p^{2e})K_X) \longrightarrow F^e_* \mathcal{O}_X(p^eL+(1-p^e)K_X) \longrightarrow \mathcal{O}_X(L).$$

Finally, using that $\Delta \geq 0$ and that $(p^e-1)\Delta$ has integer coefficients we obtain:

$$\ldots \longrightarrow F^{2e}_* \mathcal{O}_X(L+(p^{2e}-1)(L-K_X-\Delta)) \longrightarrow F^e_* \mathcal{O}_X(L+(p^e-1)(L-K_X-\Delta)) \longrightarrow \mathcal{O}_X(L).$$

Since $(p^e-1)(L-K_X-\Delta)$ is Cartier, then *locally* this gives $\mathcal{O}_X(L)$ the structure of a Cartier module (although the global structure is twisted). Suppose that $\mathscr{N} \subseteq \mathcal{O}_X(L)$ is locally a Cartier-submodule with respect to the action above. For instance, if L is Cartier, then $\tau(X,\Delta) \otimes \mathcal{O}_X(L) \subseteq \mathcal{O}_X(L)$ is such a module. Furthermore, if $\Delta = D+B$ where D is reduced and $B \geq 0$ has no common components with D, then $\tau_D(X,\Delta) \otimes \mathcal{O}_X(L)$ is also such a module where $\tau_D(X,\Delta)$ is the adjoint test ideal of **??** **??**. Regardless, we have the following.

$$\begin{array}{lll} \ldots & \longrightarrow F^{2e}_*\big(\mathscr{N} \otimes \mathcal{O}_X((p^{2e}-1)(L-K_X-\Delta))\big) & \\ & \longrightarrow F^{e}_*\big(\mathscr{N} \otimes \mathcal{O}_X((p^{e}-1)(L-K_X-\Delta))\big) & \longrightarrow \mathscr{N}. \end{array} \tag{1.10.1}$$

Definition 1.11. With notation as above then we define:

$$S^0(X,\Delta,\mathscr{N}) = \mathrm{Image}\Big(H^0\big(X, F^{ne}_*(\mathscr{N} \otimes \mathcal{O}_X((p^{ne}-1)(L-K_X-\Delta)))\big) \longrightarrow H^0\big(X,\mathscr{N}\big)\Big)$$

For $n \gg 0$.

This is independent of the choice of n since it is a submodule of a finitely generated vector space. Notice that $\mathscr{N}$ implicitly determines L since $\mathscr{N} \subseteq \mathcal{O}_X(L)$.

We also obtain the following variant of effective global generation.

Theorem 1.12. *Suppose that X is a normal projective d-dimensional variety over an F-finite field and that $\mathscr{L} = \mathcal{O}_X(L)$ is a globally generated ample line bundle. Suppose that A is a Cartier divisor, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor, and that $(p^e-1)(A-K_X-\Delta)$ is a big and nef Cartier divisor. Then*

$$\tau(X,\Delta) \otimes \mathcal{O}_X(A+dL)$$

is globally generated by $S^0(X,\Delta,\tau(X,\Delta) \otimes \mathcal{O}_X(A+dL))$.

Proof. Since $A-K_X-\Delta$ is big and nef, there exists a Cartier divisor $G>0$ so that $A-K_X-\Delta-\epsilon G$ is ample for all $1\gg\epsilon>0$, see Exercise 1.4. We can choose $\epsilon>0$ small enough so that $\tau(X,\Delta)=\tau(X,\Delta+\epsilon G)$ by Chapter 5 **??** **??**. We may also select this ϵ so that $(p^e-1)\epsilon$ is an integer after possibly replacing e by a multiple. Since

$$S^0(X,\Delta+\epsilon G,\tau(X,\Delta+\epsilon G)\otimes\mathcal{O}_X(A+dL))\subseteq S^0(X,\Delta,\tau(X,\Delta)\otimes\mathcal{O}_X(A+dL))$$

we may now assume replace Δ by $\Delta+\epsilon G$ so that $A-K_X-\Delta$ is ample.

The proof now follows that of Theorem 1.8. We must show that $F^{ne}_*\big(\tau(X,\Delta)\otimes\mathcal{O}_X(p^edL+p^eA+(p^{ne}-1)(-K_X-\Delta))\big)$ is globally generated as a $\mathcal{O}_X$-module for $n\gg 0$. But it is 0-regular by the same argument. The result follows. $\square$

Let us briefly compare the hypothesis of Theorem 1.12. We implicitly assumed that $(p^e-1)(K_X+\Delta)$ was Cartier for some $e>0$. In other words, we assumed the index of $K_X+\Delta$ was not divisible by $p>0$. If one is willing to remove the statement about S^0, we can perturb our way out this hypothesis.

Corollary 1.13. *Suppose that X is a normal projective d-dimensional variety over an F-finite field and that $\mathscr{L}=\mathcal{O}_X(L)$ is a globally generated ample line bundle. Suppose that A is a Cartier divisor, $\Delta\geq 0$ is a $\mathbb{Q}$-divisor, and that $A-K_X-\Delta$ is big and nef. Then*

$$\tau(X,\Delta)\otimes\mathcal{O}_X(A+dL)$$

is globally generated

Proof. In the proof of Theorem 1.12, we were able to assume that $A-K_X-\Delta$ is ample, and we can again use Exercise 1.4 again here to accomplish the same.

Suppose H is a Cartier divisor such that K_X+H is effective. Choose an integer $n>0$ and $e_0\geq 0$ so that $np^{e_0}(A-K_X-\Delta)$ is Cartier where p does not divide n. Then for $e>e_0$, form

$$\Gamma_e=\frac{H+K_X+\Delta}{p^e-1}.$$

For $e\gg e_0$, notice that $A-K_X-\Delta-\Gamma_e$ is still ample and that $\tau(X,\Delta+\Gamma_e)=\tau(X,\Delta+\Gamma_e)$ by Chapter 5 **??** **??**.

However,

$$\begin{array}{rl} & n(p^e-1)(K_X+\Delta+\Gamma_e)\\ = & n(p^e-1)(K_X+\Delta)+n(H+K_X+\Delta)\\ = & nH+np^e(K_X+\Delta)\end{array}$$

is Cartier. Hence we can apply Theorem 1.12 for $e \gg 0$ to complete the proof. $\square$

1.4. An application to projective space. Fix k to be an algebraically closed field and suppose that S is a collection of closed points of $\mathbb{P}^n_k$. Suppose further that $H \subseteq \mathbb{P}^n_k$ is a hypersurface of degree d that vanishes to order $\geq \ell$ for each $x \in S$. What is the smallest degree hypersurface that passes through all the points of S?

A refined version of this question that we will not state here is known as Chudnovsky's conjecture. Some notable work on this problem includes [**Wal87, EV92**], [] [] []. Our result below can was first shown by by B. Harbourne [**HH11**] where it was deduced from Chapter 4Theorem 7.8. We shall take this approach in **??**. Instead, we will explore this question via the same approach as [**Laz04b**, Proposition 10.1.1] used in characteristic zero. In fact, that is a reformulation of a part of the proof of the main result of [**EV92**] mentioned above.

Theorem 1.14. *Suppose k is an algebraically closed field of characteristic $p > 0$. Fix a reduced closed subscheme $S \subseteq \mathbb{P}^n_k$ where each irreducible component of S has codimension $\leq e$. Suppose that H is a hypersurface of degree d such that $\operatorname{mult}_x H \geq \ell$ for all $x \in S$ (including non-closed points). Then S lies on a hypersurface of degree $\lfloor \frac{de}{\ell} \rfloor$.*

Proof. Set $X = \mathbb{P}^n_k$ and $\Delta = \frac{e}{\ell} H$. Then Δ vanishes to order $e > 0$ at each point of S. Let $\mathscr{I}_S$ denote the radical ideal sheaf defining S.

Claim 1.15. $\tau(X, \Delta) \subseteq \mathscr{I}_S$

Proof of claim. Suppose $\pi : Y \longrightarrow X = \mathbb{P}^n$ is the normalization of the blowup of one of the generic points x of S. After localizing at x and replacing X by $\operatorname{Spec} \mathcal{O}_{X,x}$, we see that Y is non-singular (it is the blowup of a closed point on a non-singular variety) and also that $K_Y = (r-1)E + \pi^* K_X$ where E is the exceptional divisor and $r \leq e$ is the codimension of that component of S in $X = \mathbb{P}^n$. We know that

$$\tau(X, \Delta) \subseteq \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) \rceil).$$

by Chapter 6 Proposition 4.22. But the right side can be rewritten as

$$\pi_* \mathcal{O}_Y(\lceil \pi^* K_X + (r-1)E - \pi^* K_X - \widetilde{\Delta} - \frac{e}{\ell} m E \rceil) = \pi_* \mathcal{O}_Y(\lceil (r-1-\frac{e}{\ell} m)E - \widetilde{\Delta} \rceil)$$

where $\widetilde{\Delta}$ is the strict transform of Δ and $m = \operatorname{mult}_x H \geq \ell$. Since

$$r - 1 - \frac{e}{\ell} m \leq e - 1 - \frac{e}{\ell} \ell = -1,$$

we see that $\pi_* \mathcal{O}_Y(\lceil \pi^* K_X + (r-1)E - \pi^* K_X - \widetilde{\Delta} - \frac{e}{\ell} mE \rceil)$ is contained in the maximal ideal at x, and thus so is $\tau(X, \Delta)$. This completes the proof of the claim. $\square$

If A is a hyperplane on $X = \mathbb{P}^n$, then notice that

$$tA - K_X - \Delta \sim_{\mathbb{Q}} tA + (n+1)A - \frac{de}{\ell} A$$

is ample as long as $t > \frac{de}{\ell} - (n+1)$. If we choose t to be an integer this occurs exactly when

$$t \geq \lfloor \frac{de}{\ell} \rfloor + 1 - (n+1) = \lfloor \frac{de}{\ell} \rfloor - n.$$

We thus see that

$$\tau(X, \Delta) \otimes \mathcal{O}_X((\lfloor \frac{de}{\ell} \rfloor - n)A + nA) \cong \tau(X, \Delta) \otimes \mathcal{O}_X(\lfloor \frac{de}{\ell} \rfloor)$$

is globally generated, and so there exists a non-zero element

$$s \in H^0\big(X, \tau(X, \Delta) \otimes \mathcal{O}_X(\lfloor \frac{de}{\ell} \rfloor)\big) \subseteq H^0\big(X, \mathscr{I}_S \otimes \mathcal{O}_X(\lfloor \frac{de}{\ell} \rfloor)\big) \subseteq H^0\big(X, \mathcal{O}_X(\lfloor \frac{de}{\ell} \rfloor)\big).$$

This corresponds to a divisor of degree $\lfloor \frac{de}{\ell} \rfloor$ that vanishes on S. $\square$

1.5. Exercises.

Exercise 1.1. Suppose that X is a globally F-split variety over an F-finite field. Suppose that L is a Weil divisor so that iL is an ample Cartier divisor for some $i > 0$. Show that $S^0(X, \mathcal{O}_X(K_X + mL)) = H^0(X, \mathcal{O}_X(K_X + mL))$ for all $m \gg 0$.

Exercise 1.2. Suppose that X is a d-dimensional projective variety over a field of characteristic zero with rational singularities. Suppose that $\mathscr{L}$ is a globally generated ample divisor on X. Show that $\omega_X \otimes \mathscr{L}^{d+1}$ is globally generated by using Castelnuovo-Mumford regularity and Kodaira vanishing for rational singularities Chapter 6 Exercise 2.8.

Exercise 1.3. If $\mathscr{L}$ is a globally generated ample line bundle on X, a projective variety over a field, show that the induced map to projective space $\phi_{|\mathscr{L}|} : X \longrightarrow \mathbb{P}^N$ is finite onto its image.

Exercise 1.4. Suppose that M is a big and nef Cartier divisor on a normal projective variety X over a field. Show that there exists an effective Cartier divisor $G > 0$ so that $M - \epsilon G$ is ample for all $1 \gg \epsilon > 0$.

Hint: Choose A ample, then $nM - A$ is still big for $n \gg 0$ and so for m sufficiently divisible, there exists an effective $D \sim m(nM - A)$. Now use the fact that nef cone is the closure of the ample cone. For more on big and ample divisors see for instance [**Laz04a**]. For the second statement,

Exercise 1.5. Suppose that X is a projective variety over an F-finite field and that $\mathscr{L}$ is an ample line bundle on X. Let $S = \bigoplus_{i \geq 0} H^0(X, \mathscr{L}^i)$. If $(\mathscr{M}, \phi : F^e_* \mathscr{M} \longrightarrow \mathscr{M})$ is a Cartier module that is also a coherent $\mathcal{O}_X$-module. Use this to construct a graded Cartier module M on S. Then show that $[\underline{M}]_i = S^0(X, \phi, \mathscr{M} \otimes \mathscr{L}^i)$.

Exercise 1.6. Suppose that X is an F-rational n-dimensional projective variety over an F-finite field. Suppose that L is ample and globally generated and that N is big and nef and Cartier. Show that $\mathcal{O}_X(K_X + nL + N)$ is globally generated.

Exercise 1.7. Suppose $X = \operatorname{Proj} S$ is a normal projective variety over a field, $U = D(f) \subseteq X$ is an affine open set (the complement of the vanishing set of some homogeneous $f \in S_{>0}$), and $H_U = \operatorname{div}_U(g)$ is a principal divisor on U. Prove that there exists a Cartier divisor H on X such that $H|_U = H_U$. Furthermore, show that if H_U is effective, you can assume that H_U is effective.

Conclude that if X has characteristic $p > 0$ and Γ_U is a $\mathbb{Q}$-divisor on U such that $(p^e - 1)\Gamma$ is Cartier, then there exists a $\mathbb{Q}$-divisor Γ on X such that $\Gamma|_U = \Gamma_U$. Further show that if Γ_U is effective, you can assume that Γ is effective.

Exercise 1.8. Suppose $(R, \mathfrak{m})$ is a regular local ring of dimension r and $f \in \mathfrak{m}^a \setminus \mathfrak{m}^{a-1}$. Without blowing up, directly prove that $\tau(R, f^t) \subseteq \mathfrak{m}$ for all $t \geq \frac{r}{a}$. Use this to give an alternate proof of Theorem 1.14.

Hint: Show that $\mathfrak{m} \cdot f^{\lceil tp^e \rceil}$ is contained in $\mathfrak{m}^{rp^e+1}$ which itself is contained in $\mathfrak{m}^{[p^e]}$.

Exercise 1.9. Suppose k is an algebraically closed field and let $S = k[x_0, \ldots, x_n]$ so that $\operatorname{Proj} S = \mathbb{P}^n_k$. Suppose S is a reduced closed subscheme of $\mathbb{P}^n_k$ where every irreducible component of S has codimension $\leq e$. Let $I_S \subseteq S$ denote the corresponding saturated radical ideal. Use Chapter 4Theorem 7.8 to see that

$$I_S^{(em)} \subseteq I_S^m$$

for all $m > 0$. Now, suppose that there exists a hypersurface H of degree d that vanishes to order $\geq \ell$ for all points on S. Use the above observations to see that S lies on a hypersurface of degree $\lfloor \frac{de}{\ell} \rfloor$.

2. Lifting sections and applications to Seshadri constants

One substantial utility of the sections S^0 is that they can be "lifted" as if Kodaira vanishing holds true. Indeed, suppose that X is a smooth projective variety over a field of characteristic 0 and $D \subseteq X$ is a reduced divisor. Further suppose that $\mathscr{L}$ is ample. Then we have that

$$H^0(X, \omega_X(D) \otimes \mathscr{L}) \longrightarrow H^0(X, \omega_D \otimes \mathscr{L}) \longrightarrow H^1(X, \omega_X \otimes \mathscr{L}) = 0$$

by Kodaira vanishing. This is particularly useful because sometimes it is easier to prove that $H^0(X, \omega_D \otimes \mathscr{L})$ has a section is non-vanishing at some point $x \in D \subseteq X$. The surjectivity above then proves that $H^0(X, \omega_X(D) \otimes \mathscr{L})$ also must have a section non-vanishing at x. In characteristic $p > 0$ Kodaira vanishing is false even for smooth surfaces [**Ray78**], and so there exist examples where the map above is not surjective (since if D is sufficiently ample, we certainly have $H^1(X, \omega_X(D) \otimes \mathscr{L}) = 0$).

Our goal in this section is to prove an analog of the above surjection for S^0 and then to derive a consequence for Seshadri constants. We actually first prove a much more general statement that works for more general Cartier modules.

Theorem 2.1. *Suppose that $(\mathscr{M}, \phi) \longrightarrow (\mathscr{N}, \psi)$ is a surjective map of coherent Cartier $\mathcal{O}_X$-modules on a projective variety X over an F-finite field k. If $\mathscr{L}$ is ample, then the induced map*

$$S^0(X, \phi, \mathscr{M} \otimes \mathscr{L}) \longrightarrow S^0(X, \psi, \mathscr{N} \otimes \mathscr{L})$$

is surjective.

Proof. Fix $(\mathscr{K}, \phi|_{\mathscr{K}})$ to be the kernel of $(\mathscr{M}, \phi) \longrightarrow (\mathscr{N}, \psi)$. By Serre vanishing, we know that

$$H^1(X, \mathscr{K} \otimes \mathscr{L}^{p^e}) = 0$$

for all $e \gg 0$. Hence we have the following diagram for $e \gg 0$:

$$\begin{tikzcd}
H^0\big(X, F^e_*(\mathscr{M} \otimes \mathscr{L}^{p^e})\big) \arrow[r] \arrow[d, two heads] & H^0\big(X, F^e_*(\mathscr{N} \otimes \mathscr{L}^{p^e})\big) \arrow[r] \arrow[d, two heads] & 0 \arrow[dd] \\
S^0(X, \phi, \mathscr{M} \otimes \mathscr{L}) \arrow[r] \arrow[d, hook] & S^0(X, \psi, \mathscr{N} \otimes \mathscr{L}) \arrow[d, hook] & \\
H^0(X, \mathscr{M} \otimes \mathscr{L}) \arrow[r] & H^0(X, \mathscr{N} \otimes \mathscr{L}) \arrow[r] & H^1(X, \mathscr{K} \otimes \mathscr{L})
\end{tikzcd}$$

The desired surjectivity follows immediately from the diagram. $\square$

Example 2.2 (A simple application)**.** Suppose that X is a projective variety and $\phi : F^e_* \mathcal{O}_X \longrightarrow \mathcal{O}_X$ is a map compatible with a normal subscheme $Z \subseteq \mathcal{O}_X$. Suppose that $\phi_Z : F^e_* \mathcal{O}_Z \longrightarrow \mathcal{O}_Z$ is the induced map and that (Z, Δ_Z) is globally F-split (for instance, if Z is a finite collection of closed points). Then $S^0(Z, \phi_Z, \mathscr{L}) = H^0(Z, \mathscr{L})$ for every ample line bundle $\mathscr{L}$. Hence

$$H^0(X, \mathscr{L}) \longrightarrow H^0(Z, \mathscr{L})$$

is surjective.

Suppose now that X is a normal projective variety over an F-finite field and D is a reduced Weil divisor on X. Then the map $F^e_* \omega_X \longrightarrow \omega_X$ induces

$$\psi : F^e_* \omega_X(D) \subseteq F^e_* \omega_X(p^e D) \longrightarrow \omega_X(D)$$

making $\omega_X(D)$ into a Cartier module. By construction, we have an induced map of Cartier modules

$$(\omega_X(D), \psi) \longrightarrow (\omega_D, \psi_D)$$

which is surjective if X is Cohen-Macaulay (or at least Cohen-Macaulay near D) since the next term in the sequence is $\mathcal{H}^{-d+1}\omega_X^{\bullet}$ where $d = \dim X$. Furthermore, this map is surjective if $K_X + D$ is Cartier and D is S2 see Exercise 2.2 below. In particular, we have the following.

Corollary 2.3. *Suppose that X is a normal projective variety and $D \subseteq X$ is a reduced Weil divisor on X so that X is Cohen-Macaulay in a neighborhood of D. Then for any line ample bundle $\mathscr{L}$, we have that $S^0(X, \omega_X(D) \otimes \mathscr{L}) \longrightarrow S^0(X, \omega_D \otimes \mathscr{L})$ surjects.*

Remark 2.4 (A warning about notation)**.** It is important to point out that if $\mathscr{L} = \mathcal{O}_X(L)$, then

$$S^0(X, \omega_X(D) \otimes \mathscr{L})$$

is not quite the same as

$$S^0(X, \omega_X \otimes \mathcal{O}_X(D + L)).$$

The point is that the underlying Cartier modules are different $\omega_X(D)$ and ω_X respectively. However, we have the following containments:

$$F^e_*(\omega_X(D) \otimes \mathscr{L}^{p^e}) \subseteq F^e_*(\omega_X \otimes \mathcal{O}_X(p^e L + p^e D)).$$

After taking global sections, and assuming that $e \gg 0$, the left side surjects onto $S^0(X, \omega_X(D) \otimes \mathscr{L})$ and the right side surjects onto $S^0(X, \omega_X \otimes \mathcal{O}_X(L + D))$, hence we have that

$$S^0(X, \omega_X(D) \otimes \mathscr{L}) \subseteq S^0(X, \omega_X \otimes \mathcal{O}_X(L + D)).$$

There are other statements that do not have the Cohen-Macaulay hypothesis. We suggest the reader briefly recall the adjoint test ideal τ_D and the (F-)different from **??** **??**. In particular, Suppose X is normal, D is a reduced divisor on X with normalization $\mu : D^{\mathrm{N}} \longrightarrow D$, and $B \geq 0$ is a $\mathbb{Q}$-divisor on X, without common components with D, such that $K_X + D + B$ is $\mathbb{Q}$-Cartier. In this case, the map $\mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow \mu_* \mathcal{O}_{D^{\mathrm{N}}}$ induces a surjection

$$\tau_D(X, \Delta) \longrightarrow \mu_* \tau_{D^{\mathrm{N}}}(\mathcal{O}_{D^{\mathrm{N}}}, \mathrm{Diff}_{D^{\mathrm{N}}}(D + B))$$

by **??** **??**. We can use this to obtain the following result.

Theorem 2.5. *Suppose that X is a normal projective variety and D is a reduced divisor on X. Further suppose that $B \geq 0$ is a $\mathbb{Q}$-divisor such that $K_X + D + B$ is Cartier. Further suppose that L is a Cartier divisor such that $(p^e - 1)(L - K_X - D - B)$ is an ample Cartier divisor. Then*

$$S^0\big(X, \tau_D(X, D+B) \otimes \mathcal{O}_X(L)\big) \longrightarrow S^0\big(D^{\mathrm{N}}, \tau(D^{\mathrm{N}}, \mathrm{Diff}_{D^{\mathrm{N}}}(D+B)) \otimes \mathcal{O}_X(L)\big)$$

surjects.

Proof. Write $\mu : D^{\mathrm{N}} \longrightarrow D$ the normalization map. The idea is that the surjection $\tau_D(X, D+B) \longrightarrow \mu_* \tau(D^{\mathrm{N}}, \mathrm{Diff}_{D^{\mathrm{N}}}(D+B))$ is a surjection of twisted Cartier modules. We have the following commutative diagram where we set $L_d = (p^d - 1)(L - K_X - D - B)$:

$$\begin{tikzcd}
F^d_* \tau_D(X, D+B) \otimes \mathcal{O}_X(L - L_d) \arrow[r, two heads] \arrow[d] & F^d_* \mu_* \tau(D^{\mathrm{N}}, \mathrm{Diff}_{D^{\mathrm{N}}}(D+B)) \otimes \mu^* \mathcal{O}_X(L - L_d) \arrow[d] \\
\tau_D(X, D+B) \otimes \mathcal{O}_X(L) \arrow[r, two heads] & \mu_* \tau(D^{\mathrm{N}}, \mathrm{Diff}_{D^{\mathrm{N}}}(D+B)) \otimes \mathcal{O}_X(L)
\end{tikzcd}$$

If K is the kernel of the bottom row, then for $d \gg 0$ sufficiently divisible, we see that $H^1(X, K \otimes \mathcal{O}_X(L_e)) = 0$ by Serre vanishing. Since the images of the vertical maps are the desired S^0's, the same argument as in Theorem 2.1 completes the proof. $\square$

Indeed, the main point is that D is an F-pure center of the pair $(X, D+B)$ in the sense of **?? ??**.

The surjectivities we have produced let us compare S^0 on X and on a subscheme. We also have the following containment for blowups.

Lemma 2.6. *Suppose that X is a normal projective variety over an F-finite field of characteristic $p > 0$. Suppose that $\pi : Y \longrightarrow X$ is a proper birational map and $\mathscr{L}$ is a line bundle on X. Then we have that*

$$S^0(Y, \omega_Y \otimes \pi^* \mathscr{L}) \subseteq S^0(X, \omega_X \otimes \mathscr{L}).$$

Proof. Since $\pi_* \omega_Y \subseteq \omega_X$, we can use the projection formula to obtain that $H^0(Y, \omega_Y \otimes \pi^* \mathscr{L}^n) = H^0(X, \pi_* \omega_Y \otimes \mathscr{L}^n) \subseteq H^0(X, \omega_X \otimes \mathscr{L}^n)$. Setting $n = p^e$, pushing forward via Frobenius, and taking images then produces the desired containment. $\square$

2.1. Seshadri constants. We now briefly introduce Seshadri constants and show they can be used to show that certain sheaves are globally generated at certain points, by S^0. For a complete treatment of Seshadri constants in characteristic 0, see [**Laz04a**, Chapter 5].

Definition 2.7. Suppose X is a d-dimensional projective variety and $x \in X$ is a non-singular closed point and L is an nef divisor on X. Let $\pi : Y \longrightarrow X$ be the blowup of x with exceptional divisor $E \cong \mathbb{P}^{d-1}_{k(x)}$, see Exercise 2.4. The *Seshadri constant* of L at x, denoted $\epsilon(L; x)$ is defined to be

$$\max\{\epsilon \geq 0 \mid \pi^* L - \epsilon E \text{ is nef}\}.$$

Seshadri constants are a local measure of positivity, that is they measure how *positive* (for instance nef or ample) the line bundle $\mathcal{O}_X(L)$ is at the point x.

We point out the following facts:

Lemma 2.8 (Initial facts about Seshadri constants)**.** *With notation as in Definition 2.7:*

(a) *$\epsilon(mL; x) = m\epsilon(L; x)$ for every integer m.*
(b) *$\epsilon(L; x)$ depends only on the numerical class of L.*
(c) *If L is ample, then $\epsilon(L; x) = \sup\{\epsilon \geq 0 \mid \pi^* L - \epsilon E$ is ample$\}$.*

Proof. These are left to the reader in Exercise 2.5. ☐

Suppose first that L is ample, then we know from [**Har77**, Chapter II, Section 7] that $mL - E$ is ample (and hence nef) for $m \gg 0$. In particular, $L - \frac{1}{m}E$ is ample as well. Hence $\epsilon(L; x) \geq \frac{1}{m} > 0$.

Example 2.9. Suppose $k = \overline{k}$ and $X = \mathbb{P}^2_k$. For any closed point $x \in X$, we compute $\epsilon(H; x)$ where H is a hyperplane. Then $\epsilon(H; x) = 1$. Indeed, notice that $\mathfrak{m}_x \otimes \mathcal{O}_X(H)$ is globally generated and hence $\mathcal{O}_Y(\pi^* H - E)$ is also globally generated, and thus $\epsilon(H; x) \geq 1$. On the other hand, for any integers $m > n$, it is straightforward to see that $\pi^* nH - mE$ is not globally generated, and hence not very ample. Thus $H - \frac{m}{n}E$ cannot be ample either, which shows that $\epsilon(H; x) \leq 1$.

In characteristic zero, the following result is one of the main reasons for studying Seshadri constants.

Theorem 2.10 ([**Dem93**])**.** *Suppose that X is a d-dimensional projective smooth variety over a field of characteristic zero, $x \in X$ is a closed point and L is an ample divisor on X. If $\epsilon(L; x) > d$, then $\mathcal{O}_X(K_X + L) \cong \omega_X \otimes \mathcal{O}_X(L)$ is globally generated at the point x.*

We leave the proof to the reader in Exercise 2.6.

We will prove an analogous result in characteristic $p > 0$.

Theorem 2.11 (*cf.* [**MS12, PST17, Mur18**]). *Suppose that X is a d-dimensional projective variety over an F-finite field of characteristic $p > 0$, $x \in X$ is a non-singular closed point, and L is an ample divisor on X. If $\epsilon(L; x) > d$, then $\mathcal{O}_X(K_X + L) \cong \omega_X \otimes \mathcal{O}_X(L)$ is globally generated at the point x by $S^0(X, \omega_X \otimes \mathcal{O}_X(L))$.*

PROOF. Let $\pi : Y \longrightarrow X$ denote the blowup of $x \in X$ and set $E \cong \mathbb{P}^{d-1}_{k(x)}$ to be the reduced exceptional divisor. By hypothesis, we know that $M = \pi^* L - dE$ is ample. Since Y is non-singular and hence Cohen-Macaulay along E, we know that $\omega_Y(E) \longrightarrow \omega_E$ is a surjective. Thus we know by Corollary 2.3 that $S^0(Y, \omega_Y(E) \otimes \mathcal{O}_Y(M)) \longrightarrow S^0(Y, \omega_E \otimes \mathcal{O}_Y(M))$ surjects. Since E is globally Frobenius split, we see that

$$S^0(Y, \omega_E \otimes \mathcal{O}_Y(M)) = H^0(Y, \omega_E \otimes \mathcal{O}_Y(M)).$$

We now analyze what this means.

Notice that

$$\omega_Y(E) \otimes \mathcal{O}_Y(M) = \mathcal{O}_Y(\pi^* K_X + (d-1)E + E + \pi^* L - dE) = \mathcal{O}_Y(\pi^*(K_X + L)).$$

On the other hand, since E maps to a point and $\mathcal{O}_X(K_X + L)|_{\{x\}}$ is trivial, $\mathcal{O}_E \otimes \mathcal{O}_Y(\pi^*(K_X + L)) \cong \mathcal{O}_E$. Notice also that $E \cong \mathbb{P}^{d-1} = \mathbb{P}^{d-1}_{k(x)}$ so that

$$\omega_E \otimes \mathcal{O}_Y(M) \cong \omega_Y(E) \otimes \mathcal{O}_E \otimes \mathcal{O}_Y(M) \cong \mathcal{O}_{\mathbb{P}^{d-1}}$$

In particular, we see that

$$S^0(Y, \omega_E \otimes \mathcal{O}_Y(M)) \cong H^0(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}) = k(x). \tag{2.11.1}$$

Therefore $S^0(\omega_Y(E) \otimes \mathcal{O}_Y(M)) \subseteq H^0(Y, \mathcal{O}_Y(\pi^*(K_X + L)))$ has a section mapping to 1 in (2.11.1). In particular, $S^0(Y, \omega_Y(E) \otimes \mathcal{O}_Y(M))$ is globally generated along every point of E. Thus so is $S^0(Y, \omega_Y \otimes \mathcal{O}_Y(M + E))$, which is even larger by Remark 2.4. By Lemma 2.6 and the fact that $\pi_* \omega_Y = \omega_X$ (since X has pseudo-rational singularities at x) we see that $S^0(X, \omega_X \otimes \mathcal{O}_X(L))$ also has a global section non-vanishing at x. This completes the proof. $\square$

2.2. Exercises.

Exercise 2.1. Suppose that X is a projective variety over a perfect field k and $(\mathscr{M}, \phi : F^e_* \mathscr{M} \longrightarrow \mathscr{M})$ is a Cartier module. Let $\overline{k}$ denote the algebraic closure of k. Show that we can give the base change $\mathscr{M}_k = \mathscr{M} \otimes_k \overline{k}$ a Cartier-module structure via base change and we have a canonical identification

$$S^0(X, \mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{L}) \otimes_k \overline{k} \cong S^0(X_{\overline{k}}, \mathscr{M}_k \otimes_{\mathcal{O}_{X_k}} \mathscr{L}_k)$$

for any line bundle $\mathscr{L}$ with base change $\mathscr{L}_k$.

Hint: Exhibit an isomorphism $(F^e_* \mathscr{M}) \otimes_k k' \cong F^e_* \mathscr{M}_{k'}$ for any algebraic extension $k' \supseteq k$.

Exercise 2.2. Suppose X is a normal projective variety and $D > 0$ is a reduced S2 and G_1[2] Weil divisor on X. If $K_X + D$ is Cartier, prove that the canonical map:

$$\omega_X(D) \longrightarrow \omega_D$$

is surjective even if X is not Cohen-Macaulay.

Hint: Since $\omega_X(D)$ is locally free of rank 1, so is its image in ω_D (as a $\mathcal{O}_D$-module). Next show that the map $\omega_X(D) \longrightarrow \omega_D$ is surjective in codimension 2 on X (and so 1 on D).

Exercise 2.3. Suppose that X is a normal projective variety over an F-finite field of characteristic $p > 0$ and that $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $(p^e - 1)(K_X + \Delta)$ is Cartier. Suppose further that $Z \subseteq X$ is a normal F-pure center for (X, Δ) in the sense of **?? ??**. Let Δ_Z on Z denote the F-different of $K_X + \Delta$ on Z. For any ample Cartier divisor L, we have consider the maps $F^{ne}_* \mathcal{O}_X(p^e L + (1 - p^{ne})(K_X + \Delta)) \longrightarrow \mathcal{O}_X(L)$ as in (1.10.1). Show there is a surjection:

$$S^0(X, \Delta, \mathcal{O}_X(L)) \longrightarrow S^0(Z, \Delta_Z, \mathcal{O}_X(L)|_Z).$$

Furthermore, if Z is a closed point, show that $\Delta = 0$ and that $S^0(Z, \Delta_Z, \mathcal{O}_X(L)|_Z)$ is the residue field at x.

Exercise 2.4. Suppose that $(R, \mathfrak{m}, k)$ is a d-dimensional regular local ring and $\pi : Y \longrightarrow X = \operatorname{Spec} R$ is the blowup of $\mathfrak{m}$. Write $\mathfrak{m} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$ where E is the exceptional divisor. Show that $E \cong \mathbb{P}^{d-1}_k$.

Hint: Consider the associated graded algebra of $\mathfrak{m}$.

Exercise 2.5. Prove Lemma 2.8.

Exercise 2.6. Prove Theorem 2.10 in characteristic zero without reducing to characteristic $p > 0$. In other words, assume that X is a d-dimensional smooth projective variety over a field of characteristic zero and $x \in X$ is a closed point. Suppose additionally L is an ample line bundle and that $\epsilon(L; x) > d$. Show that

$$\mathcal{O}_X(K_X + L)$$

is globally generated at x.

Hint: Use Kodaira vanishing on the blowup $\pi : Y \longrightarrow X$. See the proof of Theorem 2.11 for some inspiration.

Exercise 2.7. Suppose that X is a projective variety over an algebraically closed field of characteristic $p > 0$. Suppose that $V = \{x_1, x_2\}$ is a set of two distinct non-singular points. Let $Y \longrightarrow X$ be the blowup of both x_i.

[2]Gorenstein in codimension 1

3. An application to discreteness of F-jumping numbers

Warning 3.1. This section will likely undergo substantial revision, eventually handling discreteness of pairs $(X, \mathfrak{a}^t)$. Suppose that X is a normal projective variety over an F-finite field of characteristic $p > 0$. Further suppose that $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and that H is any Cartier divisor on X. Consider the test ideal

$$\tau(\mathcal{O}_X, \Delta + tH)$$

as t varies. Certainly as t gets larger, $\tau(\mathcal{O}_X, \Delta + tH)$ becomes smaller. Indeed, in Chapter 4 Subsection 6.3 we showed that if X is regular and Δ is zero, then $\tau(\mathcal{O}_X, tH)$ only changed at a discrete set of rational numbers t. Recall that $t > 0$ is an *F-jumping number (for (X, Δ, H))* if $\tau(\mathcal{O}_X, \Delta+tH) \neq \tau(\mathcal{O}_X, \Delta+(t-\epsilon)H)$ for any $1 \gg \epsilon > 0$. Our goal is to generalize discreteness and rationality results to our setting. In fact, the methods of Chapter 4 Subsection 6.3 can be generalized to the setting of $\tau(\mathcal{O}_X, \Delta+tH)$. However, we can obtain a very short proof of discreteness of F-jumping numbers via the global generation results of the previous section. First we need a short

Theorem 3.2. *Suppose X is a normal projective variety over an F-finite field k, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier, and H is a Cartier divisor on X. Then the set of F-jumping numbers of $\tau(\mathcal{O}_X, \Delta+tH)$ has no accumulation points.*

Proof. First notice that if $t = n + r$ where $n \in \mathbb{Z}$ and $r \in [0, 1)$, then $\tau(\mathcal{O}_X, \Delta+tH) = \tau(\mathcal{O}_X, \Delta+rH) \cdot \mathcal{O}_X(-nH)$ by Chapter 5 Exercise 5.11. In particular, if we can show that there are only finitely many jumping numbers in the interval $[0, 1]$, then we are done.

Choose A sufficiently ample so that $A - K_X - \Delta - tH$ is ample for every $t \in [0, 1]$, note the ample cone is convex see [**Laz04a**, Section 1.4.C]. Fix L globally generated and ample and suppose that $d = \dim X$. By Corollary 1.13, we know that $\tau(X, \Delta + tH) \otimes \mathcal{O}_X(A + dL)$ is globally generated by

$$H^0(X, \tau(X, \Delta + tH) \otimes \mathcal{O}_X(A + dL)) \subseteq H^0(X, \mathcal{O}_X(A + dL)).$$

Notice that the right side is a finite dimensional vector space. As t increases, $H^0(X, \Delta, \tau(X, \Delta + tH) \otimes \mathcal{O}_X(A + dL))$ becomes smaller, and hence form a descending sequence of subspaces in a finite dimensional k-vector space. In particular, there can be only finitely many $H^0(X, \Delta, \tau(X, \Delta+tH)\otimes\mathcal{O}_X(A+dL)$ for $t \in [0, 1]$. Thus they can globally generate only finitely many test ideals for $t \in [0, 1]$. This proves the theorem. $\square$

We next need rationality. First we point out the following result on F-jumping numbers which generalizes Chapter 4 **??**.

Lemma 3.3. *Suppose that R is a normal F-finite Noetherian domain, $X = \operatorname{Spec} R$, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and H is any Cartier divisor on X. If t is an F-jumping number for (X, Δ, H), then so is tp, and so is the fractional part $\{tp\} = tp - \lfloor tp \rfloor$.*

PROOF. The second statement about the fractional part follows immediately from the first via Chapter 5 Exercise 5.11.

Write $\operatorname{div}(f) = n(K_X + \Delta)$ and set S to be the normalization of $R[f^{1/n}]$ (embedded in an algebraic closure of the total field of fractions) with induced map $\pi : Y = \operatorname{Spec} S \longrightarrow \operatorname{Spec} R = X$. Notice that $\pi^*(K_X + \Delta) = \operatorname{div}_Y(f^{1/n})$ is Cartier. Fix $T \in \operatorname{Hom}_R(S, R)$ non-zero with corresponding divisor $D_T \sim K_S - \pi^* K_R$ (if p does not divide n, you can let $T = \operatorname{Tr}$ and so D_T is the ramification divisor). Choose $G \geq 0$ Cartier so that $\pi^*(\Delta + G) \geq D_T$, the ramification divisor. Since $\tau(X, \Delta + G + tH) = \tau(X, \Delta + tH) \otimes \mathcal{O}_X(-G)$, we see that the F-jumping numbers of $(X, \Delta + G, H)$ and the F-jumping numbers of $\tau(X, \Delta, H)$ are the same and so we may replace Δ with $\Delta + G$. At this point, we have

$$\pi : Y = \operatorname{Spec} S \longrightarrow \operatorname{Spec} R = X$$

a finite map so that $\pi^*(K_X + \Delta)$ is Cartier and so that $\pi^*\Delta \geq D_T$.

By **?? ??**, we know that

$$T(\tau(S, \pi^*\Delta - D_T + t\pi^* H)) = \tau(R, \Delta + tH).$$

Hence, every F-jumping number of (X, Δ, H) is also an F-jumping number of $(Y, \pi^*\Delta - D_T, \pi^* H)$. Notice that $K_S + \pi^*\Delta - D_T \sim K_S + \pi * \Delta - K_S + \pi^* K_X = \pi^*(K_X + \Delta)$ which is Cartier. Hence, replacing X with Y, replacing Δ with $\pi^*\Delta - D_T$, and replacing H by $\pi^* H$, we may assume that $K_X + \Delta$ is Cartier.

Without loss of generality, since R is affine, we may assume that $K_X \geq 0$. Now, from Chapter 5 Theorem 6.14 and **??**, we know that

$$\tau(X, \Delta + tH) = \tau(\omega_X, K_X + \Delta + tH) = \tau(\omega_X, tH) \otimes \mathcal{O}_X(-K_X - \Delta).$$

To prove the first statement, it suffices to show that if $\tau(\omega_X, tH) \neq \tau(\omega_X, (t-\epsilon)H)$, then $\tau(\omega_X, tpH) \neq \tau(\omega_X, (tp - \epsilon)H)$.

Finally, let $T : F^e_* \omega_R \longrightarrow \omega_R$ denote the dual to Frobenius. Notice we have the transformation rule:

$$T(F^e_* \tau(\omega_R, tpH)) = \tau(\omega_R, tH)$$

by **??** (since $F^* H = pH$). Hence if $\tau(\omega_X, tH) \neq \tau(\omega_X, (t - \epsilon)H)$, then $\tau(\omega_X, tpH) \neq \tau(\omega_X, (tp - \epsilon)H)$. This completes the proof. $\square$

Theorem 3.4. *Suppose X is a normal projective variety over an F-finite field k, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and H*

is a Cartier divisor on X. Then the (finite) set of F-jumping numbers of $\tau(\mathcal{O}_X, \Delta + tH)$ is a set of rational numbers.

PROOF. Again, it suffices to show that the F-jumping numbers in $[0,1]$ are rational. Notice that if t is an F-jumping number, so is $\{tp^e\} = tp^e - \lfloor tp^e \rfloor$ by Lemma 3.3. However, there are only finitely many distinct F-jumping numbers in the range $[0,1]$ by Theorem 3.2. $\square$

Conjecture 3.5 (Discreteness)**.** *Suppose R is an F-finite normal Noetherian domain, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on $X = \operatorname{Spec} R$ and H is any Weil divisor. Then the set of F-jumping numbers of (R, Δ, H) have no accumulation points.*

Conjecture 3.5 open when $\Delta = 0$ and H is Cartier, even for R finite type over a field.

The state of the art is essentially as follows. If X is finite type over a field and $K_X + \Delta$ is $\mathbb{Q}$-Gorenstein except at finitely many points [3] then for any Cartier divisor [4] H, the jumping numbers of (X, Δ, H) have no accumulation points. The proof is a generalization of the proof we have just presented and is based upon similar results in characteristic zero, see [**Urb12, Gra16, GS18**]. For things not of finite type over a field, the strongest results are contained in [**BSTZ10**] [**STZ12**] [**?**] [**?**] where the results are proven for F-finite rings where $K_X + \Delta$ is $\mathbb{Q}$-Cartier (or satisfies the finite generation condition in the footnote above).

Question 3.6 (Rationality?)**.** Does there exist a normal F-finite domain R, and a Cartier divisor H on $X = \operatorname{Spec} R$ such that an *irrational* number t is an F-jumping number for (X, H)? That is, is it true that $\tau(X, tH) \neq \tau(X, (t-\epsilon)H)$ for $1 \gg \epsilon > 0$?

Exercises.

Exercise 3.1. Suppose X is a normal projective variety over an F-finite field k. Suppose that X is also $\mathbb{Q}$-Gorenstein. Prove that the images

$$\operatorname{Image}\Big(\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\text{eval@1}} \mathcal{O}_X \Big)$$

stabilize for $e \gg 0$. See also Chapter 4 Question 2.9.

The following exercises explore F-jumping numbers in general gives a different proof of discreteness of F-jumping numbers that works for any

[3] In fact, one only needs that ring $\bigoplus_{n \geq 0} \mathcal{O}_X(\lfloor -n(K_X + \Delta) \rfloor))$ is finitely generated except at finitely many points.

[4] It even works for triples $(X, \Delta, \mathfrak{a}^t)$ defined as in Chapter 5 Exercise 5.16.

F-finite normal domain. In particular, it does not require any finite type hypothesis over a field.

Setting 3.7. Suppose R is an F-finite normal domain, $X = \operatorname{Spec} R$, and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on X. We also suppose that H is a Cartier divisor on X.

Exercise 3.2. To show that the F-jumping numbers of (X, Δ, H) have no accumulation points, show that one may reduce to the case that the coefficients of Δ are integers.

Hint: Take a finite extension $R \subseteq S$ and argue similarly to the proof of Lemma 3.3. Note, we are not assuming that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (yet).

Exercise 3.3. Define a *test-module F-jumping number of* (ω_X, H) to be a real number $t \geq 0$ such that $\tau(\omega_X, tH) \neq \tau(\omega_X, (t - \epsilon)H)$ for all $1 \gg \epsilon > 0$. Show that to prove discreteness or rationality of F-jumping numbers for triples of the form (X, Δ, H) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, it suffices to prove discreteness or rationality of test-module F-jumping numbers for *pairs* (ω_X, H). Note the X for the triples is not necessarily the same as the X for pairs.

Exercise 3.4. Show that $\tau(\omega_X, tH) = \tau(\omega_X, (t + \epsilon)H)$ for all $1 \gg \epsilon > 0$.

Exercise 3.5. Suppose that $t_1 = \frac{a}{p^e - 1}$. Prove that there exists $t_0 < t_1$ such that $\tau(\omega_X, tH)$ is constant for all $t \in [t_0, t_1)$. Use transformation rules for finite maps to deduce the same thing for any rational t_1.

Hint: Reduce to the case where $H = \operatorname{div}(f)$. Suppose that $T^e : F^e_* \omega_X \longrightarrow \omega_X$ is the canonical map. Observe that

$$T^e(F^e_* f^a \tau(\omega_X)) = \tau(\omega_X, \frac{a}{p^e} H)$$

and that

$$T^e(F^e_* f^a \tau(\omega_X, \frac{a}{p^e} H)) = \tau(\omega_X, (\frac{a}{p^{2e}} + \frac{a}{p^e}) H)$$

etc. Continuing in this way, use Chapter 8 Theorem 2.1 to show that these images eventually stabilize.

Exercise 3.6. Use Chapter 4 Exercise 6.8 to show that the test module F-jumping numbers of (ω_X, H) are rational and have no limit points. Conclude that the F-jumping numbers of (X, Δ, H) are rational and have no limit points.

4. Applications to linear systems

This has not been written yet.

APPENDIX A

Background facts on commutative algebra

This appendix is incomplete. It will be expanded and revised.

1. Colons and submodules

Suppose R is a ring, M is a module, and $N, N' \subseteq M$ are subsets (typically submodules). Then by

$$N : N' = N :_R N'$$

we mean the set

$$\{x \in R \mid xN' \subseteq N\}.$$

Very frequently, $N \subseteq N'$ are nonzero ideals in a domain R with fraction field $\mathcal{K}(R)$. In this case

$$N :_R N' \subseteq N :_{\mathcal{K}(R)} N' \cong \operatorname{Hom}_R(N', N).$$

Indeed, there is certainly a map $N :_{\mathcal{K}(R)} N' \longrightarrow \operatorname{Hom}_R(N', N)$. On the other hand, any map $N' \longrightarrow N$, when tensored with $\otimes_R \mathcal{K}(R)$ becomes a map $\mathcal{K}(R) = N' \otimes \mathcal{K}(R) \longrightarrow N \otimes \mathcal{K}(R) = \mathcal{K}(R)$ and hence may be identified with multiplication by an element $x \in \mathcal{K}(R)$.

Lemma 1.1. *Suppose $R \longrightarrow S$ is a ring map and $N, N' \subseteq M$ are R-modules. Then*

$$(N :_R N')S \subseteq N \otimes_R S :_S N' \otimes_R S.$$

Furthermore, if $R \longrightarrow S$ is flat and N' is finitely generated, then this is an equality.

Proof. Certainly if $xN' \subseteq N$, then this is preserved after tensoring with S. Now assume that $R \longrightarrow S$ is flat and that N' is finitely generated with generators $x_1, \ldots, x_m$. Then $N :_R N'$ is the intersection of the kernels K_i of:

$$0 \longrightarrow K_i \longrightarrow R \xrightarrow{r \mapsto r x_i} \frac{N' + N}{N}.$$

The formation of these kernels K_i commutes with base change by the flat S, and so the statement reduces to the fact that *finite* intersections of ideals are preserved under flat ring extensions (ie, $M_1 \cap M_2 = \ker(M \longrightarrow M/M_1 \times M/M_2)$ and the formation of that kernel commutes with flat ring extensions). $\square$

2. Pure maps of modules

Definition 2.1. Let R be an arbitrary commutative ring. A homomorphism $M \xrightarrow{\phi} N$ of R-modules is *pure* (or *universally injective*) if for every R-module Q, the induced map
$$M \otimes_R Q \longrightarrow N \otimes_R Q$$
is injective.

Example 2.2. Two main examples of pure maps are split maps and faithfully flat maps:

(a) If the map $M \xrightarrow{\phi} N$ splits in the category of R-modules, then it is pure.
(b) If $R \longrightarrow S$ is a faithfully flat map of commutative rings, then $R \longrightarrow S$ is a pure map of R-modules.

Indeed, statement (a) is immediate, while (b) is easy as well; see [**Sta19**, Tag 05CK].

Purity of a finite map of Noetherian modules over a Noetherian ring is very close to splitting:

Proposition 2.3. *Let $M \xrightarrow{\varphi} N$ be a pure map of A-modules where A is a commutative ring. Then φ is split if the cokernel $N/\varphi(M)$ is finitely presented.*

PROOF. See [**HR76**, Corollary 5.2] or [**Sta19**, Tag 058L]. $\square$

The next lemmas are useful when checking for purity.

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let E be an injective hull of its residue field. Suppose $R \xrightarrow{\phi} M$ is an R-module homomorphism. Then ϕ is pure if and only if the map*
$$E \xrightarrow{E \otimes_R \phi} E \otimes_R M$$
induced by tensoring with E is injective.

PROOF. See [**HR76**, Prop 6.11]. $\square$

Corollary 2.5. *If $(R, \mathfrak{m})$ is a Noetherian complete local ring and $R \longrightarrow M$ is pure, then $R \longrightarrow M$ is split.*

Proof. By Lemma 2.4 $E \longrightarrow E \otimes M$ injects where E is the injective hull of the residue field. Applying the exact functor $\mathrm{Hom}_R(-, E)$ we from Hom $-\otimes$ adjointness that

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(E, E)) \cong \mathrm{Hom}_R(E \otimes M, E) \longrightarrow \mathrm{Hom}_R(E, E)$$

surjects. Since R is complete, $\mathrm{Hom}_R(E, E) \cong R$, and the surjective map above becomes

$$\mathrm{Hom}_R(M, R) \longrightarrow R$$

which one can check is the dual of $R \longrightarrow M$, in other words its the evaluation-at-1 map and $R \longrightarrow M$ splits. $\square$

Lemma 2.6. *Suppose R is a commutative ring and $M \longrightarrow N$ is a map of R-modules. The following are equivalent.*

(a) $M \longrightarrow N$ is pure.

(b) $M \longrightarrow N$ is pure if and only if

$$M \otimes_R Q \longrightarrow N \otimes_R Q$$

injects for every finitely presented *R-module Q.*

(c) For every finitely presented R-module P, the R-module map

$$\mathrm{Hom}_R(P, N) \longrightarrow \mathrm{Hom}_R(P, N/M)$$

surjects.

Proof. See [**Sta19**, Tag 058K], or prove it yourself (notice that any failure to be injective only involves finitely many equations). $\square$

The following general fact about pure maps follow immediately from the definition.

Lemma 2.7. *Let R be an arbitrary ring. Consider a composition*

$$M \xrightarrow{\phi} N \xrightarrow{\psi} P$$

of R-modules.

(a) A composition of pure R-module maps is pure. That is, if ϕ and ψ are both pure, then so is $\psi \circ \phi$.

(b) If the composition $\psi \circ \phi$ is pure, then also $M \xrightarrow{\phi} N$ is pure.

Perhaps the most commonly used consequence of pure maps in commutative algebra is the following, also see Chapter 1 Lemma 3.11.

Proposition 2.8. *Suppose that $R \longrightarrow S$ is a pure map of rings. Then for any ideal $I \subseteq R$, $(I \cdot S) \cap R = I$.*

Proof. Tensor $R \longrightarrow S$ with the R-module R/I. □

We also have the converse statement under the *approximately Gorenstein* hypothesis, see Section 11 below.

3. Normality

Definition 3.1. Let $A \hookrightarrow B$ be any inclusion of rings. We say an element $b \in B$ is **integral over** A if b satisfies a monic polynomial with coefficients in A:

$$b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0$$

where $a_i \in A$. The ring A is said to be **integrally closed in** B if whenever $b \in B$ is integral over A, then $b \in A$.

Definition 3.2. A reduced ring R is **normal** if it is integrally closed in its total quotient ring $W^{-1}R$, where W is the multiplicative set of non-zerodivisors of R.

Normalness is usually defined only for domains, in which case normal is the same as integrally closed in its field of fractions.[1]. On the other hand, for local rings, this concern about reduced vs integral domain is not an issue.

Lemma 3.3. *If a reduced local ring $(R, \mathfrak{m})$ is normal, then R is an integral domain.*

Proof. □

3.1. Weak Normality.

Maybe move stuff from Chapter 5 **??**.

4. Regular sequences and Cohen-Macaulayness

We recall what it means for a *local* Noetherian ring $(R, \mathfrak{m})$ to be Cohen-Macaulay.

First recall the definition of a regular sequences:

[1]For rings with infinitely many minimal primes (necessarily not Noetherian), there are some subtleties in defining normality, and therefore, there are competing definitions that may not agree with this one. In particular, a ring satisfying Definition 3.2 may not have the property that each R_P also does as well. See [**Sta19**, Tag 037B]

Definition 4.1. [**BH93**, Definition 1.1.1] A sequence of elements $x_1, \dots, x_d$ generating a proper ideal in a ring R is a **regular sequence** if x_1 is not a zero divisor on R, and the image of x_i in $R/(x_1, \dots, x_{i-1})$ is not a zero divisor on $R/(x_1, \dots, x_{i-1})$ for each $i = 2, 3, \dots, d$.

Definition 4.2. [**BH93**, Definition 2.1.1] A local Noetherian ring $(R, \mathfrak{m})$ is *Cohen-Macaulay* if any of the following equivalent conditions holds

(a) There is a regular sequence contained in $\mathfrak{m}$ of length equal to the dimension of R.
(b) Some system of parameters for R is a regular sequence.
(c) Every system of parameters for R is a regular sequence.
(d) The Koszul complex on some (equivalently, every) system of parameters for R is exact.
(e) The local cohomology modules $H^i_{\mathfrak{m}}(R)$ are all zero for $i < \dim R$.

If R is a Noetherian but not necessarily local ring, we say that R is *Cohen-Macaulay* if all of its localizations are Cohen-Macaulay.

We summarize below some well-known (easy to prove) facts about Cohen-Macaulayness:

Theorem 4.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring.*

(a) The ring R is Cohen-Macaulay if and only if its completion $\widehat{R}$ is Cohen-Macaulay.
(b) If R is regular, then R is Cohen-Macaulay.
(c) Let $f \in R$ be a non-zerodivisor. Then R is Cohen-Macaulay if and only if $R/(f)$ is Cohen-Macaulay.
(d) Suppose R is a finite local extension of a regular local subring[2] A. Then R is Cohen-Macaulay if and only if R is free as module over A.

4.1. Manipulations with regular sequences.

5. Maps and tensor-hom adjointness

Lemma 5.1. *Suppose $R \subseteq S$ is an extension of rings and M is an S-module. Any map $M \longrightarrow R$ factors as $M \longrightarrow \mathrm{Hom}_R(S, R) \longrightarrow R$ where the first map is S-linear. Suppose further that $\mathrm{Hom}_R(S, R) \cong S$ as S-modules. Then every*

[2] a “Noether normalization" of R

R-module homomorphism $M \longrightarrow R$ factors as $M \longrightarrow S \xrightarrow{\Phi} R$. In particular, the map

$$\begin{aligned} \mathrm{Hom}_S(M,S) \times \mathrm{Hom}_R(S,R) &\longrightarrow \mathrm{Hom}_R(M,R) \\ (\psi,\phi) &\longmapsto \psi \circ \phi \end{aligned}$$

is surjective.

Proof. This is a straightforward consequence of Hom-tensor adjointness. For instance, in the case that $S \cong S \cdot \Phi = \mathrm{Hom}_R(S,R)$, we have that

$$\mathrm{Hom}_S(M,S) \cong \mathrm{Hom}_S(M, \mathrm{Hom}_R(S,R)) \cong \mathrm{Hom}_R(M \otimes_S S, R) \cong \mathrm{Hom}_R(M,R).$$

The factorization asserted follows by tracing through the isomorphisms. □

Remark 5.2. The following generalization holds with nearly the same proof. Suppose $R \subseteq S$ is an extension of rings and M is an S-module. Then every R-module map $M \longrightarrow R$ factors through the evaluation-at-1 map $\mathrm{Hom}_R(S,R) \longrightarrow R$.

We will use Lemma 5.1 most frequently in the following setting.

Proposition 5.3. *Suppose $R \longrightarrow S \longrightarrow T$ is a sequence of finite ring maps.*

$$\mathrm{Hom}_R(S,R) = \Phi \cdot S \qquad \text{and} \qquad \mathrm{Hom}_S(T,S) = \Psi \cdot T,$$

that is, Φ and Ψ are S-module (respectively T-module) generators of their respective Hom-*sets. Then $\mathrm{Hom}_R(T,R) = (\Phi \circ \Psi) \cdot T$ as well. Furthermore, if $\mathrm{Hom}_R(S,R) \cong S$ and $\mathrm{Hom}_S(T,S) \cong T$, then $\mathrm{Hom}_R(T,R) \cong T$.*

Proof. We prove the first statement first. By Lemma 5.1, we have that for any $\gamma \in \mathrm{Hom}_R(T,R)$ that there exist $s \in S$ and $t \in T$ such that

$$\gamma(-) = \Phi(s\Psi(t-)) = (\Phi \circ \Psi)(st-).$$

The first statement follows.

For the second statement, simply notice that

$$\mathrm{Hom}_R(T,R) \cong \mathrm{Hom}_R(T \otimes_S S, R) \cong \mathrm{Hom}_S(T, \mathrm{Hom}_R(S,R)) \cong \mathrm{Hom}_S(T,S) \cong T.$$

□

The above will be particularly crucial for us for the interated Frobenius morphism.

Corollary 5.4. *Suppose R is an F-finite ring of characteristic $p > 0$ and $\mathrm{Hom}_R(F^e_* R, R) = \Phi^e \cdot (F^e_* R) \cong F^e_* R$. Then $\Phi^{ne} := \Phi \circ F^e_* \Phi \circ \cdots \circ F^{((n-1)e)}_* \Phi = \Phi^{\star n}$ satisfies*

$$\mathrm{Hom}_R(F^{ne}_* R, R) = \Phi^{ne} \cdot (F^{ne}_* R) \cong F^{ne}_* R.$$

6. Symbolic powers

Definition 6.1. Suppose R is a ring and $Q \subseteq R$ is prime. For every integer $n > 0$, we define the nth symbolic power of Q as

$$Q^{(n)} := R \cap (Q^n R_Q).$$

Here, if $R \longrightarrow R_Q$ is not injective, the intersection $\cap$ above means contraction as in [**AM69**].

More generally, if $I = Q_1 \cap \cdots \cap Q_m$ is radical, and the Q_i are its minimal primes, then we define

$$I^{(n)} := Q_1^{(n)} \cap \cdots \cap Q_m^{(n)}.$$

In other words, the nth symbolic power is the set of functions that vanish to order n after localizing at Q.

For radical ideals, it is straightforward to see that $I^{(n)}$ is simply the intersection of the *minimal* primary components in a primary decomposition of I^n, see Exercise 7.7.

We should notice that if Q is maximal, $Q^n = Q^{(n)}$.

Remark 6.2. There is an important result, due to Zariski and Nagata, which can give another interpretation of symbolic powers. With notation as above, assuming R is a polynomial ring over a field,

$$Q^{(n)} = \bigcap_{\substack{\mathfrak{m} \supseteq Q \\ \text{maximal}}} \mathfrak{m}^n.$$

See [**Zar49, Nag62, EH79**], also see [**DSGJ20**] for a generalization to mixed characteristic.

Let us consider several examples.

Example 6.3 (Principal ideals and complete intersections)**.** Suppose $f \in R$ is a non-zerodivisor. It is easy to see that $R \cap ((f)^n W^{-1} R) = (f)$ where W is the multiplicative set not containing any minimal prime of f, since $(f^n) = (f)^n$ is unmixed. More generally, if $I = (f_1, \ldots, f_l)$ where $f_1, \ldots, f_l$, is a regular sequence, then I^n is also unmixed by [**Mat89**, Exercise 17.6].

Example 6.4. Let $S = k[x, y, z]$ and $I = (x, y) \cap (x, z) \cap (y, z) = (xy, xz, yz)$. Then for degree reasons, $xyz \notin I^2$ but $xyz \in I^{(2)}$ since

$$xyz \in (x, y)^2 = (x, y)^{(2)}$$

and likewise $xyz \in (x, z)^2, (y, z)^2$.

In a non-regular ring it is even easier to construct interesting examples of symbolic powers.

Example 6.5. Let $R = k[x,y,z]/(xy - z^2)$. Set $Q = (x,z)$. Now, R_Q is a DVR with uniformizer z and $x = y^{-1}z^2$ is a unit multiple of z^2 in R_Q. Thus $Q^{(2)} = (x)$ but $x \notin Q^2$.

7. S_2-ness and reflexivity

Extension of sections over bigger open sets, maybe from Karl's notes or look in paper with

8. Gorenstein and quasi-Gorenstein rings

Definition 8.1. A local Noetherian ring $(R, \mathfrak{m})$ is **Gorenstein** if R has finite injective dimension as an R-module. A Noetherian ring in general is Gorenstein if R_Q is Gorenstein for each $Q \in \operatorname{Spec} R$.

While this definition of a Gorenstein ring we include some other characterizations that will be crucial for us (and which also appear in Appendix C).

Lemma 8.2 (Corollary 6.6)**.** *A Noetherian local ring $(R, \mathfrak{m})$ of dimension d is Gorenstein if and only if*

(a) It is Cohen-Macaulay (equivalently, if $H^i_{\mathfrak{m}}(R) = 0$ for all $i < \dim R$).
(b) $H^d_{\mathfrak{m}}(R)$ is an injective hull of the residue field $R/\mathfrak{m}$ of R.

Requiring only condition (b) yields the following definition.

Definition 8.3. Suppose $(R, \mathfrak{m})$ is an equidimensional Noetherian local ring. Then R is **quasi-Gorenstein** (or 1**-Gorenstein**) if $H^d_{\mathfrak{m}}(R)$ is an injective hull of the residue field of $R/\mathfrak{m}$. A Noetherian ring in general is **quasi-Gorenstein** if all localizations R_Q are quasi-Gorenstein.

Typically we work with quasi-Gorenstein rings with dualizing complexes, see Appendix C.

Remark 8.4. In the case that $(R, \mathfrak{m})$ is local with dualizing complex $\omega_R^{\bullet}$, quasi-Gorenstein is equivalent to the condition that the canonical module ω_R is isomorphic to R. This follows from Matlis duality; see [**BH93**, Chapter 3], Appendix C Lemma 3.14 or Corollary 6.6.

9. Complete local rings

Theorem 9.1 (Chevalley's Lemma, [**Che43**, Section 2, Lemma 7])**.** *Suppose $(R, \mathfrak{m})$ is a complete local Noetherian ring and M is a finitely generated module. Suppose further that $M_1 \supseteq M_2 \supseteq \dots$ is a descending chain of submodules such that $\bigcap_{i>0} M_i = 0$. Then for every $k > 0$ there exists $i > 0$ such that $M_i \subseteq \mathfrak{m}^k M$.*

10. Local Cohomology

10.1. Derived functors.

10.2. The Čech description of local cohomology.

10.3. Local cohomology via direct limits.

Proposition 10.1. *Suppose that $J \subseteq R$ is an ideal that can be generated by d elements. Then for any R-module N, we have a natural isomorphism:*

$$H^d_J(R) \otimes N \longrightarrow H^d_J(R).$$

11. Approximately Gorenstein rings

Approximately Gorenstein were introduced by Hochster in his study of purity (as hinted at above). Two good sources for this material are [**Hoc77**] and [**Hoc07**].

Definition 11.1 ([**Hoc77**, Definition Proposition 1.1, Proposition 2.1])**.** Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d. We say that R is **approximately Gorenstein** if there is a sequence of $\mathfrak{m}$-primary ideals $I_n \supseteq I_{n+1} \supseteq \dots$, cofinal with the powers of $\mathfrak{m}$, and such that each R/I_n is Gorenstein.

In general, a Noetherian ring is **approximately Gorenstein** if all localizations at maximal ideals are approximately Gorenstein.

Example 11.2. If $(R, \mathfrak{m})$ is a Gorenstein ring of dimension d and $f_1, \dots, f_d$ is a system of parameters. Then we can set $I_n = (f_1^n, \dots, f_d^n)$. Each quotient R/I_n is Gorenstein since we are modding out by a regular sequence.

It follows easily from the definition that Artinian rings are approximately Gorenstein if and only if they are Gorenstein (the chain of ideals is eventually 0). In higher dimensions though, most rings we will care about are approximately Gorenstein.

Lemma 11.3 (*cf.* [**Hoc77**, Theorem 5.2])**.** *Suppose that $(R, \mathfrak{m})$ is a local Noetherian ring. If R satisfies either of the following conditions*

(a) R is excellent (for instance complete) and reduced, or
(b) R is has depth ≥ 2 (for instance, is normal),

then R is approximately Gorenstein.

Suppose R and S is an R-module. Hochster defined a map $R \longrightarrow M$ to be cyclically pure **cyclically pure** if the map

$$R/I \longrightarrow M/IM$$

injects for every ideal $I \subseteq R$. Or equivalently, if $IM \cap R = I$ for every ideal $I \subseteq R$.

Theorem 11.4 ([**Hoc77**, Theorem 2.6])**.** *Suppose R is a Noetherian ring. Then the following are equivalent.*

(a) R is approximately Gorenstein.
(b) Every module extension $R \hookrightarrow M$ that is cyclically pure is also pure.
(c) Every ring extension $R \hookrightarrow S$ that is cyclically pure is also pure.

Corollary 11.5. *Suppose R is approximately Gorenstein and $R \hookrightarrow S$ is a ring extension such that $IS \cap R = I$ for every ideal $I \subseteq R$. Then $R \longrightarrow S$ is pure.*

For us, the most useful thing about approximately Gorenstein local rings $(R, \mathfrak{m}, k)$ is that we can use the sequence of ideals I_n to construct an injective hull of the residue field $E_R = E_R(k)$. Notice that

$$\operatorname{Ann}_E R/I_n = E_{R/I_n} \cong R/I_n$$

where the final isomorphism is because R/I_n is Gorenstein Artinian ([**BH93**, Proposition 3.2.12]). Hence, since every element of E is annihilated by some $\mathfrak{m}^n$, and hence some I_n, we see the following:

Proposition 11.6. *Suppose $(R, \mathfrak{m}, k)$ is an approximately Gorenstein local ring with E an injective hull of k. Then we can write:*

$$E = \bigcup_n \operatorname{Ann}_E R/I_n \cong \varinjlim_n R/I_n$$

where the direct limit is made up of injective maps so that the socle[3] of R/I_n is sent onto the socle of R/I_{n+1}.

[3]the 1-dimensional vector space annihilated by $\mathfrak{m}$

Example 11.7. Again suppose $(R, \mathfrak{m})$ is a Gorenstein ring of dimension d and $f_1, \ldots, f_d$ is a system of parameters (and hence a regular sequence since Gorenstein rings are Cohen-Macaulay). Consider the sequence $I_n = (f_1^n, \ldots, f_d^n)$. Let $f = f_1 \cdots f_d$. We can construct the sequence:

$$R/I_1 \xrightarrow{\cdot f} R/I_2 \xrightarrow{\cdot f} R/I_3$$

which we know limits to $H^d_{\mathfrak{m}}(R)$. We know that $H^d_{\mathfrak{m}}(R) = E$ (see Appendix C Corollary 6.6). In other words, we can see that the construction of E as a direct limit in Proposition 11.6, is simply another perspective on the direct limit construction for local cohomology.

APPENDIX B

Review of divisors, invertible and reflexive sheaves

This appendix is incomplete. It will be expanded and revised.

1. Weil divisors

For simplicity, we begin with a normal Noetherian integral scheme X. Let $\mathcal{K}$ denote the rational function field of X, and $\mathcal{K}^\times$ denote the units (non-zero elements) in $\mathcal{K}$. We also use the notation $\mathcal{K}$ and $\mathcal{K}^\times$ to denote the corresponding constant sheaves of rational functions on X.

Definition 1.1. A **prime divisor** on X is a reduced irreducible closed subscheme of codimension one on X. A **(Weil) divisor** on X is a formal $\mathbb{Z}$-linear combination

$$\sum_{i=1}^{t} n_i D_i$$

of prime divisors. The set of all Weil divisors on X forms a free abelian group on the prime divisors, denoted $\operatorname{Div}(X)$. We say that D is **effective** if all $n_i \geq 0$.

If D and E are two Weil divisors, we write $D \geq E$ if $D - E \geq 0$, that is if the coefficients of D are at least as large as those of E.

Remark 1.2 (A comment on codimension)**.** A Noetherian scheme (or even ring) need not have finite dimension, and so talking the codimension of a subscheme $Z \subseteq X$ is not as simple as taking $\dim X - \dim Z$. However, we can require that at each point $x \in Z \subseteq X$ we have a comparison of $\dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Z,x}$ equal to a desired value.

Definition 1.3 (Support)**.** Suppose that $D = \sum_{i=1}^{t} a_i D_i$ is a divisor on X where $a_i \neq 0$, then the **support of** D is is the closed subset of X:

$$\operatorname{Supp} D = \bigcup_{i=1}^{t} D_i.$$

Our assumption that X is normal ensures that local ring $\mathcal{O}_{X,D}$ at any prime divisor is a discrete valuation ring. The associated valuation

$$\nu_D : \mathcal{K}^\times \longrightarrow \mathbb{Z}$$

describes the "order of zero or pole of a rational function along D." Because a rational function $f \in \mathcal{K}^\times$ has at most finitely many D along which it has a zero or a pole, the following definition makes sense:

Definition 1.4. For any normal Noetherian scheme X, the **divisor associated to** $f \in \mathcal{K}^\times$ is

$$\operatorname{div}(f) := \sum_{D \text{ prime divisor}} \nu_D(f) D.$$

A divisor of this form is said to be a **principal divisor**.

Remark 1.5. A rational function ϕ on a normal Noetherian scheme is *regular* if and only if the corresponding principal divisor $\operatorname{div}(\phi)$ is effective. This follows from the fact that a normal Noetherian domain is the intersection of its localizations at all height one primes [**Sta19**, Tag 0AVB].

For any open set $U \subset X$, we can restrict a divisor $D = \sum_{i=1}^t n_i D_i$ to U by

$$D|_U := \sum_{i=1}^{t} n_i (D_i \cap U)$$

where we simply omit the term $n_i(D_i \cap U)$ if $D_i \cap U$ is empty.

Definition 1.6 (Cartier divisors)**.** A divisor D on a normal Noetherian scheme is **Cartier** (or **locally principal**) if there exists a cover $\{U_\lambda\}$ of X such that each $D|_{U_\lambda}$ is principal on U_λ, that is, such that $D|_{U_\lambda} = \operatorname{div}(\phi_\lambda)$ for some rational function ϕ_λ on U_λ. We call the rational functions $\phi_\lambda \in \mathcal{K}$ **local defining equations** for the divisor D.

Example 1.7. The prime divisor $D = \mathbb{V}(x, z) \subset \operatorname{Spec} k[x, y, z]/(xy - z^2)$ is not locally principal at the singular point $\mathfrak{m} = (x, y, z)$ since the ideal of D can not be generated by one element at $\mathfrak{m}$. In this case, however, $2D = \operatorname{div} x$ is principal.

Remark 1.8. Let X be a normal Noetherian scheme, and $U \subset X$ an open set whose complement has codimension at least two. The restriction map $D \mapsto D|_U$ clearly defines an isomorphism between the groups $\operatorname{Div}(X)$ and $\operatorname{Div}(U)$.

Definition 1.9 (Linear equivalence)**.** Two divisors D, D' on X are **linearly equivalent** if there exists $f \in \mathcal{K}^\times$ such that $D' = D + \operatorname{div}(f)$. In this case we write $D \sim D'$.

2. Sheaves associated to divisors

Associated to any divisor is a coherent subsheaf of $\mathcal{K}$, which also determines the divisor D. Indeed, when we interact with divisors, typically work with the associated sheaves.

Definition 2.1. The **sheaf associated to the divisor** D is the subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}$ defined by

$$\mathcal{O}_X(D)(U) =: \{f \in \mathcal{K}^\times \mid \mathrm{div} f|_U + D|_U \geq 0\} \cup \{0\},$$

where by ≥ 0 we mean that the divisor $\mathrm{div} f|_U + D|_U$ is *effective* on U.

Example 2.2. Let D be a Cartier divisor on a normal variety X, say with local defining equation f_λ on the open cover $\{U_\lambda\}$. Then on U_λ, the sheaf $\mathcal{O}_X(D)$ is generated as an $\mathcal{O}_X(U_\lambda)$-module by the fraction $\frac{1}{f_\lambda} \in \mathcal{K}$. This follows from Remark 1.5: $\mathrm{div}(\phi)|_{U_\lambda} + D|_{U_\lambda}$ is effective if and only if $f f_\lambda$ is regular on U_λ.

The next proposition summarizes the basic facts about the sheaves determined by Weil divisors:

Proposition 2.3. *For a Weil divisors D, D' on a normal irreducible Noetherian scheme with function field $\mathcal{K}$:*

(a) The sheaf $\mathcal{O}_X(D)$ is a coherent reflexive subsheaf of the constant sheaf $\mathcal{K}$;
(b) The sheaf $\mathcal{O}_X(D)$ is invertible if and only if D is Cartier;
(c) Every reflexive subsheaf of $\mathcal{K}$ can be identified with $\mathcal{O}_X(D)$ for some Weil divisor D.
(d) There is an isomorphism $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ of $\mathcal{O}_X$-modules if and only if D and D' are linearly equivalent.
(e) Every rank one reflexive sheaf is isomorphic to some $\mathcal{O}_X(D)$ for some Weil divisor D on X.
(f) We have that $D \leq D'$ if and only if $\mathcal{O}_X(D) \subseteq \mathcal{O}_X(D')$. In particular, D is effective if and only if $\mathcal{O}_X \subseteq \mathcal{O}_X(D)$.

Proof. All statements, with the exception of (e), can be found in [**Har77**, II §6]. We note that for (d), the isomorphism $\mathcal{O}_X(D) \to \mathcal{O}_X(D')$ is given by multiplication by f, where $D' = D + \mathrm{div}(f)$.

For (e), suppose that $\mathcal{F}$ is a rank one reflexive sheaf. Choose an open subset U of regular points of X whose complement has codimension at least two, so that $\mathcal{F}|_U$ is invertible on U. Shrinking $\mathcal{U}$ if necessary, we may assume that $\mathcal{F}$ has a global section f. Consider the map $\mathcal{O}_U \to \mathcal{F}|_U$ given by multiplication by f. □

Remark 2.4. For a normal integral scheme X, the non-singular locus X^{reg} is an open set whose complement has codimension two or more. So there is a bijection between Weil divisors on X and divisors on the non-singular locus X^{reg} given by simply restricting to X^{reg}.

Because all divisors on a regular scheme are Cartier, an arbitrary divisor D of X will become Cartier when restricted to over X^{reg}. Thus, on a normal variety, we can think of Weil divisors as the closures (in X) of locally principal divisors on the non-singular locus X^{reg}. Likewise, we can think of rank one reflexive sheaves on X as extensions of invertible sheaves on the non-singular locus of X.

Definition 2.5 (Canonical divisors)**.** Let X be a normal Noetherian scheme with a fixed canonical module ω_X (see Appendix C Section 5). Then a **canonical divisor** on X is any divisor D such that $\mathcal{O}_X(D) \cong \omega_X$. We write K_X for the canonical divisor.

Remark 2.6 (Uniqueness of canonical divisors)**.** Note canonical divisors are only defined up to linear equivalence. Indeed, if K_X and K'_X are two canonical divisors, then $\mathcal{O}_X(K_X) \cong \omega_X \cong \mathcal{O}_X(K'_X)$. However, if $\pi : Y \longrightarrow X$ is a proper birational map between normal varieties, we always pick K_Y and K_X so that agree where π is an isomorphism.

3. Global sections and effective divisors

Let D be a Weil divisor on a normal irreducible Noetherian scheme X.

Each global section $s \in H^0(X, \mathcal{O}_X(D))$, viewed as an element of $\mathcal{K}^\times$, gives rise to a unique *effective* divisor linearly equivalent to D, namely the divisor

$$D_s := \operatorname{div} s + D, \tag{3.0.1}$$

which is effective by the definition of $\mathcal{O}_X(D)$. Conversely, if D' is an effective divisor linear equivalent to D, then there is a rational function s such that $D' = D + \operatorname{div}\, s$: by definition, s is a global section of $\mathcal{O}_X(D)$. Note that the section s uniquely determines the divisor D' but the divisor D' determines s only up to unit multiple from $H^0(X, \mathcal{O}_X^\times)$. This proves:

Proposition 3.1. *For a divisor D on a normal Noetherian scheme X, the map*

$$\begin{aligned} \frac{H^0(X, \mathcal{O}_X(D))}{H^0(X, \mathcal{O}_X^\times)} &\longrightarrow \operatorname{Div}(X) \\ f &\longmapsto \operatorname{div}(f) + D \end{aligned}$$

defines a bijection onto the complete linear system of all effective divisors on X linearly equivalent to D.

We will most frequently use the previous proposition in the following way.

Corollary 3.2. *Suppose X is a Noetherian normal scheme and D is a Weil divisor on X. Then any global section $s \in H^0(X, \mathcal{O}_X(D))$ that is non-zero on every component of X determines an effective divisor linearly equivalent to D. Furthermore, two sections s, s' determine the same divisor if and only if $s = us'$ for some unit $u \in H^0(X, \mathcal{O}_X)$.*

Example 3.3. If D is effective, then $1 \in \mathcal{K}$ is a global section of $\mathcal{O}_X(D)$. The divisor determined by this section is $\operatorname{div} 1 + D = D$. So an effective divisor D determines a canonical choice of global section for $\mathcal{O}_X(D)$—namely $1 \in \mathcal{K}$—which in turn recovers D. All other effective divisors linearly equivalent to D are of the form $\operatorname{div}(s) + D$ for some global section s of $\mathcal{O}_X(D)$.

Example 3.4. Consider $\mathbb{P}^2$ with homogeneous coordinates x, y, z. Let H be the hyperplane in $\mathbb{P}^2$ defined by the homogeneous coordinate $x = 0$. Then the global sections of $\mathcal{O}_{\mathbb{P}^2}(H)$ are spanned by $\{1, \frac{y}{x}, \frac{z}{x}\}$. An arbitrary global section is $\frac{ax+by+cz}{x}$, whose associated divisor is $\operatorname{div}(\frac{ax+by+cz}{x}) + H$; this is the hyperplane defined by the vanishing of the linear form $ax + by + cz$.

If H' is some other hyperplane, say defined by the homogeneous linear form h, then the global sections of $\mathcal{O}_X(H')$ are spanned by $\{\frac{x}{h}, \frac{y}{h}, \frac{z}{h}\}$. Note that $\frac{h}{x}$ is a rational function on $\mathbb{P}^2$, and that $H' = \operatorname{div}(\frac{h}{x}) + H$.

There is another, more local, way to describe the assignment from global sections $s \in \Gamma(X, \mathcal{O}_X(D))$ to effective Weil divisors D_s. Suppose E is a prime divisor on X with generic point $\eta \in E \subseteq X$. Then we have an isomorphism $\mu : \mathcal{O}_X(D)_\eta \cong \mathcal{O}_{X,\eta}$. We claim that the coefficient of D_s along E is exactly the vanishing order of $\mu(s_\eta) \in \mathcal{O}_{X,\eta}$, that is $v_\eta(\mu(s_\eta))$. To see this we work locally and notice that $\mathcal{O}_X(D)_\eta = \frac{1}{g}\mathcal{O}_{X,\eta} \subseteq \mathcal{K}$ where $\operatorname{div}(g)$ and D have the same cofficient of along E, and $g \in \mathcal{K}$. We may also assume that the map μ sends $\frac{1}{g} \mapsto 1$, in other words μ is multiplication to g. In this case, the coefficient of D_s along E is

$$v_\eta(s) + v_\eta(g) = v_\eta(s_\eta g).$$

On the other hand $v_\eta(\mu(s_\eta)) = v(s_\eta g)$.

As a consequence, we immediately obtain the following.

Proposition 3.5. *Suppose D_1, D_2 are Weil divisors on a Noetherian normal scheme with $D_1 \sim D_2$ and let us fix an isomorphism $\rho : \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$. If $s \in \Gamma(X, \mathcal{O}_X(D_1))$ determines an effective divisor $G \sim D_1 \sim D_2$ as in Proposition 3.1, then $\rho(s) \in \Gamma(X, \mathcal{O}_X(D_2))$ also determines G.*

4. Reflexive / S2 sheaves on normal schemes

The sheaves associated to divisors are rather unique as they are determined only by codimension 1 information. The theory of reflexive or S2 sheaves on a normal Noetherian scheme comes into play and can let us work with those $\mathcal{O}_X(D)$ as if they were line bundles.

First we record some definitions and some key lemmas. Some key references for this include [**Har94**], [**Har07**] and [**Sta19**, Tag 0AVT].

Definition 4.1 (S2 sheaves)**.** Suppose X is a Noetherian scheme and $\mathscr{F}$ is a coherent sheaf on X. We say that $\mathscr{F}$ is S_2 if for each (possibly non-closed) point $x \in X$, we have that $\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathscr{F}) \geq \min(2, \dim \mathcal{O}_{X,x})$.

We now introduce some notation. Suppose X is a normal Noetherian scheme. For any sheaf $\mathscr{F}$ we write

$$\mathscr{F}^* = \mathscr{H}om(\mathscr{F}, \mathcal{O}_X) \tag{4.1.1}$$

for the $\mathcal{O}_X$-dual. Likewise, if R is a Noetherian normal ring and M is an R-module, we write

$$M^* = \operatorname{Hom}_R(M, R) \tag{4.1.2}$$

for the R-dual.

Definition 4.2 (Reflexive sheaves)**.** Suppose X is a Noetherian scheme and $\mathscr{F}$ is a coherent sheaf on X. Then we say that $\mathscr{F}$ is **reflexive** if the canonical map

$$\mathscr{F} \longrightarrow \mathscr{F}^{**} = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X), \mathcal{O}_X)$$

is an isomorphism.

Lemma 4.3 ([**Har94**, Corollary 1.8])**.** *If X is Noetherian and normal*[1] *and $\mathscr{F}$ is coherent, then $\mathscr{F}^*$ is reflexive.*

Lemma 4.4 ([**Sta19**, Tag 0AY6])**.** *Suppose X is a normal Noetherian integral scheme and $\mathscr{F}$ is a coherent sheaf on X. Then the following are equivalent:*

(a) *$\mathscr{F}$ is torsion free and* S_2.
(b) *$\mathscr{F}$ is reflexive.*
(c) *There is a closed subset $Z \subseteq X$ of codimension ≥ 2, such that if $U = X \setminus Z$ then $\mathscr{F}|_U$ is locally free and if $i : U \hookrightarrow X$ is the inclusion, then $\mathscr{F} \longrightarrow i_*\mathscr{F}|_U$ is an isomorphism.*

Remark 4.5. In fact, the first two conditions are equivalent for non-normal schemes X as long as X is Gorenstein in codimension 1 (G_1) and S_2, see [**Har94**, Theorem 1.9].

[1] or more generally S_1 and Gorenstein in dimension 0, G_0

The next statement will be extremely important for us as it implies that if X is a scheme of characteristic $p > 0$ and $\mathscr{F}$ is a S_2 sheaf on X, then the Frobenius pushforward $F_*\mathscr{F}$ is also S_2.

Corollary 4.6. *Suppose $f : Y \longrightarrow X$ is a finite surjective map of integral Noetherian schemes (assume X is either locally excellent or normal). Then a torsion-free coherent sheaf $\mathscr{F}$ on Y is S_n if and only $f_*\mathscr{F}$ is S_n on X. In particular, if X and Y are normal then $\mathscr{F}$ is reflexive if and only if $f_*\mathscr{F}$ is reflexive.*

PROOF. For any point $P \in X$ with $\dim \mathcal{O}_{X,P} = n$, we have that $f^{-1}P$ is a finite collection of points Q of Y with $\dim \mathcal{O}_{Y,Q} = n$ by either the going up theorem and the excellence of $\mathcal{O}_{X,P}$ [**Sta19**, Tag 02IJ] (actually we just need universally catenary), or by both the going up and the going down theorem for when X is normal.

Furthermore,

$$H^i_P((f_*\mathscr{F})_P) = \bigoplus_{Q \in f^{-1}(P)} H^i_Q(\mathscr{F}_Q).$$

Since S_n is a condition on depth, and so can be checked by (non-)vanishing of local cohomology, the first statement is immediate. The statement on reflexivity is a consequence of Lemma 4.4. $\square$

Phrased locally, this just says the following.

Corollary 4.7. *Suppose $(R, \mathfrak{m})$ is a Noetherian local domain and $R \subseteq S$ is a finite extension of domains. Then an S-module M is S_n if and only if it is S_n when viewed as an R-module.*

The following is a useful tool for proving maps between sheaves are isomorphisms.

Lemma 4.8. *Suppose X is a Noetherian integral scheme. Suppose that*

$$\phi : \mathscr{F} \longrightarrow \mathscr{G}$$

is a map of coherent sheaves on X. Further suppose that

(a) *$\mathscr{G}$ is torsion free,*
(b) *ϕ is an isomorphism at every $x \in X$ with $\dim \mathcal{O}_{X,x} \leq 1$ and,*
(c) *$\mathscr{F}$ is S_2.*

Then ϕ is an isomorphism.

PROOF. This follows from [**Sta19**, Tag 0AV9], or see [**Har94**]. $\square$

What we have done so far also implies the following.

Theorem 4.9 ([**Har94**, Theorem 1.12])**.** *Let X be a Noetherian normal (or $\mathrm{G}_1 + \mathrm{S}_2$) scheme. Let $Z \subseteq X$ a subscheme of codimension ≥ 2 and let $U = X \setminus Z$ with $i : U \longrightarrow X$ the inclusion. Then the restriction of a sheaf from X to U induces equivalence of categories between the reflexive coherent sheaves on X and the reflexive coherent sheaves on U. Furthermore, for any reflexive (equivalently S_2) sheaf $\mathscr{F}$ on X, $\mathscr{F} \longrightarrow i_*\mathscr{F}|_U$ is an isomorphism. In particular, if $\mathscr{G}$ is a reflexive coherent sheaf on U, then $i_*\mathscr{G}$ is a coherent reflexive sheaf on X.*

Definition 4.10 (Reflexive hulls & S_2-ification)**.** Suppose that X is a Noetherian normal scheme and $\mathscr{F}$ is a coherent sheaf on X. The target of the canonical map

$$\mathscr{F} \longrightarrow \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X), \mathcal{O}_X) = \mathscr{F}^{**}$$

is called the **reflexive hull** or **reflexification** of $\mathscr{F}$. It is also called the S_2**-ification** of $\mathscr{F}$ when $\mathscr{F}$ is torsion free, since if one chooses $i : U \hookrightarrow X$ an open dense set whose complement has codimension ≥ 2, and such that $\mathscr{F}|_U$ is free, then the above map can be identified with

$$\mathscr{F} \longrightarrow i_*(\mathscr{F}|_U).$$

From here on out, when $\mathscr{F}$ is torsion-free, we will denote target of the displayed maps as $\mathscr{F}^{\mathrm{S}_2}$. We realize that $\mathscr{F}^{**}$ is more common in the literature, but we believe the notation $\mathscr{F}^{\mathrm{S}_2}$ carries more information.

For a more general notion of S_2-ification outside the case when R is normal, see Appendix C Proposition 6.9 or [**Har07**].

The sheaves associated to divisors are always reflexive.

Lemma 4.11. *Suppose that X is a Noetherian normal scheme and D is a Weil divisor on X. Then the sheaf $\mathcal{O}_X(D)$ is reflexive (equivalently S_2).*

Proof. Since X is normal, at each point $x \in X$ with $\dim \mathcal{O}_{X,x} = 1$, we see that $\mathcal{O}_{X,x}$ is a DVR. We then see that $\mathcal{O}_X(D)$ is free at all those points. Taking $U \subseteq X$ to be the locus where $\mathcal{O}_X(D)$ is free (whose complement then has codimension ≥ 2) we see that Lemma 4.4 (c) is satisfied. The result follows. $\square$

We now see how divisor operations work with divisorial sheaves.

Proposition 4.12. *Suppose X is a normal Noetherian scheme and D, E are Weil divisors on X.*

(a) $(\mathcal{O}_X(D) \otimes \mathcal{O}_X(E))^{\mathrm{S}_2} \cong (\mathcal{O}_X(D) \cdot \mathcal{O}_X(E))^{\mathrm{S}_2} \cong \mathcal{O}_X(D + E)$.

(b) For any $n > 0$, $(\mathcal{O}_X(D)^{\otimes n})^{\mathrm{S}_2} \cong \mathcal{O}_X(nD)$.
(c) $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(E)) \cong \mathcal{O}_X(E - D)$. In particular, we have that $\mathcal{O}_X(D)^* = \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X) \cong \mathcal{O}_X(-D)$.

Proof. We first notice that all these results are straightforward for D, E Cartier and so all hold on the regular locus of X (whose complement has codimension 2 since X is normal). The results all follow from Lemma 4.8. $\square$

Example 4.13 (Symbolic powers)**.** Suppose R is a normal integral domain and P is a height-one prime corresponding to a prime divisor D. Set $X = \operatorname{Spec} R$. Then $\mathcal{O}_X(-D)$ is the sheaf corresponding to P, that is $\Gamma(X, \mathcal{O}_X(-D)) = P$. Furthermore, $\mathcal{O}_X(-nD)$ corresponds to the nth symbolic power[2] of P, that is $\Gamma(X, \mathcal{O}_X(-nD)) = P^{(n)}$. Indeed, the point is that $(-)^{\mathrm{S}_2}$ removes any non-height-one primary components of P^n by Proposition 4.12 (b) above. That is also precisely what taking a symbolic power does this case.

Another useful fact is that any rank-1 reflexive sheaf is isomorphic to a divisorial sheaf.

Proposition 4.14. *Suppose X is a Noetherian normal integral scheme. Suppose $\mathscr{F}$ is a coherent (generically) rank-1 reflexive sheaf. Then there is an inclusion of $\mathcal{O}_X$-modules into the fraction field sheaf of X:*

$$i : \mathscr{F} \hookrightarrow \mathscr{K}.$$

Furthermore, for any such inclusion i, there is a Weil divisor D on X such that $i(\mathscr{F}) = \mathcal{O}_X(D)$.

We conclude this section with a discussion of global sections of rank-1 reflexive sheaves.

Proposition 4.15. *Suppose X is a normal Noetherian scheme and $\mathscr{F}$ is a rank-1 reflexive sheaf on X. Any global section $s \in \Gamma(X, \mathscr{F})$ determines an effective Weil divisor D_s such that $\mathcal{O}_X(D_s) \cong \mathscr{F}$. Furthermore, this construction agrees with that of Proposition 3.1 in the case that $\mathscr{F} = \mathcal{O}_X(D)$.*

Proof. We give two proofs of this.

For our first proof, we define D_s as in the discussion before Proposition 3.5. For a prime divisor E on X set $\eta \in E \subseteq X$ to be it generic point. We then have $\mu : \mathscr{F}_\eta \cong \mathcal{O}_{X,\eta}$ since X is normal and $\mathscr{F}$ is rank-1 and torsion-free. The divisor D_s can then be defined by requiring that its coefficient

[2] The nth symbolic power can be defined as $P^{(n)} = P^n R_P \cap R$, or equivalently as the P-primary component in a primary decomposition of P^n.

along E is the vanishing order of $\mu(s_\eta) \in \mathcal{O}_{X,\eta}$. For the final statement we appeal to Proposition 3.5.

For the second proof, choose an embedding $\mathscr{F} \hookrightarrow \mathscr{K}$ as in Proposition 4.14 so that the image of $\mathscr{F}$ is $\mathcal{O}_X(D)$. Then the global sections of $\mathscr{F}$ are then identified with global sections of $\mathcal{O}_X(D)$ and so determine effective divisors satisfying the desired condition by (3.0.1). This is independent of the choices by Proposition 3.5. $\square$

5. $\mathbb{Q}$-divisors, $\mathbb{Z}_{(p)}$, and $\mathbb{R}$-divisors

Suppose p is a prime and let $\mathbb{Z}_{(p)}$ denote $\mathbb{Z}$ localized at the prime ideal generated by p. In other words, $\mathbb{Z}_{(p)}$ is the set of rational numbers that can be written so that p does not appear in a denominator.

Definition 5.1. Suppose X is a normal Noetherian integral scheme. A **$\mathbb{Q}$-divisor** (respectively **$\mathbb{R}$-divisor**, **$\mathbb{Z}_{(p)}$-divisor**) on X is a $\mathbb{Q}$-linear (respectively $\mathbb{R}$-linear, $\mathbb{Z}_{(p)}$-linear) formal sum of prime divisors. They form a group which we denote by $\mathrm{Div}_{\mathbb{Q}}(X)$ (respectively, $\mathrm{Div}_{\mathbb{R}}(X)$, $\mathrm{Div}_{\mathbb{Z}_{(p)}}(X)$).

Note any Weil divisor is a $\mathbb{Z}$-divisor, any $\mathbb{Z}_{(p)}$-divisor is a $\mathbb{Q}$-divisor, and any $\mathbb{Q}$-divisor is a $\mathbb{R}$-divisor.

Notation 5.2. For any $\mathbb{R}$-divisor $D = \sum_{i=1}^t a_i D_i$ where D_i are prime divisors, we write

$$\lfloor D \rfloor := \sum_{i=1}^t \lfloor a_i \rfloor D_i \text{ and } \lceil D \rceil := \sum_{i=1}^t \lceil a_i \rceil D_i$$

for the **round-down** and **round-up** of D, respectively.

Definition 5.3 ($\mathbb{Q}$-Cartier and $\mathbb{R}$-Cartier divisors)**.** A $\mathbb{Q}$-divisor D is called **$\mathbb{Q}$-Cartier** (respectively, **$\mathbb{Z}_{(p)}$-Cartier**) if there exists an integer $n > 0$ (respectively, such that n is not divisible by p) such that nD is a Cartier Weil divisor. The smallest $n > 0$ such that nD is Cartier is called the **index of** D.

A $\mathbb{R}$-divisor D is called **$\mathbb{R}$-Cartier** if there exist Cartier Weil divisors $D_1, \ldots, D_m$ such that $D = \sum_{i=1}^m r_i D_i$ for $r_i \in \mathbb{R}$.

Example 5.4. The divisor $(1/3)\,\mathrm{div}(x) + (1/4)\,\mathrm{div}(y)$ on $\mathbb{A}^2 = \mathrm{Spec}\, k[x, y]$ is $\mathbb{Q}$-Cartier. It is not a Weil divisor though.

The prime Weil divisor $D = \mathrm{div}(x, y)$ on $\mathrm{Spec}\, k[x, y, z]/(x^2 - yz)$ is $\mathbb{Q}$-Cartier, since $2D = \mathrm{div}(y)$. The $\mathbb{Q}$-divisor $D = \frac{1}{3}\,\mathrm{div}(x) = \frac{1}{3}\,\mathrm{div}(x, y) + \frac{1}{3}\,\mathrm{div}(x, z)$ is also $\mathbb{Q}$-Cartier.

Next consider $X = \operatorname{Spec} k[x,y,u,v]/(xy-uv)$. The prime divisor $D = \operatorname{div}(x,u)$ is not $\mathbb{Q}$-Cartier. Indeed, in this example $\mathcal{O}_X(-nD) = (x,u)^n$ for every integer n.

Definition 5.5. A normal scheme X with a canonical module is called **$\mathbb{Q}$-Gorenstein** if the canonical divisor is $\mathbb{Q}$-Cartier. The **index of** X is the index of the canonical divisor K_X. A normal Noetherian ring is called **$\mathbb{Q}$-Gorenstein** if $\operatorname{Spec} R$ is.

Example 5.6. Any Gorenstein normal variety is $\mathbb{Q}$-Gorenstein.

The ring $R = k[x^3, x^2y, xy^2, y^3]$ is not Gorenstein since the canonical divisor $K_R = \operatorname{div}(x^3, x^2y, xy^2)$ is a canonical divisor and the corresponding ideal (x^3, x^2y, xy^2) is not principal. On the other hand, it is $\mathbb{Q}$-Gorenstein since $3K_R = \operatorname{div}(x^3)$. From a commutative algebra perspective, $P = (x^3, x^2y, xy^2) \cong \omega_R$ is not (locally) principal but its third symbolic power is principal $P^{(3)} = (x^3)$, see Example 4.13.

6. Pulling back divisors

Suppose $f : Y \longrightarrow X$ is a map of normal integral Noetherian schemes. In this section we describe how to take divisors D on X and turn them into divisors on Y. Essentially, there are three main cases:

(a) D is $\mathbb{Q}$-Cartier and $f(Y)$ is not contained in the support of D.
(b) f is flat.
(c) f is finite and surjective.

6.1. Pulling back ($\mathbb{Q}$-)Cartier divisors. We first explain what to do when D is Cartier, and $f(Y) \not\subseteq \operatorname{Supp}(D)$ (note that this condition holds whenever f is dominant, for instance if f is birational). In this case we work locally on some open $U \subseteq X$ and assume that $D|_U = \operatorname{div}_U(g/h)$ for some $g, h \in \Gamma(U, \mathcal{O}_X)$. We may select g, h so that the of g, h under $\Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(f^{-1}(U), \mathcal{O}_Y)$ is nonzero, and so a non-zerodivisor since Y is integral. On $f^{-1}(U)$ we then set

$$f^*D|_{f^{-1}U} = \operatorname{div}_{f^{-1}(U)} g - \operatorname{div}_{f^{-1}(U)} h.$$

It is not difficult to see that this is independent of the choice of g, h and that these $f^*D|_{f^{-1}U}$ glue together on intersections. This gives us a well defined Cartier divisor f^*D on Y. Furthermore, it is not difficult to see that

$$\mathcal{O}_Y(f^*D) = f^*\mathcal{O}_X(D) \tag{6.0.1}$$

in this case when D is Cartier.

Example 6.1. If $X = \mathbb{A}^2$, $f : Y \longrightarrow X$ is the blowup of the origin and $D = \operatorname{div}(y^2 - x^3)$. Then $f^*D = D' + 2E$ where $D' = f_*^{-1}D$ is the strict transform of D (that is the divisor determined by D from the locus where f is an isomorphism) and where E is the exceptional divisor. This can be checked explicitly in local coordinates. Indeed, Y is covered by two affine charts

$$U_1 = \operatorname{Spec} k[x, y/x]$$

and

$$U_2 = \operatorname{Spec} k[x/y, y].$$

On U_1, the equation defining D factors as

$$y^2 - x^3 = x^2((y/x)^2 - x)$$

and on U_2 it factors as

$$y^2 - x^3 = y^2(1 - y(x/y)^3).$$

In either case we see that it factors as

$$\text{(something squared)} \cdot \text{(something to the first power)}$$

which is what we expected.

We now explain how to pullback $\mathbb{Q}$-Cartier (or even $\mathbb{R}$-)divisors. Indeed, if D is $\mathbb{Q}$-Cartier, then we find some $n > 0$ such that nD is Cartier. Then if $f : Y \longrightarrow X$ is as above, we simply define

$$f^*D = \frac{1}{n} f^* nD.$$

More generally, if $D = \sum_{i=1}^m r_i D_i$ for $r_i \in \mathbb{R}$ and such that D_i are Cartier that do not contain $f(Y)$ in their support, we define

$$f^*D = \sum_{i=1}^m r_i f^* D_i.$$

Example 6.2. Consider $X = \operatorname{Spec} k[x, y, z]/(x^2 - yz)$ and set $D = \operatorname{div}(x, y)$. Then $2D = Div(y)$ so that D is $\mathbb{Q}$-Cartier of index 2. Let $\pi : Y \longrightarrow X$ be the blowup of the origin (x, y, z).

To compute π^*D we simply compute $(1/2)\pi^*2D = (1/2)\operatorname{div}_Y(y)$. In particular, we need to compute the coefficient of π^*D along the exceptional divisor E as we already know what happens on the strict transform of D, $D_Y = \pi_*^{-1}D$, where it has coefficient 1. We can do this on the chart

$$U_1 = \operatorname{Spec} k[x/y, y, z/y]/(1 - (x/y)(z/y))$$

where the exceptional divisor is $E = \mathrm{div}_{U_1}(y)$. In particular, $\pi^* 2D = \pi_*^{-1} 2D + E$ where $\pi_*^{-1} 2D = 2D_Y$ is the strict transform of $2D$. Hence, dividing by 2, we see that

$$\pi^* D = D_Y + (1/2)E.$$

In particular, even though D is a Weil divisor, we see that $\pi^* D$ is a $\mathbb{Q}$-divisor.

It is natural to ask how one might be able to pull back Weil divisors that are not $\mathbb{Q}$-Cartier. This is explored in [**DH09**].

6.2. Pulling back divisor under flat maps. Next suppose $f : Y \longrightarrow X$ is flat, in this case, we see that f is dominant automatically. We have an inclusion $f^{-1}\mathcal{K}(X) \subseteq \mathcal{K}(Y)$ of fields of rational functions. Since the sheaf $\mathcal{O}_X(D) \subseteq \mathcal{K}(X)$ then pulls back to a sheaf $f^{-1}\mathcal{O}_X(D) \subseteq f^{-1}\mathcal{K}(X)$. Tensoring by $\otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$ we obtain an inclusion $f^*\mathcal{O}_X(D) \subseteq \mathcal{K}(Y)$. Next we notice that since f is flat, f^* commutes with $\mathscr{H}om$, and $f^*\mathcal{O}_X(D)$ is S2/reflexive. In other words, $f^*\mathcal{O}_X(D) \subseteq \mathcal{K}(Y)$ is a reflexive $\mathcal{O}_Y$-subsheaf, and so there exists a divisor D_Y with $f^*\mathcal{O}_X(D)\mathcal{O}_Y(D_Y)$. We set

$$f^* D := D_Y.$$

It is not difficult to see that $f^*(D + E) = f^*D + f^*E$ in this case (again, these computations may be done off a set of codimension ≥ 2 where the divisors D and E are Cartier)

If D is a $\mathbb{Q}$-divisor (or $\mathbb{R}$-divisor), we write $D = \sum_{i=1}^t a_i D_i$ with D_i prime and $a_i \in \mathbb{Q}$ (respectively $\mathbb{R}$), then we set:

$$f^* D := \sum a_i f^* D.$$

6.3. Pulling back divisors under finite surjective maps. Finally, suppose that $f : Y \longrightarrow X$ is finite surjective. Then, since X is normal, there exist closed sets $V \subseteq X$ and $W = f^{-1}V \subseteq Y$, of codimension ≥ 2 respectively such that $f' := f|_{Y \setminus W} : (Y \setminus W) \longrightarrow (X \setminus V)$ is flat. For any Weil divisor D on X, we let f^*D be the unique divisor on Y that agrees with $f'^*D|_{X \setminus V}$.

This can be made a bit more explicit as follows. Suppose we write $D = \sum_{i=1}^t a_i D_i$ where D_i is prime, and let $\eta_i \in X$ denote the generic point of D_i. We can take the normalization of $\mathcal{O}_{X,\eta_i} \subseteq \mathcal{K}(Y)$ which will be a semi-local ring where each maximal ideal corresponds to a point $y_{ij} \in Y$ mapping to η_i. The local inclusion $\mathcal{O}_{X,\eta_i} \subseteq \mathcal{O}_{Y,y_{ij}}$ is then an inclusion of DVRs. Let r, s denote uniformizers so that $r = us^{m_{ij}}$ for some unit $u \in \mathcal{O}_{Y,y_{ij}}$. If we let

E_{ij} denote the closure of $\{y_{ij}\}$, which makes it a prime divisor, we have that

$$f^*D = \sum_{i,j} a_i m_{ij} E_{ij}.$$

Example 6.3. If $F^e : X \longrightarrow X$ is e-iterated Frobenius, then $(F^e)^*D = p^e D$. Indeed, this makes sense even if Frobenius is not finite since we can work on the regular locus of X where Frobenius is flat on both the source and target.

Finally, we notice that for finite surjective maps

$$\big(f^*\mathcal{O}_X(D)\big)^{\mathrm{S}_2} = \mathcal{O}_Y(f^*D). \tag{6.3.1}$$

7. Normal and simple normal crossings divisors

Definition 7.1. Suppose X is a normal Noetherian scheme and D is a $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$-divisor on X with $D = \sum a_i D_i$ where the D_i are prime divisors.

We say that D has **simple normal crossing (SNC)** if each D_i is Cartier and as a scheme each D_i is regular. Furthermore, at each point $x \in X$ with associated stalk $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$, if $D_{i_1}, \dots, D_{i_m}$ are the D_i containing x, we have that is defined by some $f_i \in \mathfrak{m}_x$ and the f_i are part of a minimal system of generators of the maximal ideal $\mathfrak{m}_x$. It is important to observe that if (X, Δ) is SNC, then X is nonsingular at each $x \in \mathrm{Supp}(\Delta) \subseteq X$.

We say that D has **normal crossings (NC)** if for each $x \in X$ with completed stalk $(R = \widehat{\mathcal{O}_{X,x}}, \widehat{\mathfrak{m}_x})$, we have that $\mathrm{Supp}(D)|_{\mathrm{Spec}\, R} = \mathbb{V}(f)$ where $f = \prod_{i=1}^n g_i$ and the $g_1, \dots, g_n$ are part of a minimal system of generators of the maximal ideal $\widehat{\mathfrak{m}_x}$.

We say that (X, Δ) is **SNC (repsectively NC)** if X is nonsingular and Δ is SNC (respectively NC).

Example 7.2. Notice that an irreducible nodal curve (for example $\mathrm{div}(y^2 - (x^3 - x^2)) \subseteq \mathbb{A}^2$) is normal crossings but not simple normal crossings. On the other hand a reducible nodal curve (for example $\mathrm{div}(xy) \subseteq \mathbb{A}^2$) is both normal crossings and simple normal crossings.

Note that if (X, Δ) is a NC pair, and $x \in X$ is a point with $R = \widehat{\mathcal{O}_{X,x}}$, then $(\mathrm{Spec}\, R, \Delta|_{\mathrm{Spec}\, R})$ is SNC.

Definition 7.3. Suppose that $(X, \Delta = \sum D_i)$ is a SNC pair where Δ is a reduced divisor. Then the **strata (of (X, Δ))** are the set of intersections:

$$\Big\{ \bigcap_{D_i \in S} D_i \;\Big|\; S \subseteq \{D_1, \dots, D_m\} \Big\}.$$

By construction, the strata of a SNC pair are themselves nonsingular schemes (they are locally defined by a part of a minimal set of generators of the maximal ideal of a regular ring).

8. Ramification divisors and tame ramification

We begin in the ring theoretic setting.

Definition 8.1 (Wild and tame extensions)**.** Suppose that $R \subseteq R'$ is a essentially finite type, generically finite, and generically separable extension of DVRs with uniformizers $r \in R$ and $r' \in R'$ respectively. Write $r = ur'^n$ for some unit $u \in R'$ where n is the **ramification index** of the extension. We say that $R \subseteq R'$ is **tamely ramified** (or simply **tame**) if

(a) p does not divide n, and
(b) The residue field extension $R/(r) \subseteq R'/(r')$ is separable.

The extension $R \subseteq R'$ is called **wildly ramified** or (simply **wild**) if it is not tame. If $R \subseteq R'$ is tame and $n = 1$, then we say that $R \subseteq R'$ is **unramified**.

Now suppose R is a DVR and that $R \subseteq S$ is a finite extension of normal rings. We say that $R \subseteq S$ is **tame** if for each of the (finitely many) maximal ideals $\mathfrak{n} \subseteq S$ we have that $R \subseteq S_{\mathfrak{n}}$ is tame. We say that $R \subseteq S$ is **wild** if it is not tame (this means that at least one of the $R \subseteq S_{\mathfrak{n}}$ is wild).

Finally, if $R \subseteq S$ is a finite extension of normal Noetherian domains, we say that it is **tamely ramified in codimension 1**, if for each height one prime $Q \in \operatorname{Spec} R$, we have that $R_Q \longrightarrow S_Q$ is tamely ramified.

We also recall the ramification divisor.

Definition 8.2. Suppose that $R \subseteq S$ is a finite extension of normal Noetherian domains. The **ramification divisor** is the divisor Ram on $\operatorname{Spec} S$ such that for each height-one prime $Q \subseteq S$ we have that the coefficient of Ram along $\operatorname{div}(Q)$ is the same as the length of $(\Omega_{S/R})_Q$ as an S_Q-module. This notion extends to ramification divisors for finite maps of schemes normal Noetherian schemes in the natural way.

Proposition 8.3. *With notation as in Definition 8.2, suppose $P = Q \cap R$ and $R_P \subseteq S_Q$ is tamely ramified. The coefficient of* Ram *at* $\operatorname{div}(Q)$ *is equal to $n - 1$ where n is the ramification index of $R_P \subseteq S_Q$.*

PROOF. See []. □

9. Cyclic covers

In this section we briefly review cyclic covers associated to divisorial sheaves. Other good sources which cover most of the same material include [**TW92**], [**Kol97**, Section 2], [**EV92**, Section 3.5], [**KM98**, Section 2.4], [**Kol13**, Section 2.3], or [**MP20**, Appendix A.1].

Suppose X is a Noetherian normal integral scheme and D is a Weil divisor on X. Suppose we pick $z \in \Gamma(X, \mathcal{O}_X(nD))$ a nonzero global section for some $n > 0$. This section gives us a map $\mathcal{O}_X \xrightarrow{\cdot z} \mathcal{O}_X(nD)$ and hence for any other divisor E a map $\mathcal{O}_X(E) \xrightarrow{\cdot z} \mathcal{O}_X(E+nD)$. That map lets us create the following sheaf of rings:

$$\mathcal{R}(X, z \in \Gamma(X, \mathcal{O}_X(nD))) := \mathcal{O}_X \oplus \mathcal{O}_X(-D) \oplus \mathcal{O}_X(-2D) \oplus \cdots \oplus \mathcal{O}_X(-(n-1)D)$$

where the multiplication is defined by the map

$$\mathcal{O}_X(-iD) \times \mathcal{O}_X(-jD) \longrightarrow \mathcal{O}_X(-(i+j)D)$$

if $i + j < n$ and if $i + j > n$ then we define further compose the above multiplication with the inclusion $\mathcal{O}_X(-(i+j)D) \xrightarrow{\cdot z} \mathcal{O}_X(-(i+j-n)D)$.

Definition 9.1. With notation as above, the **cyclic cover of** X **associated to** $z \in \Gamma(X, \mathcal{O}_X(nD))$ is the scheme $Y := \mathbf{Spec}\mathcal{R}(X, z \in \Gamma(X, \mathcal{O}_X(nD)))$. By construction there is a map $\pi : Y \longrightarrow X$. The **index of the cyclic cover** is the integer $n > 0$.

We record the following facts.

Lemma 9.2. *Suppose X, Y, π and $z \in \Gamma(X, \mathcal{O}_X(nD))$ are as above and that X has characteristic $p > 0$. Then the following hold.*

(a) The map $\pi : Y \longrightarrow X$ is finite with
$$\pi_* \mathcal{O}_Y = \mathcal{R}(X, z \in \Gamma(X, \mathcal{O}_X(nD))).$$

(b) Y is S_2.
(c) If the divisor $D_z \sim nD$ associated to z is reduced, and p does not divide n, then Y is also normal. .
(d) If p does not divide n, then $Y \longrightarrow X$ is étale on the regular locus of $X \setminus \operatorname{Supp} D_z$.
(e) If p does not divide n and z globally generates $\mathcal{O}_X(nD)$ (so that $\mathcal{O}_X(nD) \cong \mathcal{O}_X$), then π is étale in codimension 1.
(f) If p does not divide n and D_z is reduced, then the ramification divisor of the map π is given by
$$\mathrm{Ram}_\pi = \pi^* D_z - (\pi^* D_z)_{\mathrm{red}} = \lfloor (1-\epsilon)\pi^* D_z \rfloor$$
for $1 \gg \epsilon > 0$.

Proof. (a) is immediate from the construction. (b) follows from Corollary 4.6 since it suffices to check that $\pi_* \mathcal{O}_Y$ is S_2, which is also clear since a direct sum of S_2 modules is S_2. Property (c) is proved in []. The proof of (d), which immediately implies (e), can be found in []. Part (f) is a consequence of [], or our local computations below. $\square$

Example 9.3. If $Y \longrightarrow X$ is a cylic cover of index n and $p|n$ where p is the characteristic of X, then it can happen that Y is not even reduced. For example, consider $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$ where k is a field of characteristic $p > 0$. Suppose we pick $D = \operatorname{div}(x)$ and $z = 1 \in \Gamma(X, \mathcal{O}_X(pD)) = \frac{1}{x^p} k[x]$. Now, using t as a dummy-variable, we set:

$$Y = \operatorname{Spec}\left(k[x]t^0 \oplus (x)t^1 \oplus \cdots \oplus (x^{p-1})t^{p-1}\right)$$

and we see that $(xt)^p = x^p \in k[x]$ via our choice of $z = 1$. In particular, we have that

$$Y = \operatorname{Spec} k[x,T]/(T^p - x^p) = \operatorname{Spec} k[x,T]/(T-x)^p$$

by declaring $T = xt$. Hence Y is not reduced even though X is nonsingular. Note also $D_z = p \operatorname{div}(x)$.

Now we specialize this to the local setting, generalizing the work of the previous example. Since any cyclic cover remains a cyclic cover after passing to an open cover or stalk, let us consider what happens when $X = \operatorname{Spec} R$ and $(R, \mathfrak{m} = (r))$ is a DVR with uniformizer r. In that case $D = j \operatorname{div}(r)$ for some integer j which may be negative. We then pick

$$z = ur^c \in r^{jn} R$$

with $c \geq jn$. Set $b = c - jn$ and note $D_z = (c - jn)\operatorname{div}_X(r) = b \operatorname{div}_X(r)$. We create a ring (using t as a dummy variable)

$$S := R \oplus r^{-j} Rt \oplus \cdots \oplus r^{-j(n-1)} Rt^{n-1}$$

with the relation $(r^{-j}t)^n = ur^{c-jn}$ for some unit u. In other words, setting $T := r^{-j}t$, we see:

$$S \cong R[T]/(T^n - ur^b)$$

In particular, if $b = 1$, we are just adjoining the nth root of the uniformize ur, and so S is regular. However, if $b = 0$, then we are adjoining the nth root of u, which may make S non-reduced (if u already has a pth root).

Of particular interest for us is the case when $D_z = 0$, $D = K_X$ and $X = \operatorname{Spec} R$ where R is local.

Proposition 9.4. *Suppose $(R, \mathfrak{m})$ is a Noetherian normal local domain, D is a Weil divisor on $X = \operatorname{Spec} R$ such that $nD \sim 0$ for some $n > 0$. Pick $0 \neq z \in \Gamma(X, \mathcal{O}_X(nD))$ so that $D_z = 0$ and let $\pi : Y = \operatorname{Spec} S \longrightarrow X$ be the associated cyclic cover ($S = R \oplus R(-D) \oplus \ldots R(-(n-1)D)$).*

(a) *If n is the index D (that is, $n > 0$ is the smallest integer such that $nD \sim 0$) then S is also local with unique maximal ideal $\mathfrak{m}_S = \mathfrak{m} \oplus R(-D) \oplus dots \oplus R(-(n-1)D)$ and the projection onto the degree 0 piece of S, $\rho \in \mathrm{Hom}_R(S, R)$ generates $\mathrm{Hom}_R(S, R)$ as a free S-module.*
(b) *The map ρ sends the maximal ideal $\mathfrak{m}_S$ into the maximal ideal $\mathfrak{m}$.*
(c) *$(\pi^* \omega_X)^{\S 2} = (\pi^* \mathcal{O}_X(K_X))^{\S 2}$ is a canonical module of Y. In particular, if Y is normal, then $\pi^* K_X \sim K_Y$.*
(d) *If D has index n, then $(\pi^* \mathcal{O}_X(D))^{\S 2} \cong \mathcal{O}_Y$. In particular, if Y is normal then $\pi^* D$ is a Cartier divisor.*
(e) *If $D = K_X$ has index n, then $\omega_Y \cong \mathcal{O}_Y$. In particular, Y is quasi-Gorenstein.*

PROOF. Proofs of part (a) can be found in [**TW92**, Lemma 2.1] (partially), or [**CR22**] or [**MP20**, Proposition A.5]. Part (b) follows immediately.

We now consider (c). By [**TW92**, Theorem 3.2(i)], or a direct computation, we know that

$$\omega_S = \mathrm{Hom}_R(S, \omega_R) = \bigoplus_{i=0}^{n-1} R(K_R - iD)$$

which is the same as $(\pi^* \omega_R)^{\S 2} = R(K_X) \oplus R(K_R - D) \oplus \cdots \oplus R(K_R - (n-1)D)$ (also see [**MP20**, Lemma A.7]).

Property (d) follows since $(\pi^* R(D))^{\S 2} = (R(D) \otimes_R S)^{\S 2}$ is simply

$$R(D) \oplus R(D - D) \oplus \cdots \oplus R(D - (n-1)D)$$

and we have the degree zero part $R(D) \cong zR(D) = R(D - nD) = R(-(n-1)D)$ and so $(\pi^* R(D))^{\S 2}$ is isomorphic to S as an R-module. Consider then the map $S \longrightarrow (\pi^* R(D))^{\S 2}$ sending 1 to $1 \in R(D - D) = R$ which is then easily seen to be an S-module isomorphism.

Finally, **??** is well known, variants can be found in [**TW92**, Theorem 3.2], or [**MP20**, Lemma A.7]. It also follows from (c) and (d). □

The map $\pi : Y \longrightarrow X$ as in Proposition 9.4 is typically called a **canonical cover**, even though it is *not canonical* (it is built using a *canonical* divisor however).

One key use of cyclic covers is to take a $\mathbb{Q}$-Cartier Weil divisor and pull it back to a Cartier divisor. However, as we have noted, if the index n of D is divisible by p, the associated cyclic cover Y can be non-reduced. Even in the case that S is a domain, it need not be normal, and the extension $\mathcal{K}(R) \subseteq \mathcal{K}(S)$ of fraction fields is inseparable (so there is no ramification divisor). There is a way to construct separable extensions $R \subseteq S$ of normal integral domains that preserves this property.

Lemma 9.5 ([**BST11b**, Lemma 4.5])**.** *Suppose* Γ *is a* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on a normal integral scheme* X*. Then there exists a finite* generically separable *map* $g : W \longrightarrow X$ *from a normal integral scheme* W *such that* g^*W *is a Cartier divisor.*

PROOF. Working locally with $X = \operatorname{Spec} R$ and $\Gamma = \frac{1}{n}\operatorname{div}_X(f)$. Set $K = \mathcal{K}(R)$ to be the fraction field of R and let $L = K[X]/(X^n + fX + f)$, a separable extension of K. Then let S be the normalization of R in S, $\operatorname{Spec} S$ satisfies the desired properties. $\square$

One should note that the construction above, while generically separable, is typically wildly ramified if $p|n$.

APPENDIX C

Matlis, local and Grothendieck Duality

This appendix is incomplete. It will be expanded and revised.

In this appendix we review Matlis, local and Grothendieck Duality.

1. Matlis duality

We give a brief overview of Matlis duality. Our primary reference is [**BS98**, Chapter 10] which makes a number of statements in greater generality than most other modern sources such as [**BH93**, Chapter 3], [**Hoc11**, Section 5] or [**Sta19**, Tag 08Z1].

Throughout this section of the appendix we fix the following notation.

Notation 1.1. Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian local ring. We let $E := E_R(k)$ denote the injective hull of the residue field k. We use $(-)^\vee$ to denote the functor $\mathrm{Hom}_R(-, E)$. This is called the **Matlis duality functor**. We let $\widehat{R}$ denote the completion of R.

Before stating Matlis duality, we record some preliminary helpful facts.

Lemma 1.2. *For any R-module M, there is a natural map $M \longrightarrow M^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$ defined by sending m to the function $\phi \mapsto \phi(m)$. This map is injective.*

Proof. See [**BS98**, Remark 10.2.2]. □

Lemma 1.3. *E is an Artinian R-module. Furthermore, for any Artinian R-module N, there is an injection*

$$N \hookrightarrow E^{\oplus n}$$

for some integer n.

Proof. See [**BS98**, Theorem 10.2.5, Corollary 10.2.8]. □

Lemma 1.4. *For any Artinian R-module M, we may also view M as a $\widehat{R}$-module. From this perspective, an injective hull of $R/\mathfrak{m}$ (as an R-module), when viewed as an $\widehat{R}$-module, is also an injective hull of $\widehat{R}/\widehat{\mathfrak{m}}$ (as an $\widehat{R}$-module).*

PROOF. See [**BS98**, Remark 10.2.9 and Exercise 10.2.10]. $\square$

The Matlis duality functor $\operatorname{Hom}_R(-,E)$ is certainly exact since E is injective, but it also induces a duality between Noetherian and Artinian modules.

Theorem 1.5 (Matlis Duality)**.** *Suppose that $(R,\mathfrak{m},k)$ is a Noetherian local ring with injective hull of the residue field E. Then the Matlis duality functor $(-)^\vee = \operatorname{Hom}_R(-,E)$ is a faithful functor from the category of Noetherian R-modules to the category of Artinian R-modules. Furthermore, it is a fully faithful functor from the category of Artinian R-modules to the category of Noetherian $\widehat{R}$-modules. Additionally:*

- *For any Noetherian module M, we have a natural isomorphism $M^{\vee\vee} \cong \widehat{M}$ and the canonical maps from M to both of these commute with this isomorphism.*

Finally, if $R = \widehat{R}$ is complete, then the Matlis duality functor induces two anti-equivalences[1] of categories:

$$\left\{ \begin{array}{c} \textit{Noetherian} \\ R\textit{-modules} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Artinian} \\ R\textit{-modules} \end{array} \right\}.$$

(a) From Noetherian R-modules and Artinian R-modules.
(b) From Artinian R-modules to Noetherian R-modules.

PROOF. See [**BS98**, Theorem 10.2.12, Remark 10.2.18, Theorem 10.2.19].

The final statement, when R is complete, can also be found in [**Sta19**, Tag 08Z9] or [**BH93**, Chapter 3]. $\square$

Example 1.6. Using the first bulleted property of Theorem 1.5 we see that $\operatorname{Hom}_R(E,E) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(R,E),E) \cong \widehat{R}$.

Remark 1.7. In fact, $\operatorname{Hom}(-,E)$ is a faithfully exact functor in general. It is exact since E is injective and if $M \neq 0$, there exists an injection $R/I \hookrightarrow M$ into M for some ideal I. Hence

$$\operatorname{Hom}_R(M,E) \twoheadrightarrow \operatorname{Hom}_R(R/I,E)$$

[1]In other words, an equivalence of one category with the opposite of the other.

surjects and since $\mathrm{Hom}_R(R/I, E) \neq 0$, we see that $\mathrm{Hom}_R(M, E) \neq 0$.

The following explains Matlis duality for finite ring maps, something that will be quite useful for us in the case of the Frobenius.

Lemma 1.8 ([**Hoc11**, 3.3])**.** *Suppose $(R, \mathfrak{m}, k) \longrightarrow (S, \mathfrak{n}, l)$ is a finite local morphism of Noetherian local rings. Let E_R denote an injective hull of the residue field of R. Then $\mathrm{Hom}_R(S, E_R)$ is an injective hull of the residue field of S.*

In particular, if $S = R/J$, then the injective hull of the residue field of S is identified with $0 :_E J = \{z \in E \mid Jz = 0\} \subseteq E$.

2. Derived categories and relations between derived functors

We will assume that the reader has some familiarity with derived categories and will not provide a detailed introduction or definition here. See for instance [**Wei94**, Chapter 10], [**Har66**], or [**Huy06**].

For most of the text, the key point is that the objects in the derived category are complexes of R-modules or $\mathcal{O}_X$-modules, homotopic maps of complexes are declared to be the same, and quasi-isomorphisms of complexes (maps of complexes that are isomorphisms on cohomology) are formally inverted. For us **complexes** mean cochain complexes (that is, indices increase). , If $A^\bullet$ is in some derived category of an Abelian category, by $A^\bullet[1]$ we mean $A^\bullet$ shifted 1 to the left so that $\mathcal{H}^i(A^\bullet[1]) = \mathcal{H}^{i+1}(A^\bullet)$. Recall that if $A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1]$ is a triangle in our derived category, then we have a long exact sequence:

$$\ldots \longrightarrow \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet) \longrightarrow \mathcal{H}^i(C^\bullet) \longrightarrow \mathcal{H}^{i+1}(A^\bullet) \longrightarrow \ldots$$

In various parts of the book, we will consider derived functors of $\otimes, \mathrm{Hom}, \Gamma_I = \Gamma_{V(I)}, \Gamma(X, -)$ and more. The following is a key general fact about composition of derived functors.

Theorem 2.1 (Composition of derived functors)**.** *Suppose $F : \mathbf{A} \longrightarrow \mathbf{B}$ and $G : \mathbf{B} \longrightarrow \mathbf{C}$ are left-exact functors between Abelian categories and suppose that F sends injective objects to G-acyclic objects. Then $\mathbf{R}(G \circ F) = \mathbf{R}G \circ \mathbf{R}F$*

Example 2.2. Suppose that $f : Y \longrightarrow X$ is a map of schemes and $\mathscr{F}$ is a sheaf of $\mathcal{O}_X$-modules on Y. Then for every integer i we claim that

$$H^i(Y, \mathscr{F}) = \mathcal{H}^i \mathbf{R}\Gamma(X, \mathbf{R}f_* \mathscr{F}).$$

Indeed, since f_* sends injective sheaves to flasque sheaves, and since

$$\Gamma(Y, -) = \Gamma(X, f_* -)$$

we have that

$$\mathbf{R}\Gamma(Y, -) = \mathbf{R}\Gamma(X, \mathbf{R}f_* -)$$

by Theorem 2.1. Taking ith cohomology provides the desired result. In particular, if[2] $\mathbf{R}f_*\mathscr{F} \cong f_*\mathscr{F}$ and so $H^i(Y, \mathscr{F}) = H^i(X, f_*\mathscr{F})$ for all $i \in \mathbb{Z}$.

Similarly, if $Z \subseteq X$ is a closed subscheme (for instance $X = \operatorname{Spec} R$ for $(R, \mathfrak{m})$ local and $Z = V(\mathfrak{m})$)

$$\mathcal{H}^i_{\pi^{-1}Z}(Y, \mathscr{F}) = \mathcal{H}^i\mathbf{R}\Gamma_Z(X, \mathbf{R}\pi_*\mathscr{F}).$$

We recall the following relations between various common derived functors, in the case of rings.

Proposition 2.3 (Useful relations between derived functors for rings)**.** *Suppose R is a ring and X, Y, Z are complexes of R-modules.*

(a) *There is a canonical isomorphism in $D(R)$,*

$$\mathbf{R}\operatorname{Hom}_R(X, \mathbf{R}\operatorname{Hom}_R(Y, Z)) \cong \mathbf{R}\operatorname{Hom}_R(X \otimes^{\mathbf{L}} Y, Z)$$

which is functorial in X, Y, Z, [**Sta19**, Tag 0A65].

(b) *There is a canonical morphism*

$$\mathbf{R}\operatorname{Hom}_R(X, Y) \otimes^{\mathbf{L}} Z \longrightarrow \mathbf{R}\operatorname{Hom}_R(X, Y \otimes^{\mathbf{L}} Z)$$

which is functorial in X, Y, Z, [**Sta19**, Tag 0BYN]. *Furthermore, in the case that R is Noetherian, $X \in D^-_{\mathrm{coh}}(R)$, $Y \in D^+(R)$, and Z is isomorphic to a bounded complex of flat modules ($Y \in D^b_{\mathrm{fTd}}(R)$), this map is an isomorphism* [**Har66**, Chapter II, Proposition 5.14], [**Fox77**, Proposition 1.1].

(c) *There is a canonical morphism*

$$X \otimes^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(Y, Z) \longrightarrow \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(X, Y), Z)$$

which is functorial in X, Y, Z, [**Sta19**, Tag 0A67]. *Furthermore, in the case that R is Noetherian, $X \in D^-_{\mathrm{coh}}(R)$, $Y \in D^b(R)$ and Z is isomorphic to a bounded complex of injectives ($Z \in D^b_{\mathrm{fId}}(R)$), this map is an isomorphism* [**Fox77**, Proposition 1.1].

If $f : R \longrightarrow S$ is a map of rings and $U \in D(S)$, then we also have the following.

(d) *There is a canonical isomorphism*

$$\mathbf{R}\operatorname{Hom}_R(X, U) = \mathbf{R}\operatorname{Hom}_S(X \otimes^{\mathbf{L}} S, U)$$

functorial in X and U (here we view U as an R-module via f, we could have also written f_*U), [**Sta19**, Tag 0E1W].

[2] if X is a variety of characteristic zero, π is a resolution of singularities and $\mathscr{F} = \omega_Y$, this is the statement of Grauert-Riemenschneider vanishing

(e) *There is a canonical map*

$$\mathbf{R}\operatorname{Hom}_R(X,Y)\otimes^{\mathbf{L}}_R S \longrightarrow \mathbf{R}\operatorname{Hom}_S(X\otimes^{\mathbf{L}} S, Y\otimes^{\mathbf{L}} S)$$

functorial in X, Y, [**Sta19**, Tag 0E1X]. *It is an isomorphism if X is perfect or R' is perfect as an R-module (perfect means quasi-isomorphic to a bounded complex of projectives, for instance if R is regular of finite dimension)* [**Sta19**, Tag 0A6A]. *It is also an isomorphism if R' is a flat R-module, R is Noetherian, $X \in D^-_{\mathrm{coh}}(R)$ and $Y \in D^+(R)$,* [**Har66**, Chapter II, Proposition 5.8].

3. Dualizing complexes

For this, our primary reference is [**Har66**], however the local theory, which is most important for us, is also found here [**Sta19**, Tag 08XG].

Definition 3.1 (Dualizing complexes for rings)**.** For a Noetherian ring R, we say that a complex in $\omega_R^\bullet \in D^b_{\mathrm{coh}}(R)$[3] is a **dualizing complex (for R)** if it:

(a) has finite injective dimension (meaning it is quasi-isomorphic to a bounded complex of injectives) and,
(b) the natural map $R \longrightarrow \mathbf{R}\operatorname{Hom}_R(\omega_R^\bullet, \omega_R^\bullet)$ is a quasi-isomorphism.

If $\omega_R^\bullet$ is a dualizing complex on R, then the functor which sends $C^\bullet \in D(R)$ to $\mathbf{R}\operatorname{Hom}_R(C^\bullet, \omega_R^\bullet)$ is called the **Grothendieck duality functor** (with respect to $\omega_R^\bullet$). It is denoted by $\mathbf{D}(C^\bullet)$.

Applying Grothendieck duality twice returns you to where you began.

Lemma 3.2 ([**Sta19**, Tag 0A7C])**.** *Suppose R is a Noetherian ring with a dualizing complex $\omega_R^\bullet$. Then for any $C^\bullet \in D_{\mathrm{coh}}(R)$ we have that the natural map:*

$$C^\bullet \longrightarrow \mathbf{D}(\mathbf{D}(C^\bullet))$$

is a quasi-isomorphism.

Dualizing complexes do not always exist but do for most examples we care about.

Theorem 3.3 ([**Kaw02**])**.** *A Noetherian ring R has a dualizing complex if and only if R is the homomorphic image of a finite dimensional Gorenstein ring.*

Theorem 3.4 ([**Sta19**, Tag 0A80])**.** *Suppose R is a Noetherian ring with a dualizing complex. Then R is universally catenary and of finite dimension.*

[3]The bounded derived category of R-modules with finitely generated cohomology

If R is local and regular or more generally Gorenstein[4], then R itself is a dualizing complex. However, frequently we want to shift this dualizing complex.

Definition 3.5 (Normalized dualizing complexes)**.** Suppose $(R, \mathfrak{m})$ is a Noetherian local ring. A dualizing complex $\omega_R^\bullet$ for R is called **normalized** if $\mathcal{H}^{-i}(\omega_R^\bullet) = 0$ for $i > \dim R$ and $\mathcal{H}^{-d}(\omega_R^\bullet) \neq 0$.

For local rings, normalized dualizing complexes have cohomology starting in degree $-d$, and may have cohomology up to degree 0. For instance, this follows from local duality Theorem 6.1.

Example 3.6. For example, if k is a field and $R = k[\![x_1, \dots, x_d]\!]$, then $R[d]$ is a *normalized dualizing complex for* R. This choice of normalization is particularly valuable for local duality, see Section 6. Note in particular, k is a normalized dualizing complex over itself.

Lemma 3.7. *Suppose* $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ *is a finite map of Noetherian local rings (for instance if* S *is a quotient of* R*) and* $\omega_R^\bullet$ *is a normalized dualizing complex for* R*. Then*

$$\mathbf{R}\operatorname{Hom}_R(S, \omega_R^\bullet)$$

is a normalized dualizing complex for S*.*

Example 3.8. Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring with a dualizing complex $\omega_R^\bullet$ and $f \in R$ is a non-zerodivisor. By applying the Grothendieck duality functor $\mathbf{R}\operatorname{Hom}_R(-, \omega_R^\bullet)$ to the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot f} R \longrightarrow R/(f) \longrightarrow 0$$

and using Lemma 3.7, we obtain via the triangle of normalized dualizing complexes:

$$\omega_{R/(f)}^\bullet \longrightarrow \omega_R^\bullet \xrightarrow{\cdot f} \omega_R^\bullet \xrightarrow{+1} .$$

However, when taking cohomology, because these dualizing complexes are normalized, we have the long exact sequence:

$$0 \longrightarrow \mathcal{H}^{-d}(\omega_R^\bullet) \xrightarrow{\cdot f} \mathcal{H}^{-d}(\omega_R^\bullet) \longrightarrow \mathcal{H}^{-d+1}(\omega_{R/(f)}^\bullet) \longrightarrow \mathcal{H}^{-d+1}(\omega_R^\bullet) \longrightarrow \dots$$

where $d = \dim R$.

Lemma 3.9 ([**Sta19**, Tag 0A7G], Dualizing complexes and localization)**.** *If* R *is a ring with a dualizing complex* $\omega_R^\bullet$ *and* $W \subseteq R$ *is a multiplicative set, then*

$$W^{-1}\omega_R^\bullet \cong \omega_R^\bullet \otimes_R W^{-1}R$$

is a dualizing complex for $W^{-1}R$*.*

[4]A local ring R is Gorenstein if R has finite injective dimension

Caution 3.10 (Localization does not preserve *normalized* dualizing complexes)**.** Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring with normalized dualizing complex $\omega_R^\bullet$. Then for any $Q \in \operatorname{Spec} R$ we have that $(\omega_R^\bullet)_Q$ is a dualizing complex as above, but it is not normalized since R_Q has dimension lower than R. For instance, if k is a field and $R = k[\![x]\!]$ then, the $R[1]$ is a normalized dualizing complex. If $Q = (0)$ is the zero ideal then $R[1]_Q = R_Q[1]$ is not normalized since R_Q is zero dimensional.

Remark 3.11 (Normalized dualizing complexes on non-local rings)**.** For rings and schemes of finite type over a field, there is a choice of dualizing complex that is normalized after localizing at every *closed point*, the one we discuss below in Remark 4.4. However, if $(V, (t))$ is a DVR (for instance $V = k[t]_{(t)}$ or $V = k[\![t]\!]$) then $V[x]$ is an integral domain with maximal ideals of different heights (for instance (x, t) and $(xt - 1)$) and so there cannot be a dualizing complex for $V[x]$ that is normalized after localizing at each maximal ideal.

For rings R of finite type over a field k however, there is such a dualizing complex, namely if $S \longrightarrow R$ is a k-algebra surjection from a polynomial ring over k, then $\mathbf{R}\operatorname{Hom}_S(R, S) = \omega_R^\bullet$ is a dualizing complex with the property that for each maximal ideal $\mathfrak{m} \subseteq R$, we have that $\omega_R^\bullet \otimes_R R_\mathfrak{m}$ is a normalized dualizing complex for $R_\mathfrak{m}$.

Other common ring operations also behave well with respect to dualizing complexes.

Lemma 3.12 ([**Sta19**, Tag 0E4D])**.** *Suppose R is a ring with dualizing complex $\omega_R^\bullet$. Then $\omega_R^\bullet \otimes_R R[x]$ is a dualizing complex for $R[x]$. More generally, if $R \longrightarrow S$ is faithfully flat with Gorenstein fibers, then $\omega_R^\bullet \otimes_R S$ is a dualizing complex for S.*

Lemma 3.13 (Completion and dualizing complexes, [**Sta19**, Tag 0AWD] or [**Sta19**, Tag 0DWD])**.** *Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring with a dualizing complex $\omega_R^\bullet$. Then $\omega_R^\bullet \otimes_R \widehat{R}$ is a dualizing complex for the $\mathfrak{m}$-adic completion $\widehat{R}$.*

Lemma 3.14 (Dualizing complexes and homological conditions on rings)**.** *Suppose that $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring with a normalized dualizing complex $\omega_R^\bullet$. Suppose M is a finitely generated R-module. Then*

(a) *M is Cohen-Macaulay of dimension d if and only if the Grothendieck dual $\mathbf{D}(M) = \mathbf{R}\operatorname{Hom}_R(M, \omega_R^\bullet)$ is quasi-isomorphic to a module centered in degree $-d$. In particular, R is Cohen-Macaulay if and only if $\omega_R^\bullet \cong \mathcal{H}^{-d}\omega_R^\bullet[d]$, see* [**Sta19**, Tag 0B5A].
(b) *More generally, M has depth $\geq i$ if and only if $\mathcal{H}^0\mathbf{D}(M), \ldots, \mathcal{H}^{-i+1}\mathbf{D}(M)$ are all zero.* [**Sta19**, Tag 0DWZ]

(c) R is Gorenstein if and only if $\omega_R^\bullet \cong R[d]$, [**Sta19**, Tag 0DW9].
(d) R is quasi-Gorenstein if and only if $\mathcal{H}^{-d}\omega_R^\bullet \cong R$.

The following can be thought of as a generalization of the description of the dualizing complex in a Cohen-Macaulay ring.

Proposition 3.15 (*cf.* [**KK20**, Proposition 8.1])**.** *Suppose $(R, \mathfrak{m})$ is a d-dimensional equidimensional Noetherian local ring with a normalized dualizing complex $\omega_R^\bullet$. We have that $(R, \mathfrak{m})$ is S_n for some $n \leq d$ if and only if*

$$\dim \operatorname{Supp} \mathcal{H}^{-d+i}\omega_R^\bullet \leq d - i - n$$

for every $i > 0$. Here we interpret the dimension of the empty set to be $-\infty$.

Proof. Suppose first that $(R, \mathfrak{m})$ is S_n. Fix $i > 0$ and choose some $Q \in \operatorname{Spec} R$ of height $k < d - (d - i - n) = i + n$. We need to show that $(\mathcal{H}^{-d+i}\omega_R^\bullet)_Q = 0$. Now, R_Q has dimension k and depth at least $\min(k, n)$ by hypothesis. Since R is equidimensional, we see that $(\omega_R^\bullet)_Q[-d+k]$ is then a normalized dualizing complex for R_Q, and so we have, for $j < \min(k, n)$ that

$$0 = \mathcal{H}^{-j}((\omega_R^\bullet)_Q[-d+k]) = \mathcal{H}^{-d+k-j}(\omega_R^\bullet)_Q.$$

by Lemma 3.14 (b). In particular, if we pick $j = k - i$, then we see that both $j < k$ and $j < i + n - i = n$ (since $k < i + n$) and our desired vanishing is satisfied.

For the converse, suppose that $Q \in \operatorname{Spec} R$ is of height k (and so of co-height $d - k$). We must show that R_Q has depth at least $\min(k, n)$. Since $\dim \operatorname{Supp} \mathcal{H}^{-d+i}\omega_R^\bullet \leq d - i - n$ for all $i > 0$, we see that

$$\dim \operatorname{Supp}(\mathcal{H}^{-d+i}\omega_R^\bullet)_Q \leq d - i - n - (d - k) = k - i - n.$$

In particular, if $i > k - n$ (or equivalently if $i - d > k - n - d$) then

$$(\mathcal{H}^{-d+i}\omega_R^\bullet)_Q = 0.$$

Again, since R is equidimensional, we see that $(\omega_R^\bullet)_Q[-d+k]$ is a normalized dualizing complex for R_Q. Hence for $j < \min(k, n)$, we have that $j - k < 0$ so that $k - j > 0$ and also that $k - j > k - n$, and so

$$\mathcal{H}^{-j}\big((\omega_R^\bullet)_Q[-d+k]\big) = \mathcal{H}^{-d+(k-j)}(\omega_R^\bullet)_Q = 0$$

which is what we wanted to prove by Lemma 3.14 (b). □

Example 3.16. The previous result needs the equidimensionality of R. Indeed, suppose $R = k[\![x, y, z]\!]/(xz, yz)$ with normalized dualizing complex $\omega_R^\bullet$. Then $\dim \operatorname{Supp} \mathcal{H}^{-2}\omega_R^\bullet = 2$ and $\dim \operatorname{Supp} \mathcal{H}^{-1}\omega_R^\bullet = 1$ since there are components of those dimensions. However, it is true that $\mathcal{H}^0\omega_R^\bullet = 0$ so that $\dim \operatorname{Supp} \mathcal{H}^0\omega_R^\bullet = -\infty$.

Corollary 3.17. *Suppose* $(R, \mathfrak{m})$ *is a d-dimensional equidimensional Noetherian local ring with a normalized dualizing complex* $\omega_R^{\bullet}$*. Then the non-Cohen-Macaulay locus of* $\operatorname{Spec} R$ *is the support of* $M := \bigoplus_{i=0}^{d-1} \mathcal{H}^{-i}\omega_R^{\bullet}$*. In particular, if* $I = \operatorname{Ann}_R M$ *then* $V(I)$ *defines the non-Cohen-Macaulay locus of* $\operatorname{Spec} R$*.*

The reason that the equidimensional hypothesis is needed is that if $\operatorname{Spec} R$ has irreducible components of different dimensions, say $d' < d$ with corresponding ideal $I' \subseteq R$ defining such a closed component, then there is a map of normalized dualizing complexes $\omega_{R/I'}^{\bullet} \longrightarrow \omega_R^{\bullet}$ which is an isomorphism at the generic points of $\operatorname{Spec} R/I'$. Of course, R/I' is generically Cohen-Macaulay (as is every ring) but because of that isomorphism, the dualizing complex will live in the wrong degrees relative to d to detect that.

Definition 3.18 (Dualizing complexes for schemes)**.** For a Noetherian scheme X, we say that a complex in $\omega_X^{\bullet} \in D^b_{\mathrm{coh}}(X)$[5] is a **dualizing complex for** X if it:

(a) has finite injective dimension[6] and,
(b) the natural map $\mathcal{O}_X \longrightarrow \mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X^{\bullet}, \omega_X^{\bullet})$ is a quasi-isomorphism.

Lemma 3.19 (Uniqueness of dualizing complexes, [**Har66**, Chapter V, Theorem 3.1])**.** *On a connected Noetherian scheme, if* $D^{\bullet}, E^{\bullet}$ *are both dualizing complexes, then there exists a line bundle* $\mathscr{L}$ *and an integer n such that*

$$D^{\bullet}[n] \cong E^{\bullet} \otimes \mathscr{L}.$$

In particular, if $X = \operatorname{Spec}(R, \mathfrak{m})$ *is a local scheme, then dualizing complexes are unique up to quasi-isomorphism and shift.*

We also have the following global version of Lemma 3.14.

Lemma 3.20 (Dualizing complexes and homological conditions on rings)**.** *Suppose that X is a Noetherian connected locally equidimensional scheme with a dualizing complex* $\omega_X^{\bullet}$*. Then*

(a) X is Cohen-Macaulay if and only if $\omega_X^{\bullet}$ is quasi-isomorphic to a module centered in a single degree.
(b) X is Gorenstein if and only if $\omega_X^{\bullet}$ is quasi-isomorphic to a line bundle centered in a single degree.
(c) X is quasi-Gorenstein if and only if the first non-zero cohomology of $\omega_X^{\bullet}$ is a line bundle.

[5]The bounded derived category of $\mathcal{O}_X$-modules with coherent cohomology

[6]Either in the category of $\mathcal{O}_X$-modules or equivalently the category of quasi-coherent $\mathcal{O}_X$-modules, see [**Har66**, Chapter II, Proposition 7.20]

4. Grothendieck duality

Theorem 4.1 (Grothendieck duality). *Suppose $f : Y \longrightarrow X$ is a finite type separated morphism of Noetherian schemes. Then there is a functor between the triangulated categories*

$$f^! : D^+_{\mathrm{qcoh}}(X) \longrightarrow D^+_{\mathrm{qcoh}}(Y)$$

which commutes with localization. Furthermore, if f is proper, it is right adjoint to $\mathbf{R}f^*$ in the following sense. There is a functorial isomorphism:

$$\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathbf{R}f_*A^\bullet, B^\bullet) \cong \mathbf{R}f_*\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_Y}(A^\bullet, f^!B^\bullet)$$

for $A^\bullet \in D^-_{\mathrm{coh}}(Y)$ and $B^\bullet \in D_{\mathrm{qcoh}}(X)$.

Remark 4.2 ($f^!$ in key cases). There are two key cases where the functor $f^!$ is particularly easy to understand.

Suppose $f : Y \longrightarrow X$ is a *finite map.* Then f_* induces an equivalence of categories between quasi-coherent $\mathcal{O}_Y$ modules on Y and quasi-coherent $f_*\mathcal{O}_Y$-modules on X, and their derived categories see [**Sta19**, Tag 0AVW]. By utilizing this equivalence, we may identify $f^!-$ with the functor $\mathbf{R}\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, -)$ by [**Har66**, Chapter III] or [**Sta19**, Tag 0AX2] (here we use this equivalence of categories to take sheaves on X to sheaves on Y). For maps between affine schemes $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ (corresponding to a finite ring map $R \longrightarrow S$) this is even easier. In that case we may identify $f^! : D^b_{\mathrm{qcoh}}(R) \longrightarrow D^b_{\mathrm{qcoh}}(S)$ with $\mathbf{R}\operatorname{Hom}_R(S, -)$.

On the other hand if $f : Y \longrightarrow X$ is a *smooth map* of relative dimension n, then $f^!-$ can be identified with $\Omega^n_{Y/X}[n] \otimes f^*-$ [**Har66**, Chapter III]. Combining these two facts lets us define $f^!$ for projective morphisms since given $Z \xrightarrow{g} Y \xrightarrow{f} X$ finite type maps between separated Noetherian schemes, we have

$$g^! \circ f^! = (f \circ g)^!. \tag{4.2.1}$$

The other important thing to know about is $f^!$ is that it take dualizing complexes to dualizing complexes.

Theorem 4.3 ([**Har66**], [**Sta19**, Tag 0AU3]). *Suppose that $f : X \longrightarrow Y$ is a finite type separated morphism of Noetherian schemes and that $\omega^\bullet_Y$ is a dualizing complex on Y. Then $f^!\omega^\bullet_Y$ is a dualizing complex on X. In particular, if f is finite (for instance a closed immersion), then*

$$\mathbf{R}\operatorname{Hom}_Y(f_*\mathcal{O}_X, \omega^\bullet_Y)$$

is the pushforward of a dualizing complex on X via the finite affine map f.

Remark 4.4 (Dualizing complexes when finite type over a field)**.** Because of this for schemes X of finite type over a field k, one always picks

$$\omega_X^{\bullet} = f^! k$$

Thus for varieties over a field, in view of (4.2.1), we always have a canonical choice of dualizing complex which is compatible with maps between varieties.

For instance, suppose $j : X \longrightarrow W$ is a finite map of varieties over a field and W is a smooth variety (for instance $X \subseteq \mathbb{P}^n_k = W$). Suppose $f : X \longrightarrow \operatorname{Spec} k$ and $g : W \longrightarrow \operatorname{Spec} k$ are the structural maps:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & \operatorname{Spec} k. \end{tikzcd}$$

Then we see that $\omega_W^{\bullet} = \Omega^n_{W/k}[n] = g^! k$. Hence by (4.2.1) we see that

$$\begin{aligned} j_* \omega_X^{\bullet} = {} & j_* f^! k \\ = {} & j_* (g \circ j)^! k \\ = {} & j_* j^! g^! k \\ = {} & j_* j^! \omega_W^{\bullet} \\ = {} & \mathbf{R} \mathscr{H}\mathrm{om}_{\mathcal{O}_W}(j_* \mathcal{O}_X, \omega_W^{\bullet}) \\ = {} & \mathbf{R} \mathscr{H}\mathrm{om}_{\mathcal{O}_W}(j_* \mathcal{O}_X, \omega_W^{\bullet}). \end{aligned}$$

Since j is a finite and hence affine map, we typically leave off the leading j_* and simply write

$$\omega_X^{\bullet} = \mathbf{R} \mathscr{H}\mathrm{om}_{\mathcal{O}_W}(j_* \mathcal{O}_X, \omega_W^{\bullet}).$$

Remark 4.5 (Dualizing complexes for schemes of finite type over an F-finite field)**.** Suppose that X is a quasi-projective scheme over an F-finite field k with structural map $f : X \longrightarrow \operatorname{Spec} k$. Consider the following diagram with horizontal maps the e-iterated Frobenius:

$$\begin{tikzcd} X \arrow[r, "F^e"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, "F^e"'] & \operatorname{Spec} k. \end{tikzcd}$$

A direct computation shows that $(F^e)^! k = k$ (this is just the observation that $\operatorname{Hom}_k(F^e_* k, k) \cong F^e_* k$ since both are rank-1 vector spaces). Thus we see that

$$\omega_X^{\bullet} = f^! k = (F^e \circ f)^! k = (f \circ F^e)^! k = (F^e)^! \omega_X^{\bullet}.$$

In other words:

$$F^e_* \omega_X^{\bullet} \cong \mathbf{R} \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_X, \omega_X^{\bullet})$$

and taking cohomology (assuming X is a variety and hence connected and equidimensional'') we obtain:

$$F^e_* \omega_X \cong \mathscr{H}\mathrm{om}(F^e_* \mathcal{O}_X, \omega_X)$$

Remark 4.6 (Dualizing complexes when finite type over a nice ring)**.** More generally, if $f : X \longrightarrow \operatorname{Spec} S$ is a finite type map and R has a particularly obvious choice of dualizing complex $\omega_R^\bullet$ (for instance, if S is a Gorenstein local ring one might pick $\omega_S^\bullet = S[\dim S]$) then we set $\omega_X^\bullet = f^! \omega^\bullet_{\operatorname{Spec} S}$. For instance, suppose that $R = S/I$ and $X = \operatorname{Spec} R$ with the induced closed immersion into $\operatorname{Spec} S$. In view of our description of $f^!$ for finite maps, we have that

$$\omega_R^\bullet = \mathbf{R}\operatorname{Hom}_S(R, \omega_S^\bullet). \tag{4.6.1}$$

The most common case is when $(R, \mathfrak{m})$ is a regular local ring in characteristic $p > 0$ which is then a quotient of $S = k[\![x_1, \ldots, x_n]\!]$ and we have that

$$\omega_R^\bullet = \mathbf{R}\operatorname{Hom}_S(R, S[n]).$$

Regardless, we have the following special case of Grothendieck duality:

Theorem 4.7 (Grothendieck duality with dualizing complexes)**.** *Suppose that $f : Y \longrightarrow X$ is a proper morphism of Noetherian schemes. Fix $\omega_X^\bullet$ a dualizing complex on X and $\omega_Y^\bullet = f^! \omega_X^\bullet$. In this case we have*

$$\mathbf{R}\,\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathbf{R} f_* A^\bullet, \omega_X^\bullet) \cong \mathbf{R} f_* \mathbf{R}\,\mathscr{H}\mathrm{om}_{\mathcal{O}_Y}(A^\bullet, \omega_Y^\bullet)$$

for $A^\bullet \in D^-\mathrm{coh}(Y)$.

5. Canonical sheaves and modules

Definition 5.1 (Canonical sheaves and modules)**.** Suppose $(R, \mathfrak{m})$ is a Noetherian d-dimensional local ring with normalized dualizing complex $\omega_R^\bullet$. If R is equidimensional, then we call $\omega_R = \mathcal{H}^{-d}(\omega_R^\bullet)$ the **canonical module**.

Suppose X is a Noetherian and locally equidimensional connected scheme with a fixed dualizing complex $\omega_X^\bullet$. Then we call the first non-zero cohomology

$$\omega_X := \mathcal{H}^i \omega_X^\bullet$$

the associated **canonical sheaf**. If $X = \operatorname{Spec} R$, then the associated R-module is called the **canonical module**.

Remark 5.2. If $(R, \mathfrak{m})$ is not equidimensional, then $\mathcal{H}^{-d}(\omega_R^\bullet)$ is supported on those components of $\operatorname{Spec} R$ that have dimension d, and is zero on the other components [**Sta19**, Tag 0AWN]. In particular, if X is not locally equidimensional, there cannot be a canonical *sheaf* that is defined locally corresponding to the first non-zero cohomology of a dualizing complex. Hence we avoid using the terms *canonical module* and *canonical sheaf* in such cases.

Remark 5.3. Suppose X is connected, equidimensional, and finite type over a field k (so that locally equidimensional and equidimensional coincide). Set $h : X \longrightarrow \operatorname{Spec} k$ to be the structural morphism and set $\omega_X^{\bullet} = h^! k$ as before. Then the canonical module of X is simply

$$\omega_X := \mathcal{H}^{-\dim X} \omega_X^{\bullet}.$$

This follows from Remark 4.4.

In view of Lemma 3.19, since dualizing complexes are only unique up to twisting by a line bundle, this might seem to not be a very useful notion. However, for schemes X of finite type over a field k, there are other approaches as we shall see.

The following fact about canonical modules will be used frequently.

Lemma 5.4 ([**Sta19**, Tag 0AWN])**.** *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring with a canonical module ω_R. Then ω_R is* S_2 *as an R-module and is supported on all of* $\operatorname{Spec} R$.

Corollary 5.5 ([**Har07**, Proof of Proposition 1.5])**.** *Suppose $(R, \mathfrak{m})$ is* S_1 *with canonical module ω_R. For any finitely generated R-module M, the module*

$$\operatorname{Hom}_R(M, \omega_R) = M^{\vee_\omega}$$

is S_2.

We next record a transformation rule for canonical modules with respect to finite ring maps.

Proposition 5.6. *Suppose $R \subseteq S$ is a finite inclusion of locally equidimensional rings with connected spectra. Suppose ω_R is a canonical module for R. Then*

$$\operatorname{Hom}_R(S, \omega_R)$$

is a canonical module for S. It comes with an evaluation map

$$\omega_S = \operatorname{Hom}_R(S, \omega_R) \xrightarrow{\textit{eval@1}} \omega_R$$

which may be identified with the map $\operatorname{Hom}_R(S, \omega_R) \longrightarrow \operatorname{Hom}_R(R, \omega_R)$ *and hence is the* **Grothendieck dual of** $R \hookrightarrow S$.

Proof. Suppose $\omega_R = \mathcal{H}^i \omega_R^{\bullet}$ for our fixed dualizing complex. We know that $\mathbf{R}\operatorname{Hom}_R(S, \omega_R^{\bullet})$ is a dualizing complex for S. The first degree that $\mathbf{R}\operatorname{Hom}_R(S, \omega_R^{\bullet})$ can possibly have cohomology in is degree i. Furthermore, since the inclusion is finite, we have that $\operatorname{Hom}(S, \omega_R) \neq 0$. The result follows. $\square$

We also have the following.

Proposition 5.7. *With notation as in Proposition 5.6, If R is S_1 and S is S_2, then the map $T : \omega_S \longrightarrow \omega_R$ from above, the Grothendieck dual of $R \subseteq S$, generates $\mathrm{Hom}_R(\omega_S, \omega_R)$ as an S-module.*

PROOF. By [**Har07**, Proposition 1.5], or by Proposition 6.9 below, the natural map $\mu : S \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(S, \omega_R), \omega_R)$ is an isomorphism. By construction, μ sends 1 to the $(-)^{\vee_\omega}$ of $R \hookrightarrow S$. $\square$

We also record the following basic properties of canonical modules.

Lemma 5.8. *Suppose ω_R is a canonical module for a Noetherian ring R.*

(a) If $W \subseteq R$ is a multiplicative set, the $W^{-1}\omega_R$ is a canonical module for $W^{-1}R$.
(b) If $(R, \mathfrak{m})$ is local, then the $\mathfrak{m}$-adic completion of ω_R is a canonical module for $\widehat{R}$.

PROOF. These follow from the corresponding statements for dualizing complexes, Lemma 3.9 and Lemma 3.13. $\square$

5.1. Alternate definition of canonical sheaves and modules. There is another important way to compute the canonical module for varieties over a field, which essentially follows immediately from our description for $f^!$ for smooth maps Remark 4.2 plus Lemma 5.4.

Proposition 5.9 (Dualizing complex via differential forms)**.** *Suppose X is a normal variety over a perfect field of characteristic $p > 0$. Then $\omega_X = \left(\Omega^{\dim X}_{X/k}\right)^{\mathrm{S}_2}$ is a canonical sheaf for X. In particular, if $i : U \longrightarrow X$ is the inclusion of the smooth locus of X then $i_*\Omega^{\dim X}_{U/k}$ is a canonical sheaf for X.*

6. Local duality and consequences

Local duality relates the dualizing complex of a local ring with local cohomology.

Theorem 6.1 (Local duality)**.** *Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring with normalized dualizing complex $\omega_R^{\bullet}$ and injective hull of the residue field E. For any $A^{\bullet} \in D^b_{\mathrm{coh}}(R)$ we have a functorial isomorphism in the derived category:*

$$\mathbf{R}\Gamma_m(A^{\bullet}) \cong \mathrm{Hom}_R(\mathbf{R}\,\mathrm{Hom}_R(A^{\bullet}, \omega_R^{\bullet}), E) = \left(\mathbf{R}\,\mathrm{Hom}_R(A^{\bullet}, \omega_R^{\bullet})\right)^{\vee}.$$

Here $\vee$ denotes Matlis duality as above. Alternately, if $R = \widehat{R}$ is complete, then for any $A^{\bullet} \in D_{\mathrm{coh}}(R)$, we have that

$$\big(\mathbf{R}\Gamma_m(A^{\bullet}\,)\big)^{\vee} \cong \mathbf{R}\operatorname{Hom}_R(A^{\bullet}\,, \omega_R^{\bullet}).$$

We also state the following variant that avoids the notation of derived categories.

Corollary 6.2 (Local duality)**.** *Suppose that S is a Gorenstein local ring of dimension n and $(R = S/I, \mathfrak{m})$ a quotient ring, and M is a finitely generated R-module. Then for all $i \in \mathbb{Z}$,*

$$\big(\operatorname{Ext}_S^{n-i}(M, S)\big)^{\vee} = H^i_{\mathfrak{m}}(M).$$

Restricting to the case where $A^{\bullet} = R$, we have the following special cases relating the dualizing complex and canonical modules with local cohomology.

Lemma 6.3 (The dualizing complex and local cohomology)**.** *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring with normalized dualizing complex $\omega_R^{\bullet}$. Then*

$$\big(\omega_R^{\bullet}\big)^{\vee} \cong \mathbf{R}\Gamma_{\mathfrak{m}}(R) \quad \textit{and} \quad \big(\mathbf{R}\Gamma_{\mathfrak{m}}(R)\big)^{\vee} \cong \omega_R^{\bullet} \otimes_R \widehat{R} = \omega_{\widehat{R}}^{\bullet}.$$

Lemma 6.4 (The canonical module and local cohomology)**.** *Suppose $(R, \mathfrak{m})$ is a d-dimensional equidimensional Noetherian local ring with a canonical module ω_R. Then the Matlis dual of the canonical module is the top local cohomology module of R. That is:*

$$\omega_R^{\vee} \cong H^d_{\mathfrak{m}}(R) \quad \textit{and} \quad \big(H^d_{\mathfrak{m}}(R)\big)^{\vee} \cong \omega_{\widehat{R}}.$$

Because of this, in much of the literature, a canonical module is *defined* to be any finitely generated R-module whose Matlis dual is isomorphic to $H^d_{\mathfrak{m}}(R)$ (note this makes sense even if R does not have a dualizing complex). We will work in the case that R has a dualizing complex though as that covers most cases of interest to us, notably F-finite rings and complete rings.

Combining local duality with Corollary 3.17 we obtain the following.

Corollary 6.5. *Suppose that $(R, \mathfrak{m})$ is a d-dimensional equidimensional Noetherian local ring with a dualizing complex. Then the ideal*

$$\operatorname{Ann}_R\Big(\bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(R)\Big)$$

defines the non-Cohen-Macaulay locus of $\operatorname{Spec} R$*. In particular, its annihilator is a proper ideal of* $\operatorname{Spec} R$*.*

We also obtain the following characterization of (quasi-)Gorenstein rings.

Corollary 6.6. *Suppose that $(R, \mathfrak{m})$ is a d-dimensional equidimensional Noetherian local ring with a dualizing complex and E is an injective hull of the residue field. Then*

(a) *R is quasi-Gorenstein if and only if $H^d_{\mathfrak{m}}(R)$ has a 1-dimensional socle*[7]*, in which case $H^d_{\mathfrak{m}}(R) \cong E$.*
(b) *R is Gorenstein if and only if $\mathbf{R}\Gamma_{\mathfrak{m}}(R) \cong E[-d]$.*

We also have the following interpretations of depth.

Proposition 6.7. *Suppose $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring with a dualizing complex $\omega_R^{\bullet}$ and M is a finitely generated R-module.. Then the following are equivalent:*

(a) $\operatorname{depth} M \geq t$.
(b) *$H^i_{\mathfrak{m}}(M) = 0$ for $i = 0, \ldots, t-1$.*
(c) *$\mathcal{H}^{-i}\mathbf{R}\operatorname{Hom}_R(M, \omega_R^{\bullet}) = 0$ for $i = 0, \ldots, t-1$.*

In particular, if R is Cohen-Macaulay so that $\omega_R^{\bullet} = \omega_R[d]$, then $\operatorname{depth} M \geq t$ *if and only if $\operatorname{Ext}^{d-i}_R(M, \omega_R) = 0$ for $i = 0, \ldots t-1$.*

It is worth noting that $H^i_{\mathfrak{m}}(M) = 0$ or dually that $\mathcal{H}^{-i}\mathbf{R}\operatorname{Hom}_R(M, \omega_R^{\bullet}) = 0$ for $i > \dim M$.

Proof. See [**Sta19**, Tag 0DWZ]. Note (b) and (c) are simply Matlis dual to each other. ☐

Thanks to our interpretation of depth above, it is not difficult to show the following, although we simply cite [**BH93**].

Corollary 6.8 ([**BH93**, Theorem 3.3.10])**.** *Suppose $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with canonical module ω_R. Suppose* $\operatorname{depth} M = \dim R$, *that is that M is (maximal) Cohen-Macaulay. Then* $\operatorname{Hom}_R(M, \omega_R)$ *is also Cohen-Macaulay.*

What we have done so far also lets us define a more general S_2-ification functor than what was introduced in Appendix B Definition 4.10 when R was normal.

Proposition 6.9 (General S_2-ification)**.** *Suppose $(R, \mathfrak{m})$ is a S_1 local ring with canonical module ω_R. Then for any R-module M, we have that*

$$M^{\vee_\omega \vee_\omega} = \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$$

[7]The socle of a module over a local ring $(R, \mathfrak{m})$ is the submodule annihilated by $\mathfrak{m}$.

is S_2*. Furthermore, if* M *has depth* 1 *in codimension* 1*, for instance if* M *is* S_1*, the canonical map* $M \longrightarrow M^{\vee_\omega \vee_\omega}$ *is an isomorphism in codimension 1, and so may be viewed as the* S_2*-ification map of Definition 4.10 if* R *is normal. In particular, it is an isomorphism if* M *is* S_2*.*

Finally, in the case that $M = R$*, we have that* $\operatorname{Hom}_R(\omega_R, \omega_R)$ *is the ring-theoretic* S_2*-ification of* R*.*

PROOF. Notice that $M^{\vee_\omega \vee_\omega}$ is S_2 by Corollary 5.5. To complete the proof, it suffices to prove that $M \longrightarrow M^{\vee_\omega \vee_\omega}$ is an isomorphism in codimension 1 assuming that $M_{\mathfrak{q}}$ has depth 1 for each height one prime $\mathfrak{q} \subseteq R$. Hence we may assume that R is 1-dimensional and that both R, M, and $M^{\vee_\omega}$ are Cohen-Macaulay since they are both S_1. But now $\operatorname{Hom}_R(M, \omega_R) \cong \mathbf{R}\operatorname{Hom}_R(M, \omega_R)$ and likewise we have $\operatorname{Hom}_R(M^{\vee_\omega}, \omega_R) = \mathbf{R}\operatorname{Hom}_R(M^{\vee_\omega}, \omega_R)$ by Proposition 6.7. The first result follows.

Finall, when $M = R$, we see that $\operatorname{Hom}_R(\omega_R, \omega_R)$ is an S_2 R-module. It has a ring structure under composition which commutes with localization and since it is S_2, the canonical localization map $\operatorname{Hom}_R(\omega_R, \omega_R) \longrightarrow \operatorname{Hom}_{\mathcal{K}(R)}(\mathcal{K}(R), \mathcal{K}(R)) \cong \mathcal{K}(R)$ is an injection where $\mathcal{K}(R)$ is the total ring of fractions of R. □

For any finitely generated R-module M, the natural transformation of functors $\mathrm{id}(-) \mapsto (-)^{\vee_\omega \vee_\omega} =: (-)^{S_2}$ can be viewed as an S_2-ification. While we will generally work in the normal case, this does appear in more general settings [].

Lemma 6.10. *Suppose* $(R, \mathfrak{m})$ *is a quasi-Gorenstein local ring,* $f \in \mathfrak{m}$ *is a regular element, and* $R/(f)$ *is* S_2*. Then* $R/(f)$ *is also quasi-Gorenstein.*

PROOF. By applying Grothendieck duality to the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot f} R \longrightarrow R/(f) \longrightarrow 0$$

we obtain

$$0 \longrightarrow \mathcal{H}^{-d}\omega_R^{\bullet} \xrightarrow{\cdot f} \mathcal{H}^{-d+1}\omega_R^{\bullet} \xrightarrow{\kappa} \mathcal{H}^{-d}\omega_{R/(f)}^{\bullet} \longrightarrow \mathcal{H}^{-d+1}\omega_R^{\bullet}$$

where we have zeros on the right for dimension reasons (the Matlis dual is $H^d_{\mathfrak{m}}(R/(f))$, but $R/(f)$ has dimension $d-1$). By assumption, $\mathcal{H}^{-d}(\omega_R^{\bullet}) = \omega_R \cong R$, and so it suffices to prove that the map labelled κ are surjective. Of course, if R was Cohen-Macaulay, we would be done, snce the right most column would be zero (its Matlis dual is $H^{d-1}_{\mathfrak{m}}(R)$). Since $R/(f)$ is S_2, $R/(f)$ is Cohen-Macaulay in codimension 2, and so $R \twoheadrightarrow R/(f) \hookrightarrow \omega_{R/(f)}$ is surjective at codimension-1 points of $\operatorname{Spec} R/(f)$, hence $R/(f) \longrightarrow \omega_{R/(f)}$ is an isomorphism at those codimension-1 points, and thus it is an isomorphism since both modules are S_2, see Lemma 4.8. This proves that $R/(f)$ is quasi-Gorenstein. $\square$

Notice that local duality also implies that

$$\mathbf{R}\Gamma_{\mathfrak{m}}(\omega_R^{\bullet}) = E$$

since $\mathbf{R}\operatorname{Hom}_R(\omega_R^{\bullet}, \omega_R^{\bullet}) = R$. If R is S_2, for instance if it is normal, then a spectral sequence argument also yields the following.

Corollary 6.11. *Suppose $(R, \mathfrak{m})$ is Noetherian d-dimensional, local, S_2 and has a normalized dualizing complex $\omega_R^{\bullet}$. Then $\mathcal{H}^d(\omega_R) = E$.*

Proof. Notice that $\mathcal{H}^0(\mathbf{R}\Gamma_{\mathfrak{m}}(\omega_R^{\bullet})) \cong E$. We consider the associated E_2 spectral sequence:

$$\begin{tikzcd}
0 & 0 & 0 & \cdots \\
H^d_{\mathfrak{m}}(\mathcal{H}^{-d}\omega_R^{\bullet}) \arrow[drr] & 0 & 0 & \cdots \\
H^{d-1}_{\mathfrak{m}}(\mathcal{H}^{-d}\omega_R^{\bullet}) & 0 & 0 & \cdots \\
H^{d-2}_{\mathfrak{m}}(\mathcal{H}^{-d}\omega_R^{\bullet}) & H^{d-2}_{\mathfrak{m}}(\mathcal{H}^{-d+1}\omega_R^{\bullet}) & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{tikzcd}$$

where the zeros are due the dimension of the support of the dualizing complex in the S_2 case, Proposition 3.15. The statement follows since the spectral sequence converges to $\mathcal{H}^{i+j}(\mathbf{R}\Gamma_{\mathfrak{m}}(\omega_R^{\bullet}))$. $\square$

7. Other useful results on local cohomology

Lemma 7.1. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d and*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow C \longrightarrow 0$$

is an exact sequence sequence of finitely generated R-modules such that $\dim \operatorname{Supp} K, \dim \operatorname{Supp} C \leq d - 2$. Then

$$H^d_{\mathfrak{m}}(M) \longrightarrow H^d_{\mathfrak{m}}(N)$$

is an isomorphism.

In particular, if R is normal and M is a finitely generated module such that $\operatorname{Supp} M = \operatorname{Spec} R$, *then*

$$H^d_{\mathfrak{m}}(M) \longrightarrow H^d_{\mathfrak{m}}(M^{\mathrm{S}_2})$$

is an isomorphism.

Proof. Form the short exact sequences $0 \longrightarrow K \longrightarrow M \longrightarrow B \longrightarrow 0$ and $0 \longrightarrow B \longrightarrow N \longrightarrow C \longrightarrow 0$ and use the long exact sequence of local cohomology, noting that

$$0 = H^{d-1}_{\mathfrak{m}}(K) = H^d_{\mathfrak{m}}(K) = H^{d-1}_{\mathfrak{m}}(C) = H^d_{\mathfrak{m}}(C).$$

□

Lemma 7.2. *If $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring. Then for any R-modules M, N we have a canonical homomorphism*

$$H^d_{\mathfrak{m}}(M) \otimes N \cong H^d_{\mathfrak{m}}(M \otimes N).$$

In particular, if $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ is a local homomorphism between d-dimensional Noetherian local rings, then we have a canonical isomorphism

$$H^d_{\mathfrak{m}}(R) \otimes_R S \cong H^d_{\mathfrak{m}S}(S) \cong H^d_{\mathfrak{n}}(S).$$

Proof. If we choose a system of parameters $(x_1, \ldots, x_d)$ for $\mathfrak{m} \subseteq R$, we see that

$$H^d_{\mathfrak{m}}(M) \cong \operatorname{coker}\left(\bigoplus_{i=1}^d M_{\widehat{x_i}} \longrightarrow M_{x_1 \cdots x_d}\right)$$

where $M_{\widehat{x_i}}$ is $M_{x_1 \cdots x_{i-1} x_i \cdots x_d}$. Since localization commutes with tensor products, we see immediately that

$$H^d_{\mathfrak{m}}(M) \otimes N \cong H^d_{\mathfrak{m}}(M \otimes N).$$

The question is whether this isomorphism is independent of the choice of x_i. Fix y such that $y_1 = yx_1, y_2 = x_2, \ldots, y_d = x_d$ also form a system of parameters. It is easy to see that there is a commutative diagram

$$\begin{tikzcd} \bigoplus_{i=1}^d M_{\widehat{x_i}} \arrow[r] \arrow[d] & M_{x_1 \cdots x_d} \arrow[d] \\ \bigoplus_{i=1}^d M_{\widehat{y_i}} \arrow[r] & M_{y_1 \cdots y_d}. \end{tikzcd}$$

Hence there is a canonical map (which is clearly an isomorphism) between the two cokernels, this is preserved after tensoring with N. □

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