Report on "The *FrobeniusThresholds* package for *Macaulay2*" by Hernández, et al, for JSAG

October 21, 2019

The paper is a well-written, clear, and thorough overview of a highly interesting Macaulay2 package for Frobenius thresholds, including mathematical, computational, and usage aspects (what the package does and what it means; how it does it; how to use it). The package is a valuable toolkit that will be useful for many users of Macaulay2 who are interested in positive characteristic algebra and algebraic geometry. *I recommend accepting the paper for publication, after minor revisions.* These revisions include some debugging in the package, see below.

1 Suggestions

A handful of suggestions are here:

1. Please (briefly) explain the relationship between `FThresholds` and this package. (The ArXiv *comment* addresses this, but not the paper or the documentation as far as I can see.)

    The lists of authors are unfortunately complicated. `FThresholds` seems to include several more authors (or contributors?) who are not listed in the Acknowledgements of this paper: Alberto F. Boix, Drew Ellingson, Matthew Mastroeni, Maral Mostafazadehfard. Perhaps they contributed to v1.0 but not v2.0? Or, to `TestIdeals` but not `FrobeniusThresholds`? At a minimum something like "In addition, Alberto F. Boix,..., contributed to an earlier version of this package"?

2. pg 2, paragraph "Let us outline a way...": I don't like this paragraph very much. It's vague ("some power of p^e") and doesn't seem critical for

this paper. As far as I can see this construction is only used twice: for $c^{\mathrm{m}}(f)$ and $c^{J}(I)$. That is hardly worth introducing a general approach. Of course it's valuable for commutative algebra more broadly but it just doesn't seem needed in this paper. Please consider removing the paragraph or at least rewriting it to be less vague and more directly tied to this paper.

3. Notation: why $c^{\mathrm{m}}(f)$? Why not c_f^{m}, for similarity to ν_f^{m}? (Why not c_I^{J}?) (Is it standard notation?)

4. Do the existence of $c^{\mathrm{m}}(f)$ and $c^{J}(I)$ come down to the usual Fekete lemma or is it something more special? This is not critical, more just curiosity.

5. pg 3, "natural to ask": This does not strike me as natural to ask. It seems to me to be wildly fantastical to imagine that a sequence could be recovered from its limit. Correspondingly, that the answer is yes, seems to me to go far beyond "Fortunately". It's amazing! (Well, apparently, the sequence turns out to not be so complicated. But before knowing that, it's amazing.)

6. pg 3, "We first consider": I would say "describe", but that's stylistic.

7. pg 4, paragraph "In general, the function nu works by...": This is a good paragraph, explaining computationally how nu works. It deserves a little more prominence, rather than being buried in the description of the options. Please consider moving it to its own subsection before the options; or into the "2.0" subsection; or move the UseSpecialAlgorithms later, so the explanation of how nu works can be at the head of 2.1.

8. pg 7: How do you effectively determine ϵ? Is there an effective lower bound for gaps between jumping numbers? Can there be at least a brief explanation?

9. pg 13, "Clearly": I agree that non-standard gradings are natural to consider, but they haven't been mentioned in this paper at all. So if it is "clear" then it is only so because of a reader's (assumed) prior familiarity with issues in commutative algebra. Please consider simply

deleting "Clearly". The rest of the sentence stands on its own without that!

10. Please consider using Github permalinks, `https://help.github.com/en/github/managing-files-in-a-repository/getting-permanent-links-to-files`. You can put a tag (like "v2.0") and make the links point to the v2.0 files. Right now they point to "master", i.e., the most up-to-date versions. That is a mismatch. You say things like, for example, Leykin and Tsai, *DModules, Version 1.4.0.1*, so it would make sense for the links to point to exactly that version.

    Well, perhaps there is some established answer for how to have bibliographic references to resources that can be updated later (always link to the version at the time of writing? link to most-up-to-date version?). I don't know. If the answer is to link to the same version mentioned ("Version 1.4.0.1") then the Macaulay2 team should *tag every version of M2 and packages* and make those tags available for use in bibliographic URLs.

    (This would probably work better if each package had its own repo and tags. But instead packages are within the M2 repo, so they end up using tags that apply globally to the whole M2 repo. Since that is the case, this suggestion might be impractical.)

11. You need to export "AtOrigin". I added `"AtOrigin"` to the export list in my local copy of `FrobeniusThresholds.m2`, after which it allowed me to install the package. But there were errors in some examples during installation, package checks, and article examples (at `fpt(x^5 + y^6 + z^7 + (x*y*z)^3)` on page 7). Most or all of the errors seemed to be tied to "AtOrigin" not being recognized by things in `MainFunctions.m2`.

    I was able to load the current version of `TestIdeals`, but ran into some errors with trying to install it ("error: package TestIdeals not reloaded; try Reload => true", which I don't understand). I was able to install FrobeniusThresholds, so I should have been able also to install TestIdeals.

    I wasn't able to get the documentation for FrobeniusThresholds to install. Not sure what I was doing wrong.

Here is version info. I have Macaulay2 version 1.14 (= latest release) on macOS 10.14.6. For `FrobeniusThresholds` and `TestIdeals` I grabbed files from the github repo, 6c0bccf. (I pulled to my machine, then used `loadPackage("...", FileName=>"path to local copy of git repo")`.) If I should be getting files from somewhere else, or if I need a Macaulay2 that's newer than the latest release, please let me know.

Fortunately I was able to install FrobeniusThresholds and check the article's examples up to page 7.