TESTIDEALS PACKAGE FOR *MACAULAY2*

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Abstract. This note describes a *Macaulay2* package for computations in commutative rings prime related to p^{-e}-linear and p^e-linear maps, singularities defined in terms of these maps, various test ideals and modules, and ideals compatible with a given p^{-e}-linear map.

1. Introduction

This paper constructive methods for computing various objects related to commutative rings of prime characteristic p. Such a ring R comes equipped with a built-in endomorphism, namely the Frobenius endomorphism $f : R \to R$ which is the basis for many constructions and definitions which affords a handle on many problems which is not otherwise available. Two notable examples of the use of the Frobenius endomorphism are the theory of tight closure [Add references] and the resulting theory of test ideals. [Add references]

[Add history of the package]

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2. Frobenius powers and Frobenius roots

Let S denote any commutative ring of prime characteristic p.

Definition 2.1. For any ideal $I \subseteq S$ and any integer $e \geq I$, we define the *eth Frobenius power of* I to be the ideal denoted $I^{[p^e]}$ which is generated by all p^eth powers of elements in I.

It is easy to see that, if I is generated by $g_1, \ldots, g_\ell$, $I^{[p^e]}$ is generated by $g_1^{p^e}, \ldots, g_\ell^{p^e}$.

Definition 2.2. For any ideal $I \subseteq S$ and any integer $e \geq I$, we define the *eth Frobenius root of* I to be the ideal denoted $I^{[1/p^e]}$ which is the smallest ideal J such that $I \subseteq J^{[p^e]}$, if such ideal exists.

eth Frobenius roots exist in polynomial rings (cf. [BMS08, §2]) and in power series rings (cf. [Kat08, §5]).

```
i2 :      R=ZZ/5[x,y,z]
i3 :      I=ideal(x^6*y*z+x^2*y^12*z^3+x*y*z^18)
                18    2 12 3    6
o3 = ideal(x*y*z   + x y  z  + x y*z)
o3 : Ideal of R
i4 :      frobeniusPower(1/5,I)
                2   3
o4 = ideal (x, y , z )
```

We can extend the definition of Frobenius powers as follows

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Definition 2.3. [Add references]

Let $I \subseteq S$ be an ideal.

(a) If a is a positive integer with base-p expansion $a = a_0 + a_1 p + \cdots + a_r p^r$, we define $I^{[n]} = I^{a_0} (I^{a_1})^{[p]} \ldots (I^{a_r})^{[p^r]}$.

(b) If t is a non-negative rational number of the form $t = a/p^e$, we define $I^{[t]} = (I^{[a]})^{[1/p^e]}$.

(c) If t is any non-negative rational number, and $\{a_n/p^{e_n}\}_{n\geq 1}$ is a sequence of rational numbers converging to t from above, we define $I^{[t]}$ to be the stable value of increasing chain of ideals $\{I^{[a_n/p^{e_n}]}\}_{n\geq 1}$.

```
i5 :      frobeniusPower(1/2, ideal(y^2-x^3))
o5 = ideal 1
o5 : Ideal of R
i6 :      frobeniusPower(5/6, ideal(y^2-x^3))
o6 = ideal (y, x)
o6 : Ideal of R
```

3. p^{-e}- and p^e-linear maps

Definition 3.1. Let M be an S-module and e a non-negative integer.

(a) A p^{-e}-linear map $\phi : M \to M$ is an additive map such that $\phi(s^{p^e} m) = s\phi(m)$ for all $s \in S$ and $m \in M$.

(b) A p^e-linear map $\psi : M \to M$ is an additive map such that $\phi(sm) = s^{p^e}\phi(m)$ for all $s \in S$ and $m \in M$.

The following two examples describe two prototypical p^{-e}- and p^e-linear maps.

Example 3.2. For any S-module M, we can construct a new S-module $F^e_* M$ with elements $\{F^e_* m \,|\, m \in M\}$ by defining $F^e_* m_1 + F^e_* m_2 = F^e_*(m_1 + m_2)$ for all $m_1, m_2 \in M$ and $sF^e_* m = F^e_* s^{p^e}$ for all $m \in M$ and $s \in S$.

Consider any $\phi \in \mathrm{Hom}_S(F^e_* M, M)$: if we identify $F^e_* M$ with M we can interpret ϕ as a p^{-e}-linear map.

Example 3.3. The eth Frobenius map $f : S \to S$ raising elements to their p^eth power is clearly p^e-linear. Furthermore, any ideal $I \subseteq S$, f induces an p^e-linear map $\mathrm{H}^k_I(S) \to \mathrm{H}^k_I(S)$.

Let R be a polynomial ring with irrelevant ideal $\mathfrak{m}$ and let $g \in R \setminus \{0\}$. Let $E = E_{R\mathfrak{m}}(R_{\mathfrak{m}}/\mathfrak{m})$ denote the injective hull of $R_{\mathfrak{m}}/\mathfrak{m}$.

The Frobenius map on R induces a Frobenius map S and on $\mathrm{H}^{\dim R-1}_{\mathfrak{m}}(S) = E_S(S/\mathfrak{m}S) = \mathrm{Ann}_E\, g$ and the kernel of this map is given by $\mathrm{Ann}_E(g^{p-1}R)^{[1/p]}$ (cf. [Kat08, §5]).

```
i2 :      R=ZZ/5[x,y,z]
i3 :      g=x^3+y^3+z^3
i4 :      u=g^(5-1)
i5 :      frobeniusPower(1/5,ideal(u))
o5 = ideal (z, y, x)
o5 : Ideal of R

i6 :      R=ZZ/7[x,y,z]
i7 :      g=x^3+y^3+z^3
i8 :      u=g^(7-1)
i9 :      frobeniusPower(1/7,ideal(u))
o9 = ideal 1
```

Thus we see that the induced p^e-linear map on $\mathrm{H}^2_{(x,y,z)}\left(\mathbb{K}[x,y,z]/(x^3+y^3+z^3)\right)$ is injective when the characteristic of $\mathbb{K}$ is 7 and non-injective when the characteristic is 5.

4. F-singularities

This package includes a method for checking if a ring of finite type over a prime field is F-injective or not.

Definition 4.1. A local ring $(R, \mathfrak{m})$ is called *F-injective* if the map $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R^{1/p})$ is injective for all $i > 0$. An arbitrary ring is called F-injective if each of the localizations at a prime ideal are F-injective.

```
i2 : R = ZZ/7[x,y,z]/ideal(x^3 + y^3 + z^3);
i3 : isFinjective(R)
o3 = true
i4 : R = ZZ/5[x,y,z]/ideal(x^3 + y^3 + z^3);
i5 : isFinjective(R)
o5 = false
```

The algorithm `isFinjective` determines whether the ring $R = S/I$ is F-injective or not. The algorithm works by checking the injectivity of the frobenius map using the functoriallity of Ext. The algorithm starts by computing the map $R \to F_*R$ using `frobPFMap`. This outputs a map represented as a matrix over R, using `pushFwdToAmbient` to allows this map to instead be represented over the ambient ring S. The next step computes the module $\mathrm{Ext}^i(\_\_, S)$ using the map from the previous step. Finally the algorithm checks the dimension of the cokernel of $\mathrm{Ext}^i(\_\_, S)$. If the dimension does not equal negative one then the Frobenius action is not injective and the algorithm terminates and returns false. Otherwise the algorithm continues on and checks the next degree in the same way.

The `CanonicalStrategy` tag can be used to modify the strategy the algorithm uses to check the Frobenius action on the top local cohomology. By default the algorithm is set to `CanonicalStrategy => Katzman` which then uses the strategy of Katzman [Add references]. If the tag is set to anything else `CanonicalStrategy => null` the algorithm checks the top local cohomology using the same brute force strategy used to check the injectivity at lower degrees. The Katzman strategy is typically much faster.

There are a number of options to improve the performance of the algorithm if the ring of interest is nice enough. If the ring is Cohen-Macaulay then setting `AssumeCM => true` lets the algorithm check the Frobenius action only on top cohomology (which is typically much faster). The default value is `false`. Of course, telling the algorithm to assume the ring is Cohen-Macaulay when it is not can lead to an incorrect answer if the non-injective Frobenius occurs in a lower degree. For an example of this see the documentation. If the ring is reduced then setting `AssumedReduced => true` avoids computing the bottom local cohomology, if the ring is normal then setting `AssumeNormal => true` avoids computing the bottom two local cohomologies. The default setting for both of these tags is `false`.

By default the algorithm checks for F-injectivity everywhere however one can choose to check F-injectivity only at the origin by setting the option `IsLocal => true`.

```
i2 : R = ZZ/5[x,y,z]/ideal( (x-1)^4 + y^4 + z^4 );
i3 : isFinjective(R)
o3 = false
i4 : isFinjective(R, IsLocal=>true)
o4 = true
```

Definition 4.2. A ring R is called strongly F-regular if $\tau(R) = R$.

The command `isFregular` checks whether a ring or pair is strongly F-regular. Below are two examples one F-regulare and one not.

```
i2 : R = ZZ/5[x,y,z]/ideal(x^2 + y*z);
i3 : isFregular(R)
o3 = true
i4 : R = ZZ/7[x,y,z]/ideal(x^3 + y^3 + z^3);
i5 : isFregular(R)
o5 = false
```

We can also perform these types computations for pairs.

```
i2 : R = ZZ/5[x,y];
i3 : f = y^2-x^3;
i4 : isFregular(1/2, f)
o4 = true
i5 : isFregular(5/6, f)
o5 = false
i6 : isFregular(4/5, f)
o6 = false
i7 : isFregular(4/5-1/100000, f)
o7 = true
```

The inputs for `isFregular` can be a ring, an integer and a ring element, a rational number and a ring element or a list of rational numbers and a list of ring elements.

If the input ring is $\mathbb{Q}$-Gorenstein then in each of the cases above the output is a boolean indicating if the ring is strongly F-regular. If the input ring is not $\mathbb{Q}$-Gorenstein then the algorithm can be used to determine if a ring is strongly F-regular but cannot prove a ring is not strongly F-regular

In the case that R is $\mathbb{Q}$-Gorenstein, the algorithm works by computing the test ideal of the ring (or the pair) and then uses `isSubset` to check if (1) is contained in the test ideal. In the non-$\mathbb{Q}$-Gorenstein case the algorithm checks for strong F-regularity by checking if (1) is contained in better and better approximations of the test ideal. To compute approximations of the test ideal the algorithm uses `frobeniusRoot` to compute the eth root of $c(I^{[p^e]} : I)$ (in the case we are checking strong F-regularity for a ring, appropriate modifications are made for pairs). If at any step (1) is contained in the approximation then then the algorithm returns `true`. Otherwise the algorithm continues checking for each e until a specified limit is reached. The default limit is 2 and can be changed using `DepthOfSearch => ZZ`.

The default behavior of `isFregular` checks for strong F-regularity everywhere. If the option `IsLocal => true`, the algorithm will only check at the origin (this uses a similar computation but checks for the containment of (1) instead of the test ideal plus the maximal ideal at the origin). Below is an example for both a ring and a pair.

```
i2 : R = ZZ/7[x,y,z]/ideal((x-1)^3+(y-2)^3+z^3);
i3 : isFregular(R)
o3 = false
i4 : isFregular(R, IsLocal=>true)
o4 = true
i5 : R = ZZ/13[x,y];
i6 : f = (y-2)^2 - (x-3)^3;
i7 : isFregular(5/6, f)
o7 = false
i8 : isFregular(5/6, f, IsLocal=>true)
o8 = true
```

5. Test ideals

In this section, we explain how to compute parameter test modules, parameter test ideals, test ideals and HSLG-modules[1].

5.1. **Parameter test modules.** Given an F-finite reduced ring R, the Frobenius map $R \longrightarrow R^{1/p^e}$ is dual to $T : \omega_{R^{1/p^e}} \longrightarrow \omega_R$. As before in **??**, we can represent $\omega_R \subseteq R$ as an ideal, we can write $R = S/I$, and so we can find a $u \in S^{1/p^e}$ representing the map $\omega_{R^{1/p^e}} \longrightarrow \omega_R$.

The parameter test submodule is the smallest submodule $M \subseteq \omega_R$ (and hence ideal of R since $M \subseteq R$) that agrees generically with ω_R and that satisfies

$$T(M^{1/p^e}) \subseteq M.$$

From within Macualay2, we can compute this using the `testModule` command as follows.

```
i3 : R = ZZ/5[x,y,z]/ideal(x^4+y^4+z^4);
i4 : N = testModule(R);
i5 : N#0
               2               2        2
o5 = ideal (z , y*z, x*z, y , x*y, x )
o5 : Ideal of R
i6 : N#1
o6 = ideal 1
o6 : Ideal of R
```

The output of `testModule` is a list with three items. The first entry is the test module, the second is the canonical module (as an ideal) that it lives inside, and the third is the element u described above (not listed here, since it is rather complicated). Note since this ring is Gorenstein, the canonical module is simply represented as the unit ideal.

Here is another example where the ring is not Gorenstein.

```
i2 : R = ZZ/5[x,y,z]/ideal(y*z, x*z, x*y);
i3 : N = testModule(R);
i4 : N#0
               2   2   2
o4 = ideal (z , y , x )
o4 : Ideal of R
i5 : N#1
o5 = ideal (y - z, x + z)
o5 : Ideal of R
```

We briefly explain how this is computed. First we find a *test element*.

Remark 5.1 (Computation of test elements). Roughly, we recall that an element of the Jacobian ideal that is not contained in any minimal prime is a test element []. We compute test elements by computing random partial derivatives (and linear combinations thereof) until we find an element that doesn't vanish at all the minimal primes. This method is much faster than computing the entire Jacobian ideal. If you know your ring is a domain, you can use the `AssumeDomain` flag to speed this up further.

[1] HSLG-modules can be used to give a scheme structure to the F-injective or F-pure locus.

After the test element c has been identified, we pullback the ideal ω_R to an ideal $J = S$. Then we compute the following ascending sequence of ideals

$$
(1)\qquad
\begin{array}{lll}
J_1 := & = J + (cJ)^{[1/p]} & = \mathtt{J} + \mathtt{frobeniusRoot}(1, \mathtt{c} * \mathtt{J}) \\
J_2 := & = J + (cJ)^{[1/p]} + (c^{1+p}J)^{[1/p^2]} & = \mathtt{J} + \mathtt{frobeniusRoot}(1, \mathtt{c} * \mathtt{J_1}) \\
J_3 := & = J + (cJ)^{[1/p]} + (c^{1+p}J)^{[1/p^2]} + (c^{1+p+p^2}J)^{[1/p^3]} & = \mathtt{J} + \mathtt{frobeniusRoot}(1, \mathtt{c} * \mathtt{J_2}). \\
\quad\dots & &
\end{array}
$$

As soon as this ascending sequence of ideals stabilizes, we are done. In fact, because this strategy is used in several contexts, you can call it directly for a chosen ideal J and u with the function `ascendIdeal` (this is done for test ideals below).

We can also compute parameter test modules of pairs $\tau(\omega_R, f^t)$ with $t \in \mathbb{Q}_{\geq 0}$. This is done by modifying the element u when the denominator of t is not divisible by p. When t has p in the denominator, we rely on the fact (see []) that

$$
\begin{array}{rcl}
T(\tau(\omega_R, f^a)) & = & \tau(\omega_R, f^{a/p}) \\
\mathtt{frobeniusRoot}(1, \mathtt{u} * \mathtt{I_1}) & = & \mathtt{I_2}
\end{array}
$$

Where the second line roughly explains how this is accomplished internally, here $\mathtt{I_1}$ is $\tau(\omega_R, f^a)$ pulled back to S and likewise I_2 defines $\tau(\omega_R, f^a)$ modulo the defining ideal of R.

Remark 5.2 (Optimizations in `ascendIdeal` and other `testModule` computations). Throughout the computations described above, we very frequently use the following fact.

5.2. Parameter test ideals.

5.3. Test ideals.

5.4. HSLG modules.

6. Ideals compatible with given p^{-e}-linear map

Throughout this section let R denote a polynomial $\mathbb{K}[x_1, \dots, x_n]$. In this section we address the following question: given a p^{-e} linear map $\phi : R \to R$, what are all ideals $I \subseteq R$ such that $\phi(I) = I$? We call these ideals ϕ-compatible.

Recall that in section 3 we identified p^{-e} linear maps with elements of $\operatorname{Hom}_R(F^e_* R, R)$ and it turns out that this is a principal $F^e_* R$-module generated the *trace map* $\pi \in \operatorname{Hom}_R(F^e_* R, R)$, constructed as follows. [Add references]
Let B be a $\mathbb{K}^{p^e}$-basis for $\mathbb{K}$; $F^e_* R$ is a free R-module with basis

$$\{F^e_* b x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid 0 \leq \alpha_1, \dots, \alpha_n < p^e\}$$

and the trace map π is the projection onto the free summand $RF^e_* x_1^{p^e-1} \dots x_n^{p^e-1} \cong R$.

We can now write our given ϕ as multiplication by some $F^e_* u$ followed by π and it is not hard to see that an ideal $I \subseteq R$ is ϕ-compatible if and only if $uI \subseteq I^{[p^e]}$.

Theorem 6.1 ([Add references]). *If ϕ is surjective, there are finitely many ϕ-compatible ideals, consisting of all possible intersections of ϕ-compatible prime ideals.*

In general, there are finitely many ϕ-compatible prime ideals not containing $\sqrt{\operatorname{Image} \phi}$.

The method *compatibleIdeals* produces the finite set of ϕ-compatible prime ideals in the second statement of the theorem above.

```
i2 :      R = ZZ/3[u,v];
i3 :      u = u^2*v^2;
i4 :      compatibleIdeals(u)
o4 = {ideal v, ideal (v, u), ideal u}
o4 : List
```

The defining condition $uI = I^{[p]}$ for ϕ-compatible ideals allows us to think of these in a dual form: write $\mathfrak{m} = (x_1, \dots, x_n)R$, $E = E_{R\mathfrak{m}}(R_{\mathfrak{m}}/\mathfrak{m}) = \mathrm{H}^n_{\mathfrak{m}}(R)$, and let $T : E \to E$ be the p^e-linear map induced from the Frobenius map on R. If $\psi = uT$, then $\psi \operatorname{Ann}_E I \subseteq \operatorname{Ann}_E I$ if and only if $uI = I^{[p]}$! Thus finding all R-submodules of E compatible with $\psi = uT$ also amounts to finding all $\phi = \pi \circ F^e_* u$ ideals.

Let $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$, $R = \mathbb{K}[x_1, x_2, x_3, x_4, x_5]$, $\mathfrak{m} = (x_1, \dots, x_5)R$, and let I be the ideal of 2×2 minors of

$$\begin{bmatrix} x_1 & x_2 & x_2 & x_5 \\ x_4 & x_4 & x_3 & x_1 \end{bmatrix}$$

In (cf. [Kat08, §9]) it is shown that there is a surjection $\operatorname{Ann}_E I \to \mathrm{H}^2_{\mathfrak{m}}(R/I)$ which is compatible with with the induced p^1-linear map on $\mathrm{H}^2_{\mathfrak{m}}(R/I)$ and the p^1-linear map uT on $\operatorname{Ann}_E I$ with

$$u = x_1^3x_2x_3 + x_1^3x_2x_4 + x_1^2x_3x_4x_5 + x_1x_2x_3x_4x_5 + x_1x_2x_4^2x_5 + x_2^2x_4^2x_5 + x_3x_4^2x_5^2 + x_4^3x_5^2.$$

```
i2 :      R=ZZ/2[x_1..x_5];
i3 :      I=minors(2, matrix {{x_1,x_2,x_2,x_5},{x_4,x_4,x_3,x_1}} )
                                                    2
o3 = ideal (x x  + x x , x x  + x x , x x  + x x , x  + x x , x x  + x x , x x
             1 4    2 4   1 3    2 4   2 3    2 4   1    4 5    1 2     4 5    1 2
     --------------------------------------------------------------------------
     + x x )
        3 5
o3 : Ideal of R
i4 :      u=x_1^3*x_2*x_3 + x_1^3*x_2*x_4+x_1^2*x_3*x_4*x_5+ x_1*x_2*x_3*x_4*x_5+
          x_1*x_2*x_4^2*x_5+ x_2^2*x_4^2*x_5+x_3*x_4^2*x_5^2+ x_4^3*x_5^2;
i5 :      compatibleIdeals(u)
             3        3        2                           2      2 2
o5 = {ideal(x x x  + x x x  + x x x x  + x x x x x  + x x x x  + x x x  +
             1 2 3    1 2 4    1 3 4 5    1 2 3 4 5    1 2 4 5    2 4 5
     --------------------------------------------------------------------------
        2 2    3 2            2
     x x x  + x x ), ideal (x  + x , x  + x x ), ideal (x , x ), ideal (x , x ,
      3 4 5    4 5            1    2    1    4 5            4    1            4    1
     --------------------------------------------------------------------------

     x ), ideal (x , x , x , x ), ideal (x , x , x , x , x ), ideal (x , x ,
      5            5    4    1    3            5    4    3    2    1            5    4
     --------------------------------------------------------------------------

     x , x ), ideal (x , x , x ), ideal (x , x , x , x ), ideal (x , x , x ),
      2    1            4    1    3            4    3    2    1            4    2    1
     --------------------------------------------------------------------------
                                     2
     ideal (x  + x , x  + x , x  + x x ), ideal (x  + x , x , x , x ), ideal
             3    4    1    2    2    4 5            3    4    2    1    5
     --------------------------------------------------------------------------
     (x , x , x )}
       2    1    5

o5 : List
```

7. Future plans

References

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