Minimum-Time and -Fuel Optimization

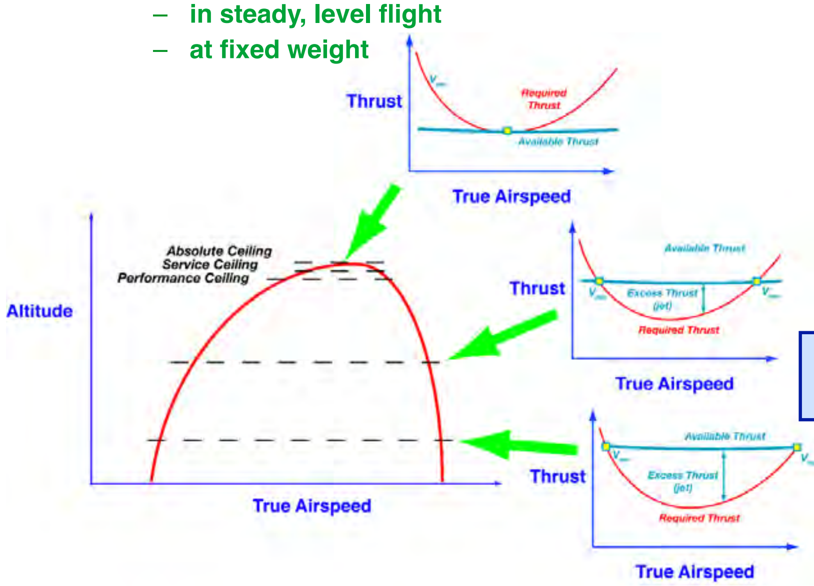
Robert Stengel
Optimal Control and Estimation MAE 546
Princeton University, 2018

- Climbing flight of an airplane
- Energy and power
- Choice of control variable
- Numerical optimization
- Minimum time-to-climb problem
- Minimum fuel-to-climb problem
- Stability and control *(supplement)*

http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Flight Envelope Determined by Available Thrust

- **Flight Envelope:** Encompasses all altitudes and airspeeds at which an aircraft can fly
  - in steady, level flight
  - at fixed weight



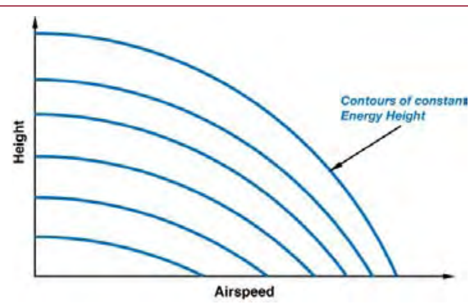
Excess thrust provides the ability to accelerate or climb

Energy Height and Specific Excess Power

Energy Height

- **Specific Energy**
  - **= (Potential + Kinetic Energy) per Unit Weight**
  - **= Energy Height**

$$\frac{Total\ Energy}{Unit\ Weight} \equiv Specific\ Energy = \frac{mgh + mV^2/2}{mg} = h + \frac{V^2}{2g}$$
$$\equiv \textbf{Energy Height}, E_h, \quad ft\ or\ m$$



Could trade altitude for airspeed with no change in energy height if thrust and drag were zero

Specific Excess Power

- **Specific Power**

$$\frac{dE_h}{dt} = \frac{d}{dt}\left(h + \frac{V^2}{2g}\right) = \frac{dh}{dt} + \left(\frac{V}{g}\right)\frac{dV}{dt}$$

$$\dot{V} = (T - D)/m - g\sin\gamma$$
$$\dot{h} = V\sin\gamma$$

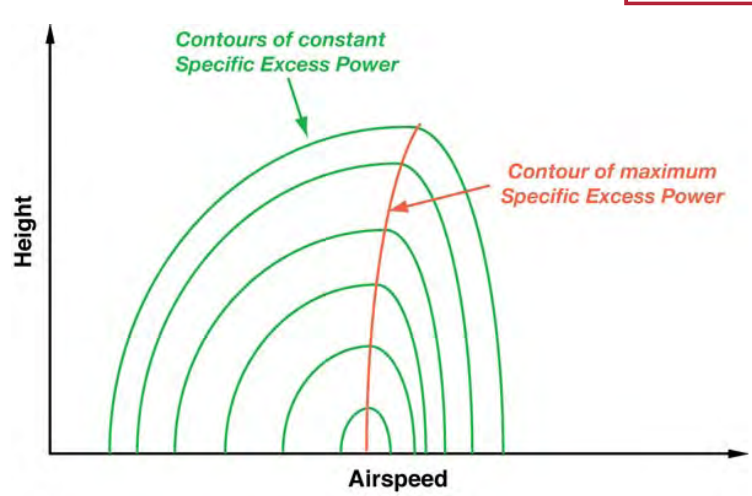
$$\frac{dE_h}{dt} = V\sin\gamma + \left(\frac{V}{g}\right)\left[\frac{(T-D)}{m} - g\sin\gamma\right]$$

$$\frac{dE_h}{dt} = V\frac{(T-D)}{W} = V\frac{(C_T - C_D)\frac{1}{2}\rho(h)V^2 S}{W}$$

$$\frac{dE_h}{dt} = \textit{Specific Excess Power}\ (SEP) = \frac{\textit{Excess Power}}{\textit{Unit Weight}} \equiv \frac{\left(P_{thrust} - P_{drag}\right)}{W}$$

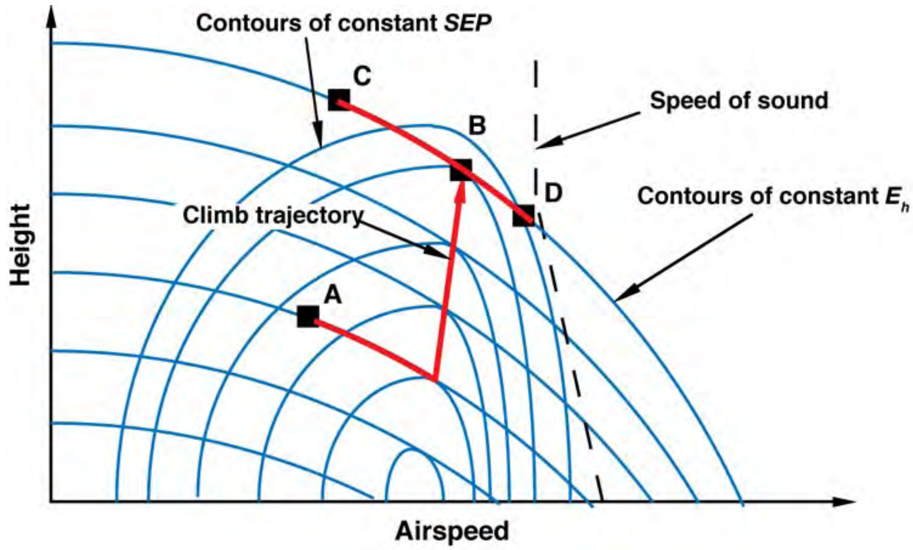
Contours of Constant Specific Excess Power

- **Specific Excess Power is a function of altitude and airspeed**
- ***SEP*** **is maximized at each altitude, *h*, when** $\frac{d[SEP(h)]}{dV} = 0$



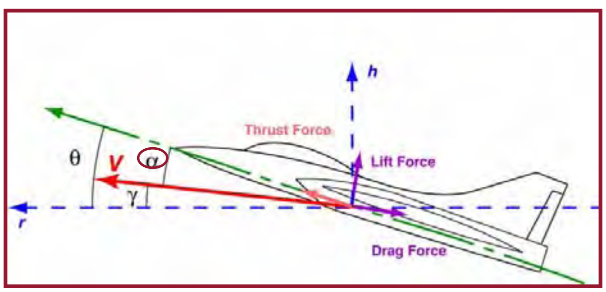
Subsonic Energy Climb

Objective: Minimize time or fuel to climb to desired altitude and airspeed



Approximate Optimal Trajectory produced by a Switching Curve

Angle of Attack Control
"Point-Mass" Model



$$\dot{V} = \left[T_{\max} - \left(C_{D_o} + \varepsilon \left[C_{L_\alpha} \alpha \right]^2 \right) \frac{1}{2} \rho V^2 S \right] \Big/ m - g \sin\gamma$$

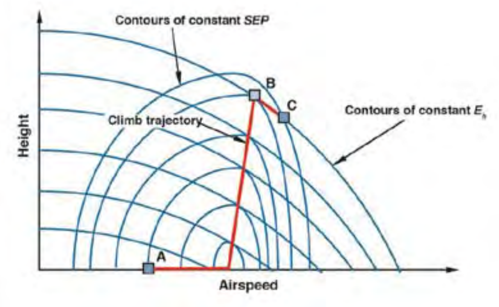
$$\dot{\gamma} = \frac{1}{V} \left[\left(C_{L_\alpha} \alpha \frac{1}{2} \rho V^2 S \right) \Big/ m - g \cos\gamma \right]$$

$$\dot{h} = V \sin\gamma$$

$$\dot{r} = V \cos\gamma$$

$$\dot{m}_{fuel} = -(SFC)(T_{\max})$$

Minimum Time-to-Climb Problem



Initial and final conditions

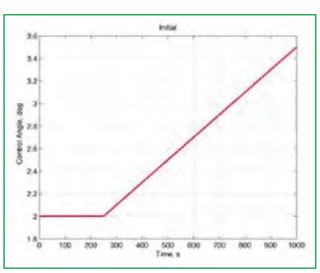
$$V_o = 100 \text{ m/s}; \quad \gamma_o = 0 \text{ rad}; \quad h_o = \text{ 0 m (sea level)}; \quad r_o = 0 \text{ m}$$

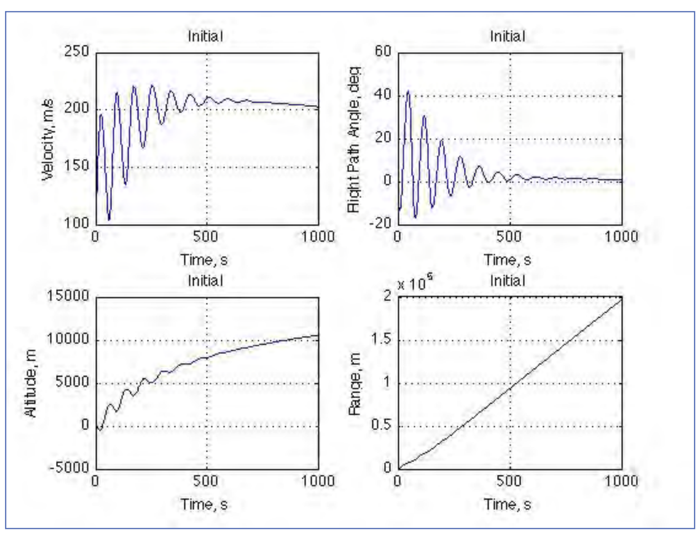
$$V_f = 200 \text{ m/s}; \quad \gamma_f = open; \quad h_f = \text{ 10,000 m}; \quad r_f = open$$

- **End time, t_f, is open, thrust takes maximum value, and control variable is angle of attack,** $\alpha(t)$
- **Fuel expended, but simulated vehicle mass held constant**

1,000-sec Trajectory, Simple Angle of Attack Control

Angle of Attack History



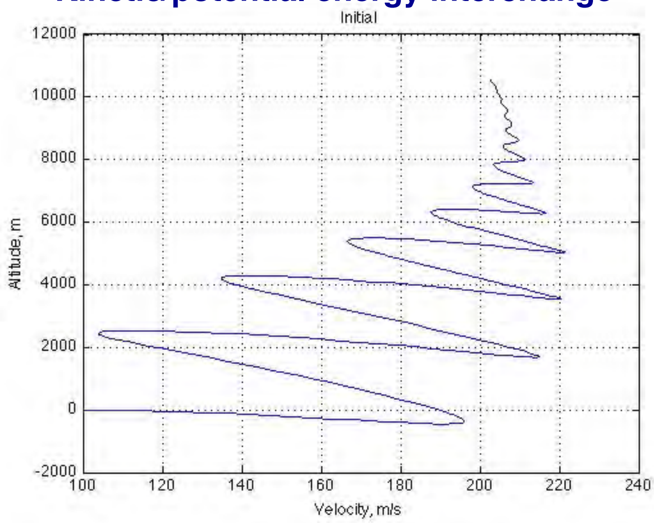


- **No optimization, open-loop control of point-mass model**
- **Interchange of kinetic and potential energy**
- **Lightly damped long-period oscillation**

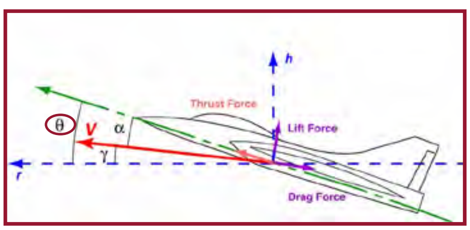
Altitude vs. Velocity,

Simple Angle of Attack Control,
t_f = 1,000 sec

- **Lightly damped, long-period oscillation**
- **Kinetic/potential energy interchange**



Alternative: Pitch Angle Control



$$\alpha = \theta - \gamma$$

$\alpha =$ Angle of Attack, rad
$\theta =$ Pitch Angle, rad
$\gamma =$ Flight Path Angle, rad

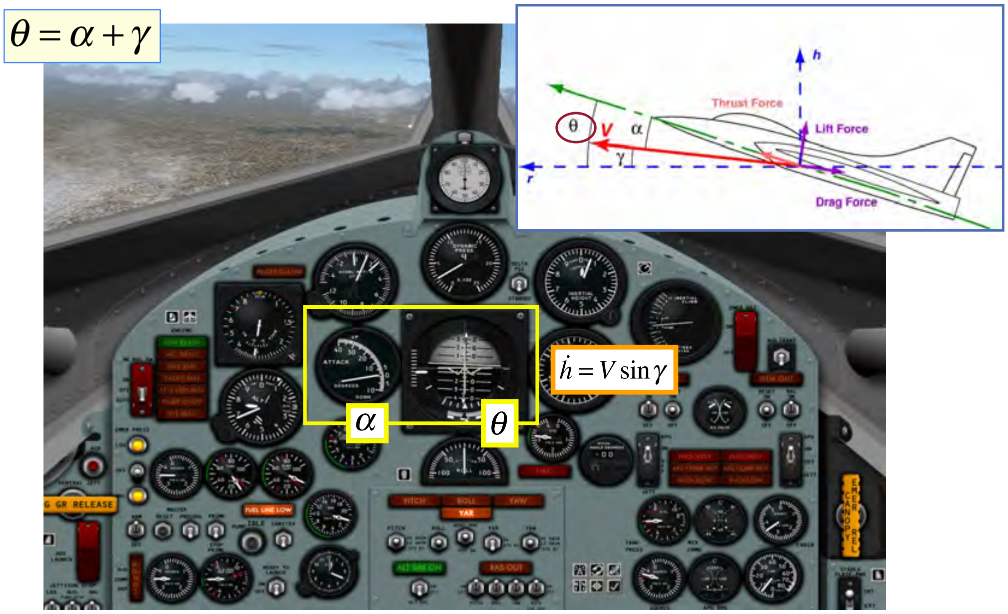
$$\dot{\gamma} = \frac{1}{V}\left[\left(C_{L_\alpha}(\theta - \gamma)\frac{1}{2}\rho V^2 S\right)\Big/ m - g\cos\gamma\right]$$

Controlling pitch angle introduces <u>flight path angle damping</u>

(see Supplemental Material for details)

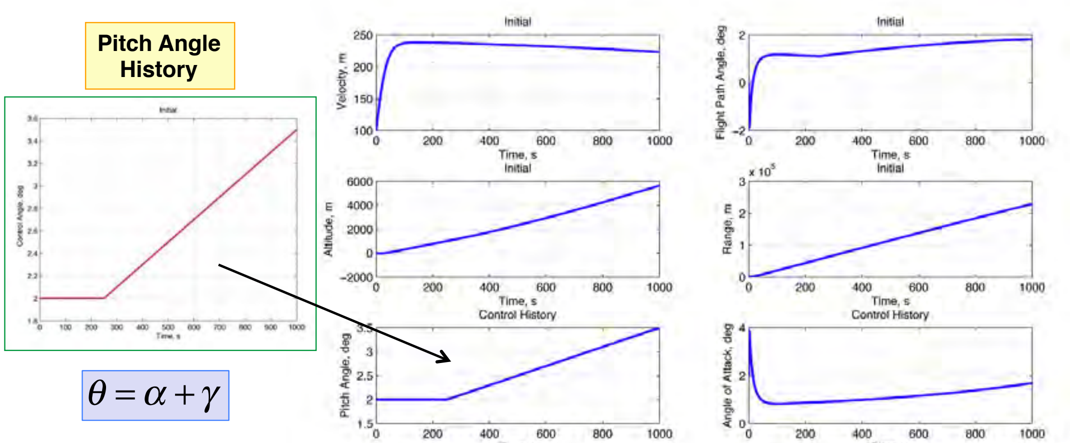
$$\dot{\gamma} = \frac{1}{V}\left[\left(C_{L_\alpha}\frac{1}{2}\rho V^2 S\theta - C_{L_\alpha}\frac{1}{2}\rho V^2 S\gamma\right)\Big/ m - g\cos\gamma\right]$$

Angle of Attack or Pitch Angle Control?



1,000-sec Trajectory, Simple Pitch Angle Control

Note difference in pitch-angle and angle-of-attack profiles

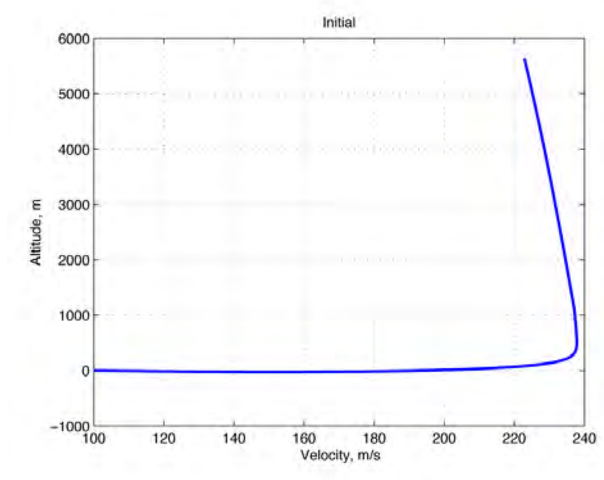


- **No optimization, open-loop control of point-mass model**
- **Inherent damping**
- **Long-period oscillation does not occur**

Altitude vs. Velocity

Simple Pitch Angle Control, t_f = 1,000 sec

- **Increased damping eliminates oscillation**
- **Pitch angle chosen to produce kinetic energy increase followed by potential energy increase**



Minimum Time-to-Climb Optimization Problem

Minimum-Time Cost Function

$$J(t_f) = \frac{1}{2}\left\{\left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]^T \mathbf{P}_f \left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]\right\} + \frac{1}{2}\int_{t_0}^{t_f} \left\{1 + \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]\right\} dt$$

$$\begin{aligned} V_0 &= 100 \text{ m/s} \\ \gamma_0 &= 0 \text{ rad} \\ h_0 &= 0 \text{ m (sea level)} \\ r_0 &= 0 \text{ m} \end{aligned}$$

Terminal cost provides trajectory objective

$$\mathbf{x}_{des}(t_f) = \begin{cases} V_f = 200 \text{ m/s} \\ \gamma_f = open \\ h_f = 10{,}000 \text{ m} \\ r_f = open \end{cases}$$

- **Integrand**
  - **"1" is the integrand for <u>minimizing time</u>**
  - **Small quadratic term**
    - **provides <u>non-singular trajectory control</u> with *ad hoc* damping and regulation [good], but**
    - **penalizes non-zero values of state and control [not so good]**

Minimum-Time Cost Function with Augmented Trajectory Damping

$$J(t_f) = \frac{1}{2}\left\{\left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]^T \mathbf{P}_f \left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]\right\} + \frac{1}{2}\int_{t_0}^{t_f} \left\{1 + \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \underline{\dot{\mathbf{x}}^T(t)\mathbf{Q}_{\dot{\mathbf{x}}}\dot{\mathbf{x}}(t)} + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]\right\} dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t)\right]$$

- **<u>State rate weighting</u>**
  - **is unbiased by non-zero state or control**
  - **provides damping**

$$J(t_f) = \frac{1}{2}\left\{\left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]^T \mathbf{P}_f \left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]\right\} + \frac{1}{2}\int_{t_0}^{t_f} \left\{1 + \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{f}^T\left[\mathbf{x}(t),\mathbf{u}(t)\right]\mathbf{Q}_{\dot{\mathbf{x}}}\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]\right\} dt$$

Weights Used in Optimization

Terminal Penalty

$$\mathbf{P} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

State Penalty

$$\mathbf{Q} = \begin{bmatrix} 10^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

State-Rate Penalty

$$\mathbf{Q}_{\dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

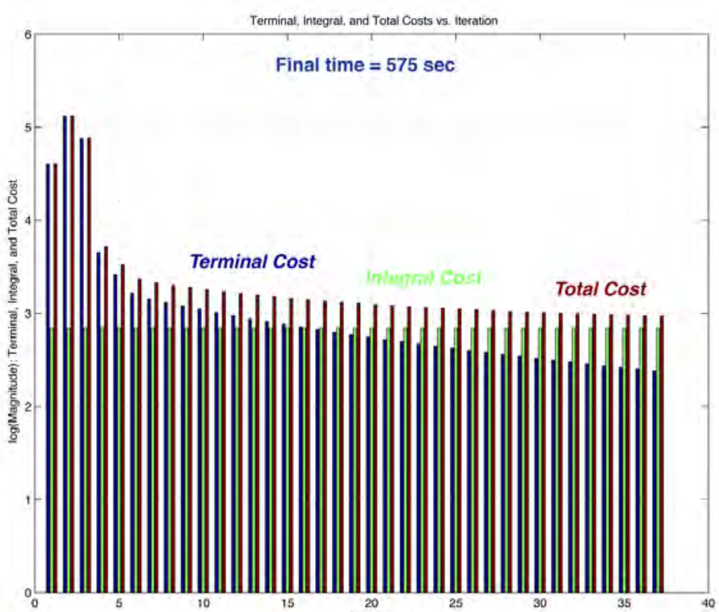
Control Penalty

$$\mathbf{R} = 1$$

$x_1 = V$: Velocity, m/s
$x_2 = \gamma$: Flight path angle, rad
$x_3 = h$: Height, m
$x_4 = r$: Range, m
$u = \theta$: Pitch angle, rad

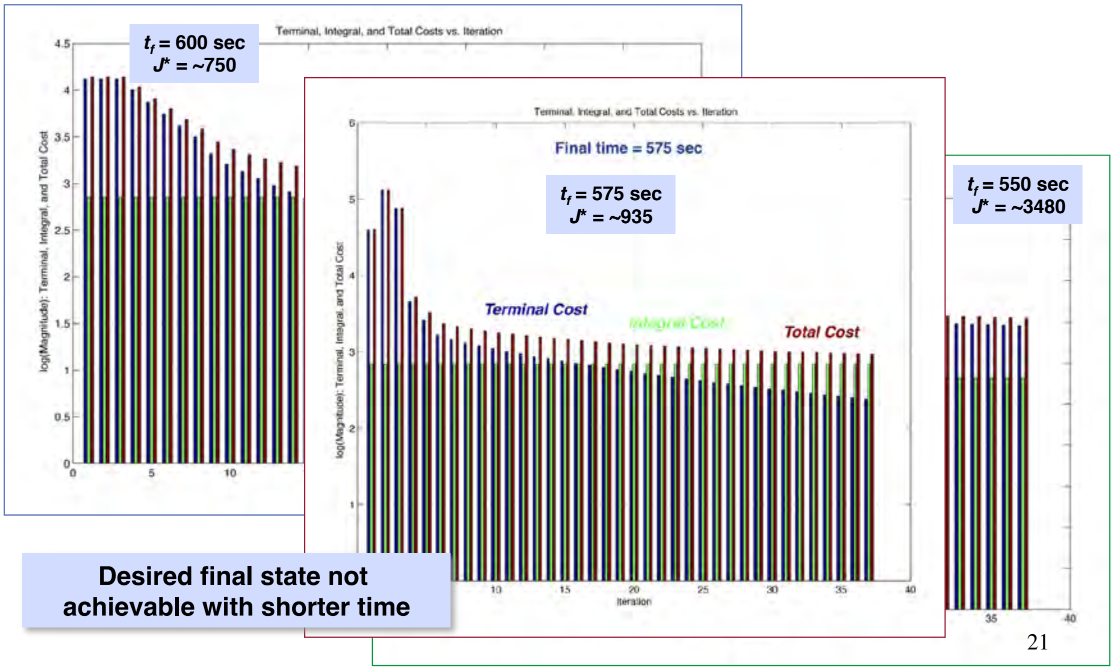
Fixed End-Time Cost History,

t_f = 575 sec, 36 Iterations



- **Ad hoc variation of final time *(TBD)***
- **Adaptive steepest-descent optimization algorithm**
- **Optimization algorithm is still minimizing at iteration cutoff**

Why is t_f = 575 sec Approximately the Minimum-Time Solution?



Desired final state not achievable with shorter time

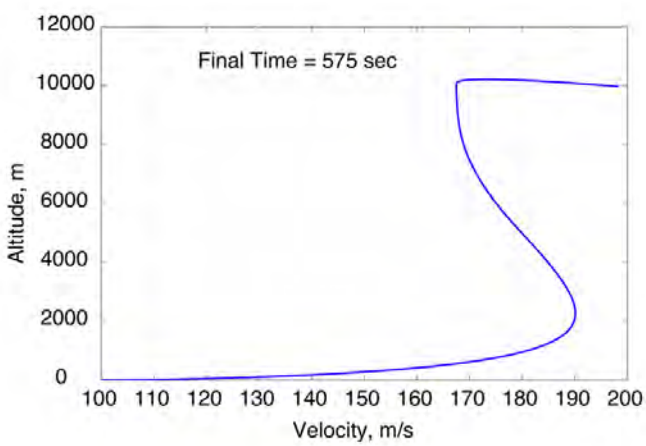
~Minimum-Time Altitude vs. Velocity,

t_f = 575 sec, 36 Iterations

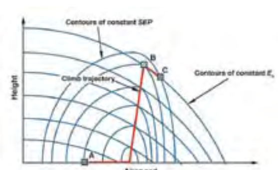
Kinetic energy increase

Trade for potential energy increase

Velocity Increase and Shallow Dive to satisfy terminal condition

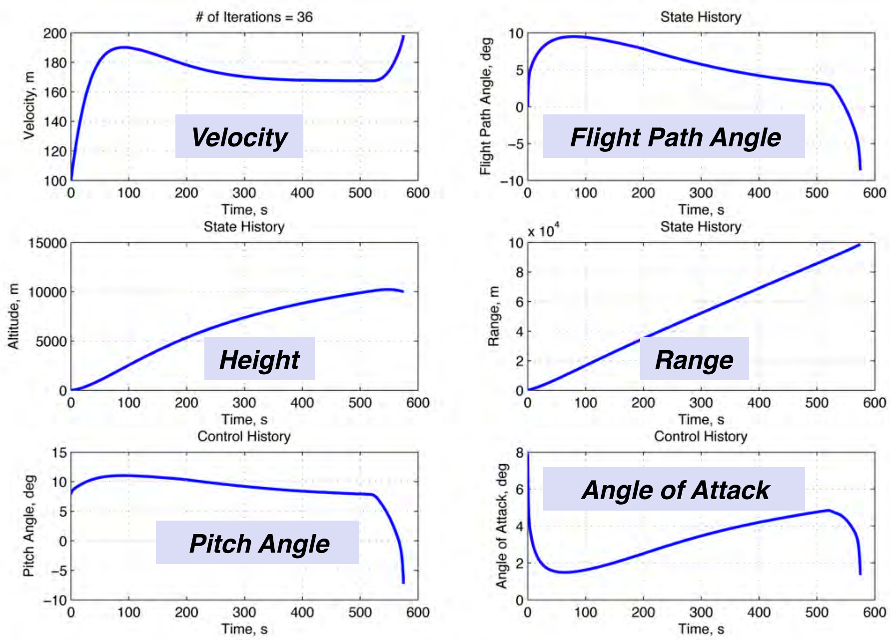


Recall Energy-Height Approximation



Power loss with altitude greater in the simulation than in E-H approximation

~Minimum-Time Trajectory,
t_f = 575 sec, 36 Iterations



Minimum Fuel-to-Climb Optimization Problem

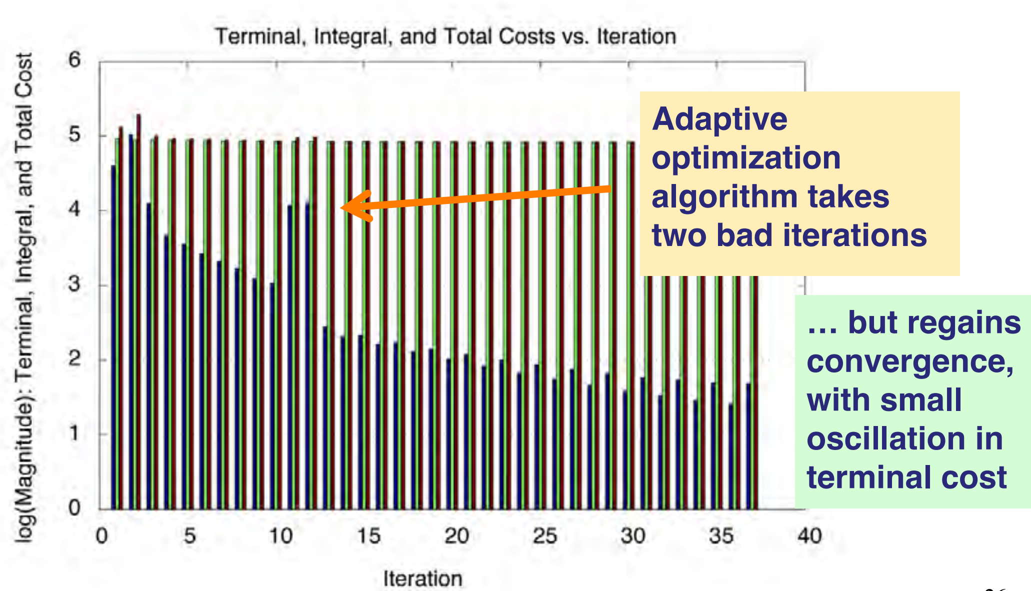
Minimum-Fuel Cost Function with Augmented Trajectory Damping

$$J(t_f) = \frac{1}{2}\left\{\left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]^T \mathbf{P}_f \left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]\right\}$$
$$+\frac{1}{2}\int_{t_0}^{t_f}\left\{\dot{m} + \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \dot{\mathbf{x}}^T(t)\mathbf{Q}_{\dot{\mathbf{x}}}\dot{\mathbf{x}}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]\right\} dt$$

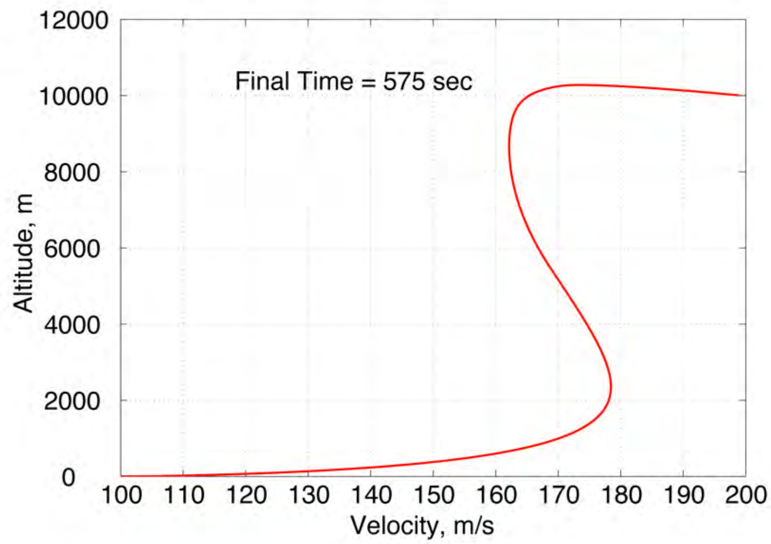
- **Same cost-function weights**
- **Different Integrand**
  - **Fuel-flow rate in the integrand for minimizing total fuel use**

Minimum-Fuel Cost History,

t_f = 575 sec, 36 Iterations

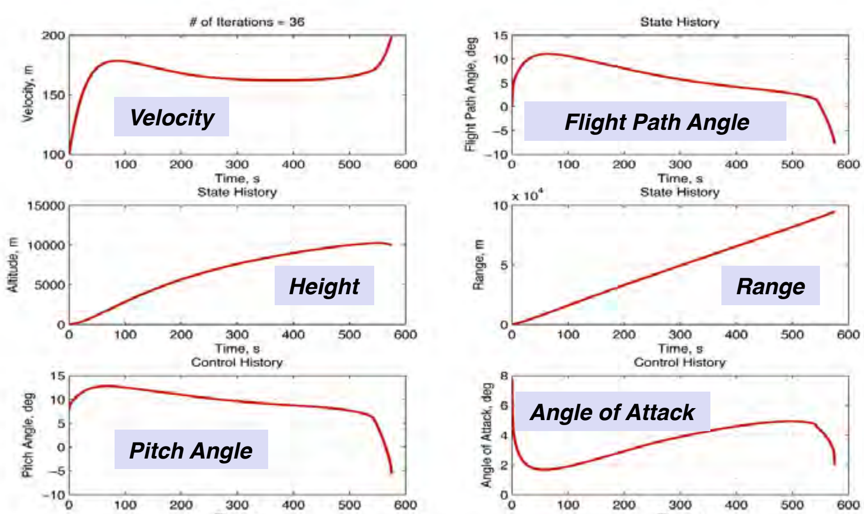


Minimum-Fuel Altitude vs. Velocity, t_f = 575 sec, 36 Iterations



~Minimum-Fuel Trajectory,

t_f = 575 sec, 36 Iterations

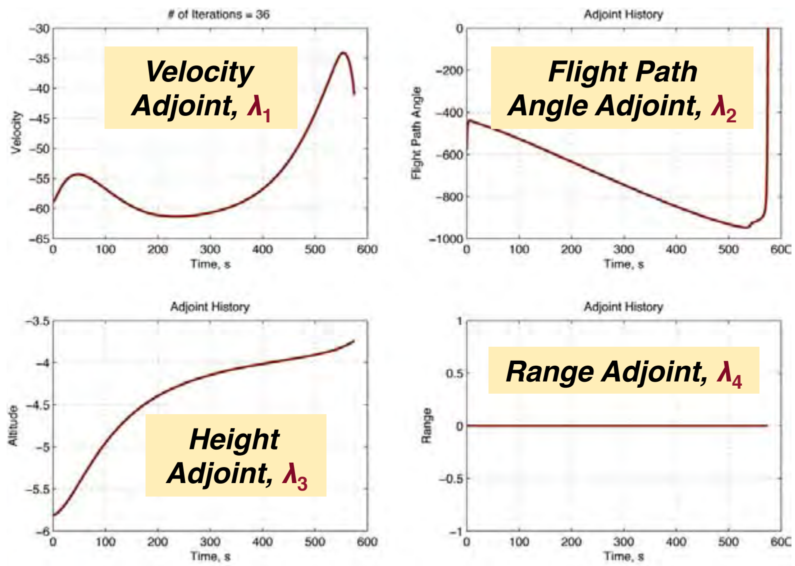


Virtually identical to minimum-time solution

… but less fuel is used
. Minimum time: 849 kg
. Minimum fuel: 831 kg

Minimum-Fuel Adjoint Vector History, $\lambda(t)$,

t_f = 575 sec, 36 Iterations



Cost sensitivity to state perturbations over time

$$\boldsymbol{\lambda}(t_f) = \left[\frac{\partial \phi}{\partial \mathbf{x}}(t_f)\right]^T = \Delta \mathbf{x}^T(t_f)\mathbf{P}_f$$

Next Time: Neighboring-Optimal Control via Linear-Quadratic Feedback

Reading

OCE: Section 3.7

Supplemental Material

Modal Properties of the System

Linearized Equations for Velocity and Flight Path Angle Perturbations, Using Angle of Attack as the Control

$$\begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \end{bmatrix} = \begin{bmatrix} (T_V - D_V) & -g \\ L_V / V_o & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix} + \begin{bmatrix} \sim 0 \\ L_\alpha / V_o \end{bmatrix} \Delta\alpha$$

$$State: \quad \Delta\mathbf{x} = \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix}$$

$$Control: \quad \Delta\mathbf{u} = \Delta\alpha$$

$T_V \triangleq \dfrac{\partial(Thrust/m)}{\partial V} =$ Sensitivity of acceleration-due-to-thrust to velocity variation

$= 0$ for this problem because [[**thrust = maximum thrust**]]

$D_V \triangleq \dfrac{\partial(Drag/m)}{\partial V} =$ Sensitivity of acceleration-due-to-drag to velocity variation > 0

$L_V \triangleq \dfrac{\partial(Lift/m)}{\partial V} =$ Sensitivity of acceleration-due-to-lift to velocity variation > 0

$L_\alpha \triangleq \dfrac{\partial(Lift/m)}{\partial \alpha} =$ Sensitivity of acceleration-due-to-lift to angle-of-attack variation > 0

Characteristic Equation and Stability, Using Angle of Attack as the Control Variable

Characteristic Equation

$$|s\mathbf{I} - \mathbf{F}| = \left| s\mathbf{I} - \begin{bmatrix} -D_V & -g \\ L_V / V_o & 0 \end{bmatrix} \right| = \begin{vmatrix} (s + D_V) & g \\ -L_V / V_o & s \end{vmatrix}$$

$$= s(s + D_V) + g\,L_V / V_o$$

$$= s^2 + D_V s + g\,L_V / V_o$$

$$= s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Natural Frequency and Damping Ratio

$$0 = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\omega_n = \sqrt{g\,L_V / V_o} \simeq \sqrt{2}\,\frac{g}{V_0} \simeq \frac{13.9}{V_o(m/s)}; \quad Period \simeq 0.453\,V_o, \sec$$

$$\zeta = \frac{D_V}{2\sqrt{g\,L_V / V_o}} \simeq \frac{\sqrt{2}}{2}\left(\frac{C_D}{C_L}\right)$$

$$\text{"Total Damping"} = 2\zeta\omega_n = D_V$$

Linearized Equations for Velocity and Flight Path Angle Perturbations, Using Pitch Angle as the Control

$$\begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \end{bmatrix} = \begin{bmatrix} -D_V & -g \\ L_V/V_o & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix} + \begin{bmatrix} \sim 0 \\ L_\alpha/V_o \end{bmatrix} \Delta\alpha$$

$$State: \quad \Delta\mathbf{x} = \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix}$$

$$Control: \quad \Delta\mathbf{u} = \Delta\theta = \Delta\alpha + \Delta\gamma$$

Replace angle of attack by pitch angle for control

$$\begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \end{bmatrix} = \begin{bmatrix} -D_V & -g \\ L_V/V_o & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix} + \begin{bmatrix} \sim 0 \\ L_\alpha/V_o \end{bmatrix} (\Delta\theta - \Delta\gamma)$$

$$= \begin{bmatrix} -D_V & -g \\ L_V/V_o & -L_\alpha/V_o \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix} + \begin{bmatrix} \sim 0 \\ L_\alpha/V_o \end{bmatrix} \Delta\theta$$

Characteristic Equation and Stability, Using Pitch Angle as the Control Variable

$$|s\mathbf{I} - \mathbf{F}| = \left| s\mathbf{I} - \begin{bmatrix} -D_V & -g \\ L_V/V_o & -L_\alpha/V_o \end{bmatrix} \right| = \begin{vmatrix} (s + D_V) & g \\ -L_V/V_o & \left(s + L_\alpha/V_o\right) \end{vmatrix}$$

$$= \left(s + L_\alpha/V_o\right)(s + D_V) + g\,L_V/V_o$$

$$= s^2 + \left(L_\alpha/V_o + D_V\right)s + \left(g\,L_V/V_o + D_V\,L_\alpha/V_o\right)$$

$$= s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$0 = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\omega_n = \sqrt{\left(g\,L_V/V_o + D_V\,L_\alpha/V_o\right)}$$

$$\zeta = \frac{\left(L_\alpha/V_o + D_V\right)}{2\sqrt{\left(g\,L_V/V_o + D_V\,L_\alpha/V_o\right)}}$$

- **Natural frequency is increased**
- **Damping is increased**

$$\text{"Total Damping"} = 2\zeta\omega_n = \left(L_\alpha/V_o + D_V\right)$$

Definitions and Numerical Values for Variables and Constants

$$T = T_{SL}\left(\frac{\rho}{\rho_{SL}}\right)\delta T = T_{SL}\left(e^{-\beta h}\right)\delta T = \text{Thrust, N;}$$

$\delta T =$ Throttle setting, %, = 100% for minimum-time/fuel problem

$SFC =$ Specific Fuel Consumption = 10 g/kN-s

$C_L = C_{L_\alpha}\alpha =$ Lift coefficient $= 5.7\alpha$

$$C_D = \frac{\left(C_{D_o} + \varepsilon C_L^{\,2}\right)}{\sqrt{1-\left(V/V_{sound}\right)^2}} = \text{Drag coefficient}$$

$$= \left(0.025 + 0.072 C_L^{\,2}\right) \Big/ \sqrt{1-\left(V/V_{sound}\right)^2}$$

$S =$ Reference area $= 21.5\ \text{m}^2$

$m_E =$ Vehicle mass = 4,550 kg ~ constant

$V =$ Velocity, m/s

$\gamma =$ Flight path angle, rad

$h =$ Altitude (or height), m

$r =$ Range, m

$m =$ Mass, kg

$\alpha =$ **Angle of attack**, rad; $\alpha_{max} = 10°$

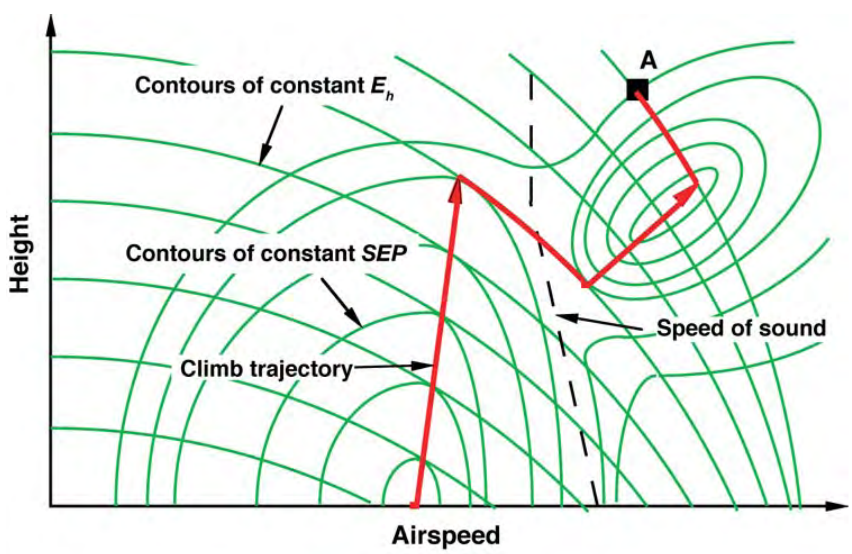
$\rho =$ Air density $= \rho_{SL} e^{-\beta h} = 1.225 e^{-h/9{,}042}\ \text{kg/m}^3$

$g =$ Gravitational acceleration $= 9.801\ \text{m/s}^2$

Angle of attack has linear effect on lift and quadratic effect on drag

Supersonic Energy Climb

Objective: Minimize time or fuel to climb to desired altitude and airspeed



Approximate Optimal Trajectory produced by a Switching Curve

Energy State Profile, $t_f = 570$ sec

$$\textbf{Energy State}: \ E = \frac{1}{mg}\left(\frac{mV^2}{2} + mgh\right) = \frac{V^2}{2g} + h, \quad meters$$

- **Optimized Energy State is monotonic and always increasing**
- **Rate of change decreases with altitude**

Specific Excess Power :

$$SEP = \frac{V}{mg}(Thrust - Drag)$$