Supporting Material for Chapter 7

Introduction

Some material that was previously in the main textbook has been moved to this supplement to allow space for new material in the textbook. All the summaries of equations for calculating the parametric analysis of the gas turbine engines with losses can be found here. In addition, you will also find the derivation of the optimum bypass ratio α^* for the turbofan engine and the optimum turbine expansion ratio τ_{tL}^* for the turboprop engine.

- 7.2.SM Summary of Equations: Turbojet Engine
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7.2.SM Summary of Equations: Turbojet Engine

Inputs:

$$M_0,\ T_0(\mathrm{K},\ ^\circ\mathrm{R}),\ \gamma_c,\ c_{pc}\left(\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^\circ\mathrm{R}}\right),\ \gamma_t,\ c_{pt}\left(\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^\circ\mathrm{R}}\right),$$

$$h_{PR}\left(\frac{\mathrm{kJ}}{\mathrm{kg}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}}\right),\ \pi_{d\,\max},\ \pi_b,\ \pi_n,\ e_c,\ e_t,\ \eta_b,\ \eta_m,\ P_0/P_9,\ T_{t4}(\mathrm{K},\ ^\circ\mathrm{R},)\ \pi_c$$

Outputs:

$$\frac{F}{\dot{m}_0}\left(\frac{\mathrm{N}}{\mathrm{kg/s}},\ \frac{\mathrm{lbf}}{\mathrm{lbm/s}}\right),f,\ S\left(\frac{\mathrm{g/s}}{\mathrm{kN}},\ \frac{\mathrm{lbm/h}}{\mathrm{lbf}}\right),\ \eta_{Th},\ \eta_P,\ \eta_O,\ \eta_c,\ \eta_t,\ \text{etc.}$$

Equations:

$$R_c = \frac{\gamma_c - 1}{\gamma_c} c_{pc} \tag{SM7.1a}$$

$$R_t = \frac{\gamma_t - 1}{\gamma_t} c_{pt} \tag{SM7.1b}$$

$$a_0 = \sqrt{\gamma_c R_c g_c T_0} \tag{SM7.1c}$$

$$V_0 = a_0 M_0 \tag{SM7.1d}$$

$$\tau_r = 1 + \frac{\gamma_c - 1}{2} M_0^2 \tag{SM7.1e}$$

$$\pi_r = \tau_r^{\gamma_t/(\gamma_t - 1)} \tag{SM7.1f}$$

$$\eta_r = 1 \quad \text{for } M_0 \leq 1 \tag{SM7.1g}$$

$$\eta_r = 1 - 0.075(M_0 - 1)^{1.35} \quad \text{for } M_0 > 1 \tag{SM7.1h}$$

$$\pi_d = \pi_{d\,\max} \eta_r \tag{SM7.1i}$$

$$\tau_\lambda = \frac{c_{pt} T_{t4}}{c_{pc} T_0} \tag{SM7.1j}$$

$$\tau_c = \pi_c^{(\gamma_c - 1)/(\gamma_c e_c)} \tag{SM7.1k}$$

$$\eta_c = \frac{\pi_c^{(\gamma_c - 1)/\gamma_c} - 1}{\tau_c - 1} \tag{SM7.1l}$$

$$f = \frac{h_{t4} - h_{t3}}{\eta_b h_{PR} - h_{t4}} \tag{SM7.1m}$$

$$\tau_t = 1 - \frac{1}{\eta_m (1 + f)} \frac{\tau_r}{\tau_\lambda} (\tau_c - 1) \tag{SM7.1n}$$

$$\pi_t = \tau_t^{\gamma_t [(\gamma_t - 1) e_t]} \tag{SM7.1o}$$

$$\eta_t = \frac{1 - \tau_t}{1 - \tau_t^{1/e_t}} \tag{SM7.1p}$$

$$\frac{P_{t9}}{P_9} = \frac{P_0}{P_9}\pi_r\pi_d\pi_c\pi_b\pi_t\pi_n \tag{SM7.1q}$$

$$M_9 = \sqrt{\frac{2}{\gamma_t - 1}\left[\left(\frac{P_{t9}}{P_9}\right)^{(\gamma_t-1)/\gamma_t} - 1\right]} \tag{SM7.1r}$$

$$\frac{T_9}{T_0} = \frac{\tau_\lambda\tau_t}{(P_{t9}/P_9)^{(\gamma_t-1)/\gamma_t}}\frac{c_{pc}}{c_{pt}} \tag{SM7.1s}$$

$$\frac{V_9}{a_0} = M_9\sqrt{\frac{\gamma_t R_t T_9}{\gamma_c R_c T_0}} \tag{SM7.1t}$$

$$\frac{F}{\dot{m}_0} = \frac{a_0}{g_c}\left[(1+f)\frac{V_9}{a_0} - M_0 + (1+f)\frac{R_t T_9/T_0}{R_c V_9/a_0}\frac{(1-P_0/P_9)}{\gamma_c}\right] \tag{SM7.1u}$$

$$S = \frac{f}{F/\dot{m}_0} \tag{SM7.1v}$$

$$\eta_{Th} = \frac{a_0^2[(1+f)(V_9/a_0)^2 - M_0^2]}{2g_c f h_{PR}} \tag{SM7.1w}$$

$$\eta_P = \frac{2g_c V_0(F/\dot{m}_0)}{a_0^2[(1+f)(V_9/a_0)^2 - M_0^2]} \tag{SM7.1x}$$

$$\eta_O = \eta_P\eta_{Th} \tag{SM7.1y}$$

7.3.SM Summary of Equations: Afterburning Turbojet Engine

Inputs:

$$M_0,\ T_0(\mathrm{K},\ {}^\circ R),\ \gamma_c,\ c_{pc}\left(\frac{\mathrm{kJ}}{\mathrm{kg\cdot K}},\ \frac{\mathrm{Btu}}{\mathrm{lbm\cdot {}^\circ R}}\right),\ \gamma_t,\ c_{pt}\left(\frac{\mathrm{kJ}}{\mathrm{kg\cdot K}},\ \frac{\mathrm{Btu}}{\mathrm{lbm\cdot {}^\circ R}}\right),$$

$$h_{PR}\left(\frac{\mathrm{kJ}}{\mathrm{kg}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}}\right),\ \gamma_{\mathrm{AB}},\ c_{p\mathrm{AB}}\left(\frac{\mathrm{kJ}}{\mathrm{kg\cdot K}},\ \frac{\mathrm{Btu}}{\mathrm{lbm\cdot {}^\circ R}}\right),\ \pi_{d\,\max},\ \pi_b,\ \pi_{\mathrm{AB}},\ \pi_n,\ e_c,\ e_t,$$

$$\eta_b,\ \eta_{\mathrm{AB}},\ \eta_m,\ P_0/P_9,\ T_{t4}(\mathrm{K},\ {}^\circ\mathrm{R}),\ T_{t7}(\mathrm{K}^\circ\mathrm{R}),\ \pi_c$$

Outputs:

$$\frac{F}{\dot{m}_0}\left(\frac{\mathrm{N}}{\mathrm{kg/s}},\ \frac{\mathrm{lbf}}{\mathrm{lbm/s}}\right), f,\ f_{\mathrm{AB}},\ S\left(\frac{\mathrm{g/s}}{\mathrm{kN}},\ \frac{\mathrm{lbm/h}}{\mathrm{lbf}}\right),\ \eta_{Th},\ \eta_P,\ \eta_O,\ \eta_c,\ \eta_t,\ \text{etc.}$$

Equations:

Equations (SM7.1a–SM7.1p) and the following:

$$R_{AB} = \frac{\gamma_{AB} - 1}{\gamma_{AB}} c_{pAB} \tag{SM7.2a}$$

$$f_{AB} = (1 + f) \frac{h_{t7} - h_{t4}\tau_t}{\eta_{AB} h_{PR} - h_{t7}} \tag{SM7.2b}$$

$$\frac{P_{t9}}{P_9} = \frac{P_0}{P_9} \pi_r \pi_d \pi_c \pi_b \pi_t \pi_{AB} \pi_n \tag{SM7.2c}$$

$$\frac{T_9}{T_0} = \frac{T_{t7}/T_0}{(P_{t9}/P_9)^{(\gamma_{AB}-1)/\gamma_{AB}}} \tag{SM7.2d}$$

$$M_9^2 = \frac{2}{\gamma_{AB} - 1} \left[\left(\frac{P_{t9}}{P_9} \right)^{(\gamma_{AB}-1)/\gamma_{AB}} - 1 \right] \tag{SM7.2e}$$

$$\frac{V_9}{a_0} = M_9 \sqrt{\frac{\gamma_{AB} R_{AB} T_9}{\gamma_c R_c T_0}} \tag{SM7.2f}$$

$$\begin{aligned} \frac{F}{\dot{m}_0} = \frac{a_0}{g_c} \Bigg[& (1 + f + f_{AB}) \frac{V_9}{a_0} - M_0 + (1 + f + f_{AB}) \\ & \times \frac{R_{AB}}{R_c} \frac{T_9/T_0}{V_9/a_0} \frac{1 - P_0/P_9}{\gamma_c} \Bigg] \end{aligned} \tag{SM7.2g}$$

$$S = \frac{f + f_{AB}}{F/\dot{m}_0} \tag{SM7.2h}$$

$$\eta_P = \frac{2 g_c V_0 (F/\dot{m}_0)}{a_0^2 [(1 + f + f_{AB})(V_9/a_0)^2 - M_0^2]} \tag{SM7.2i}$$

$$\eta_{Th} = \frac{a_0^2 [(1 + f + f_{AB})(V_9/a_0)^2 - M_0^2]}{2 g_c (f + f_{AB}) h_{PR}} \tag{SM7.2j}$$

$$\eta_O = \eta_P \eta_{Th} \tag{SM7.2k}$$

7.4a.SM Summary of Equations: Separate-Exhaust-Stream Turbofan Engine

Inputs:

$$M_0,\ T_0(\mathrm{K},\ ^\circ\mathrm{R}),\ \gamma_c,\ c_{pc}\left(\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^\circ\mathrm{R}}\right),\ \gamma_t,\ c_{pt}\left(\frac{\mathrm{kJ}}{\mathrm{kg}\cdot K},\ \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^\circ\mathrm{R}}\right),$$

$$h_{PR}\left(\frac{\mathrm{kJ}}{\mathrm{kg}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}}\right),\ \pi_{d\,\max},\ \pi_b,\ \pi_n,\ \pi_{fn},\ e_c,\ e_f,\ e_t,\ \eta_b,$$

$$\eta_m,\ P_0/P_9,\ P_0/P_{19},\ T_{t4}(\mathrm{K},\ ^\circ\mathrm{R}),\ \pi_c,\ \pi_f,\ \alpha$$

Outputs:

$$\frac{F}{\dot{m}_0}\left(\frac{\mathrm{N}}{\mathrm{kg/s}},\ \frac{\mathrm{lbf}}{\mathrm{lbm/s}}\right), f,\ S\left(\frac{\mathrm{g/s}}{\mathrm{kN}},\ \frac{\mathrm{lbm/h}}{\mathrm{lbf}}\right),\ \eta_{Th},\ \eta_P,\ \eta_O,\ \eta_c,\ \eta_t,\ \text{etc.}$$

Equations:

$$R_c = \frac{\gamma_c - 1}{\gamma_c} c_{pc} \tag{SM7.3a}$$

$$R_t = \frac{\gamma_t - 1}{\gamma_t} c_{pt} \tag{SM7.3b}$$

$$a_0 = \sqrt{\gamma_c R_c g_c T_0} \tag{SM7.3c}$$

$$V_0 = a_0 M_0 \tag{SM7.3d}$$

$$\tau_r = 1 + \frac{\gamma_c - 1}{2} M_0^2 \tag{SM7.3e}$$

$$\pi_r = \tau_r^{\gamma_c/(\gamma_c - 1)} \tag{SM7.3f}$$

$$\eta_r = 1 \quad \text{for } M_0 \le 1 \tag{SM7.3g}$$

$$\eta_r = 1 - 0.075(M_0 - 1)^{1.35} \quad \text{for } M_0 > 1 \tag{SM7.3h}$$

$$\pi_d = \pi_{d\,\max}\eta_r \tag{SM7.3i}$$

$$\tau_\lambda = \frac{c_{pt} T_{t4}}{c_{pc} T_0} \tag{SM7.3j}$$

$$\tau_c = \pi_c^{(\gamma_c - 1)/(\gamma_c e_c)} \tag{SM7.3k}$$

$$\eta_c = \frac{\pi_c^{(\gamma_c-1)/\gamma_c} - 1}{\tau_c - 1} \tag{SM7.3l}$$

$$\tau_f = \pi_f^{(\gamma_c-1)/(\gamma_c e_f)} \tag{SM7.3m}$$

$$\eta_f = \frac{\pi_f^{(\gamma_c-1)/\gamma_c} - 1}{\tau_f - 1} \tag{SM7.3n}$$

$$f = \frac{h_{t4} - h_{t3}}{\eta_b h_{PR} - h_{t4}} \tag{SM7.3o}$$

$$\tau_t = 1 - \frac{1}{\eta_m(1+f)} \frac{\tau_r}{\tau_\lambda}[\tau_c - 1 + \alpha(\tau_f - 1)] \tag{SM7.3p}$$

$$\pi_t = \tau_t^{\gamma_t/[(\gamma_t-1)e_t]} \tag{SM7.3q}$$

$$\eta_t = \frac{1 - \tau_t}{1 - \tau_t^{1/e_t}} \tag{SM7.3r}$$

Separate-stream turbofan engines are generally used with subsonic aircraft, and the pressure ratio across both primary and secondary nozzles is not very large. As a result, often convergent-only nozzles are utilized. In this case, if the nozzles are choked, we have

$$\frac{P_{t19}}{P_{19}} = \left(\frac{\gamma_c + 1}{2}\right)^{\gamma_c/(\gamma_c-1)} \quad \text{and} \quad \frac{P_{t9}}{P_9} = \left(\frac{\gamma_t + 1}{2}\right)^{\gamma_t/(\gamma_t-1)} \tag{SM7.3s}$$

Thus

$$\frac{P_0}{P_{19}} = \frac{P_{t19}/P_{19}}{P_{19}/P_0} = \frac{[(\gamma_c + 1)/2]^{\gamma_c/(\gamma_c-1)}}{\pi_r \pi_d \pi_f \pi_{fn}} \tag{SM7.3t}$$

and

$$\frac{P_0}{P_9} = \frac{P_{t9}/P_9}{P_9/P_0} = \frac{[(\gamma_t + 1)/2]^{\gamma_t/(\gamma_t-1)}}{\pi_r \pi_d \pi_c \pi_b \pi_t \pi_n} \tag{SM7.3u}$$

Note that these two expressions are valid only when both P_9 and P_{19} are greater than P_0. If these expressions predict P_9 and P_{19} less than P_0, the nozzles will not be choked. In this case, we take $P_{19} = P_0$ and/or $P_9 = P_0$.

$$\frac{P_{t9}}{P_9} = \frac{P_0}{P_9} \pi_r \pi_d \pi_c \pi_b \pi_t \pi_n \tag{SM7.3v}$$

$$M_9 = \sqrt{\frac{2}{\gamma_t - 1}\left[\left(\frac{P_{t9}}{P_9}\right)^{(\gamma_t-1)/\gamma_t} - 1\right]} \tag{SM7.3w}$$

$$\frac{T_9}{T_0} = \frac{\tau_\lambda \tau_t}{(P_{t9}/P_9)^{(\gamma_t-1)/\gamma_t}}\frac{c_{pc}}{c_{pt}} \tag{SM7.3x}$$

$$\frac{V_9}{a_0} = M_9\sqrt{\frac{\gamma_t R_t T_9}{\gamma_c R_c T_0}} \tag{SM7.3y}$$

$$\frac{P_{t19}}{P_{19}} = \frac{P_0}{P_{19}}\pi_r \pi_d \pi_f \pi_{fn} \tag{SM7.3z}$$

$$M_{19} = \sqrt{\frac{2}{\gamma_c - 1}\left[\left(\frac{P_{t19}}{P_{19}}\right)^{(\gamma_c-1)/\gamma_c} - 1\right]} \tag{SM7.3aa}$$

$$\frac{T_{19}}{T_0} = \frac{\tau_r \tau_f}{(P_{t19}/P_{19})^{(\gamma_c-1)/\gamma_c}} \tag{SM7.3ab}$$

$$\frac{V_{19}}{a_0} = M_{19}\sqrt{\frac{T_{19}}{T_0}} \tag{SM7.3ac}$$

$$\begin{aligned}\frac{F}{\dot{m}_0} = \frac{1}{1+\alpha}\frac{a_0}{g_c}\Bigg[(1+f)\frac{V_9}{a_0} - M_0 + (1+f)\\ \times \frac{R_t T_9/T_0}{R_c V_9/a_0}\frac{1-P_0/P_9}{\gamma_c}\Bigg] + \frac{\alpha}{1+1\alpha}\frac{a_0}{g_c}\\ \times \left(\frac{V_{19}}{a_0} - M_0 + \frac{T_{19}/T_0}{V_{19}/a_0}\frac{1-P_0/P_{19}}{\gamma_c}\right)\end{aligned} \tag{SM7.3ad}$$

$$S = \frac{f}{(1+\alpha)F/\dot{m}_0} \tag{SM7.3ae}$$

$$\text{Thrust ratio (FR)} = \frac{(1+f)\dfrac{V_9}{a_0} - M_0 + (1+f)\dfrac{R_t T_9/T_0}{R_c V_9/a_0}\dfrac{1-P_0/P_9}{\gamma_c}}{\dfrac{V_{19}}{a_0} - M_0 + \dfrac{T_{19}/T_0}{V_{19}/a_0}\dfrac{1-P_0/P_{19}}{\gamma_c}} \tag{SM7.3af}$$

$$\eta_P = \frac{2M_0[(1+f)V_9/a_0 + \alpha(V_{19}/a_0) - (1+\alpha)M_0]}{(1+f)(V_9/a_0)^2 + \alpha(V_{19}/a_0)^2 - (1+\alpha)M_0^2} \tag{SM7.3ag}$$

$$\eta_{Th} = \frac{a_0^2[(1+f)(V_9/a_0)^2 + \alpha(V_{19}/a_0)^2 - (1+\alpha)M_0^2]}{2g_c f h_{PR}} \tag{SM7.3ah}$$

$$\eta_O = \eta_P \eta_{Th} \tag{SM7.3ai}$$

7.4b.SM Turbofan Engine: Optimum Bypass Ratio α^*

As was true for the turbofan with no losses, we may obtain an expression that allows us to determine the bypass ratio α^* that leads to minimum thrust specific fuel consumption. For a given set of such prescribed variables $(\tau_r, \pi_c, \pi_f, \tau_\lambda, V_0)$, we may locate the minimum S by taking the partial derivative of S with respect to the bypass ratio α and set equal to zero. We consider the case where the exhaust pressures of both the fan stream and the core stream equal the ambient pressure $P_0 = P_9 = P_{19}$. Because the fuel/air ratio is not a function of bypass ratio, we have

$$S = \frac{f}{(1+\alpha)(F/\dot{m}_0)}$$

$$\frac{\partial S}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\frac{f}{(1+\alpha)(F/\dot{m}_0)}\right] = 0$$

$$\frac{\partial S}{\partial \alpha} = \frac{-f}{[(1+\alpha)(F/\dot{m}_0)]^2}\frac{\partial}{\partial \alpha}\left[(1+\alpha)\left(\frac{F}{\dot{m}_0}\right)\right] = 0$$

Thus $\partial S/\partial \alpha = 0$ is satisfied by

$$\frac{\partial}{\partial \alpha}\left[\frac{g_c}{V_0}(1+\alpha)\left(\frac{F}{\dot{m}_0}\right)\right] = 0$$

where

$$\frac{g_c}{V_0}(1+\alpha)\left(\frac{F}{\dot{m}_0}\right) = (1+f)\left(\frac{V_9}{V_0} - 1\right) + \alpha\left(\frac{V_{19}}{V_0} - 1\right)$$

Then the optimum bypass ratio is given by the following expression:

$$\frac{\partial}{\partial \alpha}\left[(1+f)\left(\frac{V_9}{V_0} - 1\right) + \alpha\left(\frac{V_{19}}{V_0} - 1\right)\right] = (1+f)\frac{\partial}{\partial \alpha}\left(\frac{V_9}{V_0}\right) + \frac{V_{19}}{V_0} - 1 = 0 \tag{i}$$

However,

$$\frac{1}{2V_9/V_0}\frac{\partial}{\partial\alpha}\left[\left(\frac{V_9}{V_0}\right)^2\right] = \frac{\partial}{\partial\alpha}\left(\frac{V_9}{V_0}\right)$$

Thus Eq. (i) becomes

$$\left(\frac{V_9}{V_0}\right)_{\alpha^*} = -\frac{1+f}{2}\frac{\partial/\partial\alpha[(V_9/V_0)^2]}{V_{19}/V_0 - 1} \tag{ii}$$

Note that

$$\left(\frac{V_9}{V_0}\right)^2 = \frac{1}{M_0^2}\left(\frac{V_9}{a_0}\right)^2 = \frac{1}{[2/(\gamma_c-1)](\tau_r-1)}\left(\frac{V_9}{a_0}\right)^2$$
$$= \frac{1}{[2/(\gamma_c-1)](\tau_r-1)}M_9^2\frac{\gamma_t R_t T_9}{\gamma_c R_c T_0}$$

Using Eqs. (7.41) and (7.42), we have

$$\left(\frac{V_9}{V_0}\right)^2 = \frac{\tau_\lambda\tau_t}{\tau_r-1}\left[1-\left(\frac{P_{t9}}{P_9}\right)^{-(\gamma_t-1)/\gamma_t}\right] \tag{iii}$$

where

$$\frac{P_{t9}}{P_9} = \pi_r\pi_d\pi_c\pi_b\pi_t\pi_n \tag{iv}$$

Combining Eqs. (iii) and (iv), we obtain

$$\left(\frac{V_9}{V_0}\right)^2 = \frac{\tau_\lambda\tau_t}{\tau_r-1}\left[1-\frac{1}{\Pi(\pi_t)^{(\gamma_t-1)/\gamma_t}}\right] \tag{v}$$

where

$$\Pi = (\pi_r\pi_d\pi_c\pi_b\pi_n)^{(\gamma_t-1)/\gamma_t} \tag{7.57}$$

Noting that

$$\pi_t^{(\gamma_t-1)/\gamma_t} = \tau_t^{1/e_t}$$

we see that then Eq. (v) becomes

$$\left(\frac{V_9}{V_0}\right)^2 = \frac{\tau_\lambda}{\tau_r-1}\left(\tau_t - \frac{1}{\Pi}\tau_t^{-(1-e_t)/e_t}\right) \tag{vi}$$

To evaluate the partial derivative of Eq. (ii), we apply the chain rule to Eq. (vi) as follows:

$$\frac{\partial}{\partial \alpha}\left[\left(\frac{V_9}{V_0}\right)^2\right] = \frac{\partial \tau_t}{\partial \alpha}\frac{\partial}{\partial \tau_t}\left[\left(\frac{V_9}{V_0}\right)^2\right]$$

$$= \frac{\partial \tau_t}{\partial \alpha}\frac{\tau_\lambda}{\tau_r - 1}\left(1 + \frac{1-e_t}{e_t}\frac{\tau_t^{-1/e_t}}{\Pi}\right) \tag{vii}$$

Because

$$\tau_t = 1 - \frac{1}{\eta_m(1+f)}\frac{\tau_r}{\tau_\lambda}[\tau_c - 1 + \alpha(\tau_f - 1)]$$

then

$$\frac{\partial \tau_t}{\partial \alpha} = -\frac{\tau_r(\tau_f - 1)}{\eta_m \tau_\lambda(1+f)} \tag{viii}$$

Combining Eqs. (ii), (vii), and (viii) yields

$$\left(\frac{V_9}{V_0}\right)_{\alpha^*} = \frac{1}{2\eta_m(\tau_r - 1)}\frac{\tau_r(\tau_f - 1)}{V_{19}/V_0 - 1}\left(1 + \frac{1-e_t}{e_t}\frac{\tau_t^{-1/e_t}}{\Pi}\right)$$

An expression for τ_t is obtained by squaring the preceding equation, substituting for $(V_9/V_0)^2$ by using Eq. (vi), and then solving for the first τ_t within parentheses on the right side of Eq. (vi). The resulting expression for the turbine temperature ratio $\tau_t{}^*$ corresponding to the optimum bypass ratio α^* is

$$\tau_t^* = \frac{\tau_t^{-(1-e_t)/e_t}}{\Pi} + \frac{1}{\tau_\lambda(\tau_r - 1)}\left[\frac{1}{2\eta_m}\frac{\tau_r(\tau_f - 1)}{V_{19}/V_0 - 1}\left(1 + \frac{1-e_t}{e_t}\frac{\tau_t^{-1/e_t}}{\Pi}\right)\right]^2 \tag{7.56}$$

Because Eq. (7.56) is an equation for τ_t^* in terms of itself, in addition to other known values, an iterative solution is required. A starting value of τ_t^*, denoted by τ_{ti}^*, is obtained by solving Eq. (7.56) for the case when $e_t = 1$, which gives

$$\tau_{ti}^* = \frac{1}{\Pi} + \frac{1}{\tau_\lambda(\tau_r - 1)}\left[\frac{1}{2\eta_m}\frac{\tau_r(\tau_f - 1)}{V_{19}/V_0 - 1}\right]^2 \tag{7.58}$$

7.5.SM Summary of Equations: Mixed-Flow Afterburning Turbofan Engine

Inputs:

$$M_0,\ T_0(\mathrm{K},\ {}^\circ\mathrm{R}),\ \gamma_c,\ c_{pc}\left(\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^\circ\mathrm{R}}\right),\ \gamma_t,\ c_{pt}\left(\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^\circ\mathrm{R}}\right),$$

$$h_{PR}\left(\frac{\mathrm{kJ}}{\mathrm{kg}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}}\right),\ \gamma_{\mathrm{AB}},\ c_{p\mathrm{AB}}\left(\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}},\ \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^\circ\mathrm{R}}\right),\ \pi_{d\,\max},\ \pi_b,\ \pi_{\mathrm{AB}},\ \pi_{M\,\max},$$

$$\pi_n,\ e_c,\ e_f,\ e_t,\ \eta_b,\ \eta_{\mathrm{AB}},\ \eta_m,\ P_0/P_9,\ T_{t4}(\mathrm{K},\ {}^\circ\mathrm{R}),\ T_{t7}(\mathrm{K},\ {}^\circ\mathrm{R}),\ \pi_c,\ \pi_f,\ M_6$$

Outputs:

$$\frac{F}{\dot{m}_0}\left(\frac{\mathrm{N}}{\mathrm{kg/s}},\ \frac{\mathrm{lbf}}{\mathrm{lbm/s}}\right),\ f,\ f_{\mathrm{AB}},\ f_O,\ S\left(\frac{\mathrm{g/s}}{\mathrm{kN}},\ \frac{\mathrm{lbm/h}}{\mathrm{lbf}}\right),\ \alpha,\ \eta_{Th},\ \eta_P,\ \eta_O,\ \eta_c,\ \eta_t,\ \text{etc.}$$

Equations:

$$R_c = \frac{\gamma_c - 1}{\gamma_c} c_{pc} \tag{SM7.4a}$$

$$R_t = \frac{\gamma_t - 1}{\gamma_t} c_{pt} \tag{SM7.4b}$$

$$R_{\mathrm{AB}} = \frac{\gamma_{\mathrm{AB}} - 1}{\gamma_{\mathrm{AB}}} c_{p\mathrm{AB}} \tag{SM7.4c}$$

$$a_0 = \sqrt{\gamma_c R_c g_c T_0} \tag{SM7.4d}$$

$$V_0 = a_0 M_0 \tag{SM7.4e}$$

$$\tau_r = 1 + \frac{\gamma_c - 1}{2} M_0^2 \tag{SM7.4f}$$

$$\pi_r = \tau_r^{\gamma_c/(\gamma_c - 1)} \tag{SM7.4g}$$

$$\eta_r = 1 \quad \text{for } M_0 \le 1 \tag{SM7.4h}$$

$$\eta_r = 1 - 0.075(M_0 - 1)^{1.35} \quad \text{for } M_0 > 1 \tag{SM7.4i}$$

$$\pi_d = \pi_{d\,\max}\eta_r \tag{SM7.4j}$$

$$\tau_\lambda = \frac{c_{pt} T_{t4}}{c_{pc} T_0} \tag{SM7.4k}$$

$$\tau_{\lambda AB} = \frac{c_{pAB} T_{t7}}{c_{pc} T_0} \tag{SM7.4l}$$

$$\tau_c = \pi_c^{(\gamma_c - 1)/(\gamma_c e_c)} \tag{SM7.4m}$$

$$\eta_c = \frac{\pi_c^{(\gamma_c - 1)/\gamma_c} - 1}{\tau_c - 1} \tag{SM7.4n}$$

$$f = \frac{h_{t4} - h_{t3}}{\eta_b h_{PR} - h_{t4}} \tag{SM7.4o}$$

$$\tau_f = \pi_f^{(\gamma_c - 1)/(\gamma_c e_f)} \tag{SM7.4p}$$

$$\eta_f = \frac{\pi_f^{(\gamma_c - 1)/\gamma_c} - 1}{\tau_f - 1} \tag{SM7.4q}$$

$$\alpha = \frac{\eta_m (1 + f)(\tau_\lambda / \tau_r)\{1 - [\pi_f / (\pi_c \pi_b)]^{(\gamma_t - 1)e_t/\gamma_t}\} - (\tau_c - 1)}{\tau_f - 1} \tag{SM7.4r}$$

$$\tau_t = 1 - \frac{1}{\eta_m (1 + f)} \frac{\tau_r}{\tau_\lambda} [\tau_c - 1 + \alpha(\tau_f - 1)] \tag{SM7.4s}$$

$$\pi_t = \tau_t^{\gamma_t / [(\gamma_t - 1) e_t]} \tag{SM7.4t}$$

$$\eta_t = \frac{1 - \tau_t}{1 - \tau_t^{1/e_t}} \tag{SM7.4u}$$

$$\frac{P_{t16}}{P_{t6}} = \frac{\pi_f}{\pi_c \pi_b \pi_t} \tag{SM7.4v}$$

$$M_{16} = \sqrt{\frac{2}{\gamma_c - 1} \left\{ \left[\frac{P_{t16}}{P_{t6}} \left(1 + \frac{\gamma_t - 1}{2} M_6^2 \right)^{\gamma_t/(\gamma_t - 1)} \right]^{(\gamma_c - 1)/\gamma_c} - 1 \right\}} \tag{SM7.4w}$$

$$\alpha' = \frac{\alpha}{1 + f} \tag{SM7.4x}$$

$$c_{p6A} = \frac{c_{pt} + \alpha' c_{pc}}{1 + \alpha'} \tag{SM7.4y}$$

$$R_{6A} = \frac{R_t + \alpha' R_c}{1 + \alpha'} \tag{SM7.4z}$$

$$\gamma_{6A} = \frac{c_{p6A}}{c_{p6A} - R_{6A}} \tag{SM7.4aa}$$

$$\frac{T_{t16}}{T_{t6}} = \frac{T_0 \tau_r \tau_f}{T_{t4} \tau_t} \tag{SM7.4ab}$$

$$\tau_M = \frac{c_{pt}}{c_{p6A}} \frac{1 + \alpha' (c_{pc}/c_{pt})(T_{t16}/T_{t6})}{1 + \alpha'} \tag{SM7.4ac}$$

$$\phi(M_6, \gamma_6) = \frac{M_6^2\{1 + [(\gamma_t - 1)/2] M_6^2\}}{(1 + \gamma_t M_6^2)^2} \tag{SM7.4ad}$$

$$\phi(M_{16}, \gamma_{16}) = \frac{M_{16}^2\{1 + [(\gamma_c - 1)/2] M_{16}^2\}}{(1 + \gamma_c M_{16}^2)^2} \tag{SM7.4ae}$$

$$\Phi = \left[\frac{1 + \alpha'}{\dfrac{1}{\sqrt{\phi(M_6, \gamma_6)}} + \alpha' \sqrt{\dfrac{R_c \gamma_t}{R_t \gamma_c} \dfrac{T_{t16}/T_{t6}}{\phi(M_{16}, \gamma_{16})}}} \right]^2 \frac{R_{6A} \gamma_t}{R_t \gamma_{6A}} \tau_M \tag{SM7.4af}$$

$$M_{6A} = \sqrt{\frac{2\Phi}{1 - 2\gamma_{6A}\Phi + \sqrt{1 - 2(\gamma_{6A} + 1)\Phi}}} \tag{SM7.4ag}$$

$$\frac{A_{16}}{A_6} = \frac{\alpha' \sqrt{T_{t16}/T_{t6}}}{\dfrac{M_{16}}{M_6} \sqrt{\dfrac{\gamma_c R_t}{\gamma_t R_c} \dfrac{1 + [(\gamma_c - 1)/2] M_{16}^2}{1 + [(\gamma_t - 1)/2] M_6^2}}} \tag{SM7.4ah}$$

$$\pi_{M\,\text{ideal}} = \frac{(1 + \alpha')\sqrt{\tau_M}}{1 + A_{16}/A_6} \frac{\text{MFP}(M_6, \gamma_t, R_t)}{\text{MFP}(M_{6A}, \gamma_{6A}, R_{6A})} \tag{SM7.4ai}$$

$$\pi_M = \pi_{M\,\max} \pi_{M\,\text{ideal}} \tag{SM7.4aj}$$

$$\frac{P_{t9}}{P_9} = \frac{P_0}{P_9} \pi_r \pi_d \pi_c \pi_b \pi_t \pi_M \pi_{AB} \pi_n \tag{SM7.4ak}$$

Afterburner *off*

$$c_{p9} = c_{p6A} \quad R_9 = R_{6A} \quad \gamma_9 = \gamma_{6A} \quad f_{AB} = 0 \tag{SM7.4al}$$

$$\frac{T_9}{T_0} = \frac{T_{t4}\tau_t\tau_M/T_0}{(P_{t9}/P_9)^{(\gamma_9-1)/\gamma_9}} \tag{SM7.4am}$$

Afterburner *on*

$$c_{p9} = c_{pAB} \quad R_9 = R_{AB} \quad \gamma_9 = \gamma_{AB} \tag{SM7.4an}$$

$$f_{AB} = \left(1 + \frac{f}{1+\alpha}\right)\frac{\tau h_{t7} - h_{t6A}}{\eta_{AB}h_{PR} - h_{t7}} \tag{SM7.4ao}$$

$$\frac{T_9}{T_0} = \frac{T_{t7}/T_0}{(P_{t9}/P_9)^{(\gamma_9-1)/\gamma_9}} \tag{SM7.4ap}$$

Continue

$$M_9 = \sqrt{\frac{2}{\gamma_9 - 1}\left[\left(\frac{P_{t9}}{P_9}\right)^{(\gamma_9-1)/\gamma_9} - 1\right]} \tag{SM7.4aq}$$

$$\frac{V_9}{a_0} = M_9\sqrt{\frac{\gamma_9 R_9 T_9}{\gamma_c R_c T_0}} \tag{SM7.4ar}$$

$$f_O = \frac{f}{1+\alpha} + f_{AB} \tag{SM7.4as}$$

$$\frac{F}{\dot{m}_0} = \frac{a_0}{g_c}\left[(1+f_O)\frac{V_9}{a_0} - M_0 + (1+f_O)\frac{R_9}{R_c}\frac{T_9/T_0}{V_9/a_0}\frac{1-P_0/P_9}{\gamma_c}\right] \tag{SM7.4at}$$

$$S = \frac{f_O}{F/\dot{m}_0} \tag{SM7.4au}$$

$$\eta_P = \frac{2g_c V_0(F/\dot{m}_0)}{a_0^2[(1+f_O)(V_9/a_0)^2 - M_0^2]} \tag{SM7.4av}$$

$$\eta_{Th} = \frac{a_0^2[(1+f_O)(V_9/a_0)^2 - M_0^2]}{2g_c f_o h_{PR}} \tag{SM7.4aw}$$

$$\eta_O = \eta_P\eta_{Th} \tag{SM7.4ax}$$

7.6.SM Summary of Equations: Turboprop Engine

Inputs:

$$M_0,\ T_0(\mathrm{K}, {}^\circ\mathrm{R}),\ \gamma_c,\ c_{pc}\left(\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}}, \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^\circ\mathrm{R}}\right),\ \gamma_t,\ c_{pt}\left(\frac{\mathrm{kJ}}{\mathrm{kg}\cdot\mathrm{K}}, \frac{\mathrm{Btu}}{\mathrm{lbm}\cdot{}^\circ\mathrm{R}}\right),$$

$$h_{PR}\left(\frac{\mathrm{KJ}}{\mathrm{kg}}, \frac{\mathrm{Btu}}{\mathrm{lbm}}\right),\ \pi_{d\,\max},\ \pi_b,\ \pi_n,\ e_c,\ e_{tH},\ e_{tL},\ \eta_b,$$

$$\eta_g,\ \eta_{mH},\ \eta_{mL},\ \eta_{\text{prop}},\ T_{t4}(\mathrm{K}, {}^\circ\mathrm{R}),\ \pi_c, \text{ and } \tau_t \quad (\text{if known})$$

Outputs:

$$\frac{F}{\dot{m}_0}\left(\frac{\mathrm{N}}{\mathrm{kg/s}}, \frac{\mathrm{lbf}}{\mathrm{lbm/s}}\right),\ \frac{\dot{W}}{\dot{m}_0}\left(\frac{\mathrm{W}}{\mathrm{kg/s}}, \frac{\mathrm{hp}}{\mathrm{lbm/s}}\right), f,$$

$$S\left(\frac{\mathrm{g/s}}{\mathrm{kN}}, \frac{\mathrm{lbm/h}}{\mathrm{lbf}}\right),\ S_P\left(\frac{\mathrm{mg/s}}{\mathrm{W}}, \frac{\mathrm{lbm/h}}{\mathrm{hp}}\right),\ \eta_{Th},\ \eta_P,\ \eta_O,\ C_C,$$

$$C_{\text{prop}},\ C_{\text{tot}},\ \tau_{tL}^*\quad (\text{if desired}),\ \text{etc.}$$

Equations:

$$R_c = \frac{\gamma_c - 1}{\gamma_c} c_{pc} \tag{SM7.5a}$$

$$R_t = \frac{\gamma_t - 1}{\gamma_t} c_{pt} \tag{SM7.5b}$$

$$a_0 = \sqrt{\gamma_c R_c g_c T_0} \tag{SM7.5c}$$

$$V_0 = a_0 M_0 \tag{SM7.5d}$$

$$\tau_r = 1 + \frac{\gamma_c - 1}{2} M_0^2 \tag{SM7.5e}$$

$$\pi_r = \tau_r^{\gamma_c/(\gamma_c - 1)} \tag{SM7.5f}$$

$$\eta_r = 1 \quad \text{for } M_0 \le 1 \tag{SM7.5g}$$

$$\eta_r = 1 - 0.075(M_0 - 1)^{1.35} \quad \text{for } M_0 > 1 \tag{SM7.5h}$$

$$\pi_d = \pi_{d\,\max}\eta_r \tag{SM7.5i}$$

$$\tau_\lambda = \frac{c_{pt} T_{t4}}{c_{pc} T_0} \tag{SM7.5j}$$

$$\tau_c = \pi_c^{(\gamma_c - 1)/(\gamma_c e_c)} \tag{SM7.5k}$$

$$\eta_c = \frac{\pi_c^{(\gamma_c - 1)/\gamma_c} - 1}{\tau_c - 1} \tag{SM7.5l}$$

$$f = \frac{h_{t4} - h_{t3}}{\eta_b h_{PR} - h_{t4}} \tag{SM7.5m}$$

$$\tau_{tH} = 1 - \frac{\tau_r(\tau_c - 1)}{\eta_{mH}(1 + f)\tau_\lambda} \tag{SM7.5n}$$

$$\pi_{tH} = \tau_{tH}^{\gamma_t/[(\gamma_t - 1)e_{tH}]} \tag{SM7.5o}$$

$$\eta_{tH} = \frac{1 - \tau_{tH}}{1 - \tau_{tH}^{1/e_{tH}}} \tag{SM7.5p}$$

If the optimum turbine temperature ratio τ_{tL}^* is desired,

$$A = \frac{[(\gamma_c - 1)/2][M_0^2/(\tau_\lambda \tau_{tH})]}{(\eta_{\text{prop}}\eta_g \eta_{mL})^2} \tag{SM7.5q}$$

$$\Pi = (\pi_r \pi_d \pi_c \pi_b \pi_n)^{(\gamma_t - 1)/\gamma_t} \tag{SM7.5r}$$

$$\tau_{tLi}^* = \frac{\tau_{tH}^{-1/e_{tH}}}{\Pi} + A \tag{SM7.5s}$$

$$\tau_{tLi}^* = \frac{\tau_{tH}^{-1/e_{tH}}}{\Pi}\tau_{tL}^{-(1-e_{tL})/e_{tL}} + A\left(1 + \frac{1 - e_{tL}}{e_{tL}}\frac{\tau_{tH}^{-1/e_{tH}}\tau_{tL}^{-1/e_{tL}}}{\Pi}\right)^2 \tag{SM7.5t}$$

Repeat calculations, using Eq. (SM7.5t), until successive values are within 0.0001.

Else, turbine temperature τ_t has been specified:

$$\tau_{tL} = \frac{\tau_t}{\tau_{tH}} \tag{SM7.5u}$$

Continue

$$\pi_{tL} = \tau_{tL}^{\gamma_t/[(\gamma_t - 1)e_{tL}]} \tag{SM7.5v}$$

$$\eta_{tL} = \frac{1 - \tau_{tL}}{1 - \tau_{tL}^{1/e_{tL}}} \tag{SM7.5w}$$

$$\frac{P_{t9}}{P_0} = \pi_r \pi_d \pi_c \pi_b \pi_{tH} \pi_{tL} \pi_n \tag{SM7.5x}$$

if

$$\frac{P_{t9}}{P_0} > \left(\frac{\gamma_t + 1}{2}\right)^{\gamma_t/(\gamma_t - 1)} \tag{SM7.5y}$$

then

$$M_9 = 1 \quad \frac{P_{t9}}{P_9} = \left(\frac{\gamma_t + 1}{2}\right)^{\gamma_t/(\gamma_t - 1)}$$

and

$$\frac{P_0}{P_9} = \frac{P_{t9}/P_9}{P_{t9}/P_0} \tag{SM7.5z}$$

else

$$M_9 = \sqrt{\frac{2}{\gamma_t - 1}\left[\left(\frac{P_{t9}}{P_0}\right)^{(\gamma_t - 1)/\gamma_t} - 1\right]} \quad \frac{P_{t9}}{P_9} = \frac{P_{t9}}{P_0}$$

and

$$\frac{P_0}{P_9} = 1 \tag{SM7.5aa}$$

$$\frac{V_9}{a_0} = \sqrt{\frac{2\tau_\lambda \tau_{tH} \tau_{tL}}{\gamma_c - 1}\left[1 - \left(\frac{P_{t9}}{P_9}\right)^{-(\gamma_t - 1/\gamma_t)}\right]} \tag{SM7.5ab}$$

$$C_{\text{prop}} = \eta_{\text{prop}} \eta_g \eta_{mL} (1 + f) \tau_\lambda \tau_{tH} (1 - \tau_{tL}) \tag{SM7.5ac}$$

$$C_c = (\gamma_c - 1) M_0 \left[(1 + f)\frac{V_9}{a_0} - M_0 + (1 + f)\frac{R_t}{R_c}\frac{T_9/T_0}{V_9/a_0}\frac{1 - P_0/P_9}{\gamma_c}\right] \tag{SM7.5ad}$$

$$C_{\text{tot}} = C_{\text{prop}} + C_C \tag{SM7.5ae}$$

$$\frac{F}{\dot{m}_0} = \frac{C_{\text{tot}} c_{pc} T_0}{V_0} \tag{SM7.5af}$$

$$S = \frac{f}{F/\dot{m}_0} \tag{SM7.5ag}$$

$$\frac{\dot{W}}{\dot{m}_0} = C_{\text{tot}} c_{pc} T_0 \tag{SM7.5ah}$$

$$S_P = \frac{f}{C_{tot} c_{pc} T_0} \tag{SM7.5ai}$$

$$\eta_{Th} = \frac{C_{tot}}{f h_{PR}/(c_{pc} T_0)} \tag{SM7.5aj}$$

$$\eta_P = \frac{C_{tot}}{C_{prop}/\eta_{prop} + \gamma_c - 1/2[(1+f)(V_9/a_0)^2 - M_0^2]} \tag{SM7.5ak}$$

7.7a.SM Turboprop Engine: Optimal Turbine Expansion Ratio τ_{tL}^*

Turboprop or prop-fan engines are designed primarily to be low-specific-fuel-consumption engines. Thus we select τ_{tL} to make S_P a minimum. From Eq. (7.85), this is equivalent to locating the maximum of C_{tot}. We will obtain an expression for the optimum low-pressure turbine temperature ratio τ_{tL}^* that gives maximum C_{tot} with all other variables constant and when $P_9 = P_0$. We have

$$C_{tot} = C_{prop} + C_C$$

with

$$C_{prop} = \eta_{prop} \eta_g \eta_{mL} (1+f) \tau_\lambda \tau_{tH} (1 - \tau_{tL})$$

and for the case $P_9 = P_0$

$$C_C = (\gamma_c - 1) M_0 \left[(1+f) \frac{V_9}{a_0} - M_0 \right]$$

Thus the total temperature ratio of the low-pressure turbine corresponding to minimum fuel consumption is obtained by finding the maximum of C_{tot} with respect to τ_{tL}. Taking the partial derivative of C_{tot} with respect to τ_{tL} (noting that only τ_{tL} is a variable in the equation for C_{prop} and that V_9/a_0 is a function of τ_{tL} in the equation for C_c) and setting the result equal to zero gives

$$\frac{\partial C_{tot}}{\partial \tau_{tL}} = -\eta_{prop} \eta_g \eta_{mL} (1+f) \tau_\lambda \tau_{tH} + (\gamma_c - 1)(1+f) M_0 \times \frac{\partial}{\partial \tau_{tL}} \left(\frac{V_9}{a_0} \right) = 0$$

or

$$\frac{\partial}{\partial \tau_{tL}} \left(\frac{V_9}{a_0} \right) = \eta_{prop} \eta_g \eta_{mL} \frac{\tau_\lambda \tau_{tH}}{(\gamma_c - 1) M_0} \tag{7.88}$$

By the chain rule, the partial derivative of the velocity ratio can be expressed as

$$\frac{\partial}{\partial \tau_{tL}}\left(\frac{V_9}{a_0}\right) = \frac{\partial(V_9/a_0)}{\partial[(V_9/a_0)^2]}\frac{\partial[(V_9/a_0)^2]}{\partial \tau_{tL}} \qquad \text{(SM7.6a)}$$

where

$$\frac{\partial(V_9/a_0)}{\partial[(V_9/a_0)^2]} = \frac{1}{2V_9/a_0} \qquad \text{(SM7.6b)}$$

The velocity ratio $(V_9/a_0)^2$ is given by

$$\left(\frac{V_9}{a_0}\right)^2 = \frac{2\tau_\lambda \tau_{tH}\tau_{tL}}{\gamma_c - 1}\left[1 - \left(\frac{P_{t9}}{P_9}\right)^{-(\gamma_t-1)/\gamma_t}\right] \qquad \text{(7.81)}$$

where

$$\frac{P_{t9}}{P_9} = \pi_r \pi_d \pi_c \pi_b \pi_{tH}\pi_{tL}\pi_n$$

Thus

$$\left(\frac{V_9}{a_0}\right)^2 = \frac{2\tau_\lambda \tau_{tH}}{\gamma_c - 1}\left[\tau_{tL} - \frac{\tau_{tL}}{\Pi(\pi_{tH}\pi_{tL})^{(\gamma_t-1)/\gamma_t}}\right] \qquad \text{(SM7.6c)}$$

where

$$\Pi = (\pi_r \pi_d \pi_c \pi_b \pi_n)^{(\gamma_t-1)/\gamma_t}$$

Using different polytropic efficiencies for the high- and low-pressure turbines, we can write

$$\pi_{tH}^{(\gamma_t-1)/\gamma_t} = \tau_{tH}^{1/e_{tH}} \quad \text{and} \quad \pi_{tL}^{(\gamma_t-1)/\gamma_t} = \tau_{tL}^{1/e_{tL}}$$

then Eq. (SM7.6c) becomes

$$\left(\frac{V_9}{a_0}\right)^2 = \frac{2\tau_\lambda \tau_{tH}}{\gamma_c - 1}\left(\tau_{tL} - \frac{\tau_{tH}^{-1/e_{tH}}}{\Pi}\tau_{tL}^{-(1-e_{tL})/e_{tL}}\right)$$

and thus

$$\frac{\partial[(V_9/a_0)^2]}{\partial \tau_{tL}} = \frac{2\tau_\lambda \tau_{tH}}{\gamma_c - 1}\left(1 + \frac{1-e_{tL}}{e_{tL}}\frac{\tau_{tH}^{-1/e_{tH}}\tau_{tL}^{-1/e_{tL}}}{\Pi}\right) \qquad \text{(SM7.6d)}$$

Substitution of Eqs. (SM7.6a), (SM7.6b), (SM7.6c), and (SM7.6d) into Eq. (7.88) gives

$$\frac{\tau_\lambda \tau_{tH}}{(\gamma_c - 1)(V_9/a_0)}\left(1 + \frac{1-e_{tL}}{e_{tL}}\frac{\tau_{tH}^{-1/e_{tH}}\tau_{tL}^{-1/e_{tL}}}{\Pi}\right) = \eta_{\text{prop}}\eta_{\text{g}}\eta_{mL}\,\frac{\tau_\lambda\tau_{tH}}{(\gamma_c - 1)M_0}$$

and solving for the velocity ratio gives

$$\frac{V_9}{a_0} = \frac{M_0}{\left(\eta_{\text{prop}}\eta_{\text{g}}\eta_{mL}\right)}\left(1 + \frac{1-e_{tL}}{e_{tL}}\frac{\tau_{tH}^{-1/e_{tH}}\tau_{tL}^{-1/e_{tL}}}{\Pi}\right) \tag{SM7.6e}$$

Equation (SM7.6e) can be most easily solved for τ_{tL}^* by squaring this equation and then equating it to Eq. (SM7.6c). Thus

$$\begin{aligned}\left(\frac{V_9}{a_0}\right)^2 &= \frac{M_0^2}{\left(\eta_{\text{prop}}\eta_{\text{g}}\eta_{mL}\right)^2}\left(1 + \frac{1-e_{tL}}{e_{tL}}\frac{\tau_{tH}^{-1/e_{tH}}\tau_{tL}^{-1/e_{tL}}}{\Pi}\right)^2\\ &= \frac{2\tau_\lambda\tau_{tH}}{\gamma c - 1}\left(\tau_{tL} - \frac{\tau_{tH}^{-1/e_{tH}}}{\Pi}\tau_{tL}^{-(1-e_{tL})/e_{tL}}\right)\end{aligned}$$

or

$$\begin{aligned}&\tau_{tL} - \frac{\tau_{tH}^{-1/e_{tH}}}{\Pi}\tau_{tL}^{-(1-e_{tL})/e_{tL}}\\ &\quad= \frac{[(\gamma_c - 1)/2][M_0^2/(\tau_\lambda\tau_{tH})]}{\left(\eta_{\text{prop}}\eta_{\text{g}}\eta_{mL}\right)^2}\left(1 + \frac{1-e_{tL}}{e_{tL}}\frac{\tau_{tH}^{-1/e_{tH}}\tau_{tL}^{-1/e_{tL}}}{\pi}\right)^2\end{aligned}$$

Solving for the first τ_{tL} gives

$$\tau_{tL}^* - \frac{\tau_{tH}^{-1/e_{tH}}}{\Pi}\tau_{tL}^{-(1-e_{tL})/e_{tL}} + A\left(1 + \frac{1-e_{tL}}{e_{tL}}\frac{\tau_{tH}^{-1/e_{tH}}\tau_{tL}^{-1/e_{tL}}}{\Pi}\right)^2 \tag{7.89a}$$

where

$$A = \frac{[(\gamma_c - 1)/2][M_0^2/(\tau_\lambda\tau_{tH})]}{\left(\eta_{\text{prop}}\eta_{\text{g}}\eta_{mL}\right)^2} \tag{7.89b}$$

Because Eq. (7.89a) is an equation for τ_{tL}^* in terms of itself, an iterative solution is required. A starting value of τ_{tL}^*, denoted by τ_{tLi}^*, is obtained by solving Eq. (7.89a) for the case when $e_{tL} = 1$:

$$\tau_{tLi}^* = \frac{\tau_{tH}^{-1/e_{tH}}}{\Pi} + A \tag{7.90}$$

This starting value can be substituted into Eq. (7.89a) and another new value of τ_{tLi}^* calculated. This process continues until the change in successive calculations of τ_{tLi}^* is less than some small number (say, 0.0001).

7.7b.SM Summary of Equations: Afterburning Turbojet with Variable Specific Heats

Inputs:

$$M_0,\ T_0(\text{K},\ ^\circ\text{R}),\ h_{PR}(\text{kJ/kg},\ \text{Btu/lbm}),\ \pi_{d\,\text{min max}},\ \pi_b,\ \pi_{\text{AB}},$$
$$\pi_n,\ e_c,\ e_t,\ \eta_b,\ \eta_{\text{AB}},\ \eta_m,\ P_0/P_9,\ T_{t4}(\text{K},\ ^\circ\text{R}),\ T_{t7}(\text{K},\ ^\circ\text{R}),\ \pi_c$$

Outputs:

$$\frac{F}{\dot{m}_0}\left(\frac{\text{N}}{\text{kg/s}},\ \frac{\text{lbf}}{\text{lbm/s}}\right), f,\ f_{\text{AB}},\ f_o,\ S\left(\frac{\text{g/s}}{\text{kN}},\ \frac{\text{lbm/h}}{\text{lbf}}\right),\ \eta_{Th},\ \eta_P,\ \eta_O,\ \eta_c,\ \eta_t,\ \text{etc.}$$

Equations:

$\text{FAIR}\big(1,\ T_0,\ h_0,\ P_{r0},\ \phi_0,\ c_{p0},\ R_0,\ \gamma_0,\ a_0,\ 0\big)$

$$V_0 = M_0 a_0 \tag{SM7.7a}$$

$$h_{t0} = h_0 + \frac{V_0^2}{2g_c} \tag{SM7.7b}$$

$\text{FAIR}\big(2,\ T_{t0},\ h_{t0},\ P_{rt0},\ \phi_0,\ c_{pt0},\ R_{t0},\ \gamma_{t0},\ a_{t0},\ 0\big)$

$$\tau_r = \frac{h_{t0}}{h_0} \tag{SM7.7c}$$

$$\pi_r = \frac{P_{rt0}}{P_{r0}} \tag{SM7.7d}$$

$$\eta_r = 1 \quad \text{for } M_0 \le 1 \tag{SM7.7e}$$

$$\eta_r = 1 - 0.075(M_0 - 1)^{1.35} \quad \text{for } M_0 > 1 \tag{SM7.7f}$$

$$\pi_d = \pi_{d\,\text{max}}\eta_r \tag{SM7.7g}$$

$$h_{t2} = h_{t0} \tag{SM7.7h}$$

$$P_{rt2} = P_{rt0} \tag{SM7.7i}$$

$$P_{rt3} = P_{rt2}\pi_c^{1/e_c} \tag{SM7.7j}$$

FAIR(3, T_{t3}, h_{t3}, P_{rt3}, ϕ_{t3}, c_{pt3}, R_{t3}, γ_{t3}, a_{t3}, 0)

$$\tau_c = \frac{h_{t3}}{h_{t2}} \tag{SM7.7k}$$

$$P_{rt3i} = P_{rt2}\pi_c \tag{SM7.7l}$$

FAIR(3, T_{t3i}, h_{t3i}, P_{rt3i}, ϕ_{t3i}, c_{pt3i}, R_{t3i}, γ_{t3i}, a_{t3i}, 0)

$$\eta_c = \frac{h_{t3i} - h_{t2}}{h_{t3} - h_{t2}} \tag{SM7.7m}$$

Set initial value of fuel/air ratio $= f_i$.
A:

FAIR(1, T_{t4}, h_{t4}, P_{rt4}, ϕ_{t4}, c_{pt4}, R_{t4}, γ_{t4}, a_{t4}, f_i)

$$f = \frac{h_{t4} - h_{t3}}{\eta_b h_{PR} - h_{t4}} \tag{SM7.7n}$$

If $|f - f_i| > 0.0001$, then $f_i = f$ and go to A; else continue.

$$\tau_\lambda = \frac{h_{t4}}{h_0} \tag{SM7.7o}$$

$$h_{t5} = h_{t4} - \frac{h_{t3} - h_{t2}}{(1+f)\eta_m} \tag{SM7.7p}$$

FAIR(2, T_{t5}, h_{t5}, P_{rt5}, ϕ_{t5}, c_{pt5}, R_{t5}, γ_{t5}, a_{t5}, f)

$$\pi_t = \left(\frac{P_{rt5}}{P_{rt4}}\right)^{1/e_t} \tag{SM7.7q}$$

$$P_{rt5i} = \pi_r P_{rt4} \tag{SM7.7r}$$

FAIR(3, T_{t5i}, h_{t5i}, P_{rt5i}, ϕ_{t5i}, c_{pt5i}, R_{t5i}, γ_{t5i}, a_{t5i}, f)

$$\eta_t \frac{h_{t4} - h_{t5}}{h_{t4} - h_{t5i}} \tag{SM7.7s}$$

Set initial value of AB fuel/air ratio $= f_{ABi}$

B:

$$f_o = f + f_{ABi} \tag{SM7.7t}$$

FAIR(1, T_{t8}, h_{t8}, P_{rt8}, ϕ_{t8}, c_{pt8}, R_{t8}, γ_{t8}, a_{t8}, f_o)

$$f_{AB} = \frac{h_{t8} - h_{t5}}{\eta_{AB} h_{PR} - h_{t8}} \tag{SM7.7u}$$

If $|f_{AB} - f_{ABi}| > 0.0001$, then $f_{ABi} = f_{AB}$ and go to B; else continue.

$$\tau_{\lambda AB} = \frac{h_{t8}}{h_0} \tag{SM7.7v}$$

$$h_{t9} = h_{t8} \tag{SM7.7w}$$

$$P_{rt9} = P_{rt8} \tag{SM7.7x}$$

$$\frac{P_{t9}}{P_9} = \frac{P_0}{P_9} \pi_r \pi_d \pi_c \pi_b \pi_t \pi_{AB} \pi_n \tag{SM7.7y}$$

$$P_{r9} = \frac{P_{rt9}}{P_{t9}/P_9} \tag{SM7.7z}$$

FAIR(3, T_9, h_9, P_{r9}, ϕ_9, c_{p9}, R_9, γ_9, a_9, f_o)

$$V_9 = \sqrt{2g_c(h_{t9} - h_9)} \tag{SM7.7aa}$$

$$\frac{F}{\dot{m}_0} = \frac{a_0}{g_c}\left[(1 + f_o)\frac{V_9}{a_0} - M_0 + (1 + f_o)\frac{R_9}{R_0}\frac{T_9/T_0}{V_9/a_0}\frac{1 - P_0/P_9}{\gamma_0}\right] \tag{SM7.7ab}$$

$$S = \frac{f_o}{F/\dot{m}_0} \tag{SM7.7ac}$$

$$\eta_{Th} = \frac{a_0^2[(1 + f_o)(V_9/a_0)^2 - M_0^2]}{2g_c f_o h_{PR}} \tag{SM7.7ad}$$

$$\eta_P = \frac{2g_c V_0(F/\dot{m}_0)}{a_0^2[(1 + f_o)(V_9/a_0)^2 - M_0^2]} \tag{SM7.7ae}$$

$$\eta_O = \eta_P \eta_{Th} \tag{SM7.7af}$$