MoviTon: Computing Photon Geodesics in GR to make Movies

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ABSTRACT

MoviTon is a software package written in Python 3 and C++ for creating educational movies. It computes and solves the evolutionary equations for photons moving through the Kerr metric. It traces paths of photons emitted from an accretion disk to an observer in free space, which is particularly helpful in understanding the different types of emission profiles observed from real-life accretion disks. Since the evolutionary equations allow interesting photon orbits to be calculated, MoviTon is exceptionally helpful in visually understanding the behavior of photons near black holes both physical and unphysical. MoviTon also has the capability to take a camera image of an accretion disk around a black hole, in both GR and Newtonian space. It is capable of producing images with physically-correct colors and visualizing non-visible wavelengths in grayscale. MoviTon generates video files visualizing what a pitching camera around the black hole system sees. It is meant to be used for educational purposes but is a powerful raytracer itself.

Keywords: Raytracing—Kerr metric—Kerr black holes—Computational astronomy

1. INTRODUCTION

Studying null geodesics in curved spacetime helps one to understand the dynamics of the accretion process as well as the strong gravitational effects expected near very heavy objects such as black holes. The Event Horizon Telescope published the first image ever of the shadow produced by a real black hole in 2019. This black hole, one of the largest known, is at the center of the elliptical M87 galaxy, around 53 million light-years away. The estimated mass of the black hole is $(6.5 \pm 0.7) * 10^9 M_\odot$. This mass was obtained by comparing the images of the real black hole with the results produced by raytraced magnetohydronamic general relativistic simulations.

There are a large number of ray-tracing codes in existence, both professional and amateur. Some amateur codes include Riccardo Antonelli's seminal program `Starless`, featuring full raytracing in Schwarzchild geometry, as well as David Madore's single-file C kerr raytracer. These codes are mostly for personal use, such as generating wallpapers and making scientific diagrams. Codes meant to be used for research and scientific computing include GRTrans, `GYOTO`, `AstroRay`, `ARCMANCER`, `OSIRIS`, `KERTAP`, and `BHAC`. MoviTon falls somewhere between these two categories, though it leans towards the former. It is capable of producing images of accretion disks with physically-correct colors, which is better than Madore, but does not account, for example, Faraday polarization like in KERTAP.

It is possible to process, analyze, and compare observational data with the help of simulations. That is one of the reasons why a large number of raytracing codes have been developed. It is possible to simulate the motion of photons around heavy compact objects to get a better understanding of the effects of the gravitational field. For example, solving null geodesic equations allows the study of the spherical photon orbits which result in the famous photon ring. It is not always possible, however, to obtain an analytical solution for arbitrary spacetime. This is why it is important to construct numerical integrators that allow finding solutions to the geodesic equations for a large number of photons. Different numerical codes stand out for different reasons, such as the spacetimes they feature, a user-friendly interface, or their speed.

In this paper we present MoviTon. MoviTon is a software package written in Python and C++ for solving photon geodesics in general relativity and creating animated movies. It solves the evolutionary equations for photons moving through the Kerr metric. It can trace the paths of photons emitted from an accretion disk in the black hole's equatorial plane to an observer in free space. This calculation is of particular importance in astronomical research, because the strong gravity of compact heavy objects can substantially modify the observed emission profiles of accretion disks. The evolutionary equations allow some interesting photon orbits to be calculated, even for unphysical systems. MoviTon can also take a camera image of a black hole's accretion disk. It produces images for a black hole disk in 'Newtonian' space as well as in a general relativistic universe. MoviTon can modify the black hole's spin and the viewing angle and combine individual camera frames into a video.

2. THE PHYSICS AND ALGORITHM

The geodesic equation is:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0 \tag{1}$$

with the Boyer-Lindquist coordinates $x^\mu = (t, r, \theta, \varphi)$. The Christoffel symbols can be derived from the Kerr metric by the following formula:

$$\Gamma^\mu_{\sigma\nu} = \frac{1}{2}g^{\mu\alpha}[\frac{\partial g_{\alpha\nu}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\nu} - \frac{\partial g_{\sigma\nu}}{\partial x^\alpha}] \tag{2}$$

MoviTon starts its computations with a photon in the observer's image plane. The position of the photon is specified by two Cartesian coordinates x_i and y_i. The observer is assumed to be at a large distance D away from the black hole and observing it at an inclination of θ_o. A visual representation of the setup is as shown:

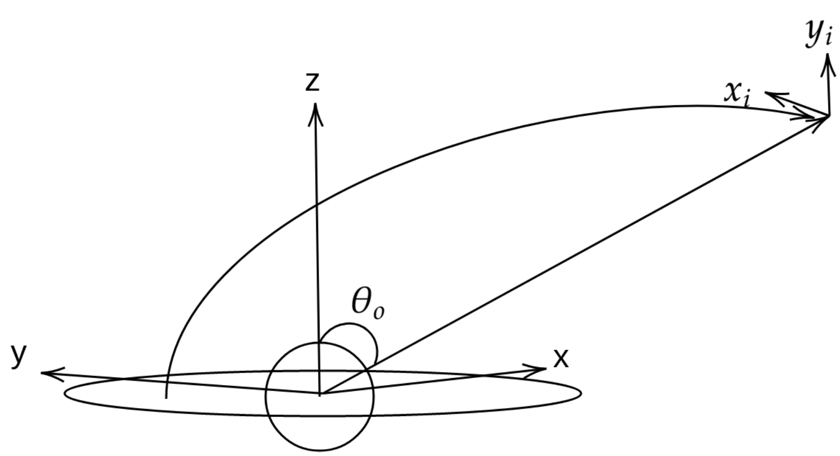

Figure 1. Observer Position

The Kerr metric is time-independent and rotation symmetric. The angular momentum and the total energy of the photon are thus conserved as the photon moves along a geodesic. This allows us to calculate both of these quantities at the beginning of the motion. By defining the following four quantities:

$$L = p_\varphi = g_{\varphi\varphi}d\varphi d\tau + g\frac{dt}{d\tau} \tag{3}$$

$$E = -p_t = -g_{tt}\frac{dt}{d\tau} - g_{t\varphi}\frac{d\varphi}{d\tau} \tag{4}$$

$$\tau' = E\tau \tag{5}$$

$$l = \frac{L}{E} \tag{6}$$

we can write two simplified equations of motion for φ and t.

$$\frac{d\varphi}{d\tau'} = \frac{lg_{tt} + g_{t\varphi}}{g_{\varphi\varphi}g_{tt} - g^2} \tag{7}$$

$$\frac{dt}{d\tau'} = -\frac{lg + g_{\varphi\varphi}}{g_{\varphi\varphi}g_{tt} - g^2} \tag{8}$$

Here l and τ' are respectively the normalized angular momentum and normalized time.

For r and θ, we write the geodesic equations:

$$\frac{d^2r}{d\tau'^2} = -\Gamma^r_{\sigma\nu}\frac{dx^\sigma}{d\tau'}\frac{dx^\nu}{d\tau'} \tag{9}$$

$$\frac{d^2\theta}{d\tau'^2} = -\Gamma^\theta_{\sigma\nu}\frac{dx^\sigma}{d\tau'}\frac{dx^\nu}{d\tau'} \tag{10}$$

where $\Gamma^r_{\sigma\nu}$ and $\Gamma^\theta_{\sigma\nu}$ are the Christoffel coefficients in Boyer-Lindquist coordinates.

2.1. *Initial conditions*

As shown in Fig1, the observer is at some large distance D from the center of the black hole and is oriented at some angel θ_o. A photon at (x_i, y_i) in the image plane will have a spherical polar coordinate given by:

$$r_i = (D^2 + x_i^2 + y_i^2)^{\frac{1}{2}} \tag{11}$$

$$cos\theta_i = \frac{1}{r_i}(Dcos\theta_o + y_i sin\theta_o) \tag{12}$$

$$tan\varphi_i = x_i(Dsin\theta_o + y_i sin\theta_o) \tag{13}$$

The only photons that will contribute to the image have 3-momentum perpendicular to the image plane. We can now specify a unique photon 4-velocity using this condition of orthogonality. Using $_o$ as a subscript for observed quantities, we get the following equations by the method used in Psaltis and Johannsen. (Psaltis and Johannsen 2012)

$$u_i^r \equiv \left(\frac{\mathrm{d}r}{\mathrm{d}\tau'}\right)_\mathrm{o} = \frac{D}{r_i} \tag{14}$$

$$u_i^\theta \equiv \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau'}\right)_\mathrm{o} = \frac{D\left(D\cos\theta_\mathrm{o} + \beta\sin\theta_\mathrm{o}\right) - r_i^2\cos\theta_\mathrm{o}}{r_i^2\left(r_i^2 - \left(D\cos\theta_\mathrm{o} + \beta\sin\theta_\mathrm{o}\right)^2\right)^{1/2}} \tag{15}$$

$$u_i^\varphi \equiv \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau'}\right)_\mathrm{o} = \frac{-\alpha\sin\theta_\mathrm{o}}{\left(D\sin\theta_\mathrm{o} - \beta\cos\theta_\mathrm{o}\right)^2 + \alpha^2}. \tag{16}$$

The final component of the 4-velocity of the photon is left. This can be found due to the fact that photons propagating through the Kerr metric have one more constraint on their motion, which arises from the fact that the norm of the photon's 4-velocity vanishes.

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau'}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau'} = 0 \tag{17}$$

To find the final component $\frac{dt}{d\tau_o'}$, we use equation [17] by plugging in the values obtained in equations [11]-[16]. This leads to the equation:

$$g_{00}\left(u_i^0\right)^2 + 2g_{\varphi 0}u_i^0 u_i^\varphi + g_{rr}\left(u_i^r\right)^2 + g_{\theta\theta}\left(u_i^\theta\right)^2 + g_{\varphi\varphi}\left(u_i^\varphi\right)^2 = 0 \tag{18}$$

The various metric components are evaluated at $(r_i, \theta_i, \varphi_i)$. To solve equations [7]-[10], we split them into six first-order equations solved by the standard RK4 method. By writing

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau'^2} = \frac{\mathrm{d}u^\theta}{\mathrm{d}\tau'} \tag{19}$$

and

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau'^2} = \frac{\mathrm{d}u^r}{\mathrm{d}\tau'} \tag{20}$$

we can express equations [7]-[10] as four first-order equations in the variables $\left(t, \varphi, u^r, u^\theta\right)$ with the definitions

$$\frac{\mathrm{d}r}{\mathrm{d}\tau'} = u^r \tag{21}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau'} = u^\theta \tag{22}$$

to completely specify the trajectory of the photon. [7]-[10], [20] and [21] form the six first-order equations to be solved by the RK4 method. For our equations, the initial conditions are given by [11]-[16]. The photon is propagated backwards until it crosses the plane of the disk, which we denote as r_p.

The adaptive step size for the RK4 implementation is set as a fixed fraction of the fastest changing variable.

$$\delta\tau' = s * \min\left[r\left(\frac{\mathrm{d}r}{\mathrm{d}\tau'}\right)^{-1}, \theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau'}\right)^{-1}, \varphi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau'}\right)^{-1}\right] \tag{23}$$

.

To find out the value of s, we define

$$K = 1 + \frac{\left[g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau'}\right)^2 + g_{\theta\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau'}\right)^2 + g_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau'}\right)^2 + 2g_{\phi 0}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau'}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau'}\right)\right]}{g_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau'}\right)^2} \tag{24}$$

If the integration was error-free, K would be 0. Practically, we set $K < 10^-5$ as our limit for accuracy to get $\delta\tau = 0.05$

MoviTon provides multiple stopping conditions for the photon. The photon can be 'swallowed' and cease to exist if it hits the black hole's event horizon, the accretion disk, or if it manages to slip through the middle. MoviTon loads the photon trajectories $r(\tau'), \theta(\tau'), \varphi(\tau'), t(\tau')$ and converts them into the Cartesian coordinates $x(\tau'), y(\tau'), z(\tau')$, which are then plotted with `matplotlib`. Axes for the X and Y coordinates of the image plane are available to toggle, but are turned off by default.

Prepackaged with MoviTon come several accretion disk profiles. The first one includes realistic exponential decay with increasing disk radius, modeled as $\frac{r}{10}^{\frac{3}{4}}$. The second one simulates 'cold spots' in the accretion disk. The third one simulates a hot spiral. These disk profiles are mostly to produce interesting videos and may or may not have real-life significance. It is trivial to write user-defined accretion disk functions that work with MoviTon as well.

3. CONCLUSION

MoviTon, a Python and C++ software package for solving photon geodesics in Kerr spacetime to create informative videos, has been presented. Users do not have to write separate codes to visualize their results. The total code kernel is slightly under 1500 lines, and can be optimized even more with CUDA and multiprocessing libraries. MoviTon is meant to be a simple educational tool, but its functionality can be extended to understand the behavior of photons in different spacetimes as well.

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