White Dwarfs Chandrashekhar Limit

Contents

1 Introduction

1.1 Stellar Life Cycle

The permanence of star is an illusion. Stars cannot last forever. It must have a beginning as well as an end. Stars form from material in interstellar space, evolve over millions or billions of years and eventually die. Stars will most easily form in regions where the interstellar material is relatively dense so that the force of gravity which tend to draw interstellar material together will overwhelm the internal pressure pushing the material apart. A clump of cloud must have a minimum mass to continue collapsing and give birth to a star. This minimum mass is called the Jeans mass. At the beginning, a protostar is formed from a large blob of gas. It matures into a star as its gases contract. Protostars with masses less than about 0.08 solar mass can never develop necessary pressure and temperature to start hydrogen fusion in their cores. Such "failed stars" end up as brown dwarfs. If the mass of the protostar is greater than about 200 solar masses, they do not become main sequence stars. A protostar becomes main-sequence star when steady hydrogen fusion begins in its core and achieves hydrostatic equilibrium. Main sequence stars have masses between 0.08 and 200 solar mass. A newly formed main-sequence star is called a ' zero age main-sequence star'. Main sequence stars are all fundamentally alike in their cores. In the cores all such stars convert hydrogen into helium by thermonuclear reactions.

1.2 Formation

When the cores of stars runs out of nuclear fuel, they can no longer convert hydrogen to helium. This is the phase where the star dies. High mass stars end their lives by becoming either black hole or neutron stars. The low or medium mass (about less than 8 times the mass of the sun) stars end their lives by becoming white dwarf stars. Before becoming a white dwarf, the star goes through a phase where it becomes a red giant. As the star leaves the main-sequence, its core contracts and the outer layer expands and surface temperature drops, which eventually gets into a phase where the star is called a Red Giant. The fusion of helium to carbon and oxygen begins at the center of a red giant. The core helium fusion comes to an end with helium flash after which the stars shrink and become less luminous. The last stage in the life of a star, the Planetary Nebula, which is an outcome of the series of bursts in luminosity. A fiercely hot, exposed core, surrounded by glowing shells of ejected gas remains. With no further thermonuclear reactions, the carbon-oxygen core cools down leaving the burnt-out relic of the star's glory which is called as White dwarfs.

1.3 Earlier Research

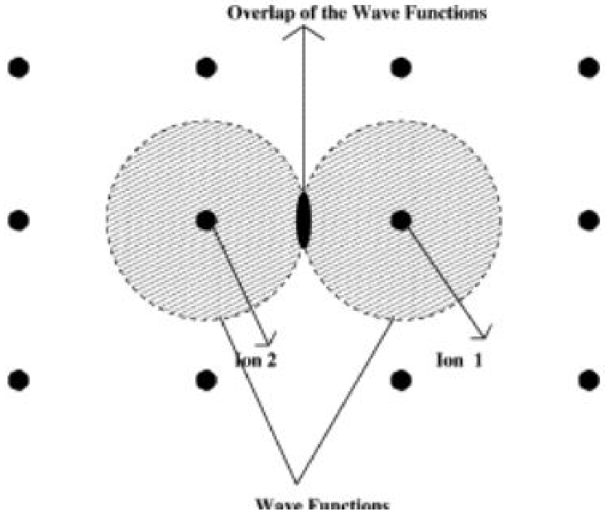
In 1914, when Walter Adams [1] was studying the companion of Sirius. He observed that the companion star had the mass of sun, half the radius of Sirius and low luminosity. It was a big mystery back then. Nine years later, in 1924, Eddington supposed that for the star to reach a stable cold state [2], its nuclei and electrons needed to recombine into atoms. But to do this the star needed to regain normal density by expanding against the force of gravity, which it could not do.According to classical physics, a cold ionised gas cannot exert any pressure, so the star is doomed to continue contracting indefinitely – a situation that Eddington regarded as absurd [3].Later, Fowler realised that, even at the absolute zero of temperature, a very dense electron gas will exert a large pressure and that this pressure could support a white dwarf star against collapse due to Pauli's exclusion principle. White dwarfs are supported from collapse by a pressure arising from electron degeneracy which is forced on the electrons by the laws of quantum mechanics. Electrons obey the Pauli's exclusion principle, which states, "No two electrons can occupy the same quantum state". The degeneracy pressure arises because only one electron can occupy a single quantum state and hence as the temperature begins to fall, the electrons start occupying the higher energy levels.

After Fowler gave his idea about the pressure of white dwarf star, Stoner took the idea and modelled a star as a sphere of ionised gas having uniform density at T=0 K. After that, Wilhelm Anderson noticed that, for masses beyond about a solar mass, the electron energies become relativistic. Anderson derived a relativistic equation of state and,despite his derivation being not strictly correct, deduced from it that there is a limiting mass for white dwarfs of 0.68 solar mass. Stoner pointed out the error in Anderson's derivation of the relativistic equation of state and derived the correct equation – the Stoner–Anderson equation [4]. Following Stoner's work, S. Chandrasekhar approached the problem by solving Lane-Emden [5] equation taking into account the relativistic effects to a system of fermions.

1.4 Death of a White Dwarf

All of the following statements are to be taken as a hypothesis. We would not be able to see these processes because of two reasons: white dwarfs have a huge lifespan, somewhere from ten to hundreds of billions of years, and also because, after they radiate most of their energy, they will stop radiating any kind of EM waves which means that they could only send gravitational waves which we cannot measure as they would be fainter than the ones which we can measure with the technology we have now. As White dwarfs don't support nuclear fusion, they don't have an energy source and so, they themselves radiate and cool to equilibrium with the cosmic microwave background radiation. As they reach equilibrium with CMB, they will stop radiating any kind of radiation, and thus, they become black. Hence the name, Black dwarfs. But the further evolution of these Black dwarfs depends on two things, their mass and the stability of the proton.

Even in our second point, we have two cases, the first being the proton decay, where we can expect all the proton to decay, which leaves these dwarfs 10^{32} to 10^{49} years to live. If the proton is stable, we can expect these Black dwarfs to enable pycnonuclear fusion, and make the core, iron-rich(Pycnonuclear fusion occurs in a cold, ultra-dense matter, where the overlapping of wave functions between neighbor ions is quite frequent). The pycnonuclear fusion reactions will tend to process the matter towards iron-56.[6]



As heavier elements take more time to fuse and form another element. Black dwarfs will eventually last longer than anything in the universe, isolated and non-radiating. Pycnonuclear reactions are density dependant, and therefore it will be maximum in the core and it will keep decreasing as we move from core to surface.The remnant which already has O/Ne/Mg will further fuse these elements to heavier elements due to pycnonuclear fusion reactions, as the density at the core of these dwarfs becomes greater. This type of fusion is slower than thermonuclear fusion and hence, heavier elements such as neon take much more time to fuse into heavier elements like silicon/iron. The estimated time silicon-28 takes to fuse into iron-56 is around 10^{1500} years which is more than the evaporating time for the largest of the black holes.

Physicists have also calculated how big these black dwarfs can be, and the conclusion that they have arrived is that the most massive black dwarfs will have to be of silicon-28 at around, 1.35 solar masses, but as the elements go to the heavier side such as iron-56, they expect these black dwarfs to be around 1.16 solar masses. This has to happen because of the decreasing Chandrashekar limit, as elements fuse to the heavier side we can expect a fall of the limit from 1.4 to 1.2 solar masses.

$$M_{Ch} \approx 1.44(2Y_e)^2 M_{\odot} \tag{1}$$

Equation 1 gives an approximate value for the Chandrashekar limit, as lighter elements fuse to heavier ones Ye decreases, decreasing the whole mass limit, where Y_e is the ratio of an atomic number to the mass number. Y_e for silicon-28 is 0.5(14/28) and Ye for iron-56 is 0.464(26/56) and therefore we expect the mass limit to drop. This indicates that as the elements in these dwarfs go to heavier sides, the mass decreases, but the fusion takes too much time that almost no heat is radiated.

So, if we were to estimate the lifespan of a black dwarf. We can use its mass to do so. For the most massive black dwarfs (the ones which are nearly 1.35 solar masses), physicists have estimated that these dwarfs will begin fusing to iron in approximately 10^{110} years. The ones that are nearly 1.24 solar masses will die at approximately 10^{1600} years. The ones that are least massive will have their lifetime set by the pycnonuclear fusion at the surface and have given an estimate of 10^{32000} years. Though at low densities the estimate can be inaccurate.

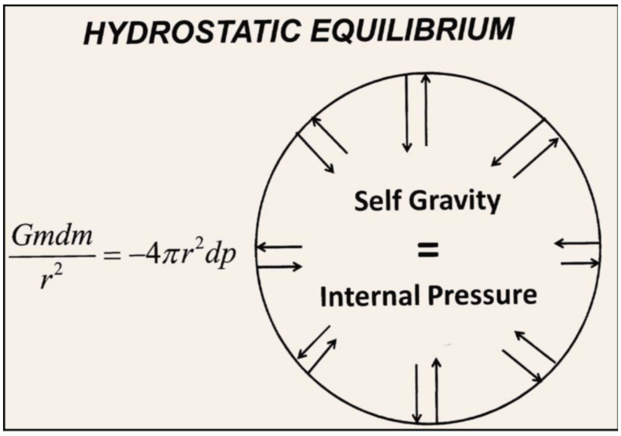
2 Problem Statement

Our work in this project is focused particularly on white dwarf stars and to calculate its limiting mass by working out the physics of the white dwarf stars, deriving a set of differential equations and using computational methods like 4th order Runge-Kutta method to arrive at a solution for the limiting mass of white dwarf stars. In our report, we give account of the abstract, theory involved in finding out the degenerate pressure, methodology and the algorithm used to derive the limiting mass.

3 Theory

3.1 Hydrostatic Equilibrium

The temperature of a main sequence star can reach upto 100 million Kelvin at the core, at this temperature, the core can start nuclear fusion of lighter elements to heavier elements to cancel out the gravitational force. This process is called hydrostatic equilibrium. In short we can say that the outward force keeps the stars from collapsing and the inward force keeps the stars from exploding. A star reaches equilibrium when the radiation pressure cancels gravitational pressure and thus it becomes stable, but when a star completes a certain stage of fusion and can not go any further, the radiation pressure decreases and the gravitational pressure takes over.



When a star completes a fusion stage such as from hydrogen to helium and if the core can no longer support fusion, then the gravitational force takes over and the star starts collapsing but as the core starts collapsing the pressures increase rapidly and therefore the temperature will also increase which might or might not trigger further fusion and if it does then the star will not collapse further. On the other hand, if a star's core becomes too heavy(fuses heavier elements) the internal pressure becomes too great for gravity to oppose and the star rapidly expands resulting in a nova.

3.2 Electron Degeneracy Pressure

When a star that is nearly 10 times as massive as sun, commonly known as red giant, fuses its core to degenerate oxygen/carbon. The shell surrounding the core will start fusing helium. This causes instability in the star and the shell is ejected as a planetary nebula. The star leaves behind a core that has degenerate matter in it. Which means that This matter cannot fuse Heavier elements but because of The degeneracy pressure released by the electrons the remnant can support itself against the increased gravitational pressure.

Electron degeneracy pressure will halt the gravitational collapse of a white dwarf star if its mass is below the Chandrasekhar limit (1.44 solar masses). This is the pressure that prevents a white dwarf star from collapsing. A star exceeding this limit and without significant thermally generated pressure will continue to collapse to form either a neutron star or black hole, because the degeneracy pressure provided by the electrons is weaker than the inward pull arises due to gravity. When Sirius B was first discovered its physical parameters were astounding. It had about the mass of the Sun confined in a volume similar to the earth. This means that the density of matter in Sirius B was much greater than ever encountered before. As it was stated that the thermal and radiation pressure that supports a normal star from gravity is no longer sufficient to counteract the enormous inward pull of gravity caused by the enormous densities present in the white dwarfs. White dwarfs are supported from collapse by a pressure arising from electron degeneracy. Electron degeneracy pressure is forced on the electrons by the laws of quantum mechanics. Electrons belong to a class of particles known as fermions. They obey the Pauli's exclusion principle, which states, No two electrons can occupy the same quantum state. The degeneracy pressure arises because only one electron can occupy a single quantum state and hence as the temperature begins to fall, the electrons start occupying the lower energy levels. At very low temperatures (T = 0 K) all the lower energy levels up to a particular level are completely filled and the higher energy levels are completely empty. Such a fermion gas is said to be completely degenerate. The pressure due to electron degeneracy can be understood in terms of wave/particle duality of electrons. Since matter is so much denser in the interior of white dwarfs the volume available for an electron becomes that much smaller. Now if we think of the electron as a wave, the reduction in volume of the space surrounding the electrons means that the wavelength of the electron becomes smaller to confine it to the smaller volume, making it more energetic. It flies about at greater speeds in its space and by bumping with other particles gives rise to the degeneracy pressure. This pressure is an unavoidable consequence of the laws of quantum mechanics. The degeneracy pressure can also be explained from Heisenberg's uncertainty principle, which can be written in the form of an equation as

$$\Delta P \Delta X \geq \frac{h}{4\pi} \tag{2}$$

Let us now interpret the uncertainty principle in a different way to get a better understanding for the the origin of degeneracy pressure. Considering 3, we infer that the minimum value for the electron momentum is ΔP. Hence as the value of ΔX becomes smaller, in other words we are confining the electron to a smaller and smaller volume, the momentum of the electron correspondingly increases and this contributes to the degenerate pressure.
In order to understand electron degeneracy pressure in terms of the fermions, we introduce fermi momentum and fermi energy. Fermi energy is defined as the energy difference between the lowest and the highest occupied single-particle state in a system of non interacting fermions at absolute zero temperature. When all the energy has been extracted from the ideal fermi gas at absolute zero, but the fermions are still moving at great speeds. Therefore, the fastest moving fermions have kinetic energy corresponding to that of fermi energy, this speed attained by fermions is called fermi velocity and the differential values of fermi velocity provide us with fermi momentum.

Unlike ordinary main sequence stars, white dwarfs are so dense that quantum effects are important – essentially because the electrons are packed so tightly together that they become a degenerate gas – i.e., one in which the momenta and energies are dictated by the "size" of the quantum states. The physical effect that arises when many electrons are confined in a small volume is called electron degeneracy pressure. This results from the random, large momenta that electrons must adopt because they cannot share the same quantum state.

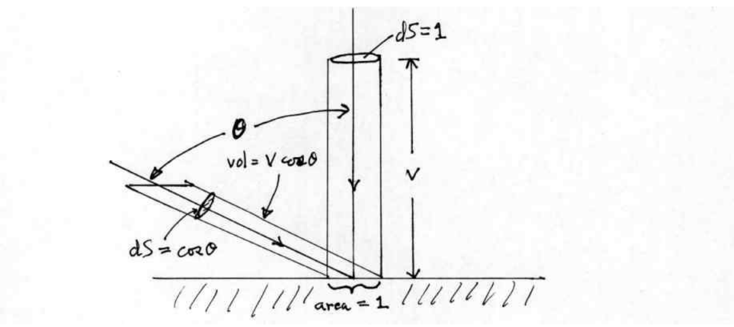
Unlike the case for the early universe when particles are freely created and destroyed, there is a fixed number of electrons inside a white dwarf, so we must include the chemical potential when we write down the occupation number. So number of electrons per unit volume of momentum space is:

$$n(p) = \frac{8\pi}{h^3} < N(E) >= \frac{8\pi}{h^3} \frac{1}{e^{(E-\mu)/kT} + 1} \tag{3}$$

In fact. white dwarfs are so dense that μ/kT is very large, and essentially all states with energy less than the Fermi energy $E_f = \mu$ are filled.

3.2.1 The Pressure of an Ideal Fermi Gas

To compute the pressure, we need to consider the rate at which momentum is transferred to a wall by the particles. Consider a cylinder which makes an angle θ with the perpendicular to the wall. Let the area of the wall cutting the cylinder be unity. Then, the cross-sectional area of this cylinder will be $ds = \cos\theta$. Let the length of the cylinder be v, the velocity of the particle. The volume of the cylinder is $v\cos\theta$. Now the momentum of the particle perpendicular to the wall is $p_\perp = |\vec{p}|\cos\theta$, and reflection from the wall will reverse the direction of $p_\perp$ so that the total momentum transferred must be $2p\cos\theta$.



To get the pressure we must multiply the number of particles per unit physical space and per unit momentum space by the factor $2vp\cos^2\theta$ and integrate over the hemisphere bounded by the wall.

$$\frac{1}{4\pi}\int_{hemisphere} 2\cos^2\theta \, d\Omega = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\pi/2} \cos^2\theta \, \sin\theta \, d\theta = \int_0^{\pi/2} \cos^2\theta \, \sin\theta \, d\theta \tag{4}$$

Let substitute $\mu = \cos\theta$ and $d\mu = -\sin\theta d\theta$, then

$$\int_0^{\pi/2} \cos^2\theta \sin\theta d\theta = \int_1^0 \mu^2(-d\mu) = \int_0^1 \mu^2 f\mu = \frac{1}{3} \tag{5}$$

So we obtain a very general equation for the pressure as the integral over the momentum p:

$$P = \frac{1}{3}\int_0^\infty 4\pi\ n(p)\ p\ v(p)\ p^2\ dp \tag{6}$$

Since there are two spin states, two electrons can occupy each cell in momentum space. Thus we have for our Fermion gas,

$$n(p) = \frac{2}{h^3}\frac{1}{e^{(E-\mu)/kT}+1} \tag{7}$$

The integral for P can be determined by writing the expression for n as the integral of $n(p)$:

$$n = \int_0^\infty 4\pi\ n(p)\ p^2\ dp = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\ dp}{e^{(E-\mu)/kT}+1} \tag{8}$$

It turns out that if the gas is non-degenerate, then $(-\mu/kT) >> 1$ so that we can ignore the $+1$ in the denominator. However, we are interested in highly degenerate conditions. We approach this by writing the pressure as

$$P = \frac{8\pi}{3h^3}\int_0^\infty F(\varepsilon)\ v(p)\ p^3\ dp \tag{9}$$

where,

$$F(\varepsilon) = \frac{1}{e^{[(\varepsilon-(\mu-mc^2))/kT]+1}} \tag{10}$$

As a result, the pressure (equation 9) becomes

$$P = \frac{8\pi}{3h^3}\int_0^{p_F} v(p)\ p^3\ dp = \frac{8\pi}{3h^3}\int_0^{p_F}\frac{c^2p^4\ dp}{\sqrt{p^2c^2+m^2c^4}} \tag{11}$$

Using $F(\varepsilon)$ in equation 8 for n, the number density of electrons

$$n = \frac{8\pi}{3h^3}\int_0^\infty F(\varepsilon)\ p^2\ dp = \frac{8\pi}{3h^3}\int_0^{p_F} p^2\ dp = \frac{8\pi}{3h^3}p_F^3 \tag{12}$$

Solving for p_F, we find that

$$p_F = \left(\frac{3}{8\pi}\right)^{1/3}\left(\frac{h}{mc}\right)\ n^{1/3} \tag{13}$$

and equation 11 can be written

$$P = \frac{8\pi c^5 m^4}{3h^3}\int_0^{x_F}\frac{x^4\ dx}{\sqrt{1+x^2}} \tag{14}$$

Let's consider the case where the degeneracy is non-relativistic (this corresponds to $x_F << 1$). Then $v(p) = p/m$. Inserting this into equation 9, we immediately obtain

$$P_{nr} = \frac{8\pi}{3mh^3}\int_0^{p_F} p^4\ dp = \frac{8\pi}{15mh^3}p_F^5 \tag{15}$$

Using equation 13, this becomes

$$P_{nr} = \left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{20m}n^{5/3} \tag{16}$$

At the other extreme, consider the pressure of a highly relativistic degenerate electron gas (this corresponds to $x_F >> 1$). Then $v = c$, and equation 9 becomes

$$\begin{aligned} P_{rel} &= \frac{8\pi c}{3h^3}\int_0^{p_F} p^3 dp = \frac{2\pi c}{3h^3}p_F^4 \\ &= \frac{2\pi c}{3h^3}\left(\frac{3}{8\pi}\right)^{4/3} h^4\ n^{4/3} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} n^{4/3} \end{aligned}$$

Now number of electrons cm^{-3} will be the density in gcm^{-3} divided by $\mu_e m_H$, the number of grams per electron:

$$n_e = \frac{p}{\mu_e m_H} = \frac{N_A}{\mu_e} \tag{17}$$

where m_H = mass of the Hydrogen atom, N_A = Avogadro number. So,

$$P_{nr} = \left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{20m}\left(\frac{N_A}{\mu_E}\right)^{5/3} p^{5/3} \tag{18}$$

and

$$P_{rel} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{N_A}{\mu_E}\right)^{4/3} p^{4/3} \tag{19}$$

3.3 The Differential Equations governing a White Dwarf

The gravitational force per unit mass acting on a layer of the star at a distance r from the centre having density ρ is given by *(using Newton's law of gravitation)*

$$F_{grav} = -\frac{GM\rho}{r^2} \tag{20}$$

We know that, for stability, the gravitational force has to be equal to the pressure gradient of the star. Thus the expression for hydrostatic equilibrium will be

$$\frac{dp}{dr} = -\frac{GM\rho}{r^2} \tag{21}$$

The total mass within that sphere will be

$$m(r) = 4\pi \int_0^r \rho(r')r'^2 dr' \tag{22}$$

The derivative of this gives the mass continuity(mass as a function of the radial distance)

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \tag{23}$$

The pressure of the degenerate electrons can be due to *Ultra relativistic* or *Non relativistic* velocities. Therefore one hase to consider all these boundary conditions. We can express the kinematics of the electron gas using Einstein's equations as:

$$\epsilon_p^2 = m_e c^2 + p^2 c^2 \tag{24}$$

Expressing the pressure in terms of the density of states as g(p) = V/$h^3 4\pi p^2$ we get

$$P = \frac{1}{3V}\int_0^{P_F} \frac{p^2 c^2 g(p)}{\epsilon_p} \tag{25}$$

One obtains

$$P = \frac{8\pi}{3h^3}\int_0^{P_F} p^4 c^2 (m_e c^2 + p^2 c^2)^{-1/2} dp \tag{26}$$

Applying a variable change: x = $\frac{p}{m_e c}$, dx = $\frac{dp}{m_e c}$

$$P = \frac{8\pi m_e^4 c^5}{3h^3}\int_0^{x_F} \frac{x^4}{(1+x^2)^{1/2}} dx \tag{27}$$

After integration, this leads to

$$P = K_{UR} n_e^{4/3} I(x_F) \tag{28}$$

where

$$K_{UR} = \frac{hc}{4}(\frac{3}{8\pi})^{1/3} \tag{29}$$

and

$$I(x) = \frac{3}{2x^4}[x(1+x^2)^{1/2}(2/3x^2 - 1) + ln[x + (1+x^2)^{1/2}] \tag{30}$$

x_F is the dimensionless variable whose value is:

$$x_F = \frac{P_F}{m_e c} = (\frac{3n_e}{8\pi})^{1/3}\frac{h}{m_e c} \tag{31}$$

n_e is the number of electrons per unit volume. This is related to the density using Y_e as the number of electrons per nucleons

$$n_e = \frac{Y_e \rho}{m_H} \tag{32}$$

Fermi momentum as a function of density will be

$$x_F = (\frac{3Y_e}{8\pi m_H})^{1/3}\frac{h}{m_e c}\rho^{1/3} \tag{33}$$

This pressure is now expressed in terms of the dimensionless *Fermi momentum* x_F , which depends on the density. When considering higher densities $x_f >> 1$ then $I(x_F) \rightarrow 1$, thus $P \rightarrow K_{UR} n_e^{4/3}$

Now we can find out how ρ changes as a function of radial coordinate and x_F. We start by taking the derivative of equation (9)

$$\frac{dp}{dr} = \frac{d}{dr} K_{UR} n_e^{4/3} I(x_F) \tag{34}$$

$$\frac{dp}{dr} = \frac{d}{dr} K_{UR} \frac{Y_e}{m_H}^{4/3} \rho^{4/3} I(x) \tag{35}$$

$$= K_{UR} \frac{Y_e}{m_H}^{4/3} (4/3 \rho^{1/3} \frac{d\rho}{dr} I(x) + \rho^{4/3} \frac{dI}{dx}|_{x_F} \frac{dx_F}{dr}) \tag{36}$$

where

$$\frac{dI}{dx} = \frac{4x^2+6}{2x^4(1+x^2)^{1/2}} - \frac{6ln[(1+x^2)^{1/2}+x]}{x^5} \tag{37}$$

and

$$\frac{dx_F}{dr} = 1/3 K_F \rho^{-2/3} \frac{d\rho}{dr} \tag{38}$$

From here, we assume that $Y_e = 0.5$ and the star mainly consists of ^{12}C Writing the pressure gradient now gives us the following equation:

$$\frac{dP}{dr} = K_{UR} (\frac{0.5}{m_H})^{4/3} (\frac{4\rho^{1/3}}{3} \frac{d\rho}{dr} I(x) + \rho^{2/3} \frac{K_F}{3} \frac{d\rho}{dr} \frac{dI}{dx}) \tag{39}$$

where

$$K_F = (\frac{3 * 0.5}{8\pi m_H})^{1/3} \frac{h}{m_e c} \tag{40}$$

We can rewrite this equation using a *hidden* x_F and K_F, thus the pressure gradient equation boils down to:

$$\frac{dP}{dr} = \frac{hc}{4} (\frac{3}{8\pi})^{1/3} (\frac{0.5}{m_H})^{4/3} \rho^{1/3} \frac{4x}{3(1+x^2)^{1/2}} \frac{d\rho}{dr} \tag{41}$$

The pressure gradient for a degenerate electron gas will be :

$$\frac{dP}{dr} = \frac{0.5 m_e c^2}{3 m_H} \frac{x^2}{(1+x^2)^{1/2}} \frac{d\rho}{dr} \tag{42}$$

With this pressure gradient and *mass continuity equation* we get the governing equations for a white dwarf

$$\frac{d\rho}{dr} = \frac{-Gm(r)\rho(r)}{r^2 \gamma(r)} \frac{m_H}{0.5 c^2 m_e} \tag{43}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \tag{44}$$

where

$$\gamma(r) = \frac{x^2}{3(1+x^2)^{1/2}} \tag{45}$$

3.4 Obtaining the Dimensionless Equations

In order to solve the set of equations properly we introduce the following dimensionless variables

$$\mu = \frac{\rho}{\rho_c} \tag{46}$$

$$\varepsilon = \frac{r}{R_\odot} \tag{47}$$

$$\zeta = \frac{m}{M_\odot} \tag{48}$$

where, $\rho_c, R_\odot, M_\odot$ stands as the Sun's central density, radius and mass respectively

Rewriting e the former set of equations by using our new set of variables μ, ε and N
Lets start with the mass continuity equation. It follows directly from the definition of the Fermi momentum the following

$$x_F = (K_F(\rho_c \mu))^{1/3} \tag{49}$$

From (45) we can directly obtain our first dimensionless equation:

$$\frac{d\zeta}{d\varepsilon} = \frac{4\pi\varepsilon^2\mu R_\odot^3\rho_c}{M_\odot} \tag{50}$$

or it can be introduced as

$$\frac{d\zeta}{d\varepsilon} = C_m\varepsilon^2\mu \tag{51}$$

where,

$$C_m = \frac{4\pi R_\odot^3\rho_c}{M_\odot} \tag{52}$$

For the remaining let us consider the equation:

$$\frac{d\rho}{dr} = \frac{-2Gm(r)\rho(r)}{r^2N(x)}\frac{m_H}{c^2m_c} \tag{53}$$

where, $N(x) = \frac{x^2}{3(1+x^2)^{1/2}}$. Considering the pressure gradient for a degenerate electron gas,

$$\frac{dP}{dr} = \frac{1}{2}\frac{m_ec^2}{m_H}\frac{x^2}{(1+x^2)^2}\frac{d\rho}{dr} = \frac{1}{2}\frac{m_ec^2}{m_H}N(x)\frac{d\rho}{dr} \tag{54}$$

By a quick inspection we can pass from N(x) to N(ρ)

$$N_\rho = \frac{K_F^2\rho^{2/3}}{3(1+K_F^2\rho_c^{2/3})^{1/2}} \tag{55}$$

From here we get,

$$N_\mu = \frac{K_F^2\rho^{2/3}\mu^{2/3}}{3(1+K_F^2\rho_c^{2/3})^{1/2}\mu^{2/3}} \tag{56}$$

Thus the pressure gradient from (53) to

$$\frac{dP}{d\varepsilon} = \frac{1}{2}\frac{m_ec^2N(\mu)}{m_H}\frac{d\rho}{d\varepsilon} \tag{57}$$

From the condition of Hydrostatic equilibrium, we can easily get the same equilibrium condition in terms of μ and ε

$$\frac{dP}{d\varepsilon} = \frac{-Gm(\varepsilon)\mu\rho_c}{\varepsilon^2R_\odot} \tag{58}$$

Now we can rewrite the following,

$$\frac{dP}{d\varepsilon} = \frac{-2Gm_Hm(\varepsilon)\mu\rho_c}{\varepsilon^2N(\mu)m_ec^2R_\odot} \tag{59}$$

From here it is easy to get the remaining dimensionless equation as

$$\frac{dP}{d\varepsilon} = \frac{-2Gm_H\mu M_\odot\zeta}{\varepsilon^2N(\mu)m_ec^2R_\odot} \tag{60}$$

As before this can be arranged as

$$\frac{dP}{d\varepsilon} = C_\rho\frac{\zeta\mu}{\varepsilon^2N(\mu)} \tag{61}$$

where, $C_\rho = \frac{-2Gm_HM_\odot}{m_ec^2R_\odot}$. Finally we obtained the set of dimensionless equations with their respective dimensionless constants needed for an improved numerical computation as shown.

$$\frac{d\zeta}{d\varepsilon} = C_m\varepsilon^2\mu \tag{62}$$

$$\frac{d\mu}{d\varepsilon} = C_m\frac{\zeta\mu}{\varepsilon^2N(\mu)} \tag{63}$$

where

$$C_m = \frac{4\pi R_\odot^3\rho_c}{M_\odot},\ C_\rho = \frac{-2Gm_HM_\odot}{m_ec^2R_\odot} \tag{64}$$

and

$$N(\mu) = \frac{K_F^2\rho^{2/3}\mu^{2/3}}{3(1+K_F^2\rho_c^{2/3}\mu^{2/3})^{1/2}} \tag{65}$$

.

4 Solution to the Differential Equations

To solve the system of differential equation, we implemented the fourth order Runge-Kutta Method. To use the system, we specify an initial core density as a multiple of the central density of the sun and an initial mass. This initial mass it would be equal to zero. This value of the initial mass implies that the density gradient will vanish at the origin of the radial coordinate. Thus the profile of the density as a function of radius should approach its maximum smoothly at the zero value of the radial coordinate. The radius of the star can be defined as the distance when the density reaches a zero value. I an similar way. the mass of the star can be defined as the value of the mass when the density reaches a zero value.

By using the Runge-Kutta of fourth order Method to solve the differential equa- tions numerically we will not obtain a density profile that reaches zero but one that tends to. Thus we "cut" the density values at a certain point and after that we imposed that the rest of the values were zero. That non-zero value would correspond to the radius of the star.

4.0.1 RK4 Method

For a given differential equation:

$$\frac{dy(t)}{dt} = f(y(t), t), \ y(t_0) = y_0 \tag{66}$$

We have the slopes:

$$k_1 = f\left(y(t_0), t_0)\right) \tag{67}$$

$$k_2 = f\left(y(t_0) + k_1\frac{h}{2}, t_0 + \frac{h}{2}\right) \tag{68}$$

$$k_3 = f\left(y(t_0) + k_2\frac{h}{2}, t_0 + \frac{h}{2}\right) \tag{69}$$

$$k_4 = f\left(y(t_0) + k_3 h, t_0 + h\right) \tag{70}$$

A weighted combination of these slopes gives us our final approximation:

$$y(t_0 + h) = y(t_0) + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} h \tag{71}$$

4.1 Naturalization

The scales of the problem cause an issue while running the simulation. The initial simulations all encountered a precision problem while running, due to Python's double floats being too large to hold the Planck constant. The code can run one of two ways; either find constants that don't trigger Python's precision errors, such as CGS units, or naturalize the constants. The equations are naturalized as shown in the presentation.

The equation of state being in natural units require initial values for the numerical IVP solver, u0,x0,q0. The solutions utilize stepped integration over an initial domain, with a step size ranging from a millionth to a thousandth of the unit. Doing this avoids division-by-zero errors, and running the simulation at thousandth-step-sizes does not occupy too much RAM.

Since the system does not have an analytical solution, and the RK4 algorithm does not provide deviations from higher IVPs, errors have to be gathered from the deviation of results for different integration step lengths.

The star data has central density values in natural units, randing from 10^-9 to 10^19. This set is integrated outwards until a given density is found. This value is set as $qc * 10^-4$ and is treated as the edge of the star, the best for practical purposes. The radius and mass values for the edge of the star are stored for later use in the graphs.

5 Results

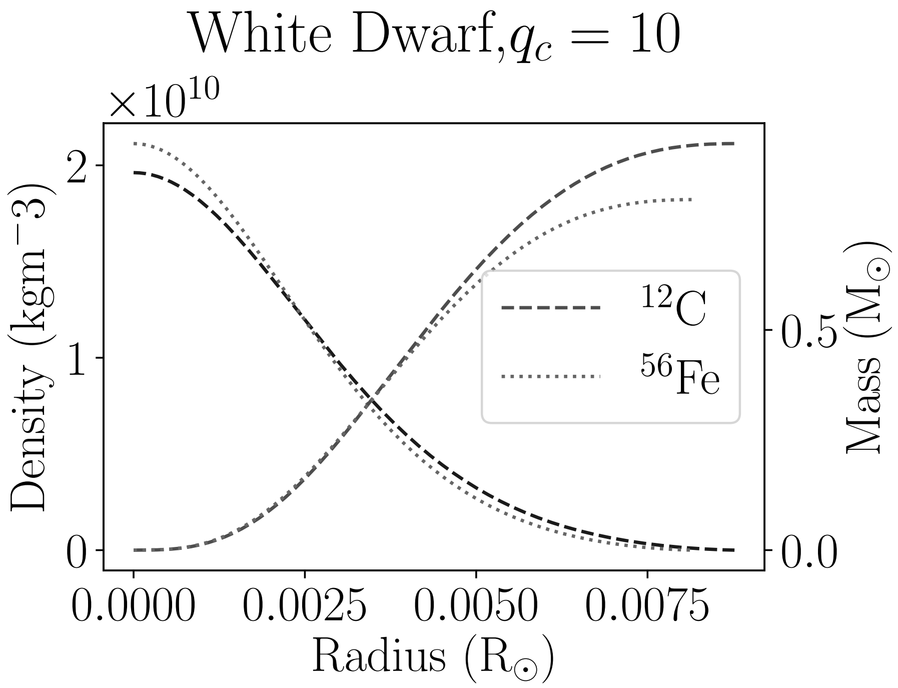


Figure 1: Variation of density and mass vs radius for a white dwarf

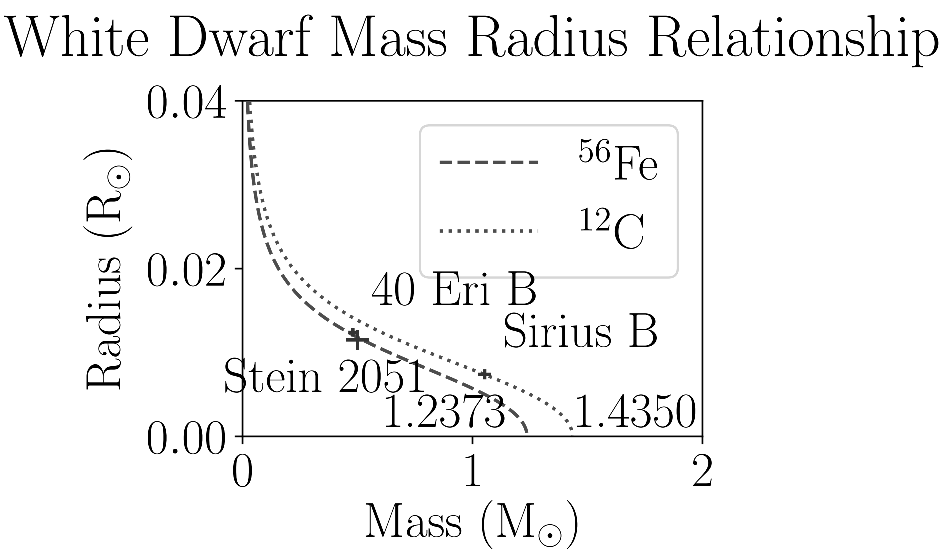


Figure 2: Mass vs Radius

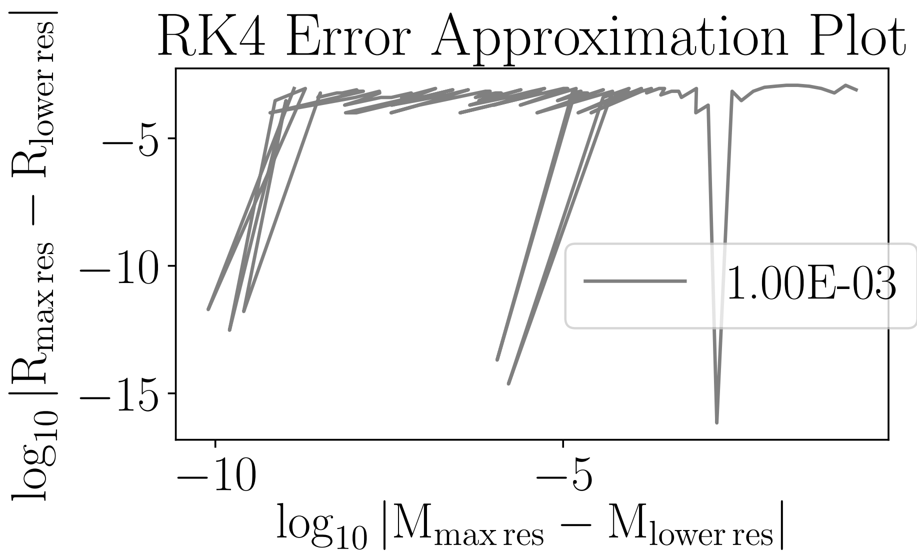


Figure 3: Error plot for RK4 Method

5.1 Code

5.1.1 Simulation

```
from astropy.constants.iau2015 import R_sun, M_sun #chose this because it avoids the classerro
from tqdm import tqdm #progressbars woohoo
import scipy.constants as sciconst
import matplotlib.pyplot as plt
import numpy as np
import math


#used _f to indicate files because a lot of overlap occurs otherwise, really annoying
_f_m_e=sciconst.electron_mass
#made sense to use scipy because it works with numpy
#instead of going through extra conversion processes later
_f_m_p=sciconst.proton_mass
_f_c=sciconst.speed_of_light
_f_pi=sciconst.pi
_f_big_g=sciconst.gravitational_constant
_f_hbar=sciconst.hbar


def runkut(n, x, y, h): #standard runge-katta 4 solver
    #advances the solution of a differential equation defined by derivatives from x to x+h
    y0=y[:]
    k1=derivative_field(n, x, y)
    for i in range(1, n+1):
        y[i]=y0[i]+0.5*h*k1[i]
    k2=derivative_field(n, x+0.5*h,y)
    for i in range(1, n+1):
        y[i] = y0[i]+h*(0.2071067811*k1[i]+0.2928932188*k2[i])
    k3=derivative_field(n,x+0.5*h,y)
    for i in range(1,n+1):
        y[i] = y0[i]-h*(0.7071067811*k2[i]-1.7071067811*k3[i])
    k4 = derivative_field(n,x+h,y)
    for i in range(1,n+1):
        a = k1[i]+0.5857864376*k2[i]+3.4142135623*k3[i]+k4[i]
        y[i]=y0[i]+0.16666666667*h*a
    x+=h
```

```
    return x,y

#finishing the runge-katta solver
def derivative_field(_,x_val,state_vals):
    _,mu,q=state_vals
    y_prime=[
        0.0,
        dmu_by_dx(x_val, q),
        dq_by_dx(x_val, q, mu)
    ]
    return y_prime


def dq_by_dx(x_val,q_val,mu_val):
    if q_val<0 or mu_val<0 or x_val==0:
        return 0.0
    gamma_func_result = (
        math.pow(q_val,2/3)/(3*math.sqrt(1+math.pow(q_val,2/3)))
    )
    return(
        (-1*q_val*mu_val)/(x_val * x_val * gamma_func_result)
    )


def dmu_by_dx(x_val,q_val):
    return 3*q_val*x_val*x_val


def rho_0(y_e_val):
    return (
        (_f_m_p*_f_m_e*_f_m_e*_f_m_e*_f_c*_f_c*_f_c)/(3 * _f_pi * _f_pi * \
        _f_hbar * _f_hbar * _f_hbar * y_e_val)
    )


def big_r_0(y_e_val):
    return math.sqrt(
        (3*y_e_val*_f_m_e*_f_c*_f_c)/(4*_f_pi*_f_big_g*_f_m_p*rho_0(y_e_val))
    )


def mu_0(y_e_val):
    big_r_0_val=big_r_0(y_e_val)
    return (
        (4*_f_pi*rho_0(y_e_val)*big_r_0_val*big_r_0_val*big_r_0_val)/3
    )


def sim_star(step_length,q_c):
    x_val=0.0
    state_vec=[0.0,0.0,q_c]
    x_list=[]
    mu_list=[]
    q_list=[]
    while state_vec[2]>1e-4*q_c:
        (x_val, state_vec)=runkut(2,x_val,state_vec,step_length)
        x_list.append(x_val)
        mu_list.append(state_vec[1])
        q_list.append(state_vec[2])
    return x_list, mu_list, q_list


def __test_star_simulation(y_e_val):
    q_c_range=[-10, 20]
```

```
    q_c_set=np.around(np.exp(np.linspace(*q_c_range,num=60)),10)
    star_list=[]
    star_edge_radius_list=[]
    star_edge_mass_list=[]
    for q_c in tqdm(q_c_set):
        x_val=0.0
        state_vec=[0.0,0.0,q_c]
        x_list, mu_list, q_list = sim_star(1e-4, q_c)
        r_list=np.multiply(x_list,big_r_0(y_e_val))
        m_list=np.multiply(mu_list,mu_0(y_e_val))
        rho_list=np.multiply(q_list,rho_0(y_e_val))
        star_list.append((r_list,m_list,rho_list))
        if len(r_list)>0:
            star_edge_radius_list.append(r_list[-1])
            star_edge_mass_list.append(m_list[-1])
    fig, ax=plt.subplots()
    ax.plot(star_list[5][0],star_list[5][2])
    ax.set(xlabel="Radius (m)",ylabel="Density (kgm$^{-3}$)",
           title=f"Density Radius Relation {q_c_set[5] * rho_0(y_e_val)}:.4f")
    fig.savefig("./iron_star_test.png")
    fig, ax=plt.subplots()
    ax.plot(np.divide(star_edge_mass_list,M_sun.value),
            np.divide(star_edge_radius_list,R_sun.value))
    ax.set(xlabel="Mass", ylabel="Radius",
           title="White Dwarf Mass Radius Relation")
    fig.savefig("./iron_star_mass_radius_test.png")


if __name__ =="__main__":
    plt.tight_layout()
    print(f"Test Iron Star Y_e")
    fe_y_e_val = 26/56
    __test_star_simulation(fe_y_e_val)
```

5.1.2 Test Suite

```
import sim
import plot_gen
from tqdm import tqdm
import matplotlib.pyplot as plt
import numpy as np
import multiprocessing
import json
import os
import functools



def generate_star_data():
    q_c_range=[-9, 19]
    q_c_set=np.around(np.exp(np.linspace(*q_c_range, num=100)),10)
    data_ret=[]
    for step_length in [1e-4,1e-3]:
        with multiprocessing.Pool(12) as w_pool:
            sim_star_partial=functools.partial(sim.sim_star, step_length)
            results=w_pool.imap(sim_star_partial, q_c_set)
            star_edge_list_x=[]
            star_edge_list_mu=[]
            star_result_set=[]
            for x_list,mu_list,q_list in tqdm(results,total=len(q_c_set),desc="Solve ODEs"):
                star_edge_list_x.append(x_list[-1])
                star_edge_list_mu.append(mu_list[-1])
                star_result_set.append([x_list, mu_list, q_list])
        data_ret.append((step_length, \
        list(q_c_set),star_result_set,star_edge_list_x,star_edge_list_mu))
```

```
    return data_ret


if __name__ == "__main__":
    if not os.path.exists("star_data.json"):
        out_data=generate_star_data()
        file_formatted_data = [
            {"step_length": step_length,
             "q_c_set": q_c_set,
             "star_radii": star_edge_list,
             "star_masses": star_mass_list}
            for step_length,q_c_set, _,star_edge_list,star_mass_list in out_data
        ]
        with open("star_data.json","w") as json_obj:
            json.dump(file_formatted_data,json_obj, indent=4)
    with open("star_data.json","r") as json_obj:
        saved_run_data=json.load(json_obj)
    q_c_vals=saved_run_data[0]["q_c_set"]
    radial_edge_list=saved_run_data[0]["star_radii"]
    star_mass_list=saved_run_data[0]["star_masses"]
    mass_radius_step_length_tuple_list = [
        (simulation_data["step_length"],simulation_data["star_masses"], \
        simulation_data["star_radii"])
        for simulation_data in saved_run_data
    ]
    plot_gen.plot_mass_radius_relation(radial_edge_list, star_mass_list)
    plot_gen.plot_mass_radius_relation_relations(mass_radius_step_length_tuple_list)
    x_list, mu_list, q_list=sim.sim_star(1e-6, 10)
    plot_gen.plot_single_white_dwarf_values(x_list,q_list,mu_list,10)
    exit(0)
```

5.1.3 Main Plot

```
import sim

from matplotlib import rc
import matplotlib.pyplot as plt
import numpy as np
from astropy.constants.iau2015 import R_sun, M_sun
import matplotlib.font_manager

plt.tight_layout()

rc('font', **{
    'family': 'sans',
    'serif': ['Computer Modern'],
    'size': '22',
})
rc('text', usetex=True)
rc('figure', **{'autolayout': True})

# Uses solar units
_f_known_white_dwarf_data = {
    "names": ["Sirius B", "40 Eri B", "Stein 2051"],
    "masses": [1.053, 0.48, 0.50],
    "mass_err": [0.028, 0.02, 0.05],
    "radii": [0.0074, 0.0124, 0.0115],
    "radius_err": [0.0006, 0.0005, 0.0012],
    "annotation_configs": [
        {
            "xytext": (8, 8),
            "va": "bottom"
        },
        {
```

```
            "xytext": (8, 8),
            "va": "bottom"
        },
        {
            "xytext": (-60, -8),
            "va": "top"
        },
    ]
}

fe_y_e = 26/56 #standard ratio for iron
c_y_e = 0.5
fe_big_r_0 = sim.big_r_0(fe_y_e)
c_big_r_0 = sim.big_r_0(c_y_e)
fe_mu_0 = sim.mu_0(fe_y_e)
c_mu_0 = sim.mu_0(c_y_e)
fe_rho_0 = sim.rho_0(fe_y_e)
c_rho_0 = sim.rho_0(c_y_e)
rho_sun = M_sun.value/(4/3*np.pi*R_sun*R_sun*R_sun)


def plot_mass_radius_relation_relations(mass_radius_sets_with_step_length):
    fig,ax=plt.subplots()
    linestyles=[":","--","-"]
    colours=["0.5","0.4","0.1"]
    max_res_data=mass_radius_sets_with_step_length[0]
    for (step_length,mass_set,radius_set),linestyle,colour in \
    zip(mass_radius_sets_with_step_length[1:],linestyles, colours):
        star_mass_diff = np.log10(np.abs(np.subtract(max_res_data[1], mass_set)))
        star_rad_diff = np.log10(np.abs(np.subtract(max_res_data[2], radius_set)))
        ax.plot(star_mass_diff, star_rad_diff, label=f"{step_length:.2E}", color=colour)
    ax.set(xlabel=r"$\log_{10}\mathrm{|M_{max\,res}-M_{lower\,res}|}$",
           ylabel=r"$\log_{10}\mathrm{|R_{max\,res}-R_{lower\,res}|}$",
           title="RK4 Error Approximation Plot")
    fig.legend(loc="center right")
    fig.savefig("./test_error_plot.pdf")


def plot_mass_radius_relation(edge_x_set,edge_mu_set):

    fe_edge_r_set=np.multiply(edge_x_set,fe_big_r_0/R_sun.value)
    c_edge_r_set=np.multiply(edge_x_set,c_big_r_0/R_sun.value)

    fe_edge_m_set=np.multiply(edge_mu_set,fe_mu_0/M_sun.value)
    c_edge_m_set=np.multiply(edge_mu_set,c_mu_0/M_sun.value)

    fig,ax=plt.subplots()
    ax.plot(fe_edge_m_set,fe_edge_r_set,
            color="0.3",linestyle="--",label="$^{56}$Fe")
    ax.plot(c_edge_m_set,c_edge_r_set,
            color="0.3",linestyle=":",label="$^{12}$C")
    ax.set(ylabel=r"Radius (R$_\odot$)",xlabel=r"Mass (M$_\odot$)",
           ylim=(0.0, 0.04),xlim=(0.0, 2.0))
    ax.set_title("White Dwarf Mass Radius Relationship",pad=20)
    ax.annotate(r"{:.4f}".format(fe_edge_m_set[-1]),(fe_edge_m_set[-1],0.0),
                textcoords="offset points",va="bottom",xytext=(-65, 0),ha="left")
    ax.annotate(r"{:.4f}".format(c_edge_m_set[-1]),(c_edge_m_set[-1],0.0),
                textcoords="offset points",va="bottom",ha="left",xytext=(0,0))
    names=[name for name in _f_known_white_dwarf_data["names"]]
    mass_vals=[mass for mass in _f_known_white_dwarf_data["masses"]]
    radius_vals=[radius for radius in _f_known_white_dwarf_data["radii"]]
    mass_errs=[mass_err for mass_err in _f_known_white_dwarf_data["mass_err"]]
    radius_errs=[radius_err for radius_err in _f_known_white_dwarf_data["radius_err"]]
    ax.errorbar(mass_vals,radius_vals,xerr=mass_errs,yerr=radius_errs,ls="none",
```

```
            color="0.2",capsize=0,elinewidth=1.5,zorder=10)
    for star_index in range(len(names)):
        ax.annotate(names[star_index],(mass_vals[star_index],radius_vals[star_index]),
                    textcoords="offset points",zorder=20,
                    **_f_known_white_dwarf_data["annotation_configs"][star_index])
    ax.legend()
    fig.savefig("./mass_radius_relation.png", dpi=600)
    fig.savefig("./mass_radius_relation.svg")
    fig.savefig("./mass_radius_relation.pdf")


def plot_single_white_dwarf_values(radial_steps,density_steps,mass_steps,central_density):
    fig, ax_density=plt.subplots()
    ax_density.set(xlabel=r"Radius (R$_\odot$)",ylabel=r"Density ($\mathrm{kgm^-3}$)")
    ax_density.set_title(r"White Dwarf, $q_c=10$",pad=28)
    ax_mass=ax_density.twinx()
    ax_mass.set(ylabel=r"Mass (M$_\odot$)")
    c_radius_solar=np.multiply(radial_steps,c_big_r_0/R_sun.value)
    fe_radius_solar=np.multiply(radial_steps,fe_big_r_0/R_sun.value)
    c_mass_solar=np.multiply(mass_steps,c_mu_0/M_sun.value)
    fe_mass_solar=np.multiply(mass_steps,fe_mu_0/M_sun.value)
    c_density_solar=np.multiply(density_steps,c_rho_0)
    fe_density_solar=np.multiply(density_steps,fe_rho_0)
    ax_mass.plot(c_radius_solar,c_mass_solar,label=r"$^{12}$C",color="0.3",
               linestyle="--")
    ax_mass.plot(fe_radius_solar,fe_mass_solar,label=r"$^{56}$Fe",color="0.4",
               linestyle=":")
    ax_mass.legend(loc="center right")
    ax_density.plot(c_radius_solar, c_density_solar, color="0.1", linestyle="--")
    ax_density.plot(fe_radius_solar, fe_density_solar, color="0.4", linestyle=":")
    fig.savefig(f"./example.pdf")
```

6 Bibliography

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