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Lecture 10: Totally Unimodular Matrices and Applications

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

Recall that a polyhedron $P \subseteq \mathbb{R}^n$ is said to be *integral* if $P = P_I$ where $P_I :=$ conv-hull$(P \cap \mathbb{Z}^n)$. We will see properties of A and b which are sufficient conditions for the polyhedron $P = \{x : Ax \leq b\}$ to be integral.

Question. Given A, b *which are integral*, can we verify if $P = \{x : Ax \leq b\}$ is integral?

Note that we will assume the constraint matrix A and the RHS vector b to be integral—i.e., all their entries are integers. This assumption is without loss of generality since the entries are rational anyway and hence, they can be scaled without changing the polyhedron.

10.1 Totally Unimodular Matrices

The notion of totally unimodular matrices will help us obtain sufficient conditions for a polyhedron to be integral.

Definition 1. A matrix A is *totally unimodular* (TU) if every square submatrix of A has determinant $0, 1$, or -1.

Note. All entries of a TU matrix should be in $\{0, \pm 1\}$.

The next proposition is a simple, yet powerful property of totally unimodular matrices. Recall that a square matrix is nonsingular if its determinant is non-zero (i.e., it is full-ranked).

Proposition 2. *If A is TU and U is a nonsingular square submatrix of A, then U^{-1} is integral.*

Proof. By Cramer's rule,

$$|U^{-1}[i,j]| = \frac{|det(U_{\overline{ij}})|}{|det(U)|},$$

where $U_{\overline{ij}}$ is the submatrix of U obtained by dropping row i and column j. Note that the denominator is in $\{\pm 1\}$ by TU and non-singularity. The numerator is also integral since the determinant of an integral matrix will be integral. Hence all entries of U^{-1} are integral. □

This powerful property leads us to the following important result:

Lemma 2.1. *Let $A \in \mathbb{R}^{m \times n}$. If A is TU, then for every $b \in \mathbb{Z}^n$, the polyhedron*

$$P(b) := \{x : Ax \leq b\}$$

is an integral polyhedron.

Proof. Fix an integral vector b and the polyhedron $P = P(b)$. By the characterization of integral polyhedron, it suffices to show that each minimal face of P contains an integral point. Let F be a minimal face of P. Hence, $F \neq \emptyset$. By the characterization of minimal faces, we know that $F = \{x : A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$. Let $A' = [U\ V]$, where U is a full rank square matrix—such a U exists since F is a minimal face and hence, $\text{rank}(A') = \text{rank}(A)$ (recall the lectures on minimal faces of a polyhedron). Since U is full-ranked, it follows that U is non-singular. Consider the point

$$\bar{x} = \begin{bmatrix} U^{-1}b \\ 0 \end{bmatrix}.$$

We note that $\bar{x} \in F$. By Proposition 2 and the fact that $b \in \mathbb{Z}^n$, we have that $\bar{x} \in \mathbb{Z}^n$. Hence, the minimal face F contains an integral point. $\square$

Note that the lemma shows that *every* polyhedron defined by a TU constraint matrix A with integral RHS vector b is integral. In fact, the converse of the lemma is also true:

Theorem 3 (Hoffman-Kruskal)**.** *Let $A \in \mathbb{Z}^{m\times n}$. The matrix A is TU iff the polyhedron $\{x : Ax \leq b\}$ is integral for all $b \in \mathbb{Z}^n$.*

Significance of Lemma 2.1. If we want to solve the IP $\max\{c^T x : Ax \leq bx \in \mathbb{Z}^n\}$, where A is TU and b is integral, then it is sufficient to solve the LP $\max\{c^T x : Ax \leq b\}$.

Now that we know that TU matrices correspond to lucky case of IPs (i.e., LP-relaxation optimum will be an optimum to the IP), the next natural question is the following:

Question. Can we efficiently verify if a given matrix is TU?

We will now see some sufficient conditions for a matrix to be TU.

10.2 Properties of TU matrices

We begin with some basic properties of TU matrices.

Proposition 4. *Let $A \in \mathbb{R}^{m\times n}$. Then,*

1. *A is TU $\implies$ $A_{ij} \in \{-1, 0, +1\}$ for all $i \in [m]$ and $j \in [n]$.*
2. *A is TU $\iff$ $-A$ is TU $\iff$ A^T is TU.*
3. *A is TU $\iff$ $\begin{bmatrix} A & I \end{bmatrix}$ is TU $\iff$ $\begin{bmatrix} A & -A & I & -I \end{bmatrix}$ is TU (**Exercise**).*

Using property 3 in conjunction with Theorem 3 leads to the following observation:

Observation. A is TU iff the polyhedron $\{x : a \leq Ax \leq b, l \leq x \leq u\}$ is integral for all integral vectors a, b, l, u.

We will see a sufficient condition for a matrix to be TU followed by a few applications of these conditions.

Theorem 5 (Sufficient conditions for TU)**.** *Let $A \in \{0, 1, -1\}^{m\times n}$ such that*

(i) Each column of A contains at most two non-zeros and

(ii) There exists a partition $M_1 \cup M_2 = [m]$ of the rows such that every column j with two non-zero entries satisfies

$$\sum_{i \in M_1} A_{ij} = \sum_{i \in M_2} A_{ij}.$$

Then A is TU.

Proof. Suppose not. Let A be a counterexample with smallest number of rows + number of columns. Then, A is a square matrix—otherwise, we can obtain a smaller counterexample which would be a contradiction. Therefore, A is a *smallest* square matrix such that A satisfies (i) and (ii), but $\det(A) \notin \{0, 1, -1\}$.

If A has a column with only one non-zero entry, then removing that column will give a smaller counterexample which contradicts the choice of A. Therefore, we may assume that every column of A has exactly two non-zeros.

Condition (ii) implies that $\sum_{i \in M_1} A_{ij} - \sum_{i \in M_2} A_{ij} = 0$ for every column j. So the rows of A are linearly dependent which implies that $\det(A) = 0$. This is a contradiction. □

10.3 Applications of TU matrices

10.3.1 Min cost perfect matching in bipartite graphs (assignment problem)

Given a bipartite graph $G = (V = L \cup R, E)$, with cost function $c : E \to \mathbb{R}$ on the edges, we are interested in finding an optimum solution to the IP

$$\min \left\{ \sum_{e \in E} c_e x_e : \sum_{e \in \delta(v)} x_e = 1 \ \forall v \in V, x_e \geq 0 \ \forall e \in E, x \in \mathbb{Z}^E \right\}$$

where $\delta(v)$ is the set of edges incident to vertex v.

The LP relaxation of this IP is

$$\mathrm{LP_{rel}} : \min \left\{ \sum_{e \in E} c_e x_e : \sum_{e \in \delta(v)} x_e = 1 \ \forall v \in V, x_e \geq 0 \ \forall e \in E \right\}.$$

The constraints in $\mathrm{LP_{rel}}$ are of the form $Ax = \mathbb{1}, x \geq 0$ which is equivalent to

$$\begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \leq \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \\ 0 \end{bmatrix}. \tag{10.1}$$

Proposition 6. *The matrix A appearing in system (10.1) is TU. I.e., the vertex-edge incidence matrix of a bipartite graph is TU.*

Proof. Rows of A correspond to nodes of G and columns of A correspond to edges of G. In particular,

$$A[v, e] = \begin{cases} 1 & \text{if } e \in \delta(i), \\ 0 & \text{otherwise.} \end{cases}$$

So, $A \in \{0, 1\}^{V \times E}$. We also have the following properties for A:

(i) The number of ones per column is two (since each edge is adjacent to exactly two nodes).

(ii) Taking $M_1 := L$ and $M_2 := R$ we have that

$$\sum_{v \in M_1} A[v, e] = 1 = \sum_{v \in M_2} A[v, e] \quad \forall e \in E.$$

Therefore, by Theorem 5, the matrix A is TU.

□

By property 3 of TU matrices we have that $\begin{bmatrix} A \\ -A \\ -I \end{bmatrix}$ is also TU. The RHS vector of LP_{rel} is integral. Consequently, LP_{rel} has an integral optimum solution for every objective direction c.

10.3.2 Maximum weight matching in bipartite graphs

Given a bipartite graph $G = (V = L \cup R, E)$ with weight function $w : E \to \mathbb{R}$ on the edges, we would like to find a maximum weight matching in G. This is formulated by the following IP:

$$\max \left\{ \sum_{e \in E} w_e x_e : \sum_{e \in \delta(v)} x_e \le 1 \ \forall v \in V, x_e \ge 0 \ \forall e \in E, x \in \mathbb{Z}^E \right\}.$$

The LP relaxation of this IP is

$$\text{LP}_{\text{rel}} : \max \left\{ \sum_{e \in E} w_e x_e : \sum_{e \in \delta(v)} x_e \le 1 \ \forall v \in V, x_e \ge 0 \ \forall e \in E \right\}.$$

If G is bipartite, then the constraint matrix of LP_{rel} is TU (by Proposition 6). Moreover, the RHS of LP_{rel} is also integral. Therefore, by Theorem 3, LP_{rel} has an integral optimum solution. Thus, the above two problems have the same optimum value.

We now focus on the strong dual problem when the weight function is unit. Consider the maximum cardinality matching problem in bipartite graphs formulated by the IP:

$$\max \left\{ \sum_{e \in E} x_e : \sum_{e \in \delta(v)} x_e \le 1 \ \forall v \in V, x_e \ge 0 \ \forall e \in E, x \in \mathbb{Z}^E \right\}$$
$$= \max \left\{ \sum_{e \in E} x_e : \sum_{e \in \delta(v)} x_e \le 1 \ \forall v \in V, x_e \ge 0 \ \forall e \in E \right\}.$$

The equality above is because we have already seen that the LP relaxation has an integral optimum solution. The right side problem above is an LP. By LP-duality, we know that its optimum objective value is equal to that of its dual. What is the dual? The dual is

$$\min \left\{ \sum_{v \in V} y_v : y_u + y_v \ge 1 \ \forall (u, v) \in E, y_v \ge 0 \ \forall v \in V \right\}.$$

Call the dual problem as P_1.

Note that the constraint matrix of the dual is also TU and hence P_1 also has an integral optimum solution!

Optimal solution to P_1 will be a solution $y \in \{0,1\}^V$ where $\text{support}(y) := \{v : y_v = 1\}$ is a vertex cover of G. Therefore, P_1 formulates the minimum cardinality vertex cover problem. Recall that the primal and dual optimum values of the LP are the same. Thus, we have derived the following theorem.

Theorem 7 (König). *Let G be a bipartite graph. Then the maximum cardinality matching size in G is equal to the minimum cardinality vertex cover size in G.*

Consequently, the minimum cardinality vertex cover problem is a strong dual to the max cardinality matching problem in bipartite graphs.