TIN093 Algorithms - Assignment 7

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Problem 12

The flaw is in the assumption that a CNF formula can be reduced to a DNF formula in polynomial time which would be necessary in order to solve CNF in polynomial time. However we know that the conversion from CNF to DNF can take much longer than polynomial time. This conversion has a worst case time complexity which is exponential. Hence CNF is not polynomial time reducible to DNF. Therefore solving CNF by first converting it to DNF will not take polynomial time, since the conversion/reduction can take exponential time.

Problem 13

To prove Half-half subset sum is NP complete we must confirm that:

1. Subset sum is NP complete
2. Subset sum is polynomial time reducible to Half-half subset sum
3. The reduced instance of Subset sum to Half-Half subset sum is equivalent with the original subset-sum problem. I.e the answer in the original Subset Sum is yes/no if and only if the answer is yes/no in the reduced problem to Half-half subset sum.

We already know that Subset Sum is NP-complete. So we must show a polynomial time reduction to Half-Half subset sum which gives an equivalent instance of the problem as described in 3. We propose the following polynomial time reduction which depends on 3 cases.

1. $W = \sum_{i=1}^{n} w_i/2$ If $W = \sum_{i=1}^{n} w_i/2$ then our subset sum problem is already an instance of Half-Half subset sum and we solve it.
2. $W > \sum_{i=1}^{n} w_i/2$ In this case, we add an item x with a weight w_x such that the total weight $W_{tot} = \sum_{i=1}^{n} w_i$ added with w_x when divided by 2 gives W. That is:
$$\frac{W_{tot} + w_x}{2} = \frac{\sum_{i=1}^{n} w_i + w_x}{2} = W \tag{1}$$

So what is this weight w_x equal to? It becomes:

$$w_x = 2W - W_{tot} \tag{2}$$

Now, why is this equivalent to the original Subset sum problem? Assume a Subset S with weight $W > \frac{W_{tot}}{2}$ existed. In that case, a subset with weight $W = \frac{W_{tot}+w_x}{2} = W_{half}$ would also exist in the Half-Half subset sum problem, namely the subset S. And assume a subset with weight W did not exist in Subset sum. Then no subset with weight W exists in Half-half subset sum which does not contain w_x. A subset with weight W could still exist in Half-half subset sum, however if it existed it would be because it contained w_x. We are only interested in subsets in Half-half subset sum which do not contain w_x.

3. $W < \sum_{i=1}^{n} w_i/2$ In this case, we have to be more creative. We use the following equation:

$$W + w_x = \frac{W_{tot} + w_x}{2} = W_{half} \implies w_x = W_{tot} - 2W > 0 \tag{3}$$

But what does this mean? It means that we add an item w_x to W_{tot} such that half of their total weight equals $W + w_x$ which is indeed possible. Remember that it was impossible to add an item w_x to W_{tot} such that half of their total weight became W since $W < 0.5W_{tot}$. But this other way works. But now we have to be careful how we analyze it. Since we have:

$$W + w_x = W_{half} \tag{4}$$

This means that we must check if there is a subset in Half-half subset sum which includes w_x. Notice how we in situation 2) looked for a subset which did not include w_x. Here we indeed look for a solution S which includes w_x because we know that the weight of $S \setminus x$ is indeed W because $W + w_x = W_{half}$. So again, lets prove equivalence. If a subset S exists in Subset sum with weight $W < 0.5W_{tot}$ then a solution/subset S' exists in Half-half subset sum with weight $W + w_x = W_{half}$. If a subset with weight $W < 0.5W_{tot}$ does not exist in Subset sum, then no solution S' in Half-half subset sum including w_x exists. Note that there might still be a solution without w_x that has a weight of W_{half}. But no solution of the form $W + w_x = W_{half}$ will exist.