ITEC201006

natural number]→ integer
negative number
정수

rational number => 유리수 (분수가능)

Introduction to Computer Science & Engineering

Lecture 2: Binary Number System

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Numeral Systems

- Main types
  - Natural numbers: $\mathbb{N}$
  - Integers: $\mathbb{Z}$
  - Rational numbers: $\mathbb{Q}$
  - Complex numbers: $\mathbb{C}$

A quotient of two integers

- Why do we care anyway?
  - Numbers are crucial to computing
  - A language of a computing system

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Morning Brain Teaser

- Contrary to rational numbers, irrational numbers cannot be expressed as a quotient of two integers
  - Those two integers are relatively prime
- Then, prove $\sqrt{2}$ is a irrational number

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Positional Notation

- Decimal number
  - We are so familiar with positional notation of a decimal system that we probably don't think about it:
  - $943 = 9 * 10^2 + 4 * 10^1 + 3$

Our natural number system is base = 10

- Other base
  - $943 = 1 * 8^3 + 6 * 8^2 + 5 * 8 + 7 \quad = 1657_8$
  - $x = \sum_{n=1}^{N} d_n * R^{n-1}$

A general form with base = R

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Positional Notation (Contd.)

- Some notes
  - $$x = \sum_{n=1}^{N} d_n * R^{n-1}$$
    $$\begin{cases} x : \text{a value} \\ d_n : \text{a digit} \\ R : \text{base} \end{cases}$$
  - $R \in \mathbb{N} \backslash \{1\}$
  - $0 \leq d_n < R, \ \forall n$
- Base larger than 10
  - We use an ***alphabet***
    - $0, 1, 2, ..., 9, \mathrm{A}, \mathrm{B}, ..., \mathrm{F}$ in **hexagonal (=16) base**
    - For example, $943 = 3\mathrm{AF}_{16}$

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Base Conversion

- Algorithm pseudocode
  - Given x, convert the new base $\tilde{R}$

1. Find the maximum n such that $0 < \left\lfloor \frac{x}{\tilde{R}^n} \right\rfloor < \tilde{R}$
2. Update $x \leftarrow x - \tilde{R}^n * \left\lfloor \frac{x}{\tilde{R}^n} \right\rfloor$
3. Iterate the step 1 ~ 2 until $x = 0$

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Binary Number System

- Binary number system
  - A number system with base = 2
- Why so special?
  - The base-2 number system is particularly important in computing
  - Natural representation of **bits**
  - $943 = 1110101111_2$
  - Computers' storage unit only deals with 0 (low voltage) or 1 (high voltage)

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Relationship to other bases (1)

- Octal (=8) base
  - $1\ 1\ 1\ 0\ 1\ 0\ 1\ \underset{2^2}{1}\ \underset{2^1}{1}\ \underset{2^0}{1}$
  - $\underbrace{1}_{=1}\ \underbrace{1\ 1\ 0}_{=6}\ \underbrace{1\ 0\ 1}_{=5}\ \underbrace{1\ 1\ 1}_{=7} \rightarrow 1657_8$
  - Why is this possible?: $8 = 2^3$

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Relationship to other bases (2)

- Hexagonal (=16) base
  - $\underbrace{1\ 1}_{=3}\ \underbrace{1\ 0\ 1\ 0}_{=A}\ \underbrace{1\ 1\ 1\ 1}_{=F} \rightarrow 3AF_{16}$
- Most computer engineers commonly use hexagonal base in their design
  - E.g., considering a 16 bits system,

$$0111111111111111 = 7FFF$$
$$(= 32767)$$

Sign bit

Could be the maximum number that can be represented

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Arithmetic in Binary

- Same with decimal

  - Addition

```
 1 0 1 1 1 1 1
   1 0 1 0 1 1 1
  +1 0 0 1 0 1 1
 1 0 1 0 0 0 1 0
```

  - Subtraction

```
0 1 2
      0 2
1 0 1 0 1 1 1
-  1 1 1 0 1 1
0 0 1 1 1 0 0
```


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Some Notes

- What is the maximum length of the bits after summing two 10 bits numbers?
  - The answer is 11 bits (Why?)

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Fractional Bits

- How to represent a number smaller than 1?
  - Fractional bits $1\,1\,.\,1\,0\,1$

    $2^1 \quad 2^0 \qquad 2^{-1} \quad 2^{-2} \quad 2^{-3}$

    $(= 1 * 2^1 + 1 * 2^0 + 1 * 2^{-1} + 0 * 2^{-2} + 1 * 2^{-3})$

    $(= 3.625)$
  - There exist some numbers that cannot be represented in fractional bits (when the bits length is limited)
  - This is called ***quantization error***

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Fractional Bits (Contd.)

- T/F: without any error, all ***the integers*** can be expressed in binary
  - **F** when the bits length is limited
- T/F: without any error, all ***the rational numbers*** can be expressed in binary
  - **F** when the bits length is limited
- Back to a 16 bits machine
  - How to use fractional bits in 16 bits machines?
  - There are multiple formats in 16 bits representation
  - I.e., we do not know the exact value only with $7\mathrm{FFF}$

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S16.XX

- Consider our 16 bits system is as follows
  - $\mathrm{7FFF} = 0111111111111111 \quad 2^{0}$
  - In this case, our system might cover integers
  - This format is called **S16.00**
- Now, consider that
  - $\mathrm{7FFF} = 0111111111111111 \quad 2^{-15}$
  - In this case, our system might cover rational numbers < 1
  - This format is called **S16.15**

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Some Notes

- Theoretically, any format is possible, e.g., S16.-10 or S16.99
  - What will change depending on the format?
  - What will be an efficient way to determine this format?
- What is the feasible range of S16.XX?

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Conversion Algorithm

- Assume that we have a 16 bits machine
- Given an arbitrary value (could be an integer or rational number), design a pseudo algorithm that finds
    - Appropriate format (S16.??)
    - Exact hexagonal representation
    - Minimizing the quantization error

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Quiz:
Blind Separation

- Let assume we have 10 coins, where 4 of them are flipped
    - Assume that we cannot distinguish them by seeing or touching them
- Provide a method that
    - separates coins into two groups, and makes each group have same number of flipped coins
    - Any process is possible to be used
- Try to think algorithmically!

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Negative Numbers

- Assume that the format is S16.15
  - $8000 = 1\underbrace{000000000000000}_{\text{Non-sign bits}}$ (2^{-15})

    Sign bit

  - Non-sign bits value - Sign bit value
    - $0 - 1 * 2^0 = -1$
  - Other examples:
    - $7\mathrm{FFF} = 0111111111111111$
      $= 2^{-1} + 2^{-2} + ... + 2^{-15}$
    - $\mathrm{FFFF} = 1111111111111111$
      $= 2^{-1} + 2^{-2} + ... + 2^{-15} - 2^0$

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Feasible Range

- What is a feasible range of S16.15?
  - Maximum value: $7FFF = 0111111111111111$

$$= 2^{-15} + 2^{-14} + \ldots + 2^{-1}$$

등비수열

$$= 1 - 2^{-15}$$

$$\sum_{k=0}^{14} 2^{-15} \cdot 2^{k} \Rightarrow \frac{2^{-15}(1-2^{14-0+1})}{1-2} \Rightarrow \frac{2^{-15}-1}{-1} \Rightarrow 1-2^{-15}$$

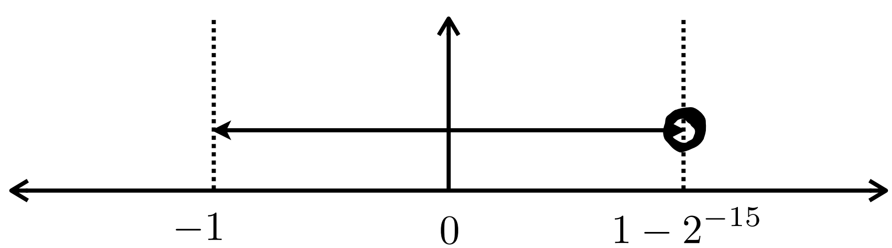
  - Minimum value: $8000 = 1000000000000000$

$$= -2^{0}$$

$$= -1$$

8000 (= -1)
< FFFF (= -2^{-15})

  - Feasible range:



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