Sliding down a Sliding Plane

Consider the case of a particle of mass m sliding down a smooth inclined plane of mass M which is, itself, free to slide on a smooth horizontal surface, as shown in Figure 34. This is a two degree of freedom system, so we need two coordinates to specify the configuration. Let us choose x, the horizontal distance of the plane from some reference point, and x', the parallel displacement of the particle from some reference point on the plane.

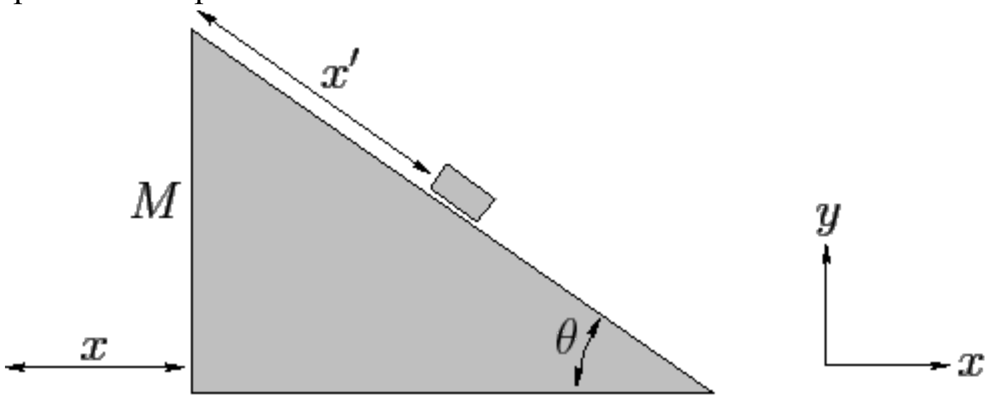

Figure 34: *A sliding plane.*

Defining x- and y-axes, as shown in the diagram, the x- and y-components of the particle's velocity are clearly given by

$$v_x = \dot{x} + \dot{x}' \cos\theta, \tag{637}$$

$$v_y = -\dot{x}' \sin\theta, \tag{638}$$

respectively, where θ is the angle of inclination of the plane with respect to the horizontal. Thus,

$$v^2 = v_x^2 + v_y^2 = \dot{x}^2 + 2\,\dot{x}\,\dot{x}' \cos\theta + \dot{x}'^2. \tag{639}$$

Hence, the kinetic energy of the system takes the form

$$K = \frac{1}{2}\,M\,\dot{x}^2 + \frac{1}{2}\,m\,(\dot{x}^2 + 2\,\dot{x}\,\dot{x}' \cos\theta + \dot{x}'^2), \tag{640}$$

whereas the potential energy is given by

$$U = -m\,g\,x'\,\sin\theta + \text{const.} \tag{641}$$

It follows that the Lagrangian (which is **K – U**) is written

$$L = \frac{1}{2}\,M\,\dot{x}^{2} + \frac{1}{2}\,m\,(\dot{x}^{2} + 2\,\dot{x}\,\dot{x}'\,\cos\theta + \dot{x}'^{2}) + m\,g\,x'\,\sin\theta + \text{const.} \tag{642}$$

The equations of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{643}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} = 0, \tag{644}$$

thus yield

$$M\,\ddot{x} + m\,(\ddot{x} + \ddot{x}'\,\cos\theta) = 0, \tag{645}$$

$$m\,(\ddot{x}' + \ddot{x}\,\cos\theta) - m\,g\,\sin\theta = 0. \tag{646}$$

Finally, solving for $\ddot{x}$ and $\ddot{x}'$, we obtain

$$\ddot{x} = -\frac{g\,\sin\theta\,\cos\theta}{(m+M)/m - \cos^{2}\theta}, \tag{647}$$

$$\ddot{x}' = \frac{g\,\sin\theta}{1 - m\,\cos^{2}\theta/(m+M)}.$$

Conservation Of T+U

A smooth right angled wedge **ABC** with mass **M**, with **B** being 90deg and the side **AC** being 1meter, is placed on a smooth horizontal table and is constrained not to move. A smooth particle with mass **m** is held at **A**, in preparation for sliding down the side **AC**. When the particle is released at **A**, the constraint on the wedge is removed. Show that the total energy is conserved when the particle reaches **C**.

Refer to “Sliding_Down_A_Sliding_Plane_A.docx”. Here, for readability, we will denote the co-ordinates **x** as **x1** and **x’** as **x2.**

So **x1** is the distance from a fixed point on the table to the wedge and **x2** is the distance from **A** to the particle along **AC**. Then, from the above reference. these results follow:

$$\ddot{x}1 = -\frac{mgsin(\theta)\cos(\theta)}{M + msin^2(\theta)}$$

$$\ddot{x}2 = \frac{(M+m)gsin(\theta)}{(M + msin^2(\theta))}$$

where dots denote differentiation with respect to time.

Let **t** be the time since **m** left **A**. Then **t = 0** when **m** leaves **A**. Let **t = T** when **m** reaches **B**.

Then

$$T = \sqrt{\frac{2(M + msin^2(\theta))}{(M+m)gsin(\theta)}}$$

The speed of **m** with respect to **M** at **T** is

$$\dot{x}2 = \sqrt{\frac{2(M+m)gsin(\theta)}{(M + msin^2(\theta))}}$$

The speed of **M** with respect to the horizontal table at **T** is

$$\dot{x}1 = \frac{mgsin(\theta)\cos(\theta)}{(M + msin^2(\theta)}\sqrt{\frac{2(M + msin^2(\theta))}{(M+m)gsin(\theta)}}$$

Choose a frame of reference that is moving at a fixed speed equal to and in the same direction as that of **M** at **T**.

In this frame, at **t = 0** the system will have a Potential Energy of

$$mgsin(\theta)$$

and a Kinetic Energy of:

$$\frac{(M+m)\dot{x}1^2}{2}$$

And at **t = T**, the system will have a Potential Energy of zero

And a Kinetic Energy of

$$\frac{m\dot{x}2^2}{2}$$

At both **t = 0** and at **t = T** the sum of the potential and kinetic energies add up to:

$$\boldsymbol{T + U = \frac{(M+m)mgsin(\theta)}{(M + msin^2(\theta))}}$$