Задача 1.

$$\bar a \times \bar b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\vec i + (a_3 b_1 - a_1 b_3)\vec j + (a_1 b_2 - a_2 b_1)\vec k$$

$a_i' = \alpha_{in} a_n$

$b_i' = \alpha_{im} b_m$

$$\bar a' \times \bar b' = \begin{vmatrix} i' & j' & k' \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} \\ \alpha_{1m} & \alpha_{2m} & \alpha_{3m} \end{vmatrix} a_n b_m =$$

$$= (\alpha_{2n}\alpha_{3m} - \alpha_{3n}\alpha_{2m}) a_n b_m \vec i' + (\alpha_{3n}\alpha_{1m} - \alpha_{1n}\alpha_{3m}) a_n b_m \vec j + (\alpha_{1n}\alpha_{2m} - \alpha_{2n}\alpha_{1m}) a_n b_m \vec k$$

$a \times b = c$

$a' \times b' = c'$

$c_1' = (\alpha_{2n}\alpha_{3m} - \alpha_{3n}\alpha_{2m}) a_n b_m$

$c_1 = a_2 b_3 - a_3 b_2$

$$c_1' = \alpha_{2n}\alpha_{3m} a_n b_m - \alpha_{3n}\alpha_{2m} a_n b_m = \alpha_{2n}\alpha_{3m} a_n b_m - \alpha_{3m}\alpha_{2n} a_m b_n = \underbrace{\alpha_{2n}\alpha_{3m}}_{\alpha_{1k}} \underbrace{(a_n b_m - a_m b_n)}_{c_k}$$

$c_1' = \alpha_{1k} c_k$ Аналогично для c_2' и c_3'

Изменение по векторному закону

Задача 2

2.1.

$c_i = e_{ijk} a_j$

$c_k = e_{ijk} a$

$\vec C = \vec a \times \vec b =$

$c_x = e_{xj}$

$= a_y b_z -$

$c_y = e_{yj}$

$= a_z b_x -$

$c_z = e_z$

$= a_x b_y$

$c_k = e_{ij}$

$c_x = e_{ij}$

$= a_y$

$c_y = e_{ij}$

$c_z = e_i$

$= a_x$

2.2

$\nabla \times \nabla$

r

$(a_3b_1 - a_1b_3)\vec{j} +$

$+$

$n\alpha_{2m}$ -

$=$

n =

e_2' и e_3'

закону

Задача 2.

2.1.

$c_i = \varepsilon_{ijk} a_j b_k$

$c_k = \varepsilon_{ijk} a_i b_j$

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = e_x(\overbrace{a_y b_z - a_z b_y}^{c_x}) + \vec{e}_y(\overbrace{a_z b_x - a_x b_z}^{c_y}) + \vec{e}_z(\overbrace{a_x b_y - a_y b_x}^{c_z})$$

$c_x = \varepsilon_{xjk} a_j b_k = \varepsilon_{xyz} a_y b_z + \varepsilon_{xzy} a_z b_y =$
$= a_y b_z - a_z b_y$

$c_y = \varepsilon_{yjk} a_j b_k = \varepsilon_{yxz} a_x b_z + \varepsilon_{yzx} a_z b_x =$
$= a_z b_x - a_x b_z$

$c_z = \varepsilon_{zjk} a_j b_k = \varepsilon_{zxy} a_x b_y + \varepsilon_{zyx} a_y b_x =$
$= a_x b_y - a_y b_x$

$c_k = \varepsilon_{ijk} a_i b_j$

$c_x = \varepsilon_{ijx} a_i b_j = \varepsilon_{yzx} a_y b_z + \varepsilon_{zyx} a_z b_y =$
$= a_y b_z - a_z b_y$

$c_y = \varepsilon_{ijy} a_i b_j = \varepsilon_{xzy} a_x b_z + \varepsilon_{zxy} a_z b_x = a_z b_x - a_x b_z$

$c_z = \varepsilon_{ijz} a_i b_j = \varepsilon_{xyz} a_x b_y + \varepsilon_{yxz} a_y b_x =$

$= a_x b_y - a_y b_x$

2.2 $\nabla \times \nabla f = 0$

$\nabla \times \nabla f = \text{rot grad}$

$$\text{rot grad} f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} =$$

$$i\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z}-\frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right)\ldots\ldots$$

т.к. порядок дифференцирования не важен

$$\nabla\times\nabla f=0$$

2.3.

$$\nabla\times r=0$$

$$\mathrm{rot}\,\bar r=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=$$

$$=i\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)^{=0}\ldots\ldots=0.$$

2.4.

$$\nabla f^2$$

$$\mathrm{grad}\,f^2=\left(2f\frac{\partial f}{\partial x};2f\frac{\partial f}{\partial y};2f\frac{\partial f}{\partial z}\right)$$

$$\frac{\partial(f^2)}{\partial x}=2f\frac{\partial f}{\partial x}.$$

$$\mathrm{grad}\,f^\alpha=\left(\alpha f^{\alpha-1}\frac{\partial f}{\partial x};\alpha f^{\alpha-1}\frac{\partial f}{\partial y};\alpha f^{\alpha-1}\frac{\partial f}{\partial z}\right)$$

2.5. ∇r^2, где $r=\sqrt{x^2+y^2+z^2}$

$$\frac{\partial r^2}{\partial x}=2x\qquad\frac{\partial r^2}{\partial y}=2y\qquad\frac{\partial r^2}{\partial z}=2z$$

$$\nabla r^2=(2x;2y;2z)$$

$\nabla r^{\frac12}$

$\mathrm{grad}\,r^{\frac12}=($

$\frac12 r^{-\frac12}\frac{\partial r}{\partial z})$

2.6. При перехо

меняется $\Rightarrow$

x_i, x_j, g_{ij} в

$A'=\sum_{mp}$ Lind

g_i

$=\sum_{lmnp}\delta_{kn}\delta_p$

$=\sum_{km}g_{km}x_k$

Задача 3.

$e_{ijl}e_{ijm}=$

$e_{123}e_{123}+2$

$+e_{321}e_{321}$

$2\delta_{jm}=2$

2. $\nabla\times(\bar a\times\bar b)$

$\vec\nabla\times(\bar a\times\bar b)$

По прав

$\nabla(\bar a\times\bar b)=$

$\nabla(\bar a\times\bar b)=$

3. $(\vec a\cdot\nabla\vec r)$

...
овение не важен

$=0.$

$2f\frac{\partial f}{\partial z}\Big)$

$\frac{\partial f}{\partial y}, 2f^{2-1}\frac{\partial f}{\partial z}\Big)$

$=2z$

$\nabla r^{\frac{1}{2}}$

$grad\, r^{\frac{1}{2}} = \left(\frac{1}{2} r^{-\frac{1}{2}} \frac{\partial r}{\partial x}; \frac{1}{2} r^{-\frac{1}{2}} \frac{\partial r}{\partial y};\right.$
$\left.\frac{1}{2} r^{-\frac{1}{2}} \frac{\partial r}{\partial z}\right) = \left(\frac{1}{2\sqrt{r}}\frac{\vec r}{r}; \frac{1}{2\sqrt{r}}\frac{\vec r}{r}; \frac{1}{2\sqrt{r}}\frac{\vec r}{r}\right) = \frac{\vec r}{2\sqrt{r^3}}$

2.6 При переходе в другую СК скаляр не меняется $\Rightarrow$ $A = A'$ Запишем ~~вект~~ преобразование x_i, x_j, g_{ij} в предположении, что g_{ij} – тензор.

$A' = \sum_{lmnp} \underbrace{\alpha_{in}\alpha_{jp}}_{g'_{ij}} \underbrace{\sum_k \alpha_{ik} x_k}_{x'_i} \underbrace{\sum_m \alpha_{jm} x_m}_{x'_j} =$

$= \sum_{lmnp} \delta_{kn}\delta_{pm} g_{np} x_k x_m = \sum_{nkm} \delta_{kn} g_{km} x_k x_m =$

$= \sum_{km} g_{km} x_k x_m = \sum_{ij} g_{ij} x_i x_j$ Утверждение верно

Задача 3.

$e_{ijl}\, e_{ijm} = 2\delta_{lm}$

$e_{123}e_{123} + e_{213}e_{213} + e_{312}e_{312} + e_{132}e_{132} + e_{321}e_{321} + e_{231}e_{231} = 6$

$2\delta_{jm} = 2(\delta_{11} + \delta_{22} + \delta_{33}) = 6$

2. $\nabla \times (\vec a \times \vec b) = \vec a(\nabla \vec b) - (\nabla \vec a)\vec b + (\vec b \nabla \vec a) - (\vec a \nabla)\vec b$

$\vec\nabla \times (\vec a \times \vec b) = \nabla \times (\vec a^{\downarrow} \times \vec b) + \nabla(\vec a \times \vec b^{\downarrow})$

По правилу бац - цаб

$\nabla(\vec a^{\downarrow} \times \vec b) = \vec a(\nabla \vec b) - \vec b(\nabla \vec a)$

$\nabla(\vec a \times \vec b^{\downarrow}) = \vec a(\nabla \vec b) - \vec b(\nabla \cdot \vec a)$

3. $(\vec a \cdot \nabla \vec r) = \vec a$

$$\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)\vec r =$$

$$=\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)x\vec i +$$

$$+\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)y\vec j +$$

$$+\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)z\vec k =$$

$$=\left(\frac{\partial x}{\partial x}a_x\vec i+\frac{\partial}{\partial y}a_y\vec j+\frac{\partial z}{\partial z}a_z\vec k\right)=\vec a$$

4. $\nabla\cdot(\vec a\times\nabla)\vec r=\nabla\times\vec a$

$$\vec a\times\nabla=\begin{vmatrix} i & j & k\\ a_x & a_y & a_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix}=-\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_x & a_y & a_z\end{vmatrix}=$$

$$=\vec i\left(\frac{\partial a_y}{\partial z}-\frac{\partial a_z}{\partial y}\right)+\vec j\left(\frac{\partial a_z}{\partial x}-\frac{\partial a_x}{\partial z}\right)+$$

$$+\vec k\left(\frac{\partial a_x}{\partial y}-\frac{\partial a_y}{\partial x}\right)$$

$$(\vec a\times\nabla)\vec r=\left(\frac{\partial a_y}{\partial z}-\frac{\partial a_z}{\partial y}\right)x+$$

$$+\left(\frac{\partial a_z}{\partial x}-\frac{\partial a_x}{\partial z}\right)y+\left(\frac{\partial a_x}{\partial y}-\frac{\partial a_y}{\partial x}\right)z$$

$$\mathrm{div}\,(a\times\nabla)\vec r=\left(\frac{\partial a_y}{\partial z}-\frac{\partial a_z}{\partial y}\right)\vec i+$$

$$+\left(\frac{\partial a_z}{\partial x}-\frac{\partial a_x}{\partial z}\right)\vec j+\left(\frac{\partial a_x}{\partial y}-\frac{\partial a_y}{\partial x}\right)\vec k=$$

$$=$$

$$\nabla \times (\nabla \times \vec{a}) = \nabla \cdot (\nabla \cdot a) - \nabla^2 \vec{a} =$$

$$= \operatorname{grad} \operatorname{div} \vec{a} - \nabla^2 \vec{a}$$

$$\operatorname{rot} \vec{a} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = i\left(\frac{\partial a}{\partial y} - \frac{\partial a_y}{\partial z}\right) -$$

$$- j\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \vec{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$\operatorname{rot} \operatorname{rot} \vec{a} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) & \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) & \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right)\right) +$$

$$+ j\left(\frac{\partial}{\partial x}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\right) +$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\right) =$$

$$= \vec{i}\left(\frac{\partial^2 a_y}{\partial y \partial x} - \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_z}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z^2}\right) -$$

$$- \vec{j}\left(\frac{\partial^2 a_y}{\partial x^2} - \frac{\partial^2 a_x}{\partial x \partial y} - \frac{\partial^2 a_z}{\partial z \partial y} + \frac{\partial^2 a_y}{\partial z^2}\right) +$$

$$+ \vec{k}\left(\frac{\partial^2 a_x}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial x^2} - \frac{\partial^2 a_z}{\partial y^2} + \frac{\partial a_y}{\partial z \partial y}\right) =$$

$$= \vec{i}\left(\frac{\partial^2 a_y}{\partial y \partial x} - \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_z}{\partial z \partial x} + \frac{\partial^2 a_x}{x^2} - \frac{\partial^2 a_x}{\partial x^2}\right) +$$

$$+ \vec{j}\left(\frac{\partial^2 a_x}{\partial x \partial y} + \frac{\partial^2 a_z}{\partial z \partial y} - \frac{\partial^2 a}{\partial x^2} - \frac{\partial^2 a_y}{\partial z^2} - \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_y}{\partial y}\right)$$

$$+ \vec{k}\left(\frac{\partial^2 a_x}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial x^2} - \frac{\partial^2 a_z}{\partial y^2} + \frac{\partial a_y}{\partial z \partial y} - \frac{\partial^2 a_z}{\partial z^2} + \frac{\partial^2 a_z}{\partial z^2}\right)$$

$$-\left(\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a}{\partial} \ldots\right.$$

$$+ \frac{\partial^2 a_y}{\partial y^2} + \ldots$$

$$\vec{i} + \left(\frac{\partial^2 a_y}{\partial y^2} \ldots\right.$$

$$+ \left(\frac{\partial^2 a_z}{\partial z^2} + \frac{\partial}{\partial} \ldots\right.$$

$= \text{T.r. grad div} \ldots$

$$= \left(\frac{\partial}{\partial x}\left(\frac{\partial a_x}{\partial x} \ldots\right.\right.$$

$$+ \frac{\partial a_y}{\partial y} + \ldots$$

$\cancel{\nabla^2 a_z}$ ...

5. $\operatorname{grad} f(r) \ldots$

$$\frac{\partial f(r)}{\partial x} = \ldots$$

$$\frac{\partial f(r)}{\partial z} = \ldots$$

$$= \left(\frac{df}{dr} \ldots\right.$$

$\operatorname{grad} r \ldots$

6. $\nabla \cdot (\nabla \times \ldots$

$(\nabla \times a) = | \ldots$

$+$

$) - \nabla^2\vec{a} =$

$= i\left(\frac{\partial a}{\partial y} - \frac{\partial a_y}{\partial z}\right) -$

$\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$

$\frac{\partial}{\partial z}$

$\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$

$\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) +$

$\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}) +$

$y\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)) =$

$- \frac{\partial^2 a_x}{\partial z^2}) -$

$+ \frac{\partial^2 a_y}{\partial z^2}) +$

$\frac{a_y}{z\partial y}) =$

$\frac{\partial^2 a_x}{x^2} - \frac{\partial^2 a_x}{\partial x^2}) +$

$\frac{a_y}{z^2} - \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_y}{\partial y}$

$\frac{\partial a_y}{\partial z\partial y} - \frac{\partial^2 a_z}{\partial z^2} +$

$$-\left(\frac{\partial^2 a_x}{\partial x^2}+\frac{\partial^2 a_x}{\partial y^2}+\frac{\partial^2 a_x}{\partial z^2}\right)\bar{i} - \left(\frac{\partial^2 a_y}{\partial x^2} +\right.$$

$$\left.+\frac{\partial^2 a_y}{\partial y^2}+\frac{\partial^2 a_y}{\partial z^2}\right)\bar{j} - \left(\frac{\partial^2 a_z}{\partial x^2}+\frac{\partial^2 a_z}{\partial y^2}+\frac{\partial^2 a_z}{\partial z^2}\right)$$

$$\bar{k} + \left(\frac{\partial^2 a_y}{\partial y^2}+\frac{\partial^2 a_x}{\partial x\partial y}+\frac{\partial^2 a_z}{\partial z\partial y}\right)\vec{j} +$$

$$+\left(\frac{\partial^2 a_z}{\partial z^2}+\frac{\partial^2 a_x}{\partial x\partial z}+\frac{\partial^2 a_y}{\partial y\partial z}\right)\bar{k} =$$

$$= \text{т.к. } \operatorname{grad}\operatorname{div}\vec{a} = \operatorname{grad}\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right); \frac{\partial}{\partial y}\left(\frac{\partial a_x}{\partial x}+\right.\right.$$

$$\left.+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}; \frac{\partial}{\partial z}\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)\right)$$

$$\cancel{\nabla^2 a} = \operatorname{grad}\operatorname{div}\vec{a} - \nabla^2\vec{a}.$$

5. $\operatorname{grad} f(r) =$

$$\frac{\partial f(r)}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} \quad \frac{\partial f(r)}{\partial y}=\frac{df}{dr}\frac{\partial r}{\partial y}$$

$$\frac{\partial f(r)}{\partial z} = \frac{df}{dr}\frac{\partial r}{\partial z}$$

$$= \left(\frac{df}{dr}\frac{\partial r}{\partial x}, \frac{df}{dr}\frac{\partial r}{\partial y}, \frac{df}{dr}\frac{\partial r}{\partial z}\right) = \frac{df}{dr}\operatorname{grad} r.$$

$$\operatorname{grad} r = \frac{\vec{r}}{r} = \frac{df}{dr}\frac{\bar{r}}{r}.$$

6. $\nabla\cdot(\vec{\nabla}\times\vec{a}) = 0$

$$(\nabla\times a) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \vec{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) +$$

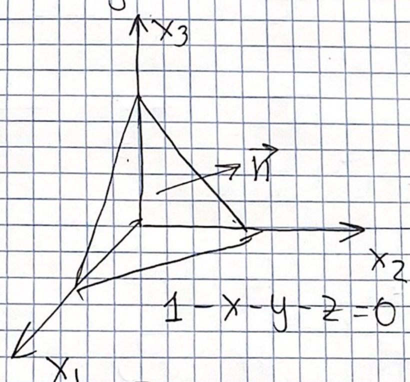
$$+\vec{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \bar{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$\operatorname{div}(\operatorname{rot}\vec{a}) = \frac{\partial}{\partial x}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial a_y}{\partial x} + \frac{\partial a_x}{\partial y}\right) =$$

$$= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_z}{\partial y \partial z} - \frac{\partial^2 a_y}{\partial x \partial z} + \frac{\partial^2 a_y}{\partial x \partial z} + \frac{\partial a_x}{\partial y \partial z} - \frac{\partial a_x}{\partial z \partial y} = 0$$

Т.к. порядок дифференцирования не важен.

Задача 4.



$$\Pi = p_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 2 \end{pmatrix}$$

$$\vec{n} = (1;1;1)$$

$$\vec{S}_n = \vec{S}_x \cos(n;x) + \vec{S}_y \cos(n;y) + \vec{S}_z \cos(n;z)$$

$$\cos(n;x) = \cos(n;y) = \cos(n;z) = \frac{1 \cdot 1}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

~~cos~~

$\vec{S}_x = (1\ 0\ 0)p_0$

$\vec{S}_y = (0; 1; \frac{1}{2})p_0$

$\vec{S}_z = (0\ \frac{1}{2}; 2)p_0$

$$\vec{f}_n = \left(\frac{\sqrt{3}}{3}p_0 i + \frac{\sqrt{3}}{3}p_0 j + \frac{1}{2}\frac{\sqrt{3}}{3}p_0 k + \right.$$
$$\left. + \frac{\sqrt{3}}{3}\cdot\frac{1}{2}p_0 j + \frac{\sqrt{3}}{3}\cdot 2p_0 k\right)$$

$$\bar{f}_n = \left(\frac{\sqrt{3}}{3}p_0;\ \frac{\sqrt{3}}{2}p_0 ;\ \frac{5\sqrt{3}}{6}p_0\right)$$

$$\cos(f_n;n) = \frac{\cancel{p_0}\left(\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{2}+\frac{5\sqrt{3}}{6}\right)}{\sqrt{3}\cdot\cancel{p_0}\sqrt{\left(\frac{\sqrt{3}}{3}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{5\sqrt{3}}{6}\right)^2}} =$$
$$= \frac{5\sqrt{114}}{5\cdot 7}$$

Нормальная составляющая.

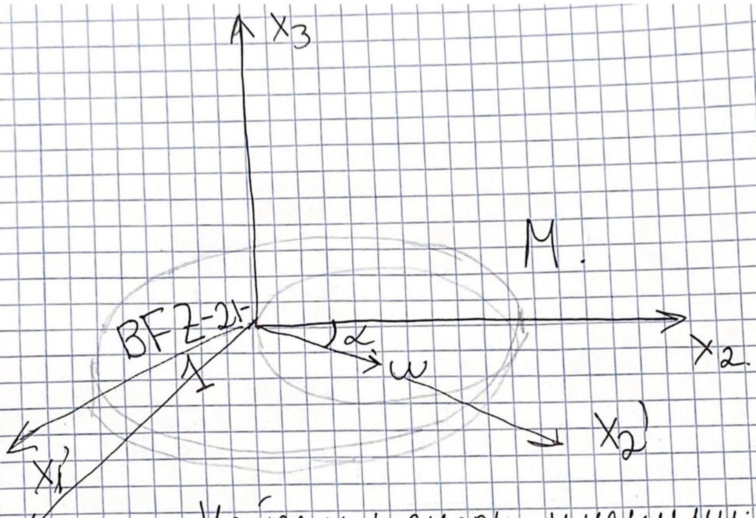
$$\vec{f}_{nn} = \vec{f}_n \cos(f_n;n) =$$
$$= \left(p_0\frac{5\sqrt{38}}{57};\ \frac{5\sqrt{38}}{38}p_0;\ \frac{25\sqrt{38}\,p_0}{114}\right)$$

$$\cos\left(90^\circ - \arccos\left(\frac{5\sqrt{114}}{57}\right)\right) = \cos(90 - 20^\circ 30' 50'') =$$
$$= \cos(69{,}49^\circ) = 0{,}35 = \frac{\sqrt{399}}{57}$$

$\cos(n;z)$

$$\bar{f}_{nk} = \left(\frac{\sqrt{133}}{57}p_0;\ \frac{\sqrt{133}}{38}p_0;\ \frac{5\sqrt{133}}{144}p_0\right)$$

Задача 5.



Найдем тензор инерции:

$$I_{kj} = \int dm\,(r^2 \delta_{jk} - x_j x_k)$$

$$I_{11} = \int dm\,(\underbrace{r^2 - x_1^2}_{x_2^2}) = \int dm\, x_2^2 =$$

(без учета дырки) $dm = \frac{M}{\pi R^2} = \rho\, dS = \frac{M}{\pi R^2}\, r\, dr\, d\varphi$

$$= \iint \frac{M}{\pi R^2}\, r^2 \sin^2\varphi\, r\, dr\, d\varphi =$$

$$= \frac{M}{\pi R^2}\int_0^R r^3\, dr \int_0^{2\pi} \sin^2\varphi\, d\varphi = \frac{M}{R^2}\cdot\frac{R^4}{4} = \frac{MR^2}{4}$$

Дыра:

$$I_{11} = -\frac{M}{\pi R^2}\int_0^{R/2} r^3\, dr \int_0^{2\varphi} \sin^2\varphi\, d\varphi =$$

$$dm = \rho\, dS = \frac{4M}{\pi R^2}\cdot\frac{1}{4}\, dS = \frac{M}{\pi R^2}\, r\, dr\, d\varphi$$

$$= -\frac{M}{\pi R^2}\cdot\frac{R^4 \pi}{64} = -\frac{MR^2}{64}.$$

$I_{11} = \frac{15}{64} M$

$I_{22} = \frac{15}{64} MR$

$I_{33} = \frac{M}{\pi R^2}\int_0^R r^3$

$= I_{11}$

$I_{12} = \int dm$

(без учета ды

Дырка

$I_{12} = +$

Аналоги

$i \neq j$

$I = \frac{15}{64}$

Момент

Запишем

$I' = \alpha$

$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} =$

$$I_{11} = \frac{15}{64} MR^2$$

$$I_{22} = \frac{15}{64} MR^2.$$

$$I_{33} = \frac{M}{\pi R^2}\int_0^R r^3 dr \int_0^{2\varphi} (\sin^2\varphi + \cos^2\varphi) d\varphi = $$

$$= I_{11} + I_{22} = \frac{30}{64} MR^2$$

$$- \frac{M}{\pi R^2}\int_0^{\frac{R}{2}} r^3 dr \int_0^{2\varphi} (\sin^2\varphi + \cos^2\varphi) d\varphi$$

$$I_{12} = \int dm(-X_1 X_2) = -\int_0 r^3 \int_0 \frac{\sin 2\varphi}{2} d\varphi = 0.$$

(без дырки)

Дырка

$$I_{12} = + \frac{M}{\pi R^2}\int_0^{\frac{R}{2}} r^3 d\varphi \int_0^{2\varphi} \frac{\sin 2\varphi}{2} d\varphi = 0.$$

Аналогично для других компонент с $i \neq j$

$$I = \frac{15}{64}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} MR^2$$

Момент сил со стороны крепления оси.

Запишем тензор в новой СК:

$$I' = \alpha I \alpha^T$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$=\alpha$

$$I' = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot$$

$$\times \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{MR^2 15}{64} =$$

$$= MR^2 \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \frac{MR^2 15}{64} \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$

$$= \frac{MR^2 15}{64} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Найдем момент импульса

$$L = \frac{MR^2 15}{64} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = \frac{15 MR^2}{64} (0\ \omega\ 0)$$

$$\vec{M} = \frac{dL}{dt} = \frac{15 MR^2}{64} \vec{\varepsilon} = \left(0\ \ 15MR\right.$$

где $\vec{\varepsilon}$ – угловое ускорение

Задача 6

$f(r) = g_{ij} x_i x$

$\frac{\partial(g_{ij} x_i x_j)}{\partial x_i}$

$\text{grad}\, f(r) =$

Задача 7

1. $S_{ij} = a_i b_j -$

Заметим

$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Компоненты

полученные

векторов, по

$\vec{a}$ и $\vec{b}$.

2. $\omega_i =$

$\omega_1 =$

$= \frac{1}{2}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{matrix} \ldots s\alpha & \sin\alpha & 0 \\ \ldots n\alpha & \cos\alpha & 0 \\ & 0 & 1 \end{matrix} =$$

Задача 6

$$f(r) = g_{ij} x_i x_j$$

$$\frac{\partial (g_{ij} x_i x_j)}{\partial x_i} = g_{ij} \frac{\partial (x_i x_j)}{\partial x_i} = (\delta_{ij} x_j + x_i \delta_{ii}) g_{ij}$$

$$\mathrm{grad}\, f(r) = (\delta_{ij} x_j + x_i \delta_{ii}) g_{ij}$$

Задача 7.

1. $S_{ij} = a_i b_j - a_j b_i = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$

Заметим, что

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \Rightarrow i\overbrace{(a_2 b_3 - a_3 b_2)}^{=S_{23}} + j\overbrace{(a_3 b_1 - a_1 b_3)}^{=S_{31}} + k\underbrace{(a_1 b_2 - a_2 b_1)}_{S_{12}}$$

Компоненты представляют собой скаляры, полученные проецированием на оси x, y, z вектора, полученного векторным произведением $\vec{a}$ и $\vec{b}$.

2. $\omega_i = \frac{1}{2} e_{ijk} S_{jk} =$

$$\omega_1 = \frac{1}{2} e_{123} S_{23} + \frac{1}{2} e_{132} S_{32} = \frac{1}{2}(a_2 b_3 - a_3 b_2 - a_3 b_2 + a_2 b_3) = S_{23}.$$

$\omega_2 = S_{31}$

$\omega_3 = S_{12}$.

~~ω~~ Модули проекций вектора $\vec{c} = \vec{a} \times \vec{b}$.

Задача 8.

$$\nabla(\bar{a}\,\vec{b}) = (\vec{a}\nabla)\vec{b} + (\bar{b}\nabla)\bar{a} + b \times (\nabla \times \bar{a}) + \vec{a} \times (\nabla \times \vec{b}) =$$

бац - цаб

$$= \cancel{(\vec{a}\nabla)\vec{b}} + \cancel{(\vec{b}\nabla)\vec{a}} + \nabla(\vec{b}\,\bar{a}) - \cancel{\bar{a}(\vec{b}\nabla)} + \nabla(\vec{a}\,\vec{b}) - \cancel{\vec{b}(\vec{a}\nabla)} =$$

$$= \nabla(\vec{b}\,\bar{a}) + \nabla(\bar{a}\,\vec{b}) = \nabla(\vec{a} \cdot \vec{b})$$

Пункт 2.2

$$\nabla \times \nabla f \equiv 0 \quad \nabla \frac{\partial f}{\partial x_k} e_{ijk} \frac{\partial}{\partial x_j} \cdot \frac{\partial f}{\partial x_k} =$$

$$= e_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k}; \quad \underbrace{e_{123} \frac{\partial^2 f}{\partial x_2 \partial x_3} + e_{132} \frac{\partial f}{\partial x_3 \partial x_2}}_{=0} =$$

i-ая компонента

Аналогично для других компонент.

Порядок дифференцирования не важен.

Задача 2.3

$$\nabla \times r = e_{ijk} x_k \frac{\partial}{\partial x_j} = e_{123} \frac{\partial x_3}{\partial x_2} + \ldots = 0$$

аналогично для других компонент

Задача 3.

2. $\nabla(\vec{a} \times \vec{b}) = a(\nabla \cdot b) - (\nabla \cdot a) b + (b \cdot \nabla) a - (a \cdot \nabla) b$

$$\{\nabla \times (\vec{a} \times \vec{b})\}_i = e_{ijk} \frac{\partial (a \times b)_k}{\partial x_j} = e_{ijk} e_{klm}$$

$$\frac{\partial (a_l b_m)}{\partial x_j} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial (a_l b_m)}{\partial x_j} =$$

$$= \frac{\partial (a_i b_j)}{\partial x_j} - \frac{\partial (a_j b_i)}{\partial x_j} = a_i \frac{\partial b_j}{\partial x_j} +$$

$$+ b_j \frac{\partial a_i}{\partial x_j} - a_j \frac{\partial b_i}{\partial x_j} - b_i \underbrace{\frac{\partial a_i}{\partial x_j}}_{\nabla b} =$$

$$= a_i (\nabla b) - b_i (\nabla a) + b_j \frac{\partial a_i}{\partial x_j} \nabla a - a_j \frac{\partial b_i}{\partial x_j} =$$

$$= a_i (\nabla b) - b_i (\nabla a) + a_i (\vec{b} \cdot \nabla) - b_i (\vec{a} \cdot \nabla)$$

3. $\frac{\partial a_i}{\partial x_j} a_j x_i = a_i$

при $i=j$

5. $\nabla \times (\nabla \times a) = \nabla(\nabla \cdot a) - \nabla^2 \cdot a$

$\nabla(\nabla a) - \nabla^2 a = \frac{\partial}{\partial x_m} \frac{\partial a_k}{\partial x_k} - \frac{\partial^2 a_k}{\partial x_k^2} =$

$= e_{pmi} e_{pjk} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} a_k =$

$= -(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} a_k =$

$= \delta_{mk}\delta_{ij} \frac{\partial}{\partial x_m} \frac{\partial}{x_j} a_k - \delta_{mj}\delta_{ik} \frac{\partial}{\partial m} \frac{\partial}{\partial x_j} a_k =$

$= \frac{\partial}{\partial x_m} \frac{\partial a_k}{\partial x_j} a_m - \frac{\partial^2 a_k}{\partial x_k^2}$ (в силу немости индексов равенство выполняется)

$\nabla f(r) = \frac{\partial f}{\partial r} \frac{\vec{r}}{r}$

$\frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial x_i} = \frac{\partial f}{\partial r} \frac{\vec{r}}{r}$

$\nabla(\nabla \times a) = 0$

$\frac{\partial}{\partial x_i} e_{ijk} \frac{\partial}{\partial x_j} a_k = e_i$

$e_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} a_k = -e_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} a_k.$