Задача 1

$z = \frac{1}{a^3}(xy^3 + x^3y) = \frac{r^4}{a^3}(\cos\varphi \cdot \sin^3\varphi + \cos^3\varphi \cdot \sin\varphi) =$

$x = r\cos\varphi$

$y = r\sin\varphi$

$= \frac{r^4}{a^3}\cos\varphi\sin\varphi \cdot =$

$= \frac{r^4}{2a^3}\sin 2\varphi$

$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$g_{11} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 =$

$= \cos^2\varphi + \sin^2\varphi + \left(\frac{2r^3}{a^3}\sin 2\varphi\right)^2 =$

$= 1 + \left(2\frac{r^3}{a^3}\sin 2\varphi\right)^2 = 1 + \frac{4r^6}{a^6}\sin^2 2\varphi$

$g_{12} = g_{21} = \frac{\partial x}{\partial r}\cdot\frac{\partial x}{\partial\varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial\varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial\varphi} =$

$= -\cos\varphi \cdot r\sin\varphi + r\sin\varphi\cos\varphi + \frac{2r^3}{a^3}\sin 2\varphi \cdot \frac{r^4}{a^3}\cos 2\varphi = \frac{\cancel{2}\, r^7}{a^6}\sin 4\varphi$

$g_{22} = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 + \left(\frac{\partial z}{\partial\varphi}\right)^2 =$

$= r^2\sin^2\varphi + r^2\cos^2\varphi + \frac{r^8}{a^6}\cos^2 2\varphi$

$g_{ij} = \begin{pmatrix} 1 + \frac{4r^6}{a^6}\sin^2 2\varphi & \frac{r^7}{a^6}\sin 4\varphi \\ \frac{r^7}{a^6}\sin 4\varphi & r^2 + \frac{r^8}{a^6}\cos^2 2\varphi \end{pmatrix}$

Задача 5.

$V = \det \Big(A_1 \; B_1 \; C_1$ …

$V_1 = \det$ …

$g_{ki} A^i = A_k$ …

$V = g V_1$

$= V_1$ …

по с-ву

$c_{ij} = a_{i}$ …

$~~c_{ij} = a_{i1}~~$

$c_{11} = g_1$ …

(т.к. A …

$= \Big| A_1 \; A$ … B_1 … C_1 …

$$\bar{A}_k \bar{B}^p = \cancel{\frac{\partial x_i}{\partial \bar{x}_k}} \cancel{\frac{\partial \bar{x}^p}{\partial x_i}} \cancel{A^i} B_i$$

$$\bar{A}_k = \frac{\partial x_i}{\partial \bar{x}_k} A_i$$

$$\bar{B}^k = \frac{\partial \bar{x}_k}{\partial x_j} B^j$$

$$\bar{A}_k \bar{B}^k = \underbrace{\frac{\partial x_i}{\partial \bar{x}_k} \frac{\partial \bar{x}_k}{\partial x_j}}_{\delta^j_i} A_i B^j = S$$

При переходе в новые координаты S не меняется $\Rightarrow$ S-скаляр

Задача 3

$R_{ijkl}T^{kl}$

$$\bar R_{ijkl} = \frac{\partial x_m}{\partial \bar x_i}\frac{\partial x_p}{\partial \bar x_j}\frac{\partial x_\alpha}{\partial \bar x_k}\frac{\partial x_\beta}{\partial \bar x_l}\cdot R_{mp\alpha\beta}$$

$$\bar T^{kl} = \frac{\partial \bar x_k}{\partial x_\gamma}\cdot\frac{\partial \bar x_l}{\partial x_\theta}\, T^{\gamma\theta} \qquad \delta^{\gamma}_{\alpha}$$

$$\bar R_{ijkl}\bar T^{kl} = \frac{\partial x_m}{\partial \bar x_i}\frac{\partial x_p}{\partial \bar x_j}\cdot\left(\frac{\partial x_\alpha}{\partial \bar x_k}\cdot\frac{\partial \bar x_k}{\partial x_\gamma}\right)\cdot$$

$$\times \underbrace{\frac{\partial x_\beta}{\partial \bar x_l}\cdot\frac{\partial \bar x_l}{\partial x_\theta}}_{\delta^{\beta}_{\theta}} R_{mp\alpha\beta}\cdot T^{\gamma\theta} =$$

$$= \frac{\partial x_m}{\partial \bar x_i}\frac{\partial x_p}{\partial \bar x_j}\, R_{mp\alpha\beta}T^{\alpha\beta}$$

Закон преобразования ковариантного тензора 2-го ранга.

Задача 4.

$$\cancel{A_iA^i = \underbrace{\frac{\partial x_i}{\partial \bar x_k}\frac{\partial \bar x_k}{\partial x_j}}_{\delta^{j}_{i}} A_i A^j = S}$$

не меняется

$$A_i = \frac{\partial x_k}{\partial \bar x_i}A_k$$

$$A^i = \frac{\partial \bar x_i}{\partial x_p}A^p$$

$$A_iA^i = \underbrace{\frac{\partial x_k}{\partial \bar x_i}\frac{\partial \bar x_i}{\partial x_p}}_{\delta^{k}_{p}} A_kA^p = S$$

не меняется при переходе.

$$\Gamma_{ik} = \frac{\partial A_i}{\partial x_k}$$

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^k} = \frac{\partial}{\partial x^l}(\bar{A}_i) = \frac{\partial}{\partial x^l}\left(\frac{\partial x^j}{\partial \bar{x}_i} A_j\right) =$$

$$= \underbrace{\frac{\partial A_j}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^l}\cdot\frac{\partial x^j}{\partial \bar{x}_i}} + A_j \frac{\partial^2 x^j}{\partial x^l \partial \bar{x}_i} - \text{нет}$$

тензорный закон.

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}_i} A_j$$

Закон преобразования не тензорный!

$$\bar{T}_{kl} = \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^l} A_{ij}$$

3. $\Gamma_{ik} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x_i} = \frac{\partial A_\alpha}{\partial x_\beta}\frac{\partial x_\beta}{\partial \bar{x}_k}\frac{\partial x^\alpha}{\partial \bar{x}_i} +$

$$+ A_\alpha \cancel{\frac{\partial^2 x_\alpha}{\partial \bar{x}_i \partial \bar{x}_k}} - \frac{\partial A_\alpha}{\partial x_\beta}\frac{\partial x_\beta}{\partial \bar{x}_i}\frac{\partial x^\alpha}{\partial \bar{x}_k} -$$

$$- A_\alpha \cancel{\frac{\partial^2 x_\alpha}{\partial \bar{x}_k \partial \bar{x}_i}}$$ (порядок дифференцирования не важен) $=$

Поменяем местами немые индексы α и β

$$= \frac{\partial A_\alpha}{\partial x_\beta}\frac{\partial x_\beta}{\partial \bar{x}_k}\frac{\partial x^\alpha}{\partial \bar{x}_i} - \frac{\partial A_\beta}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial \bar{x}_i}\frac{\partial x^\beta}{\partial \bar{x}_k} =$$

$$= \underbrace{\left(\frac{\partial A_\alpha}{\partial x_\beta} - \frac{\partial A_\beta}{\partial x_\alpha}\right)}_{\Gamma_{\alpha\beta}}\left(\frac{\partial x_\beta}{\partial \bar{x}_k}\frac{\partial x^\alpha}{\partial \bar{x}_i}\right) \cancel{\frac{\partial x^\alpha}{\partial \bar{x}_i}\frac{\partial x^\beta}{\partial \bar{x}_k}}$$

$$\bar{T}_{kl} = \frac{\partial x_i}{\partial \bar{x}_k}\frac{\partial x_j}{\partial \bar{x}_l} T_{ij}$$ тензорный закон преобразования

Задача 5.

$$V = \det\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

$$V_1 = \det\begin{pmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{pmatrix}$$

$$g_{ki} A^i = A_k$$

$$V = g V_1$$

$$V = \begin{pmatrix} g_{1k}A^k & g_{2k}A^k & g_{3k}A^k \\ g_{1k}B^k & g_{2k}B^k & g_{3k}B^k \\ g_{1k}C^k & g_{2k}C^k & g_{3k}C^k \end{pmatrix} =$$

= | по с-ву

$c_{ij} = a_{ij} b_{jk}$

$c_{ij} = a_{i1} b_{j1} + a_{i2} b_{2j} + a_{i3} b_{3j}$

$c_{11} = \underbrace{g_{1k} A_1} + \underline{g_{2k} B_{21}} + \underline{g_{3k} C_{31}}$

$A_k,\ k = 1, 2, 3$ | =

$= (\text{т.к. } A_k = g_{ki} A^i) =$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\Gamma_{ik} = \frac{\partial A_i}{\partial x_k}$$

$$VV_1 = \det \begin{pmatrix} A^i A_i & \cancel{B} A^i B_i & A^i C_i \\ B^i A_i & B^i B_i & B^i C_i \\ C^i A_i & C^i B_i & C^i C_i \end{pmatrix} =$$

$$= \left| \begin{matrix} c_{ij} = a_{ij} b_{ji} \\ C_1 = A_1 A^1 + A_2 A^2 + A_3 A^3 \\ A^i A_i \end{matrix} \right| =$$

$$= \begin{vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix} \cdot \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_1 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$