1. $z(r) = \sqrt{1-r^2} + \ln \frac{r}{1+\sqrt{1-r^2}}$

$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$

$dz = \frac{1-r^2}{r\sqrt{1-r^2}} dr$

$ds^2 = \left(1 + \left(\frac{1-r^2}{r\sqrt{1-r^2}}\right)^2\right) dr^2 + r^2 d\varphi^2 = \frac{1}{r^2} dr^2 + r^2 d\varphi^2$

$g_{ij} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & r^2 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$

$\Gamma_{i,nk} = \frac{1}{2}\left(\frac{\partial g_{in}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^n} - \frac{\partial g_{nk}}{\partial x^i}\right)$ (пусть $1 \equiv r$, $2 \equiv \varphi$)

$\Gamma_{1,11} = \frac{1}{2}\frac{\partial g_{11}}{\partial r} = -\frac{1}{r^3}$

$\Gamma_{1,21} = \Gamma_{1,21} = 0$

$\Gamma_{1,22} = \frac{1}{2}\left(\cancel{\frac{\partial g_{12}}{\partial \varphi}}^{0} + \cancel{\frac{\partial g_{12}}{\partial r}}^{0} - \frac{\partial g_{22}}{\partial r}\right) = -\frac{1}{2}\frac{\partial g_{22}}{\partial r} = -\frac{1}{2}\cdot 2r = -r$

$\Gamma_{2,11} = 0$

$\Gamma_{2,12} = \Gamma_{2,21} = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial \varphi} + \frac{\partial g_{22}}{\partial r} - \frac{\partial g_{12}}{\partial r}\right) = \frac{1}{2}\cdot 2r = r$

$\Gamma_{2,22} = 0$

$\Gamma^{\lambda}{}_{\alpha\beta} = g^{i\lambda}\Gamma_{i,\alpha\beta}$

получим

$\Gamma^1{}_{,11} = g^{11}\cdot\Gamma_{1,11} = r^2\left(-\frac{1}{r^3}\right) = -\frac{1}{r}$

$\Gamma^1{}_{,12} = \Gamma^1{}_{,21} = \Gamma^2{}_{,11} = \Gamma^2{}_{,22} = 0$

$\Gamma^1{}_{,22} = g^{11}\Gamma_{1,22} = r^2(-r) = -r^3$

$\Gamma^2{}_{,12} = g^{22}\Gamma_{2,12} = \frac{1}{r^2}\cdot r = \frac{1}{r} = \Gamma^2{}_{,21}$

$$\begin{cases} \frac{d^2r}{ds^2} - \frac{1}{r}\left(\frac{dr}{ds}\right)^2 - r^3\left(\frac{d\varphi}{ds}\right)^2 = 0 \\ \frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} = 0 \end{cases}$$

$\frac{dr}{ds} \equiv \dot{r}$

$\frac{d\varphi}{ds} \equiv \dot{\varphi}$

$$\begin{cases} \ddot{r} - \frac{1}{r}(\dot{r})^2 - r^3(\dot{\varphi})^2 = 0 \\ \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \end{cases}$$

$$\frac{d}{ds}(\ln \dot\varphi + 2\ln r) = 0 \quad \text{(из 2 ур-я)}$$

$$\ln\dot\varphi + 2\ln r = const$$

$$\dot\varphi = \frac{e^{const}}{r^2} = \frac{\alpha}{r^2}$$

$$\ddot r - \frac{1}{r}(\dot r)^2 - r^3\frac{\alpha^2}{r^4} = 0$$

$$\ddot r - \frac{1}{r}(\dot r)^2 - \frac{\alpha^2}{r} = 0$$

решим однородное ДУ:

$$\ddot r - \frac{1}{r}(\dot r)^2 = 0$$

$$\frac{\ddot r}{\dot r} = \frac{\dot r}{r} \Rightarrow \ln\dot r - \ln r = const$$

$$\frac{\dot r}{r} = C \Rightarrow \dot r = C(s)\,r(s)$$

решим неоднородное ДУ:

$$\dot C r + C\dot r - \frac{1}{r}(Cr)^2 - \frac{\alpha^2}{r} = 0$$

$$\dot C r + \cancel{C^2 r} - \cancel{C^2 r} - \frac{\alpha^2}{r} = 0 \Rightarrow \dot C r - \frac{\alpha^2}{r} = 0 \Rightarrow \dot C = \frac{\alpha^2}{r^2}$$

т.к $\frac{dC}{ds} = \frac{dC}{dr}\frac{dr}{ds} = \frac{dC}{dr}Cr$

$$\frac{dC}{dr}\cdot Cr = \frac{\alpha^2}{r^2} \Rightarrow \int C\,dC = \alpha^2\int\frac{dr}{r^3}$$

$$\frac{C^2}{2} = -\frac{\alpha^2}{2r^2} + D \Rightarrow C^2 + \frac{\alpha^2}{r^2} = D \Rightarrow$$

$$\Rightarrow C = \sqrt{D - \frac{\alpha^2}{r^2}} = \frac{\sqrt{Dr^2 - \alpha^2}}{r}$$

подставим C в $\underline{\dot r = C(r)\,r}$

$$\dot r = \frac{dr}{d\varphi}\cdot\frac{d\varphi}{ds} = \frac{dr}{d\varphi}\cdot\frac{\alpha}{r^2}$$

$$\frac{dr}{d\varphi}\cdot\frac{\alpha}{r^2} = \frac{\sqrt{Dr^2-\alpha^2}}{\cancel{r}}\cdot\cancel{r} \Rightarrow \frac{dr}{d\varphi} = \frac{r^2}{\alpha}\sqrt{Dr^2-\alpha^2}$$

$$\int \frac{dr}{r^2}\frac{\alpha}{\sqrt{Dr^2-\alpha^2}} = \int d\varphi$$

$$\int \frac{\alpha\, dr}{r^3\sqrt{D - \frac{\alpha^2}{r^2}}} = -\int \frac{1}{2}\frac{d\left(\frac{1}{r^2}\right)\alpha}{\sqrt{D-\frac{\alpha^2}{r^2}}} = -\frac{1}{2\alpha}\int \frac{d\left(\frac{\alpha^2}{r^2}\right)}{\sqrt{D-\frac{\alpha^2}{r^2}}}$$

пусть $\frac{\alpha^2}{r^2} = t$

$$\Rightarrow -\frac{1}{2\alpha}\int\frac{dt}{\sqrt{D-t}} = \frac{1}{\alpha}\sqrt{D-t} = \sqrt{\frac{D}{\alpha^2}-\frac{1}{r^2}}$$

получим $\quad \varphi - \varphi_0 = \sqrt{\frac{D}{\alpha^2} - \frac{1}{r^2}}$

$$(\varphi-\varphi_0)^2 = \frac{D}{\alpha^2} - \frac{1}{r^2} \Rightarrow \frac{1}{r^2} = \frac{D}{\alpha^2} - (\varphi-\varphi_0)^2$$

$$\Rightarrow r^2 = \frac{1}{\frac{D}{\alpha^2} - (\varphi-\varphi_0)^2}$$

$$r(\varphi) = \frac{1}{\sqrt{\frac{D}{\alpha^2} - (\varphi-\varphi_0)^2}}$$ – семейство ур-й геодезических линий

$z(r) = \arcsin\sqrt{r} + \sqrt{r(1-r)}$

$$dS^2 = dr^2 + r^2 d\varphi^2 + dz^2$$

$$dz = \frac{\sqrt{r-r^2}}{r}\,dr$$

$$dS^2 = dr^2 + r^2 d\varphi^2 + \frac{(r-r^2)}{r^2}dr^2 = \left(1 + \frac{r-r^2}{r^2}\right)dr^2 + r^2 d\varphi^2$$

$$= \left(\frac{\cancel{r^2} + r - \cancel{r^2}}{r^2}\right)dr^2 + r^2 d\varphi = \frac{1}{r}dr^2 + r^2 d\varphi^2$$

$$g_{ij} = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & r^2 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

$$\Gamma_{i,nk} = \frac{1}{2}\left(\frac{\partial g_{in}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^n} - \frac{\partial g_{nk}}{\partial x^i}\right) \qquad (1 \equiv r,\ 2 \equiv \varphi)$$

$$\underline{\Gamma_{1,11}} = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial r}\right) = \underline{-\frac{1}{2r^2}}$$

$$\Gamma_{1,21} = \Gamma_{1,12} = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial r} + \frac{\partial g_{11}}{\partial \varphi} - \frac{\partial g_{21}}{\partial r}\right) = 0$$

$$\underline{\Gamma_{1,22}} = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial \varphi} + \frac{\partial g_{12}}{\partial \varphi} - \frac{\partial g_{22}}{\partial r}\right) = \underline{-r}$$

$$\Gamma_{2,22} = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial \varphi}\right) = 0$$

$$\underline{\Gamma_{2,12} = \Gamma_{2,21}} = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial \varphi} + \frac{\partial g_{22}}{\partial \varphi} - \frac{\partial g_{12}}{\partial \varphi}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial r} = \underline{r}$$

$$\Gamma_{2,11} = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial \varphi} + \frac{\partial g_{21}}{\partial r} - \frac{\partial g_{11}}{\partial \varphi}\right) = 0$$

$$\Gamma^{\lambda}_{,\alpha\beta} = g^{i\lambda}\,\Gamma_{i,\alpha\beta}$$

$$\underline{\Gamma^1_{,11}} = g^{11}\,\Gamma_{1,11} = r\cdot\left(-\frac{1}{2r^2}\right) = \underline{-\frac{1}{2r}}$$

$$\underline{\Gamma^1_{,22}} = g^{11}\,\Gamma_{1,22} = r\cdot(-r) = \underline{-r^2}$$

$$\underline{\Gamma^2_{,12}} = \Gamma^2_{,21} = g^{22}\,\Gamma_{2,12} = \frac{1}{r^2}\cdot r = \underline{\frac{1}{r}}$$

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

$$\begin{cases} \dfrac{d^2r}{ds^2} - \dfrac{1}{2r}\left(\dfrac{dr}{ds}\right)^2 - r^2\left(\dfrac{d\varphi}{ds}\right)^2 = 0 \\ \dfrac{d^2\varphi}{ds^2} + \dfrac{2}{r}\dfrac{dr}{ds}\dfrac{d\varphi}{ds} = 0 \end{cases} \qquad \frac{dr}{ds} \equiv \dot{r} \qquad \frac{d\varphi}{ds} \equiv \dot{\varphi}$$

$$\begin{cases} \ddot{r} - \frac{1}{2r}(\dot{r})^2 - r^2(\dot{\varphi})^2 = 0 \\ \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \end{cases}$$

$$\frac{d}{ds}(\ln\dot{\varphi} + 2\ln r) = 0$$

$$\ln\dot{\varphi} + 2\ln r = const$$

$$\dot{\varphi} = \frac{e^{const}}{r^2} = \frac{\alpha}{r^2}$$

$$\ddot{r} - \frac{1}{2r}(\dot{r})^2 - \frac{r^2\alpha^2}{r^4} = 0$$

$$\ddot{r} - \frac{(\dot{r})^2}{2r} - \frac{\alpha^2}{r^2} = 0$$

решим однородное ДУ:

$$\ddot{r} - \frac{(\dot{r})^2}{2r} = 0 \qquad \frac{\ddot{r}}{\dot{r}} = \frac{\dot{r}}{2r} \Rightarrow \frac{d}{ds}\left(\ln\dot{r} - \ln\sqrt{r}\right) = 0$$

$$\ln\dot{r} - \ln\sqrt{r} = C \Rightarrow \frac{\dot{r}}{\sqrt{r}} = C \Rightarrow \dot{r} = C(s)\sqrt{r}$$

решим неоднородное ДУ

$$\ddot{r} = \dot{C}\sqrt{r} + \frac{C\dot{r}}{2\sqrt{r}}$$

$$\dot{C}\sqrt{r} + \frac{C\dot{r}}{2\sqrt{r}} - \frac{C^2 r}{2r} - \frac{\alpha^2}{r^2} = 0$$

$$\dot{C}\sqrt{r} + \frac{C^2\cancel{\sqrt{r}}}{\cancel{2}\cancel{\sqrt{r}}} - \frac{C^2}{\cancel{2}} - \frac{\alpha^2}{r^2} = 0 \Rightarrow \dot{C}\sqrt{r} - \frac{\alpha^2}{r^2} = 0 \Rightarrow \dot{C} = \frac{\alpha^2}{r^2\sqrt{r}}$$

т.к. $\frac{dC}{ds} = \frac{dC}{dr}\cdot\frac{dr}{ds} = \frac{dC}{dr}C\sqrt{r}$

$$\frac{dC}{dr}\cdot C\sqrt{r} = \frac{\alpha^2}{r^2\sqrt{r}} \Rightarrow \int C\,dC = \int\frac{\alpha^2\,dr}{r^3}$$

$$\frac{C^2}{2} = -\frac{\alpha^2}{2r^2} + D \Rightarrow C = \sqrt{D - \frac{\alpha^2}{r^2}} = \frac{\sqrt{Dr^2 - \alpha^2}}{r}$$

$$\frac{dr}{ds} = \frac{\sqrt{Dr^2 - \alpha^2}}{r} \cdot \sqrt{r} = \sqrt{\frac{Dr^2 - \alpha^2}{r}}$$

$$\int \frac{dr \cdot \sqrt{r}}{\sqrt{Dr^2 - \alpha^2}} = \int ds \quad \Rightarrow \quad S = \int \frac{\sqrt{r}}{\sqrt{Dr^2 - \alpha^2}}\, dr + \beta$$