① $z = \frac{1}{a^3}(xy^3 + x^3y)$

перейдем к полярным координатам: $x = r\cos\varphi$, $y = r\sin\varphi$

$$z = \frac{1}{a^3}\left(r\cos\varphi\, r^3\sin^3\varphi + r^3\cos^3\varphi\, r\sin\varphi\right) = \frac{r^4}{a^3}\cos\varphi\cdot\sin\varphi = \frac{r^4\sin 2\varphi}{2a^3}$$

значит: $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = \dfrac{r^4\sin 2\varphi}{2a^3} \end{cases}$

метрический тензор в коор. x, y, z имел вид $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $g_{ik} = \delta_{ik}$

Найдем метрический тензор в полярном коор.

$$\bar{g}_{em} = \frac{\partial x_i}{\partial \bar{x}^e}\cdot\frac{\partial x^k}{\partial \bar{x}^m}\, g_{ik}$$

$$\bar{g}_{11} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial r}g_{11} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r}g_{22} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r}g_{33} + \frac{\partial x}{\partial r}\frac{\partial y}{\partial r}\cancel{g_{12}} + \ldots +$$

все остальные слагаемые будут равны 0, т.к. $g_{ik} = 0$ при $i \neq k$

$$\bar{g}_{11} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$\bar{g}_{11} = \cos^2\varphi + \sin^2\varphi + \left(\frac{4r^3\sin 2\varphi}{2a^3}\right)^2 = 1 + \frac{4r^6\sin^2 2\varphi}{a^6}$$

$$\bar{g}_{22} = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 + \left(\frac{\partial z}{\partial\varphi}\right)^2 =$$

$$= (-r\sin\varphi)^2 + (r\cos\varphi)^2 + \left(\frac{r^4\cos 2\varphi\cdot\cancel{2}}{\cancel{2}a^3}\right)^2 =$$

$$= r^2 + \frac{r^8\cos^2 2\varphi}{a^6}$$

$$\bar{g}_{12} = \frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial\varphi}\frac{\partial z}{\partial r} =$$

$$= -\cancel{\cos\varphi\cdot r\sin\varphi} + \cancel{r\cos\varphi\sin\varphi} + \frac{r^4\cdot\cos 2\varphi}{a^3}\cdot\frac{\sin 2\varphi\cdot\overset{2}{\cancel{4}}r^3}{\cancel{2}a^3} =$$

$$= \frac{2r^7\cos 2\varphi\cdot\sin 2\varphi}{a^6} = \frac{r^7\sin 4\varphi}{a^6}$$

$$\bar{g}_{21} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial\varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial\varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial\varphi} = \bar{g}_{12} = \frac{r^7\sin 4\varphi}{a^6}$$

получим $\begin{pmatrix} 1 + \dfrac{4r^6\sin^2 2\varphi}{a^6} & \dfrac{r^7\sin 4\varphi}{a^6} \\ \dfrac{r^7\sin 4\varphi}{a^6} & r^2 + \dfrac{r^8\cos^2 2\varphi}{a^6} \end{pmatrix}$

(2) · контр. B^{μ}

$$\bar{B}^i = \frac{\partial \bar{x}^i}{\partial x^{\mu}} B^{\mu}$$

предположим, что A_i – ковариантный вектор, тогда $\bar{A}_i = \frac{\partial x^p}{\partial \bar{x}^i} A_p$

получим, что $\bar{S} = \bar{A}_i \bar{B}^i = \frac{\partial \bar{x}^i}{\partial x^{\mu}} \cdot \frac{\partial x^p}{\partial \bar{x}^i} A_p B^{\mu} \ominus$

$$\left(\text{т.к } \frac{\partial \bar{x}^i}{\partial x^{\mu}} \cdot \frac{\partial x^p}{\partial \bar{x}^i} = \frac{\partial x^p}{\partial x^{\mu}} = \delta^p_{\mu} \right) \quad \delta^p_{\mu} = \begin{cases} 1, & p = \mu \\ 0, & p \neq \mu \end{cases}$$

$\ominus \; \delta^p_{\mu} A_p B^{\mu} = A_{\mu} B^{\mu}$

мы получили $\bar{S} = \bar{A}_i \bar{B}^i = A_{\mu} B^{\mu} = S$

значит наше предположение было верно и A_i – ковариантный вектор, т.к скаляр S при переходе в другие координаты не меняется ($\bar{S} = S$)

(3) · $R_{ijkl} T^{kl}$, где R_{ijkl} – ковар. тензор 4 ранга

T^{kl} – контрвар. тензор 2-ого ранга

перейдем в др. систему коор.

$$\bar{R}_{ijkl} = \frac{\partial x^m}{\partial \bar{x}^i} \cdot \frac{\partial x^p}{\partial \bar{x}^j} \cdot \frac{\partial x^{\alpha}}{\partial \bar{x}^k} \cdot \frac{\partial x^{\beta}}{\partial \bar{x}^l} R_{mp\alpha\beta}$$

$$\bar{T}^{kl} = \frac{\partial \bar{x}^k}{\partial x^{\gamma}} \cdot \frac{\partial \bar{x}^l}{\partial x^{\theta}} = T^{\gamma\theta}$$

$$\bar{R}_{ijkl} \cdot \bar{T}^{kl} = \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^j} \underline{\underline{\frac{\partial x^{\alpha}}{\partial \bar{x}^k}}} \frac{\partial x^{\beta}}{\partial \bar{x}^l} \underline{\underline{\frac{\partial \bar{x}^k}{\partial x^{\gamma}}}} \cdot \frac{\partial \bar{x}^l}{\partial x^{\theta}} R_{mp\alpha\beta} \cdot T^{\gamma\theta}$$

$$= \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^j} \delta^{\alpha}_{\gamma} \delta^{\beta}_{\theta} R_{mp\alpha\beta} T^{\gamma\theta} =$$

$$= \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^j} \underbrace{R_{mp\alpha\beta} T^{\alpha\beta}}_{M_{mp}} = \bar{M}_{ij}$$

получим $\bar{M}_{ij} = \bar{R}_{ijkl} \bar{T}^{kl} = \underline{\underline{\frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^j} M_{mp}}}$ – ковариантный тензор.

④ A_i – ковариантный вектор

• док-ть $A_i A^i$ – скаляр

$$\bar{A}_i = \frac{\partial x^k}{\partial \bar{x}^i} A_k$$

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^e} A^e$$

$$\bar{A}_i \bar{A}^i = \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial \bar{x}^i}{\partial x^e} A_k A^e = \delta^k_e A_k A^e = A_k A^k$$

мы получили, что $A_i A^i$ – скаляр, т.к он не меняется при переходе из одних координат в другие

• $\Gamma_{ik} = \frac{\partial A_i}{\partial x^k}$ $\qquad$ $\bar{A}_j = \frac{\partial x^i}{\partial \bar{x}^j} A_i$

$$\underline{\bar{\Gamma}_{je}} = \frac{\partial \bar{A}_j}{\partial \bar{x}_e} = \frac{\partial}{\partial \bar{x}_e}\left(\frac{\partial x^i}{\partial \bar{x}^j} A_i\right) = \underline{\frac{\partial A_i}{\partial x^\beta} \cdot \frac{\partial x^\beta}{\partial \bar{x}^e} \cdot \frac{\partial x^i}{\partial \bar{x}^j} +}$$

$$\underline{+ A_i \frac{\partial^2 x^i}{\partial \bar{x}_e \partial \bar{x}^j}}$$

мы получили, что наш элемент Γ_{ik} при переходе в другие координаты зависит от 2 производной, а это не тензорной закон преобразования.

• док-ть $\Gamma_{ik} = \frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i}$ – ковариантный тензор 2-ого ранга

$$\bar{A}_i \cancel{\frac{\partial A_i}{\partial x_k}} = \frac{\partial x^p}{\partial \bar{x}^i} A_p$$

$$\bar{A}_k = \frac{\partial x^m}{\partial \bar{x}^k} A_m$$

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^k} = \frac{\partial}{\partial \bar{x}^k}(\bar{A}_i) = \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial x^p}{\partial \bar{x}^i} A_p\right) =$$

$$= \frac{\partial A_p}{\partial x^\beta} \cdot \frac{\partial x^\beta}{\partial \bar{x}^k} \cdot \frac{\partial x^p}{\partial \bar{x}^i} + A_p \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^k}$$

$$\frac{\partial \bar{A}_k}{\partial \bar{x}_i} = \frac{\partial A_m}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}_i} \cdot \frac{\partial x^m}{\partial \bar{x}^k} + A_m \frac{\partial^2 x^m}{\partial \bar{x}^k \cdot \partial \bar{x}_i} =$$

$$= \frac{\partial A_p}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}_i} \cdot \frac{\partial x^p}{\partial \bar{x}^k} + A_p \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}_i}$$

$$\underline{\bar{\Gamma}_{ik}} = \frac{\partial \bar{A}_i}{\partial \bar{x}_k} - \frac{\partial \bar{A}_k}{\partial \bar{x}_i} = \frac{\partial A_p}{\partial x^\beta} \cdot \frac{\partial x^\beta}{\partial \bar{x}^k} \cdot \frac{\partial x^p}{\partial \bar{x}^i} + \cancel{A_p \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^k}}$$

$$- \frac{\partial A_p}{\partial x^\alpha} \cdot \frac{\partial x^\alpha}{\partial \bar{x}_i} \cdot \frac{\partial x^p}{\partial \bar{x}^k} - \cancel{A_p \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^i}} =$$

$$= \frac{\partial A_p}{\partial x^\beta} \cdot \frac{\partial x^\beta}{\partial \bar{x}^k} \cdot \frac{\partial x^p}{\partial \bar{x}^i} - \frac{\partial A_p}{\partial x^\alpha} \cdot \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^k} =$$

$$= \frac{\partial A_p}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}^k} \cdot \frac{\partial x^p}{\partial \bar{x}^i} - \frac{\partial A_p}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^k} =$$

$$= \frac{\partial A_\alpha}{\partial x^p} \frac{\partial x^p}{\partial \bar{x}^k} \frac{\partial x^\alpha}{\partial \bar{x}^i} - \frac{\partial A_p}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^p}{\partial \bar{x}^k} = \underbrace{\left(\frac{\partial A_\alpha}{\partial x^p} - \frac{\partial A_p}{\partial x^\alpha} \right)}_{\Gamma_{\alpha p}} \frac{\partial x^p}{\partial \bar{x}^k} \frac{\partial x^\alpha}{\partial \bar{x}^i}$$

$$= \underline{\Gamma_{\alpha p} \frac{\partial x^p}{\partial \bar{x}^k} \frac{\partial x^\alpha}{\partial \bar{x}^i}}$$ – ковариантный тензор 2-ого ранга

(5) $A_i\ B_i\ C_i$ – контрвариантные вектора
$A^i\ B^i\ C^i$ – ковариантные вектора

мы знаем, что $A_i = g_{ik} A^k$

$$\underbrace{\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}}_{=g} \cdot \underbrace{\begin{vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix}}_{=V_1}$$ = (по правилу умножения определителей:

$a_{ij} \cdot b_{ij} = c_{ij}$ $\quad c_{ij} = a_{i1} b_{j1} + a_{i2} b_{j2} + \ldots + a_{ik} b_{jk}$)
где a_{ij}, b_{ij}, c_{ij} – определители

$$= \begin{vmatrix} g_{1k} A^k & g_{2k} A^k & g_{3k} A^k \\ g_{1k} B^k & g_{2k} B^k & g_{3k} B^k \\ g_{1k} C^k & g_{2k} C^k & g_{3k} C^k \end{vmatrix} = (\text{т.к. } A_i = g_{ik} A^k) = \underbrace{\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}}_{=V}$$

получим $gV_1 = V$