Задача 1.5

$$ds^2 = d\tau^2 - a^2(\tau)\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2)\}$$

$$d\tau = a(\eta)d\eta \quad ; \quad d\tau^2 = a(\eta)^2 d\eta^2$$

$$ds^2 = a^2(\eta)\{d\eta^2 - d\chi^2 - \sin\chi(d\theta^2 + \sin\theta\, d\varphi^2)\}$$

$$g_{ij} = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2\sin^2\chi & 0 \\ 0 & 0 & 0 & -a^2\sin^2\chi\sin^2\theta \end{pmatrix}$$

пусть $\eta \equiv 1$, $\chi \equiv 2$, $\theta \equiv 3$, $\varphi \equiv 4$

Найдем символ Кристоффеля $\Gamma_{i,jk}$ $\left(\Gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right)\right)$

если $i \neq j \neq k$ $\Gamma_{i,jk} = 0$

$$\Gamma_{1,11} = \frac{1}{2}\frac{\partial g_{11}}{\partial x^1} = -\frac{1}{2}\cdot 2a(\eta)\cdot a'(\eta) = \underline{-a\,a'}$$

$$\Gamma_{1,12} = \Gamma_{1,21} = -\frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0$$

$$\Gamma_{1,22} = \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = \underline{-a\cdot a'}$$

$$\Gamma_{1,13} = \Gamma_{1,31} = -\frac{1}{2}\frac{\partial g_{11}}{\partial x^3} = 0$$

$$\Gamma_{1,33} = \frac{1}{2}\frac{\partial g_{33}}{\partial x^1} = \underline{-a\,a'\sin^2\chi}$$

$$\Gamma_{1,14} = \Gamma_{1,41} = -\frac{1}{2}\frac{\partial g_{11}}{\partial x^4} = 0$$

$$\Gamma_{1,44} = \frac{1}{2}\frac{\partial g_{44}}{\partial x^1} = \underline{-a\,a'\sin^2\chi\sin^2\theta}$$

$$\Gamma_{2,11} = \frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0$$

$$\Gamma_{2,21} = \Gamma_{2,12} = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = \underline{a\,a'}$$

$$\Gamma_{2,22} = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^2} = 0$$

$$\Gamma_{2,23} = \Gamma_{2,32} = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^3} = 0$$

$$\Gamma_{2,33} = \frac{1}{2}\frac{\partial g_{33}}{\partial x^2} = \underline{-a^2\sin\chi\cos\chi}$$

$$\Gamma_{2,24} = \Gamma_{2,42} = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^4} = 0$$

$$\Gamma_{2,44} = \frac{1}{2}\frac{\partial g_{44}}{\partial x^2} = \underline{-a^2\sin\chi\cdot\cos\chi\sin^2\theta}$$

$$\Gamma_{3,11} = \frac{1}{2}\frac{\partial g_{11}}{\partial x^3} = 0$$

$$\Gamma_{3,13} = -\frac{1}{2}\frac{\partial g_{33}}{\partial x^1} = aa' \sin^2 \chi = \Gamma_{3,31}$$

$$\Gamma_{3,22} = \frac{1}{2}\frac{\partial g_{22}}{\partial x^3} = 0$$

$$\Gamma_{3,23} = \Gamma_{3,32} = -\frac{1}{2}\frac{\partial g_{33}}{\partial x^2} = a^2 \sin\chi \cos\chi$$

$$\Gamma_{3,33} = \frac{1}{2}\frac{\partial g_{33}}{\partial x^3} = 0$$

$$\Gamma_{3,34} = \Gamma_{3,43} = -\frac{1}{2}\frac{\partial g_{33}}{\partial x^4} = 0$$

$$\Gamma_{3,44} = \frac{1}{2}\frac{\partial g_{44}}{\partial x^3} = -a^2 \sin^2\chi \cdot \sin\theta \cos\theta$$

$$\Gamma_{4,11} = \frac{\partial g_{11}}{\partial x_4} = 0$$

$$\Gamma_{4,14} = \Gamma_{4,41} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x^1} = aa' \sin^2\chi \sin^2\theta$$

$$\Gamma_{4,22} = \frac{1}{2}\frac{\partial g_{22}}{\partial x^4} = 0$$

$$\Gamma_{4,24} = \Gamma_{4,42} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x^2} = a^2 \sin\chi \cos\chi \sin^2\theta$$

$$\Gamma_{4,33} = \frac{1}{2}\frac{\partial g_{33}}{\partial x^4} = 0$$

$$\Gamma_{4,34} = \Gamma_{4,43} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x^3} = a^2 \sin^2\chi \sin\theta \cos\theta$$

$$\Gamma_{4,44} = 0$$

получим:

$$\Gamma_{2,21} = \Gamma_{2,12} = aa'$$

$$\Gamma_{1,11} = \Gamma_{1,22} = -aa'$$

$$\Gamma_{1,33} = -aa' \sin^2\chi$$

$$\Gamma_{1,44} = -aa' \sin^2\chi \sin^2\theta$$

$$\Gamma_{2,44} = -a^2 \sin\chi \cos\chi \sin^2\theta$$

$$\Gamma_{2,33} = -a^2 \sin\chi \cdot \cos\chi$$

$$\Gamma_{3,32} = \Gamma_{3,23} = a^2 \sin\chi \cdot \cos\chi$$

$$\Gamma_{3,31} \quad \Gamma_{3,13} = aa' \sin^2\chi$$

$$\Gamma_{3,44} = -a^2 \sin^2\chi \sin\theta \cos\theta$$

$$\Gamma_{4,14} = aa' \sin^2\chi \sin^2\theta$$

$$\Gamma_{4,42} = \Gamma_{4,24} = a^2 \sin\chi \cos\chi \sin^2\theta$$

$$\Gamma_{4,34} = \Gamma_{4,43} = a^2 \sin^2\chi \sin\theta \cos\theta$$

Задача 1.2.

- $x^i \equiv x^i(y_\alpha)$ y_α - евкл. коор.

$$\Gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

$$g_{ik} = \sum_\alpha \frac{\partial y_\alpha}{\partial x^i}\frac{\partial y_\alpha}{\partial x^k} \qquad g_{jk} = \sum_\alpha \frac{\partial y_\alpha}{\partial x^j}\frac{\partial y_\alpha}{\partial x^k}$$

$$g_{ij} = \sum_\alpha \frac{\partial y_\alpha}{\partial x^i}\frac{\partial y_\alpha}{\partial x^j}$$

$$\Gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial}{\partial x^j}\left(\frac{\partial y_\alpha}{\partial x^i}\frac{\partial y_\alpha}{\partial x^k}\right) + \frac{\partial}{\partial x^k}\left(\frac{\partial y_\alpha}{\partial x^i}\frac{\partial y_\alpha}{\partial x^j}\right) - \frac{\partial}{\partial x^i}\left(\frac{\partial y_\alpha}{\partial x^j}\frac{\partial y_\alpha}{\partial x^k}\right)\right) =$$

$$= \frac{1}{2}\left(\cancel{\frac{\partial}{\partial x^i}\left(\frac{\partial y_\alpha}{\partial x^j}\frac{\partial y_\alpha}{\partial x^k}\right)} + \frac{\partial}{\partial x^k}\left(\frac{\partial y_\alpha}{\partial x^i}\frac{\partial y_\alpha}{\partial x^j}\right) - \cancel{\frac{\partial}{\partial x^i}\left(\frac{\partial y_\alpha}{\partial x^j}\frac{\partial y_\alpha}{\partial x^k}\right)}\right) =$$

$$= \frac{1}{2}\left(\frac{\partial^2 y_\alpha}{\partial x^i \partial x^k}\frac{\partial y_\alpha}{\partial x^j} + \frac{\partial^2 y_\alpha}{\partial x^k \partial x^j}\frac{\partial y_\alpha}{\partial x^i}\right) = \frac{1}{2}\left(\frac{\partial^2 y_\alpha}{\partial x^j \partial x^k}\frac{\partial y_\alpha}{\partial x^i} + \frac{\partial^2 y_\alpha}{\partial x^k \partial x^j}\frac{\partial y_\alpha}{\partial x^i}\right)$$

$$= \sum_\alpha \frac{\partial^2 y_\alpha}{\partial x^j \partial x^k}\cdot\frac{\partial y_\alpha}{\partial x^i}$$

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• $\nabla_\alpha(A^\mu)B_\mu + A^\mu \nabla_\alpha(B_\mu) = \left(\frac{\partial A^\mu}{\partial x^\alpha} + A^\lambda \Gamma^\mu_{\lambda\alpha}\right)B_\mu +$

$+ A^\mu\left(\frac{\partial B_\mu}{\partial x^\alpha} - B_\lambda\Gamma^\lambda_{\mu\alpha}\right) = \frac{\partial A^\mu}{\partial x^\alpha}B_\mu + A^\mu\frac{\partial B_\mu}{\partial x^\alpha}$

$+ A^\lambda\Gamma^\mu_{\lambda\alpha}B_\mu - A^\mu B_\lambda \Gamma^\lambda_{\mu\alpha} =$

$= \frac{\partial}{\partial x^\alpha}(A^\mu B_\mu) + \cancel{A^\mu \Gamma^\lambda_{\mu\alpha}B_\lambda} - \cancel{A^\mu B_\lambda\Gamma^\lambda_{\mu\alpha}} = \nabla_\alpha(A^\mu B_\mu)$

$\nabla_\alpha(A^\mu B_\mu) + A^\mu\nabla_\alpha(B_\mu) = \underline{A^\mu\nabla_\alpha B_\mu} + B_\mu\nabla_\alpha A^\mu + \underline{A^\mu\nabla_\alpha B_\mu} =$

$= 2A^\mu\nabla_\alpha B_\mu + B_\mu\nabla_\alpha A^\mu = 2A^\mu\left(\frac{\partial B_\mu}{\partial x^\alpha} - B_\lambda\Gamma^\lambda_{\mu\alpha}\right) + B_\mu\left(\frac{\partial A^\mu}{\partial x^\alpha} + A^\lambda\Gamma^\mu_{\lambda\alpha}\right)$

$= 2A^\mu\frac{\partial B_\mu}{\partial x^\alpha} - 2A^\mu B_\lambda\Gamma^\lambda_{\mu\alpha} + B_\mu\frac{\partial A^\mu}{\partial x^\alpha} + B_\mu A^\lambda\Gamma^\mu_{\lambda\alpha} =$

$= 2A^\mu\frac{\partial B_\mu}{\partial x^\alpha} - 2A^\mu B_\lambda\Gamma^\lambda_{\mu\alpha} + B_\mu\frac{\partial A^\mu}{\partial x^\alpha} + B_\lambda A^\mu\Gamma^\lambda_{\mu\alpha} =$

$= A^\mu B_\lambda \Gamma^\lambda_{\mu\alpha} + 2A^\mu\frac{\partial B_\mu}{\partial x^\alpha} + B_\mu\frac{\partial A^\mu}{\partial x^\alpha}$

Задача 1.3.

$\frac{\partial g}{\partial x^l} = \underbrace{\frac{\partial g}{\partial g^{ij}}}_{G^{ij}}\frac{\partial g_{ij}}{\partial x^l}$, G^{ij} - алгебраич. дополнение.

$\{g^{-1}\}_{ij} = \frac{G_{ji}}{g} = g^{ij} \Rightarrow G^{ij} = g g^{ji}$

$\Rightarrow \frac{\partial g}{\partial x^l} = g g^{ji}\frac{\partial g_{ij}}{\partial x^l}$

Задача 1.1

док-ть $e_{ijk} a_{i\alpha} a_{j\beta} a_{k\gamma} = e_{\alpha\beta\gamma} \det a$

$\det a = e_{ijk} \cdot a_{1i} a_{2j} a_{3k} = \frac{1}{n!} e_{ijk} e_{\alpha\beta\gamma} a_{i\alpha} a_{j\beta} a_{k\gamma} = \frac{1}{6} e_{ijk} e_{\alpha\beta\gamma} a_{i\alpha} a_{j\beta} a_{k\gamma}$

домножим $\det a$ на $e_{\alpha\beta\gamma}$

получим. $\underbrace{\frac{e_{\alpha\beta\gamma} e_{\alpha\beta\gamma}}{6}}_{=6} e_{ijk} a_{i\alpha} a_{j\beta} a_{k\gamma} = \underline{e_{ijk} a_{i\alpha} a_{j\beta} a_{k\gamma}}$

$= \underline{e_{\alpha\beta\gamma} \det a}$

$\Rightarrow e_{ijk} a_{i\alpha} a_{j\beta} a_{k\gamma} = e_{\alpha\beta\gamma} \det a$

Задача. 1.4.

док-ть:

$E^{ijk} = \frac{1}{\sqrt{g}} e_{ijk}$

рассмотрим. $E^{ijk} E_{ijk} = \frac{1}{\sqrt{g}} e_{ijk} \cdot \sqrt{g}\, e_{ijk} = e_{ijk} \cdot e_{ijk} = 6$

$\Rightarrow E^{ijk}$ - контрвариантный тензор 3-его ранга

• $\Gamma^i_{i\alpha} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^\alpha}$ - док-ть

$$\Gamma_{\alpha,\beta\gamma} = -\frac{1}{2}\left(\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right)$$

$$\Gamma_{\beta,\alpha\gamma} = -\frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)$$

$$\Gamma_{\alpha,\beta\gamma} + \Gamma_{\beta,\alpha\gamma} = \cancel{-\frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}} \cancel{+\frac{1}{2}\frac{\partial g_{\alpha\gamma}}{\partial x^\beta}} \cancel{-\frac{1}{2}\frac{\partial g_{\alpha\gamma}}{\partial x^\beta}} + \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{1}{2}\frac{\partial g_{\beta\alpha}}{\partial x^\gamma} \cancel{+\frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}}$$

$$= \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$$

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = \Gamma_{\alpha,\beta\gamma} + \Gamma_{\beta,\alpha\gamma} \Rightarrow \frac{\partial g_{ik}}{\partial x^\alpha} = \Gamma_{i,k\alpha} + \Gamma_{k,i\alpha}$$

из предыд. пункта

$$\frac{\partial g}{\partial x^\alpha} = g g^{ki} \frac{\partial g_{ik}}{\partial x^\alpha} = g g^{ki}\left(\Gamma_{i,k\alpha} + \Gamma_{k,i\alpha}\right) =$$

$$= g g^{ki}\Gamma_{i,k\alpha} + g g^{ki}\Gamma_{k,i\alpha} = g\,\Gamma^k_{k\alpha} + g\,\Gamma^i_{i\alpha} = 2g\,\Gamma^i_{i\alpha}$$

$$\Rightarrow \Gamma^i_{i\alpha} = \frac{\partial g}{\partial x^\alpha}\cdot\frac{1}{2g} = \frac{1}{2}\frac{\partial g}{\partial x^\alpha}\cdot\frac{1}{g} = \frac{1}{2}\frac{\partial g}{\partial x^\alpha}\cdot\frac{1}{\sqrt{g}\cdot\sqrt{g}} =$$

$$\Rightarrow \Gamma^i_{i\alpha} = \frac{1}{2}\cdot 2\frac{\partial \sqrt{g}}{\partial x^\alpha}\cdot\frac{1}{\sqrt{g}} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^\alpha}$$

• $$\nabla_i A^i = \frac{\partial A^i}{\partial x^i} + A^i\Gamma^i_{i\alpha} = \frac{\sqrt{g}}{\sqrt{g}}\frac{\partial A^i}{\partial x^i} + \frac{A^i}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i} =$$

$$= \frac{1}{\sqrt{g}}\left(\sqrt{g}\cdot\frac{\partial A^i}{\partial x^i} + \frac{\partial\sqrt{g}}{\partial x^i}\cdot A^i\right) = \frac{1}{\sqrt{g}}\frac{\partial(A^i\sqrt{g})}{\partial x^i}$$