Задача 1.

$$\Rightarrow c_m = a_n b_k - a_k b_n$$

$$\vec{c} = \bar{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}(a_2 b_3 - b_2 a_3) - \vec{j}(a_1 b_3 - a_3 b_1) + \vec{k}(a_1 b_2 - a_2 b_1)$$

т.к $\bar{a}$ и $\bar{b}$ - векторы: $a_i' = \sum_n \alpha_{in} a_n \quad b_j' = \sum_k \alpha_{jk} b_k$

$$\bar{c}' = \vec{a}' \times \bar{b}' = \begin{vmatrix} i & j & k \\ \sum_n \alpha_{1n} a_n & \alpha_{2n} a_n & \alpha_{3n} a_n \\ \alpha_{1k} b_k & \alpha_{2k} b_k & \alpha_{3k} b_k \end{vmatrix} = \vec{i}(\alpha_{2n} a_n \cdot \alpha_{3k} b_k - \alpha_{3n} a_n \alpha_{2k} b_k) - \vec{j}(\alpha_{1n} a_n \alpha_{3k} b_k - \alpha_{3n} a_n \alpha_{1k} b_k)$$

$$+ \vec{k}(\alpha_{1n} a_n \alpha_{2k} b_k - \alpha_{2n} a_n \alpha_{1k} b_k)$$ {поменяем „немного" индексы}

$$= \vec{i}\, \alpha_{2n}\alpha_{3k}(a_n b_k - a_k b_n) - \vec{j}\, \alpha_{1n}\alpha_{3k}(a_n b_k - a_k b_n) +$$

$$+ \vec{k}\, \alpha_{1n}\alpha_{2k}(a_n b_k - a_k b_n) \Rightarrow c_\ell' = \underbrace{\alpha_{in}\alpha_{jk}}_{\alpha_{\ell m}} \underbrace{(a_n b_k - a_k b_n)}_{c_m}$$

$$c_\ell' = \alpha_{\ell m} c_m$$

Задача 2.

2.1 показать что $\{c\}_i \equiv c_i = e_{ijk} a_j b_k$ и $\{c_k\} \equiv c_k = e_{ijk} a_i b_j$

т.к $$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \bar{e}_x \overbrace{(a_y b_z - a_z b_y)}^{c_x} + \vec{e}_y \overbrace{(a_z b_x - a_x b_z)}^{c_y} + \vec{e}_z \overbrace{(a_x b_y - a_y b_x)}^{c_z}$$

- пусть $c_i = e_{ijk} a_j b_k$ тогда

$$\underline{c_x} = e_{xjk} a_j b_k = e_{xyz} a_y b_z + e_{xzy} a_z b_y = \underline{a_y b_z - a_z b_y}$$

$$\underline{c_y} = e_{yjk} a_j b_k = e_{yxz} a_x b_z + e_{yzx} a_z b_x = \underline{a_z b_x - a_x b_z}$$

$$\underline{c_z} = e_{zjk} a_j b_k = e_{zxy} a_x b_y + e_{zyx} a_y b_x = \underline{a_x b_y - a_y b_x}$$

мы получили одинаковое выражение для c_i, значит действительно $c_i = e_{ijk} a_j b_k$ при $\bar{c} = \bar{a} \times \bar{b}$

- пусть $c_k = e_{ijk} a_i b_j$

$$c_x = e_{ijx} a_i b_j = e_{yzx} a_y b_z + e_{zyx} a_z b_y = a_y b_z - a_z b_y$$

$$c_y = e_{ijy} a_i b_j = e_{xzy} a_x b_z + e_{zxy} a_z b_x = a_z b_x - a_x b_z$$

$$c_z = e_{ijz} a_i b_j = e_{xyz} a_x b_y + e_{yxz} a_y b_x = a_x b_y - a_y b_x$$

мы также получили одинаковое выражение для c_i

2.2 док-ть $\nabla \times \nabla f \equiv 0$ (<u>смотреть другое док-во в конце</u>)

$\nabla \times \nabla f = \text{rot grad } f$

т.к $\text{grad } f = \left(\frac{\partial f}{\partial x} ; \frac{\partial f}{\partial y} ; \frac{\partial f}{\partial z} \right)$

получим, что $\text{rot grad } f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} =$

$$= \vec{i} \left(\cancel{\frac{\partial}{\partial y} \frac{\partial f}{\partial z}} - \cancel{\frac{\partial}{\partial z} \frac{\partial f}{\partial y}} \right) + \vec{j} \left(\cancel{\frac{\partial}{\partial x} \frac{\partial f}{\partial z}} - \cancel{\frac{\partial}{\partial x} \frac{\partial f}{\partial z}} \right) + \vec{k} \left(\cancel{\frac{\partial}{\partial x} \frac{\partial f}{\partial y}} - \cancel{\frac{\partial}{\partial y} \frac{\partial f}{\partial x}} \right)$$

$= 0$, т.к. порядок дифференцирования не важен

2.3 - доказать, что $\nabla \times r = 0$ (<u>смотреть другое док-во в конце</u>).

$$\text{rot } \vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i} \left(\cancel{\frac{\partial z}{\partial y}}^{0} - \cancel{\frac{\partial y}{\partial z}}^{0} \right) - \vec{j} \left(\cancel{\frac{\partial z}{\partial x}}^{0} - \cancel{\frac{\partial x}{\partial z}}^{0} \right) + $$

$$+ \vec{k} \left(\cancel{\frac{\partial y}{\partial x}}^{0} - \cancel{\frac{\partial x}{\partial y}}^{0} \right) = 0$$

2.4. ∇f^2

$\text{grad } f^2 = \left(2f \frac{\partial f}{\partial x} ; 2f \frac{\partial f}{\partial y} ; 2f \frac{\partial f}{\partial z} \right)$

$\frac{\partial (f^2)}{\partial x} = 2f \frac{\partial f}{\partial x}$;

$\frac{\partial (f^2)}{\partial y} = 2f \frac{\partial f}{\partial y}$; $\frac{\partial (f^2)}{\partial z} = 2f \cdot \frac{\partial f}{\partial z}$

$\text{grad } f^{\alpha} = \left(\alpha f^{\alpha - 1} \frac{\partial f}{\partial x} ; \alpha f^{\alpha - 1} \frac{\partial f}{\partial y} ; \alpha f^{\alpha - 1} \frac{\partial f}{\partial z} \right)$

2.5. Вычислим ∇r^2, где $r = \sqrt{x^2 + y^2 + z^2} \Rightarrow$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$\frac{\partial r^2}{\partial x} = 2x$ $\quad \frac{\partial r^2}{\partial z} = 2z$

$\frac{\partial r^2}{\partial y} = 2y$

$\Rightarrow \nabla r^2 = (2x ; 2y ; 2z)$

с испол. пункта 2.4

$$\nabla r^{1/2} = \left(\frac{1}{2} r^{-\frac{1}{2}} \frac{\partial r}{\partial x} ; \frac{1}{2} r^{-\frac{1}{2}} \frac{\partial r}{\partial y} ; \frac{1}{2} r^{-\frac{1}{2}} \frac{\partial r}{\partial z}\right)$$

$$= \frac{1}{2} r^{-\frac{1}{2}} \left(\frac{\partial r}{\partial x} ; \frac{\partial r}{\partial y} ; \frac{\partial r}{\partial z} \right) = \frac{1}{2\sqrt{r}} \cdot \frac{\vec{r}}{r} = \frac{\vec{r}}{2\sqrt{r^3}}$$

2.6. при переходе в другую систему координат скаляр не меняется $A = A'$

запишем преобразовани преобразование, в предположении, что g_{ij} — тензор

тогда

$$x'_i = \sum_k \alpha_{ik} x_k \qquad x'_j = \sum_m \alpha_{jm} x_m \qquad g'_{ij} = \sum_{np} \alpha_{in} \alpha_{jp} g_{np}$$

$$A' = \sum_{kmnp} \underline{\alpha_{in}}\, \alpha_{jp}\, g_{np}\, \underline{\alpha_{ik}}\, x_k\, \alpha_{jm}\, x_m = \sum_{kmnp} \underline{\delta_{nk}}\, \delta_{pm}\, x_k x_m\, \underline{g_{np}} =$$

$$= \sum_{kmp} \underline{\delta_{pm}}\, x_k x_m\, \underline{g_{kp}} = \sum_{km} g_{km} x_k x_m$$

мы получили, что $A' = A$, значит наше предположение было верно и g_{ij} — тензор 2-го ранга

Задача 3.

$$e_{ijl} e_{ijm} = \cancelto{1}{e_{123}} e_{123} + \cancelto{1}{e_{213}} e_{213} + \cancelto{1}{e_{312}} e_{312} + \cancelto{1}{e_{132}} e_{132} +$$
$$+ \cancelto{1}{e_{321}} e_{321} + \cancelto{1}{e_{231}} e_{231} = 6$$

$$2\delta_{lm} = 2(\delta_{11} + \delta_{22} + \delta_{33}) = 6$$

$$\Rightarrow e_{ijl} e_{ijm} = 2\delta_{lm}$$

(смотреть другое док-во в конце)

• $\vec\nabla \times (\vec a \times \vec b) = \vec a(\vec\nabla \vec b) - (\vec\nabla \vec a)\vec b + (\vec b \vec\nabla)\vec a - (\vec a \vec\nabla)\vec b$

$$\vec\nabla \times (\vec a \times \vec b) = \vec\nabla \times (\overset{\downarrow}{\vec a} \times \vec b) + \vec\nabla(\vec a \times \overset{\downarrow}{\vec b})$$

$$\vec\nabla \times (\overset{\downarrow}{\vec a} \times \vec b) = \vec a(\vec\nabla \cdot \vec b) - \vec b(\vec\nabla \cdot \vec a)$$

$$\vec\nabla \times (\overset{\downarrow}{\vec a} \times \overset{\downarrow}{\vec b}) = \vec a(\vec\nabla \cdot \vec b) - \vec b(\vec\nabla \cdot \vec a)$$

$$\vec\nabla \times (\vec a \times \vec b) = \vec a(\vec\nabla \cdot \vec b) - (\vec\nabla \cdot \vec a)\vec b + (\vec\nabla \vec b)\vec a - (\vec\nabla \vec a)\vec b$$

• $(\vec a \cdot \nabla)\vec r = \vec a$ (смотреть другое решение в конце)

$$\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)\vec r = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) x\,\vec i +$$

$$+ \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) y\,\vec j + \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) z\,\vec k$$

$$= \left(\frac{\partial}{\partial x} a_x \cdot x + \frac{\partial}{\partial y} a_y \cdot x + \frac{\partial}{\partial z} \cdot a_z \cdot x\right)\vec i + \left(\frac{\partial}{\partial x} a_x y + \frac{\partial}{\partial y} a_y \cdot y + \frac{\partial}{\partial z} \cdot a_z y\right)\vec j$$

$$+ \left(\frac{\partial}{\partial x} a_x z + \frac{\partial}{\partial y} a_y z + \frac{\partial}{\partial z} a_z \cdot z\right)\vec k = a_x \vec i + a_y \vec j + a_z \vec k$$

$$= \vec a$$

• $\nabla \cdot (\vec a \times \nabla)\vec r = \nabla \times \vec a$

$$\vec a \times \nabla = -\begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = -\vec i\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \vec j\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \vec k\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$(\vec a \times \nabla)\vec r = -\vec i\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)x + \vec j\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right)y - \vec k\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)z$$

$$\nabla \cdot (\vec a \times \nabla)\vec r = \frac{\partial}{\partial x}\left(-\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)x\right)\vec i + \frac{\partial}{\partial y}\left(-\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)y\right)\vec j +$$

$$+ \frac{\partial}{\partial z}\left(-\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)z\right)\vec k =$$

$$= -\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec i + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec j + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec k$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{i} + \left(-\frac{\partial a_x}{\partial z} + \frac{\partial a_z}{\partial x}\right)\vec{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{k}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \nabla \times \vec{a}$$

- $\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a} = \operatorname{grad} \operatorname{div} \vec{a} - \nabla^2 \vec{a}$

(смотреть другое решение в конце)

$\operatorname{rot} \operatorname{rot} \vec{a}$

$$\operatorname{rot} \vec{a} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \vec{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \vec{j}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \vec{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$\operatorname{rot} \operatorname{rot} \vec{a} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) & -\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) & \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) \end{vmatrix} = \vec{i}\left(\frac{\partial}{\partial y}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right)\right)$$

$$- \vec{j}\left(\frac{\partial}{\partial x}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\right) + \vec{k}\left(\frac{\partial}{\partial x}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\right)$$

$$= \vec{i}\left(\frac{\partial^2 a_y}{\partial y \partial x} - \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_z}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z^2}\right) - \vec{j}\left(\frac{\partial^2 a_y}{\partial x^2} - \frac{\partial^2 a_x}{\partial x \partial y} - \frac{\partial^2 a_z}{\partial z \partial y} + \frac{\partial^2 a_y}{\partial z^2}\right)$$

$$+ \vec{k}\left(\frac{\partial^2 a_x}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial x^2} - \frac{\partial^2 a_z}{\partial y^2} + \frac{\partial^2 a_y}{\partial z \partial y}\right) =$$

$$= \vec{i}\left(\frac{\partial^2 a_y}{\partial y \partial x} - \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_z}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z^2} + \frac{\partial^2 a_x}{\partial x^2} - \frac{\partial^2 a_x}{\partial x^2}\right) +$$

$$+ \vec{j}\left(\frac{\partial^2 a_x}{\partial x \partial y} + \frac{\partial^2 a_z}{\partial z \partial y} - \frac{\partial^2 a_y}{\partial x^2} - \frac{\partial^2 a_y}{\partial z^2} - \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_y}{\partial y^2}\right)$$

$$+ \vec{k}\left(\frac{\partial^2 a_x}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial x^2} - \frac{\partial^2 a_z}{\partial y^2} + \frac{\partial^2 a_y}{\partial z \partial y} - \frac{\partial^2 a_z}{\partial z^2} + \frac{\partial^2 a_z}{\partial z^2}\right)$$

$$-\left(\frac{\partial^2 a_x}{\partial x^2}+\frac{\partial^2 a_x}{\partial y^2}+\frac{\partial^2 a_x}{\partial z^2}\right)\vec{i}-\left(\frac{\partial^2 a_y}{\partial x^2}+\frac{\partial^2 a_y}{\partial y^2}+\frac{\partial^2 a_y}{\partial z^2}\right)\vec{j}-$$

$$-\left(\frac{\partial^2 a_z}{\partial x^2}+\frac{\partial^2 a_z}{\partial y^2}+\frac{\partial^2 a_z}{\partial z^2}\right)\vec{k}+\left(\frac{\partial^2 a_y}{\partial x\partial y}+\frac{\partial^2 a_z}{\partial x\partial z}+\frac{\partial^2 a_x}{\partial x^2}\right)\vec{i}$$

$$+\left(\frac{\partial^2 a_y}{\partial y^2}+\frac{\partial^2 a_x}{\partial x\partial y}+\frac{\partial^2 a_z}{\partial z\partial y}\right)\vec{j}+\left(\frac{\partial^2 a_z}{\partial z^2}+\frac{\partial^2 a_x}{\partial x\partial z}+\frac{\partial^2 a_y}{\partial y\partial z}\right)\vec{k} \;=\;,$$

т.к. $\operatorname{grad}\operatorname{div}\vec{a}=\operatorname{grad}\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)=$

$$=\left(\frac{\partial}{\partial x}\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right),\frac{\partial}{\partial y}\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right),\frac{\partial}{\partial z}\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)\right)$$

а $\nabla^2\vec{a}=\left(\frac{\partial^2 a_x}{\partial x^2}+\frac{\partial^2 a_x}{\partial y^2}+\frac{\partial^2 a_x}{\partial z^2}\right)\vec{i}+\left(\frac{\partial^2 a_y}{\partial x^2}+\frac{\partial^2 a_y}{\partial z^2}+\frac{\partial^2 a_y}{\partial y^2}\right)\vec{j}+$

$$+\vec{k}\left(\frac{\partial^2 a_z}{\partial z^2}+\frac{\partial^2 a_z}{\partial y^2}+\frac{\partial^2 a_z}{\partial x^2}\right)$$

$= \operatorname{grad}\operatorname{div}\vec{a}-\nabla^2\vec{a}$

- док-ть $\nabla f(r)=\frac{df}{dr}\frac{\vec{r}}{r}$

$\operatorname{grad} f(r) =$

$$\frac{\partial f(r)}{\partial x}=\frac{df}{dr}\frac{\partial r}{\partial x}\qquad \frac{\partial f(r)}{\partial y}=\frac{df}{dr}\frac{\partial r}{\partial y}\qquad \frac{\partial f(r)}{\partial z}=\frac{df}{dr}\frac{\partial r}{\partial z}$$

$$=\left(\frac{df}{dr}\frac{\partial r}{\partial x};\frac{df}{dr}\frac{\partial r}{\partial y};\frac{df}{dr}\frac{\partial r}{\partial z}\right)=\frac{df}{dr}\operatorname{grad} r =$$

$$\operatorname{grad} r=\operatorname{grad}\sqrt{x^2+y^2+z^2}=\left(\frac{x}{\sqrt{x^2+y^2+z^2}};\frac{y}{\sqrt{x^2+y^2+z^2}};\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$=\frac{\vec{r}}{r}$$

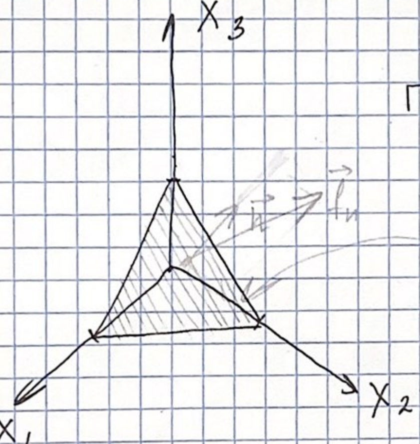
$$= \frac{df}{dr}\operatorname{grad} r=\frac{df}{dr}\frac{\vec{r}}{r}$$

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) = 0$ (смотреть другое решение в конце)

$$(\vec{\nabla}\times\vec{a}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \vec{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \vec{j}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \vec{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$\operatorname{div}(\operatorname{rot}\vec{a}) = \frac{\partial}{\partial x}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$
$$= \cancel{\frac{\partial^2 a_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 a_y}{\partial x \partial z}} + \cancel{\frac{\partial^2 a_x}{\partial y \partial z}} - \cancel{\frac{\partial^2 a_z}{\partial y \partial x}} + \cancel{\frac{\partial^2 a_y}{\partial x \partial z}} - \cancel{\frac{\partial^2 a_x}{\partial z \partial y}} = 0$$

Задача 4.



$$\Pi = p_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 2 \end{pmatrix} \begin{matrix} f_x \\ f_y \\ f_z \end{matrix}$$ – тензор напряжений

$1 - x - y - z = 0$

Найти: компоненты вектора напряжения.

Вектор нормали $\vec{n} = (1; 1; 1)$

$$\vec{f}_n = \vec{f}_x \cos(n, x) + \vec{f}_y \cos(n, y) + \vec{f}_z \cos(n, z)$$

$$\cos(n, x) = \frac{(\vec{n}\cdot\vec{x})}{|\vec{n}|\,|\vec{x}|} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cos(n, y) = \frac{\sqrt{3}}{3} \qquad \cos(n, z) = \frac{\sqrt{3}}{3}$$

т.к. $\vec{f}_x = p_0\vec{i}$ $\quad \vec{f}_y = p_0\vec{j} + \frac{1}{2}p_0\vec{k}$ $\quad \vec{f}_z = \frac{1}{2}\vec{j}p_0 + 2p_0\vec{k}$

$$\vec{f}_n = \frac{\sqrt{3}}{3}\vec{f}_x + \frac{\sqrt{3}}{3}\vec{f}_y + \frac{\sqrt{3}}{3}\vec{f}_z$$

$$\vec{f}_n = \frac{\sqrt{3}}{3}p_0\vec{i} + \underbrace{\frac{\sqrt{3}}{3}p_0\vec{j}} + \underline{\frac{\sqrt{3}}{6}p_0\vec{k}} + \underbrace{\frac{1\sqrt{3}}{2\cdot 3}p_0\vec{j}} + \underline{\frac{2\sqrt{3}}{3}p_0\vec{k}}$$
$$= \frac{\sqrt{3}}{3}p_0\vec{i} + \frac{\sqrt{3}}{2}p_0\vec{j} + \frac{5\sqrt{3}}{6}p_0\vec{k}$$

$$\vec{f}_n = \left(\frac{\sqrt{3}}{3}p_0;\ \frac{\sqrt{3}}{2}p_0;\ \frac{5\sqrt{3}}{6}p_0\right)$$

$$\cos(f_n, n) = \frac{\frac{\sqrt{3}}{3}p_0 + \frac{\sqrt{3}}{2}p_0 + \frac{5}{6}\sqrt{3}\,p_0}{p_0\sqrt{\left(\frac{\sqrt{3}}{3}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{5\sqrt{3}}{6}\right)^2}\cdot\sqrt{3}} = \frac{\frac{1}{3}+\frac{1}{2}+\frac{5}{6}}{\sqrt{\left(\frac{\sqrt{3}}{3}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{5\sqrt{3}}{6}\right)^2}} =$$

$$\frac{5\sqrt{114}}{57} = 0{,}937$$

Нормальная ~~на~~ составляющая:

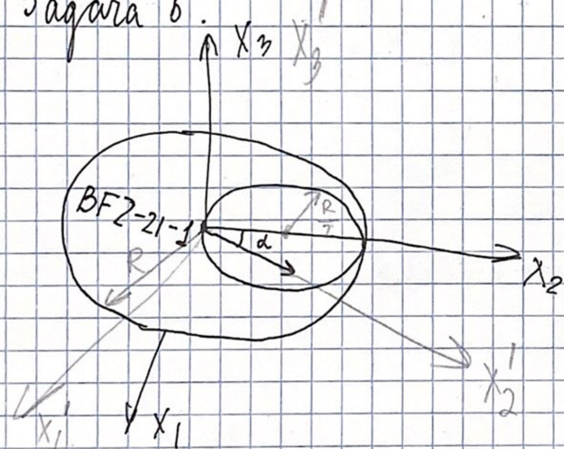
$$f_n\cdot\cos(f_n, n) = \left(\frac{5\sqrt{38}}{57}p_0;\ \frac{5\sqrt{38}}{38}p_0;\ \frac{25\sqrt{38}}{114}p_0\right)$$

$$\cos(90 - \arccos(f_n, n)) = \cos(90-(20)) = \frac{\sqrt{399}}{57} = \cos\alpha$$

Касательная составляющая:

$$f_n\cdot\cos\alpha = \left(\frac{\sqrt{133}}{57}p_0;\ \frac{\sqrt{133}}{38}p_0;\ \frac{5\sqrt{133}}{114}p_0\right)$$

Задача 6.



$$dm = \rho dS = \frac{M}{\pi R^2} r\,dr\,d\varphi$$

Найдем тензор инерции

$$I_{ki} = \int dm\,(r^2\delta_{ik} - X_iX_k)$$

без „дорог"

$$1)\ I_{11} = \int dm(r^2\delta_{11} - X_1^2) = \int dm\,(\cancel{X_1^2} + X_2^2 + \cancel{X_3^2} - \cancel{X_1^2}) = \int X_2^2\,dm$$

$$= \int_0^{2\pi}\int_0^R \frac{M}{\pi R^2} r^2\sin^2\varphi\cdot r\,dr\,d\varphi = \frac{M}{\pi R^2}\int_0^R r^3 dr \underbrace{\int_0^{2\pi}\sin^2\varphi\,d\varphi}_{\pi} = \frac{M}{R^2}\int_0^R r^3 dr =$$

$$= \frac{M}{R^2}\frac{r^4}{4}\Big|_0^R = \frac{MR^4}{4R^2} = \frac{MR^2}{4}$$

„дырка"

$$2)\ I_{11} = \int dm\,X_2^2 = \left(dm = \rho ds = \frac{M}{\pi R^2} r\,dr\,d\varphi\right) =$$

$$= \int_0^{R/2}\int_0^{2\pi} r^2 \sin^2\varphi \cdot \frac{M}{\pi R^2} r\, dr\, d\varphi = \frac{M}{\pi R^2}\int_0^{R/2} r^3 dr \underbrace{\int_0^{2\pi} \sin^2\varphi\, d\varphi}_{=\pi} =$$

$$= \frac{M}{R^2}\int_0^{R/2} r^3 dr = \frac{M}{R^2}\frac{r^4}{4}\Big|_0^{R/2} = \frac{M}{4R^2}\cdot\frac{R^4}{16} = \frac{MR^2}{64}$$

$$I_{11} = \frac{MR^2}{4} - \frac{MR^2}{64} = \frac{15MR^2}{64}$$

$$I_{22} = \frac{15MR^2}{64}$$

$$I_{33} = \int dm\,(x_2^2 + x_1^2) = \underbrace{\int dm\, x_2^2}_{I_{11}} + \underbrace{\int x_1^2\, dm}_{I_{22}} =$$

$$= I_{11} + I_{22} = \frac{15}{64}MR^2 + \frac{15}{64}MR^2 = \frac{15}{32}MR^2$$

$$I_{12} = \int dm\,(r^2 \cancel{\delta_{12}}^{\,0} - x_1 x_2) = -\int dm\, x_1 x_2 =$$

без „дырки“ $I_{12} = -\int_0^R\int_0^{2\pi} r\cos\varphi \cdot r\sin\varphi\, r \frac{M}{\pi R^2} dr\, d\varphi =$

$$-\frac{M}{\pi R^2}\int_0^R r^3 dr \int_0^{2\pi}\frac{2}{2}\cos\varphi\sin\varphi\, d\varphi = -\frac{M}{\pi R^2}\int_0^R r^3 dr \int_0^{2\pi}\cancel{\frac{\sin 2\varphi}{2}}^{\,0} d\varphi = 0$$

„дырка“: $I_{12} = -\int_0^{R/2}\int_0^{2\pi} r\cos\varphi\, r\sin\varphi\, r\cdot\frac{M}{\pi R^2} dr\, d\varphi =$

$$= -\frac{M}{\pi R^2}\int_0^{R/2} r^3 dr \int_0^{2\pi}\cancel{\frac{\sin 2\varphi}{2}}^{\,0} d\varphi = 0$$

$$I_{12} = 0 \qquad I_{21} = I_{12} = I_{31} = I_{13} = I_{32} = I_{23} = 0$$

$$I = \frac{15}{64}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} MR^2$$

запишем наш тензор в новой системе координат

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$I' = \lambda I \lambda^T$$

$$I' = \frac{15MR^2}{64}\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{15MR^2}{64}\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \frac{15MR^2}{64}\begin{pmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \cos\alpha\sin\alpha & 0 \\ \sin\alpha\cos\alpha - \sin\alpha\cos\alpha & \sin\alpha^2 + \cos^2\alpha & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$

$$= \frac{15MR^2}{64}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

посчитаем момент импульса

$$L = I \cdot \omega \qquad L = \frac{15MR^2}{64}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} =$$

$$= \frac{15MR^2}{64}\begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{15MR^2}{64}\omega \\ 0 \end{pmatrix}$$

момент сил:

$$\vec{M} = \frac{dL}{dt} = \begin{pmatrix} 0 \\ \frac{15MR^2}{64}\varepsilon \\ 0 \end{pmatrix}$$, где ε - угловое ускорение

Задача 6

$f(\vec{r}) = g_{ij} x_i x_j$ $\qquad$ g_{ij} – симметр. тензор.

$$\frac{\partial(g_{ij} x_i x_j)}{\partial x_i} = g_{ij} \frac{\partial(x_i x_j)}{\partial x_i} = \delta_{ij} x_j + x_i \delta_{ii}$$

$$\operatorname{grad} f(\vec{r}) = \delta_{ij} x_j + x_i \delta_{ii}$$

Задача 7

- $S_{ij} = a_i b_j - a_j b_i = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}\,\underbrace{(a_2 b_3 - a_3 b_2)}_{S_{23}} - \vec{j}\,\underbrace{(a_1 b_3 - a_3 b_1)}_{S_{31}} + \vec{k}\,\underbrace{(a_1 b_2 - b_1 a_2)}_{S_{12}}$$

$\parallel S_{13}$

компоненты S_{ij} представляют собой скаляры, полученные проецированием на оси вектора, полученного векторным произведением $\bar{a}$ и $\vec{b}$

Задача

- $\omega_i = \frac{1}{2} e_{ijk} S_{jk}$

$$\omega_1 = \frac{1}{2} e_{1jk} S_{jk} = \frac{1}{2} e_{123} S_{23} + \frac{1}{2} \cancel{e_{132}}^{-1} S_{32} + \cancel{\tfrac{1}{2}e} =$$

$$= \frac{1}{2} S_{23} - \frac{1}{2} S_{32} = \frac{1}{2} S_{23} + \frac{1}{2} S_{23} = S_{23} = a_2 b_3 - b_2 a_3$$

$$\omega_2 = \cancel{S_{13} = (a_1 b_3 - a_3 b)} = \frac{1}{2} \cancel{e_{231}}^{1} S_{31} + \frac{1}{2} \cancel{e_{213}}^{-1} S_{13} = S_{31} = a_3 b_1 - a_1 b_3$$

$$\omega_3 = S_{12} = a_1 b_2 - b_1 a_2$$

* ω_i – модули проекций вектора $\vec{c} = \bar{a} \times \bar{b}$

Задача 8.

$$\nabla(\vec{a}\,\vec{b}) = (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\,\vec{\nabla})\vec{a} + \vec{b}\times(\vec{\nabla}\times\vec{a}) + \vec{a}\times(\vec{\nabla}\times\vec{b})$$

$$(\vec{a}\,\vec{\nabla})\vec{b} + (\vec{b}\,\vec{\nabla})\vec{a} + \vec{b}\times(\vec{\nabla}\times\vec{a}) + \vec{a}\times(\vec{\nabla}\times\vec{b}) =$$

$$= \cancel{(\vec{a}\,\vec{\nabla})\vec{b}} + \cancel{(\vec{b}\,\vec{\nabla})\vec{a}} + \vec{\nabla}(\vec{b}\cdot\overset{\downarrow}{\vec{a}}) - \cancel{\vec{a}(\vec{b}\cdot\vec{\nabla})} + \nabla(\vec{a}\cdot\overset{\downarrow}{\vec{b}}) - \cancel{\vec{b}(\vec{a}\cdot\vec{\nabla})}$$

$$= \vec{\nabla}(\vec{b}\,\overset{\downarrow}{\vec{a}}) + \vec{\nabla}(\vec{a}\cdot\overset{\downarrow}{\vec{b}}) = \nabla(\vec{a}\cdot\vec{b})$$

Другие док-ва

2.2 rot grad f

$\mathrm{grad}\, f = \frac{\partial f}{\partial x_i}$

$c_i = \mathrm{rot}\,\mathrm{grad}\, f = e_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k} = e_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k}$

$c_1 = \overset{1}{e_{123}} \frac{\partial^2 f}{\partial x_2 \partial x_3} + \overset{-1}{e_{132}} \frac{\partial^2 f}{\partial x_3 \partial x_2} \overset{=0}{=}$ (т.к. порядок диф. не важен)

аналогично для других $\Rightarrow c_i = 0 \Rightarrow \mathrm{rot}\,\mathrm{grad}\, f = 0$

2.3. $\bar{\nabla} \times \bar{r} = 0$

$\{\bar{\nabla} \times \bar{r}\}_i = e_{ijk} \frac{\partial}{\partial x_j} x_k$

рассмотрим при $i=1$

$\{\nabla \times r\}_1 = e_{123} \frac{\partial x_3}{\partial x_2} - e_{132} \frac{\partial x_2}{\partial x_3} = 0$

получим, что выражение $\bar{\nabla} \times \bar{r}$ равно нулю

т.к. при $j \neq k$ $\frac{\partial x_k}{\partial x_j} = 0$, а при $j = k$ $e_{ijj} = 0$

Задача 3.

$\vec{\nabla} \times (\vec{a} \times \vec{b}) = \bar{a}(\bar{\nabla} \cdot \bar{b}) - (\bar{\nabla} \cdot \bar{a})\bar{b} + (\bar{b}\bar{\nabla})\bar{a} - (\bar{a} \cdot \bar{\nabla})\bar{b}$

$\{\nabla \times (\vec{a} \times \vec{b})\}_i = e_{ijk} \frac{\partial (\vec{a} \times \vec{b})_k}{\partial x_j} = e_{ijk} e_{klm} \frac{\partial (a_l b_m)}{\partial x_j}$

$= \{e_{ijk} e_{lmk} = \delta_{jm}\delta_{il} - \delta_{im}\delta_{jl}\}$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial (a_l b_m)}{\partial x_j} = \frac{\partial (a_l b_m)}{\partial x_j} \delta_{il}\delta_{jm} -$

$- \delta_{im}\delta_{jl} \frac{\partial (a_l b_m)}{\partial x_j} = \frac{\partial (a_i b_j)}{\partial x_j} - \frac{\partial (a_j b_i)}{\partial x_j} =$

$= a_i \frac{\partial b_j}{\partial x_j} + b_j \frac{\partial a_i}{\partial x_j} - a_j \frac{\partial b_i}{\partial x_j} + \frac{\partial a_j}{\partial x_j} b_i =$

$a_i (\nabla \cdot b) + b_j \frac{\partial a_i}{\partial x_j} - a_j \frac{\partial b_i}{\partial x_j} + b_i (\nabla \cdot a) =$

$= a_i (\nabla \cdot b) + \frac{\partial a_i}{\partial \vec{b}} - \frac{\partial b_i}{\partial \vec{a}} - b_i (\nabla \cdot a) =$

$= a_i (\nabla \cdot b) + a_i (b \cdot \nabla) - b_i (a \nabla) - b_i (\nabla \cdot a) =$

• 3.3 $(\bar{a}\cdot\bar{\nabla})\vec{r} = \vec{a}$

$$(\bar{a}\cdot\vec{\nabla}) = \frac{\partial}{\partial x_i}\cdot a_i$$

$$\{(\vec{a}\cdot\vec{\nabla})\vec{r}\}_i = \frac{\partial}{\partial x_i}\cdot a_i\cdot x_i = a_i\frac{\partial x_i}{\partial x_i} = a_i$$

3.5. $\operatorname{rot}\operatorname{rot}\vec{a}$

$$\{\nabla\times\vec{a}\}_p = e_{pjk}\frac{\partial}{\partial x_j}a_k$$

$$\{\nabla\times\{\nabla\times\vec{a}\}\}_i = e_{imp}\frac{\partial}{\partial x_m}e_{pjk}\frac{\partial}{\partial x_j}a_k =$$

$$= -e_{pmi}e_{pjk}\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_j}a_k = -(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij})\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_j}a_k = \delta_{mk}\delta_{ij}\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_j}a_k - \delta_{mj}\delta_{ik}\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_j}a_k$$

$$= \underbrace{\frac{\partial^2 a_k}{\partial x_k x_i}}_{\text{grad div }\vec{a}} - \underbrace{\frac{\partial^2 a_k}{\partial x_m^2}}_{\nabla^2 a} = \operatorname{grad}\operatorname{div}\vec{a} - \nabla^2 a =$$

$$\left\{\operatorname{grad}\operatorname{div}\vec{a} = \frac{\partial}{\partial x_i}\left(\frac{\partial a_k}{\partial x_k}\right) = \frac{\partial^2 a_k}{\partial x_i\partial x_k}\right\}$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2 a$$

3.6. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) = 0$

т.к. $\{(\vec{\nabla}\times\vec{a})\}_i = e_{ijk}\frac{\partial}{\partial x_j}a_k$

$$c_i = \{\nabla\cdot(\vec{\nabla}\times\vec{a})\}_i = \frac{\partial}{\partial x_i}e_{ijk}\frac{\partial}{\partial x_j}a_k$$

т.к. $\frac{\partial}{\partial x_i}e_{ijk}\frac{\partial}{\partial x_j}a_k = -e_{ikj}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}a_k$

то $c_i = 0$, т.к. они будут сокращатся