Матрица перехода

$\mathbb{R}^2$: $e_1=(1,4)$, $e_2=(2,9)$ $\qquad C_{e\to e'}$

$e_1'=(3,1)$, $e_2'=(-1,-2)$

$$\boxed{\left(\begin{array}{cc|cc} 1 & 2 & 3 & -1 \\ 4 & 9 & 1 & -2 \end{array}\right)} \longrightarrow \text{кан. ступ. вид}$$

$$(e_1' \;\; e_2') = (e_1 \;\; e_2)\cdot C$$

$$\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}\cdot C \qquad \Big| \cdot \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}^{-1} \text{ слева}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}^{-1}\cdot\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = C$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}^{-1} = \frac{1}{1\cdot 9-2\cdot 4}\begin{pmatrix} 9 & -4 \\ -2 & 1 \end{pmatrix}^{T} = \begin{pmatrix} 9 & -2 \\ -4 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 9 & -2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \left(\begin{array}{c|c} 25 & -5 \\ -11 & 2 \end{array}\right)$$

$e_1=(1,4)$, $e_2=(2,9)$ $\qquad e_1' = 25e_1 - 11e_2$

$e_1'=(3,1)$, $e_2'=(-1,-2)$ $\qquad e_2' = -5e_1 + 2e_2$

№1.

$u_1=(1,2,3)$, $u_2=(0,1,1)$, $u_3=(0,0,1)$ $\leftarrow$ лин. нез.

$\downarrow \qquad \downarrow \qquad \downarrow$

$v_1=(2,3,2)$, $v_2=(0,0,0)$, $v_3=(0,0,3)$

$$\underset{3\times 3}{A}u_1 = v_1,\quad Au_2=v_2,\quad Au_3=v_3$$

поиск матр. лин. отобр.

$$A\,(u_1\;u_2\;u_3) = (v_1\;v_2\;v_3)$$

$$A\cdot\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 0 & 3 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}^{-1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{III}-3\text{I}]{\text{II}-2\text{I}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\xrightarrow{\text{III}-\text{II}} \left(\begin{array}{c|ccc} & 1 & 0 & 0 \\ E & -2 & 1 & 0 \\ & -1 & -1 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 0 & 3 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ -1 & -3 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\to\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix},\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\to\begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\to\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

№2. $\qquad \varphi:\mathbb{R}^2\to\mathbb{R}^2$

$u_1=(1,2)$, $u_2=(0,3)$, $u_3=(1,4)$, $u_4=(2,5)$

$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$

$v_1=(5,4)$, $v_2=(6,3)$, $v_3=(9,6)$, $v_4=(12,8)$

$$\underset{2\times 2}{A}u_1 = v_1,\quad \underset{2\times 2}{A}u_2 = v_2$$

$$A\cdot(u_1|u_2) = (v_1|v_2)$$

$$A\cdot\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 4 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2/3 & 1/3 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 6 \\ 4 & 3 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 6 \\ 4 & 3 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ -2/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$Au_3 = v_3\ ? \qquad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} = v_3$$

$$Au_4 = v_4\ ? \qquad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \end{pmatrix} \neq v_4$$

Ответ: нет

№3.

$V = \mathbb{R}[x]_{\leq 2}$ $\qquad ax^2+bx+c$

$e=\{1, x, x^2\}$ $\qquad \varphi: V\to V$ $\qquad$ матр.

$$A = \begin{array}{c} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{array}{c} 1 \\ x \\ x^2 \end{array} \\ \varphi(1)\ \ \varphi(x)\ \ \varphi(x^2) \\ \uparrow\ \ \uparrow\ \ \uparrow \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

$\varphi(1) = 1\cdot x^2 = x^2$

$\varphi(x) = x$

$\varphi(x^2) = 1\cdot 1 + 1\cdot x^2 = x^2+1$

$$e' = \{3x^2+2x+1,\ x^2+3x+2,\ 2x^2+x+3\}$$

$$A' = C^{-1}_{e\to e'}\,A\,C_{e\to e'}$$

$$C = \begin{array}{c} 1 \\ x \\ x^2 \end{array}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{III}-3\text{I}]{\text{II}-2\text{I}} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\xrightarrow{(\text{III}-5\text{II})/18} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{7}{18} & \frac{-5}{18} & \frac{1}{18} \end{array}\right) \longrightarrow$$

$$\xrightarrow{-\text{II}} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & +1 & +5 & +2 & -1 & 0 \\ 0 & 0 & 1 & \frac{7}{18} & \frac{-5}{18} & \frac{1}{18} \end{array}\right) \xrightarrow[\text{II}-5\text{III}]{\text{I}-3\text{III}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1/6 & 5/6 & -1/6 \\ 0 & 1 & 0 & 1/18 & 7/18 & -5/18 \\ 0 & 0 & 1 & 7/18 & -5/18 & 1/18 \end{array}\right)$$

$$\xrightarrow{\text{I}-2\text{II}} \left(\begin{array}{c|ccc} & -5/18 & 1/18 & 7/18 \\ E & 1/18 & 7/18 & -5/18 \\ & 7/18 & -5/18 & 1/18 \end{array}\right)$$

$-1/6 - 2/18 = -\frac{5}{18}$

$5/6 - 14/18 = 1/18$

$-1/6 + 10/18 = 7/18$

$$C^{-1} = \frac{1}{18}\begin{pmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{pmatrix}$$

$$C^{-1}\cdot A\cdot C$$

$$\frac{1}{18}\begin{pmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} = A'$$

№4.

а) $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = A$

$$\det(tE-A) = \det\begin{pmatrix} t-2 & 1 \\ -1 & t \end{pmatrix} = (t-2)t+1 =$$

$$= t^2-2t+1 = (t-1)^2 \qquad \text{корень: } t=1$$

с.з. : 1

$$E-A:\ \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad -x_1+x_2=0 \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

ФСР: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$V_1 = \left\{\lambda\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}\ \Big|\ \lambda\in\mathbb{R}\right\} = \underline{\left\langle\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\rangle} \quad \dim = 1$$

$\dim V_1 < 2 \Rightarrow$ не диагонализ.

б) $\begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix} = A$ $\qquad \operatorname{tr} A = 5$

$\det A = -5+9+9-15-3+9 = 27-23 = 4$

$$\chi_A(t) = t^3 - (\operatorname{tr}A)t^2 + (\ldots\ldots)t - (\det A)$$

$$\begin{vmatrix} -1 & 3 \\ -3 & 5 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -3 & 1 \end{vmatrix} = 4+8-4 = 8$$

$$t^3-5t^2+8t-4 = 0$$

$t=1$ $\quad\Big|\quad (t-1)(t^2-4t+4) = (t-1)(t-2)^2$

$4t+4t=8t$

с.з.: 1, 2

$\downarrow$

$\dim V_1 = 1$

$$\begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix} - E = \begin{pmatrix} -2 & 3 & -1 \\ -3 & 4 & -1 \\ -3 & 3 & 0 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} -2 & 3 & -1 \\ -3 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{\text{I}-\text{II}} \begin{pmatrix} 1 & -1 & 0 \\ -3 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix} \to$$

$$\xrightarrow{\text{II}+3\text{I}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & -1 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1-x_2=0 \\ x_2-x_3=0 \end{cases}$$

ФСР: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$V_1 = \left\langle\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\rangle$$

$$\begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix} - 2E = \begin{pmatrix} -3 & 3 & -1 \\ -3 & 3 & -1 \\ -3 & 3 & -1 \end{pmatrix} \to (-3\ \ 3\ \ -1)$$

$-3x_1+3x_2-x_3=0$

ФСР: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$

$$V_2 = \left\langle\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}\right\rangle$$

$$\dim V_1 + \dim V_2 = 3 = \dim\mathbb{R}^3$$

$$\underset{\text{ст.}}{A} \rightsquigarrow \underset{e'}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}} = C^{-1}AC$$

$$C_{\text{ст.}\to e'} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$