Midterm 1 Review

Objects

- Sets and Subsets
- Strings and Permutations
- Graphs
- Trees and Rooted Trees

Skills

- Counting
  - strings/permutations over an alphabet
  - rearrangements of a word
  - inclusion-exclusion
- Graph Basics
  - drawing
  - finding walks/paths
  - connectivity
- Recurrence Relations
  - solving first order
  - setting them up

Note: Know the vocabulary from the course notes!

Strings

Theorem. If $\mathcal{A}$ is an alphabet with k symbols, there are k^n strings of size n over $\mathcal{A}$ for all $n \geq 0$.

Exercise: If A is a set of size a and B is a set of size b, how many functions are there from A to B?

Theorem. If $\mathcal{A}$ is an alphabet with k symbols, there are $k!$ permutations over $\mathcal{A}$.

Theorem. If n and k are positive integers with $0 \leq k \leq n$, the number of ways to choose k elements from a set of size n is equal to

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Exercise: Let A be a set of size a. How many functions from A to $\{0, 1\}$ have the property that there are exactly k elements of A that map to 1?

Exercise: How many nonnegative solutions are there to the equation

$$x_1 + x_2 + \ldots x_k = n?$$

Binomial Theorem. If x and y are variables and n is a positive integer:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Example: $(x+y)^3 =$

Multinomials. Let $(n_1, n_2, \ldots, n_k)$ be a sequence of k non-negative numbers summing to n. The number of strings over an alphabet of size k with content $(n_1, n_2, \ldots, n_k)$ is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

Exercise: How many ways can the letters of the word $SUSPENSE$ be arranged?

Exercise: How many ways can the letters of this word be arranged so that they contain the substring PUN?

Exercise: How many functions from the set $\{1,2,3,4,5,6,7\}$ to $\{A,B,C\}$ have the property that there are exactly 3 elements mapping to A, 2 elements mapping to B, and 2 elements mapping to C?

Graphs

Exercise: Draw the graph $G = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}, \{1, 5\}, \{2, 5\}\}$.

Questions about G

1. What is the order of G?
2. What is the size of G?
3. Is G connected?
4. How many subgraphs of G are cycles?
5. How many induced subgraphs does G have?
6. How many spanning subgraphs of G have exactly 3 edges?

Trees

Definition. A **tree** is a connected acyclic graph. A **forest** is an acyclic graph. Note: if G is a forest, every connected component of G is a tree.

Example:

Theorem. If $T = (V, E)$ is a tree

1. There is a unique path between any two vertices of T.
2. If $|V| \geq 2$ there are at least 2 leaf vertices.
3. $|V| = |E| + 1$

Theorem. Let $G = (V, E)$ be a graph and consider the following properties:

1. G is connected
2. G is acyclic
3. $|V| = |E| + 1$

If G has any two of these properties, then it has the third (and G is a tree)

Rooted Trees

Definition. A **rooted tree** is a pair (T, r) where T is a tree and $r \in V(T)$ is a distinguished vertex called the **root**.

Example:

Note: In the book rooted trees are developed using digraphs.

Exercise: Draw all ordered binary rooted trees with height 2.

Inclusion-Exclusion

A way of counting all elements that satisfy **none** of a list of properties.

Notation.

- Let $\mathcal{S}$ be the ground set (i.e. all possible elements) and assume $|\mathcal{S}| = N$.
- Let $C_1, \dots, C_t$ be properties that elements of $\mathcal{S}$ may or may not satisfy.
- We let $\overline{N}$ denote the number of elements of $\mathcal{S}$ satisfying none of the properties $C_1, C_2, \dots, C_t$
- For a subset of properties, say $\{C_1, C_3, C_6\}$ we let $N(C_1C_3C_6)$ denote the number of elements of $\mathcal{S}$ that satisfy C_1 and C_3 and C_6. For any $0 \le k \le t$ we define the number S_k as:

$$\begin{aligned} S_0 &= N \\ S_k &= \sum_{i_1 < i_2 \dots < i_k} N(C_{i_1} C_{i_2} \cdots C_{i_k}) \end{aligned}$$

Inclusion-Exclusion Theorem.

$$\overline{N} = \sum_{k=0}^{t} (-1)^k S_k = S_0 - S_1 + S_2 - S_3 \dots + (-1)^t S_t$$

Examples:

- For two conditions C_1, C_2 we have

$$\begin{aligned} \overline{N} &= S_0 - S_1 + S_2 \\ &= N - \Big(N(C_1) + N(C_2)\Big) + N(C_1C_2) \end{aligned}$$

- For three conditions C_1, C_2, C_3 we have

$$\begin{aligned} \overline{N} &= S_0 - S_1 + S_2 - S_3 \\ &= N - \Big(N(C_1) + N(C_2) + N(C_3)\Big) \\ &\quad + \Big(N(C_1C_2) + N(C_1C_3) + N(C_2C_3)\Big) - N(C_1C_2C_3) \end{aligned}$$

Note: inclusion-exclusion problems come in two varieties:

1. There are few conditions and you must evaluate each term individually.
2. The terms contributing to each S_k are equal.

Example of type 1: How many numbers from 1 to 70 are not divisible by 2, not divisible by 5, and not divisible by 7.

Example of type 2: How many derangements are there of the set $\{1, 2, 3, 4, 5\}$?

Recurrence Relations

Definition. An infinite sequence $a_0, a_1, a_2, \dots$ of integers satisfies a **recurrence relation** of **order** k if there exist functions f and g so that the following equation holds for all $n \geq k$

$$f(a_n, a_{n-1}, \dots, a_{n-k}) = g(n)$$

(we will always assume that f is a polynomial)

Notation:

- **linear** means that f is a polynomial of degree 1
- **homogeneous** means the $g(n) = 0$
- **constant coefficients** means that the coefficients of the a_i's in the function f are constants not depending on n.

Theorem. If $a_0, a_1, \dots$ is a sequence satisfying the first order recurrence relation

$$\begin{aligned} a_0 &= A \\ a_n &= da_{n-1} \end{aligned}$$

Then we have $a_n = Ad^n$.

Exercise: Let s_n be the number of strings over the alphabet $\{A, B, C, D\}$ with the special property that every A is immediately followed by a B. Find a recurrence for the sequence $s_0, s_1, \dots$

Exercise: Let t_n be the number of strings over the alphabet $\{A, B, C, D\}$ with the special property that every A is eventually followed by a B. Find a recurrence for the sequence $t_0, t_1, \ldots$