MACM 201 Homework 1 Solutions

Instructor question(s): In these question we consider sequences $A_1, A_2, \ldots, A_k$ where each A_i is a subset of $\{1, 2, \ldots, n\}$.

1. How many sequences $A_1, A_2, \ldots, A_k$ have the property that $\cup_{i=1}^k A_i = \{1, 2, \ldots, n\}$?

   Solution: We will determine the sets $A_1, \ldots, A_k$ by determining for each element $i \in \{1, \ldots, n\}$ which of the sets $A_1, \ldots, A_k$ contain i. There are 2^k ways to decide which sets $A_1, \ldots, A_k$ contain i (since i can either appear or not appear in each one). However, i must appear in at least one of the sets in order for i to be contained in the union $\cup_{i=1}^k A_i$. Taking this into account, we see that there are $2^k - 1$ valid ways to choose which sets $A_1, \ldots, A_k$ contain i (all of the possibilities minus the one forbidden one). Applying this logic to each element $i \in \{1, \ldots, n\}$ we deduce that there are $(2^k - 1)^n$ sequences $A_1, \ldots, A_k$ satisfying $\cup_{i=1}^k A_i = \{1, \ldots, n\}$.

2. How many sequences $A_1, A_2, \ldots, A_k$ have the property that $A_1 = \emptyset$, $A_k = \{1, 2, \ldots, n\}$ and $A_i \subseteq A_{i+1}$ holds for every $1 \leq i \leq k - 1$?

   Solution: Again, we determine the sets $A_1, \ldots, A_k$ by determining for each element $i \in \{1, \ldots, n\}$ which of the sets $A_1, \ldots, A_k$ contain i. For each element i, if i appears in the set A_j then in must also be contained in $A_{j+1}, A_{j+2}, \ldots, A_k$. So, in other words, the only choice to make is the first set containing i. Since $A_1 = \emptyset$, the first set containing i must be one of the $k - 1$ sets $A_2, \ldots, A_k$, giving us $k - 1$ possibilities. Applying this logic to each element $i \in \{1, \ldots, n\}$ we find that there are $(k-1)^n$ sequences $A_1, \ldots, A_k$ satisfying the above description.