MACM 201 Assignment 1 Solutions September 10, 2018

Section 1.1-2

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16. (a) With repetitions allowed there are 40^{25} distinct messages.
(b) By the rule of product there are $40 \times 30 \times 30 \times \ldots \times 30 \times 30 \times 40 = (40^2)(30^{23})$ messages.

20. (a) Since there are three A's, there are $8!/3! = 6720$ arrangements.

(b) Here we arrange the six symbols D,T,G,R,M, AAA in $6! = 720$ ways.

24. $P(n+1,r) = (n+1)!/(n+1-r)! = [(n+1)/(n+1-r)] \cdot [n!/(n-r)!] = [(n+1)/(n+1-r)]P(n,r)$.

30. (a) For five letters there are $26 \times 26 \times 26 \times 1 \times 1 = 26^3$ palindromes. There are $26 \times 26 \times 26 \times 1 \times 1 \times 1 = 26^3$ palindromes for six letters.
(b) When letters may not appear more than two times, there are $26 \times 25 \times 24 = 15,600$ palindromes for either five or six letters.

32. (a) For positive integers n, k, where $n = 3k$, $n!/(3!)^k$ is the number of ways to arrange the n objects $x_1, x_1, x_1, x_2, x_2, x_2, \ldots, x_k, x_k, x_k$. This must be an integer.
(b) If n, k are positive integers with $n = mk$, then $n!/(m!)^k$ is an integer.

Section 1.3

6.

$$\binom{n}{2}+\binom{n-1}{2}=(\frac{1}{2})(n)(n-1)+(\frac{1}{2})(n-1)(n-2)=(\frac{1}{2})(n-1)[n+(n-2)]=$$

$(\frac{1}{2})(n-1)(2n-2)=(n-1)^2$.

8. (a) $\binom{4}{1}\binom{13}{5}$ (b) $\binom{4}{4}\binom{48}{1}$ (c) $\binom{13}{1}\binom{4}{4}\binom{48}{1}$ (d) $\binom{4}{3}\binom{4}{2}$

(e) $\binom{4}{3}\binom{12}{1}\binom{4}{2}$ (f) $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}=3744$

(g) $\binom{13}{1}\binom{4}{3}\binom{48}{1}\binom{44}{1}/2$ (Division by 2 is needed since no distinction is made for the order in which the other two cards are drawn.) This result equals $54,912=\binom{13}{1}\binom{4}{3}\binom{48}{2}-3744=\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$.

(h) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$.

18. (a) $10!/(4!3!3!)$ (b) $\binom{10}{8}2^2+\binom{10}{9}2+\binom{10}{10}$

(c) $\binom{10}{4}$ (four 1's, six 0's) + $\binom{10}{2}\binom{8}{1}$ (two 1's, one 2, seven 0's) + $\binom{10}{2}$ (two 2's, eight 0's)

28. a) $\displaystyle\sum_{i=0}^{n}\frac{1}{i!(n-i)!}=\frac{1}{n!}\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}=\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}=2^n/n!$

b) $\displaystyle\sum_{i=0}^{n}\frac{(-1)^i}{i!(n-i)!}=\frac{1}{n!}\sum_{i=0}^{n}\frac{(-1)^i n!}{i!(n-i)!}=\frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}=\frac{1}{n!}(0)=0.$

Section 1.4

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10. Here we want the number of integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100$, $x_i \geq 3$, $1 \leq i \leq 6$. (For $1 \leq i \leq 6$, x_i counts the number of times the face with i dots is rolled.) This is equal to the number of nonnegative integer solutions there are to $y_1+y_2+y_3+y_4+y_5+y_6 = 82$, $y_i \geq 0$, $1 \leq i \leq 6$. Consequently the answer is $\binom{6+82-1}{82} = \binom{87}{82}$.

12. (a) The number of solutions for $x_1 + x_2 + \ldots + x_5 < 40, x_i \geq 0,\ 1 \leq i \leq 5$, is the same as the number for $x_1 + x_2 + \ldots + x_5 \leq 39, x_i \geq 0,\ 1 \leq i \leq 5$, and this equals the number of solutions for $x_1 + x_2 + \ldots + x_5 + x_6 = 39, x_i \geq 0,\ 1 \leq i \leq 6$. There are $\binom{6+39-1}{39} = \binom{44}{39}$ such solutions.
(b) Let $y_i = x_i+3$, $1 \leq i \leq 5$, and consider the inequality $y_1 + y_2 + \ldots + y_5 \leq 54$, $y_i \geq 0$. There are [as in part (a)] $\binom{6+54-1}{54} = \binom{59}{54}$ solutions.

26. Each such composition can be factored as k times a composition of m. Consequently, there are 2^{m-1} compositions of n, where $n = mk$ and each summand in a composition is a multiple of k.

28. (a) For $n \geq 4$, consider the strings made up of n bits – that is, a total of n 0's and 1's. In particular, consider those strings where there are (exactly) two occurrences of 01. For example, if $n = 6$ we want to include strings such as 010010 and 100101, but not 101111 or 010101. How many such strings are there?
(b) For $n \geq 6$, how many strings of n 0's and 1's contain (exactly) three occurrences of 01?
(c) Provide a combinatorial proof for the following:

$$\text{For } n \geq 1, \quad 2^n = \binom{n+1}{1} + \binom{n+1}{3} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even.} \end{cases}$$

(a) A string of this type consists of x_1 1's followed by x_2 0's followed by x_3 1's followed by x_4 0's followed by x_5 1's followed by x_6 0's, where,

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = n, \quad x_1, x_6 \geq 0, \quad x_2, x_3, x_4, x_5 > 0.$$

The number of solutions to this equation equals the number of solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = n - 4, \text{ where } y_i \geq 0 \text{ for } 1 \leq i \leq 6.$$

This number is $\binom{6+(n-4)-1}{n-4} = \binom{n+1}{n-4} = \binom{n+1}{5}$.

(b) For $n \geq 6$, a string with this structure has x_1 1's followed by x_2 0's followed by x_3 1's ... followed by x_8 0's, where

$$x_1 + x_2 + x_3 + \cdots + x_8 = n, \quad x_1, x_8 \geq 0, \quad x_2, x_3, \ldots, x_7 > 0.$$

The number of solutions to this equation equals the number of solutions to

$$y_1 + y_2 + y_3 + \cdots + y_8 = n - 6, \text{ where } y_i \geq 0 \text{ for } 1 \leq i \leq 8.$$

This number is $\binom{8+(n-6)-1}{n-6} = \binom{n+1}{n-6} = \binom{n+1}{7}$.

(c) There are 2^n strings in total and $n+1$ strings where there are k 1's followed by $n-k$ 0's, for $k = 0, 1, 2, \ldots, n$. These $n+1$ strings contain no occurrences of 01, so there are $2^n - (n+1) = 2^n - \binom{n+1}{1}$ strings that contain at least one occurrence of 01. There are $\binom{n+1}{3}$ strings that contain (exactly) one occurrence of 01, $\binom{n+1}{5}$ strings with (exactly) two occurrences, $\binom{n+1}{7}$ strings with (exactly) three occurrences, ... , and for

(i) n odd, we can have at most $\frac{n-1}{2}$ occurrences of 01. The number of strings with $\frac{n-1}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1 + x_2 + \cdots + x_{n+1} = n, \; x_1, x_{n+1} \geq 0, \quad x_2, x_3, \ldots, x_n > 0.$$

This is the same as the number of integer solutions for

$$y_1 + y_2 + \cdots + y_{n+1} = n - (n-1) = 1, \;\text{ where }\; y_1, y_2, \ldots, y_{n+1} \geq 0.$$

This number is $\binom{(n+1)+1-1}{1} = \binom{n+1}{1} = \binom{n+1}{n} = \binom{n+1}{2(\frac{n-1}{2})+1}$.

(ii) n even, we can have at most $\frac{n}{2}$ occurrences of 01. The number of strings with $\frac{n}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1 + x_2 + \cdots + x_{n+2} = n, \;\; x_1, x_{n+2} \geq 0, \quad x_2, x_3, \ldots, x_n > 0.$$

This is the same as the number of integer solutions for

$$y_1 + y_2 + \cdots + y_{n+2} = n - n = 0, \;\text{ where }\; y_i \geq 0 \;\text{ for }\; 1 \leq i \leq n+2.$$

This number is $\binom{(n+2)+0-1}{0} = \binom{n+1}{0} = \binom{n+1}{n+1} = \binom{n+1}{2(\frac{n}{2})+1}$.

Consequently,

$$2^n - \binom{n+1}{1} = \binom{n+1}{3} + \binom{n+1}{5} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even}, \end{cases}$$

and the result follows.