Generating Functions 1 - Inverses

Recall the definition of generating functions:

Definition. A **generating function** is a formal expression of the form

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots = \sum_{i=0}^{\infty} a_i x^i$$

This generating function may be viewed as another way of encoding the infinite sequence $a_0, a_1, a_2, \ldots$, and we say that $A(x)$ is the generating function **for** the sequence $a_0, a_1, \ldots$.

Note: One of the reasons that generating functions are so useful is that they have a nice algebraic structure. We can add them and multiply them.

Example: Suppose $A(x) = 1+x+x^2+x^3+\ldots$ and $B(x) = 2+2x+2x^2+2x^3+\ldots$.

1. What is $A(x) + B(x)$?

$$= 3 + 3x + 3x^2 + \ldots$$

2. What is $A(x) \times B(x)$?

$$(1 + x + x^2 + x^3 + \ldots)(2 + 2x + 2x^2 + \ldots)$$

$$= 2 + 4x + 6x^2 + 8x^3 + \ldots$$

$$= \sum_{n=0}^{\infty} 2(n+1)x^n$$

Formally the definition of addition and multiplication are as follows:

Definition. If $A(x) = a_0 + a_1x + \ldots$ and $B(x) = b_0 + b_1x + \ldots$ are generating functions,

- **Sum:** $A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots = \sum_{n=0}^{\infty}(a_n + b_n)x^n$
- **Product:** $A(x) \times B(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} x^n$

Example: Consider the following generating functions:

- $N(x) = 1 + x^5 + x^{10} + x^{15} + \ldots$ (for nickels)
- $D(x) = 1 + x^{10} + x^{20} + x^{30} + \ldots$ (for dimes)
- $Q(x) = 1 + x^{25} + x^{50} + x^{75} + \ldots$ (for quarters)

Fact: The coefficient of x^k in the product $N(x) \times D(x) \times Q(x)$ tells us how many ways you can express k cents as a sum of nickels, dimes, and quarters.

Example: Consider the coefficient of x^{30} in $N(x) \times D(x) \times Q(x)$

$$\left(1 + x^5 + x^{10} + x^{15} + x^{20} + x^{25} + x^{30} + \ldots\right)\left(1 + x^{10} + x^{20} + x^{30} + \ldots\right)\left(1 + x^{25} + \ldots\right)$$

Upshot: Generating functions naturally encode many counting problems.

Issues: To take advantage of this framework we need:

1. Efficient ways to describe generating functions like $N(x)$, $D(x)$, and $Q(x)$
2. Tools to extract the coefficients of the generating functions we construct.

in these notes we focus on 1

Multiplication

Real numbers. There are two features of real number multiplication we wish to recall:

- [identity] The number 1 has the property that $1 \cdot x = x$ for every x.
- [inverses] Every nonzero real number x has an inverse $\frac{1}{x}$ so that $x \cdot \frac{1}{x} = 1$

Generating Functions.

- [identity] The generating function 1 has the property that $1 \times A(x) = A(x)$ for every generating function $A(x)$.
- [inverses] We say that two generating functions $A(x)$ and $B(x)$ are inverses if $A(x) \times B(x) = 1$. In this case we may write $B(x) = \frac{1}{A(x)}$

Note: Unlike numbers, not all nonzero generating functions have inverses!

Generating Functions with easy inverses

1. Let $A(x)$ be the generating function for the sequence $(1, 1, 1, 1, \ldots)$ so
$$A(x) = 1 + x + x^2 + x^3 + \ldots$$
Verify that $A(x) = \frac{1}{1-x}$ i.e verify that $A(x)$ and $1-x$ are inverses

$$(1-x)A(x) = (1-x)(1+x+x^2+x^3+x^4+\ldots)$$
$$= 1 + (1-1)x + (1-1)x^2 + \ldots$$
$$= 1 \quad \checkmark$$

2. Let $B(x)$ be the generating function for the sequence $(0, 1, 2, 3, \ldots)$ so
$$B(x) = 0 + x + 2x^2 + 3x^3 + \ldots$$
Find $A(x) + B(x)$ $\quad A(x) = 1 + x + x^2 + x^3 + \ldots$

$$1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n$$

What is $x(A(x) + B(x))$?

$$x + 2x^2 + 3x^3 + 4x^4 + \ldots = \sum_{n=0}^{\infty} n x^n = B(x)$$

$$xA(x) + xB(x) = B(x)$$

Solve the above equation for $B(x)$

$$xA(x) = (1-x)B(x)$$

$$\frac{x}{1-x}A(x) = B(x)$$

$$A(x) = \frac{1}{1-x} \quad \text{so} \quad B(x) = \frac{x}{(1-x)^2}$$

3. Let $C(x)$ be the generating function for the sequence $2^0, 2^1, 2^2, 2^3, \ldots$.

$$C(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots = \sum_{n=0}^{\infty} 2^n x^n$$

What is $2xC(x)$?

$2x + 4x^2 + 8x^3 + \ldots$

Solve for $C(x)$

$C(x) - 2xC(x) = 1$

$(1-2x)C(x) = 1$

$C(x) = \frac{1}{1-2x}$

4. Let $N(x)$ be the generating function for Nickels we saw earlier

$$N(x) = 1 + x^5 + x^{10} + x^{15} + \ldots$$

What is $x^5N(x)$?

$x^5 + x^{10} + x^{15} + \ldots$

Solve for $N(x)$

$N(x) - x^5N(x) = 1$

$(1-x^5)N(x) = 1$

$N(x) = \frac{1}{1-x^5}$

Key Idea: Although generating functions are generally infinite objects, many important ones can be expressed very compactly using the concept of an inverse.

Example: The nickel, dime, and quarter generating functions are

$$N(x) = \frac{1}{1-x^5} \qquad D(x) = \frac{1}{1-x^{10}} \qquad Q(x) = \frac{1}{1-x^{25}}$$

So the generating function that counts the number of ways we can write k cents as a sum of nickels, dimes, and quarters is given by:

$$N(x) \times D(x) \times Q(x) = \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{25}}$$

At this point we have a very compact and convenient description of our coin-possibility-counting generating function. However, in order to utilize it, we still need tools to extract (i.e. determine) its coefficients!