Generating Functions 2 - Rational GF's

So far we have seen that some generating functions can be compactly expressed using inverses. For example

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

In this set of notes we will develop further tools for working with generating functions like this, called rational generating functions.

Definition. A generating function $A(x)$ is called **rational** if it can be expressed as

$$A(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

Note: The coefficients in our polynomials and the coefficients of our generating functions may be real numbers (not just integers).

GF represented as

Inf seq of coeff $\longleftrightarrow$ rational function

Two useful GF's

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$B(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

For the purposes of this set of notes (only!) we will always use the names $A(x)$ and $B(x)$ to refer to these particular generating functions.

Note: Multiplying a generating function $C(x) = c_0 + c_1x + c_2x^2 + \ldots = \sum_{n=0}^{\infty}$ by a power of x, say x^k, just shifts the coefficients by k.

$$x^kC(x) = c_0x^k + c_1x^{k+1} + c_2x^{k+2} + \ldots = \sum_{n=0}^{\infty} c_nx^{k+n}$$

Using this it is easy to multiply a generating function by a polynomial.

Problem. Determine the sequence for each generating function.

1. $\frac{x^3-2}{1-x}$ $= (x^3-2)\frac{1}{1-x} = (x^3-2)A(x)$

$= x^3A(x) - 2A(x)$

$(x^3+x^4+x^5+\ldots) + (-2-2x-2x^2-2x^3 \ldots)$

$= -2-2x-2x^2-x^3-x^4 \ldots$

Sequence $(-2,-2,-2,-1,-1,-1,-1,\ldots)$

2. $\frac{2x^2+5}{(1-x)^2}$ $= (2x^2+5)\frac{1}{(1-x)^2} = (2x^2+5)B(x)$

$= 2x^2B(x) + 5B(x)$

$= \sum_{n=0}^{\infty} 2(n+1)x^{n+2} + \sum_{n=0}^{\infty} 5(n+1)x^n$

$= (0+0x+2x^2+4x^3+\ldots) + (5+10x+15x^2+\ldots$

Seq $(5, 10, 2+15, 4+20, 6+25$

$(5, 10, 10+7(n-1)\ldots)$

Substitution

Although generating functions are **not** functions, if $C(x)$ is a generating function, we can make a new generating function by substituting another term in for x (as with functions)

Definition. Let $C(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots = \sum_{n=0}^{\infty} c_n x^n$ be a generating function. Then we define

$$C(kx^m) = c_0 + c_1(kx^m) + c_2(kx^m)^2 + c_3(kx^m)^3 + \ldots = \sum_{n=0}^{\infty} c_n k^n x^{mn}$$

Example: The GF for nickels $N(x) = 1 + x^5 + x^{10} + x^{15} + \ldots = \sum_{n=0}^{\infty} x^{5n}$ is obtained from the GF $A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$ by substitution:

$$A(x^5) = \sum_{n=0}^{\infty} (x^5)^n = \sum_{n=0}^{\infty} x^{5n} = N(x)$$

We can then use the formula $A(x) = \frac{1}{1-x}$ to express $N(x)$ as a rational function

$$N(x) = A(x^5) = \frac{1}{1-x^5}$$

(we already found this formula for $N(x)$, but this is an easier way)

$$F(x) = 1 + \overset{(ax^k)^1}{ax^k} + \overset{(ax^k)^2}{a^2 x^{2k}} + \overset{(ax^k)^3}{a^3 x^{3k}} + \ldots$$

$$F(x) = A(ax^k)$$

$$F(x) = \frac{1}{1-ax^k}$$

Substituting in $-x$

(1) Using the GF $A(x)$ and substituting in $-x$ for x gives

$$\begin{aligned}
\frac{1}{1+x} &= \frac{1}{1-(-x)} \\
&= A(-x) \\
&= 1 + (-x) + (-x)^2 + (-x)^3 + \dots \\
&= 1 - x + x^2 - x^3 + \dots \\
&= \sum_{n=0}^{\infty} (-1)^n x^n
\end{aligned}$$

Problem. Express $C(x) = 1 - 2x + 4x^2 - 8x^3 + 16x^4 \dots$ as a rational function.

$C(x) = A(-2x)$

$C(x) = \frac{1}{1-(-2x)} = \frac{1}{1+2x}$

(2) Using the GF $B(x)$ and substituting in $-x$ for x gives

$$\begin{aligned}
\frac{1}{(1+x)^2} &= \frac{1}{(1-(-x))^2} \\
&= B(-x) \\
&= 1 + 2(-x) + 3(-x)^2 + 4(-x)^3 + \dots \\
&\boxed{= 1 - 2x + 3x^2 - 4x^3 + \dots} \\
&= \sum_{n=0}^{\infty} (n+1)(-1)^n x^n
\end{aligned}$$

Problem. Express $D(x) = x - 2x^2 + 3x^3 - 4x^4 + \dots$ as a rational function.

$D(x) = \frac{x}{(1+x)^2}$

Finding Coefficients

Using substitution and our $A(x)$ and $B(x)$ GF's, we can now determine the coefficients for any GF that can be expressed as

$$\frac{p(x)}{ax+b} \quad \text{or} \quad \frac{p(x)}{(ax+b)^2}$$

where $p(x)$ is a polynomial

Problem. Find the coefficient of x^k in the generating function $[x^k]C(x)$

$$C(x) = \frac{x^2}{2x+3} = x^2 \cdot \frac{1}{2x+3} = \frac{1}{3}x^2 \cdot \frac{1}{1+\frac{2}{3}x}$$

$$= \frac{1}{3}x^2 \cdot \frac{1}{1-(-\frac{2}{3}x)} = \frac{1}{3}x^2 \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n x^n = \sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{2}{3}\right)^n x^{n+2}$$

$k = n+2$

$n = k-2$

$$[x^k]C(x) = \begin{cases} 0 & \text{if } k=0,1 \\ \frac{1}{3}\left(-\frac{2}{3}\right)^{k-2} & k \geq 2 \end{cases}$$

Problem. Find the coefficient of x^k in the generating function

$$D(x) = \frac{x^2+1}{(5x+2)^2}$$

$$= (x^2+1)\frac{1}{(5x+2)^2} = \frac{1}{4}(x^2+1)\frac{1}{(1+\frac{5}{2}x)^2} = \frac{1}{4}(x^2+1)\frac{1}{(1-(-\frac{5}{2}x))^2}$$

$$= \frac{1}{4}(x^2+1)\, B\left(-\frac{5}{2}x\right) = \frac{1}{4}(x^2+1)\sum_{n=0}^{\infty}(n+1)\left(-\frac{5}{2}\right)^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{4}(n+1)\left(-\frac{5}{2}\right)^n x^{n+2} + \sum_{n=0}^{\infty} \frac{1}{4}(n+1)\left(-\frac{5}{2}\right)^n x^n$$

$$[x^k]D(x) = \begin{cases} \frac{1}{4}(k+1)\left(-\frac{5}{2}\right)^k & k=0,1 \\ \frac{1}{4}(k+1)\left(-\frac{5}{2}\right)^k + \frac{1}{4}(k-1)\left(-\frac{5}{2}\right)^{k-2} & k \geq 2 \end{cases}$$

General Form

We have been working with the generating functions

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$B(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

There is a more general generating function as follows:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

Combining this formula with substitution allows us to determine the coefficients of any rational polynomial of the form $\frac{p(x)}{(ax+b)^k}$.