Lecture outline: Graph coloring

A graph coloring is a mapping of the vertices of a graph to a set of colors, which is usually represented by integers. We are interested here in colorings where two adjacent vertices have different colors.

We will see a few simple properties of graph colorings, together with a way to encode the number of proper colorings for all possible number of colors using the chromatic polynomial.

Graph coloring

Definition. Let $G = (V, E)$ be a connected graph or multigraph.

A **coloring** c of G with k colors, also called a k-**coloring** of G, is a mapping

$$c : V \to \{1, 2, \dots, k\}$$

A k-coloring c of G is a **proper coloring** if there is no edge $\{u, v\} \in E$ such that $c(u) = c(v)$.

The minimum number k needed for a proper k-coloring of a graph G is the **chromatic number** of G denoted by $\chi(G)$.

Examples.

Note. A k-coloring does not require that all k colors are used.

Application of graph coloring to scheduling.

We are given a set of n tasks $T_1, \ldots, T_n$ (for example exams) and a list of conflicts between these tasks, i.e. pairs of tasks that can not be scheduled at the same time (for example because both tasks require a resource such as a special exam room that is available once at a time or if many students take part in both of the exams).

We want to know how many time slots it will take to get all tasks done, assuming that each task requires a single time slot.

We can model this as a graph coloring problem: Let $G = (V, E)$ where

$$V = \{T_1, \ldots, T_n\}$$

$$E = \{\{T_i, T_j\} \mid T_i \text{ and } T_j \text{ are in conflict}\}$$

Then the number of time slots required to complete all the tasks is exactly $\chi(G)$.

The four-color theorem. The faces of any planar embedding of a planar map can be colored with four colors in such a way that no two faces sharing a common edge have the same color.

Why is it a vertex coloring result? Take the dual graph. Note that a "map" does not have a region bordering itself, which means that the dual graph is loop-less.

The four-color theorem, dual version. Every loop-free planar graph has chromatic number at most 4.

Remark. The four-color theorem was introduced around 1850, and it was an open problem for a very long time. Despite its simplicity it was only proved in 1976, and the proof relies on the use of a computer to check a large (too large to be checked by a human) number of "cases".

Examples.

Computing the chromatic number of graphs

Theorem. Deciding if the chromatic number of a graph G is at most k, with $k \geq 3$, is NP-complete.

A few simple examples where we can find the chromatic number.

What are the chromatic numbers of the following graphs:

$$K_n,\ \overline{K_n},\ Q_n,\ P_n,\ C_n,\ \text{the Petersen graph, bipartite graphs}$$

where P_n is the path of length $n-1$ and C_n is the cycle of length n.

The chromatic polynomial

Definition. Let $G = (V, E)$ be a graph or multigraph. The **chromatic polynomial** of G

$$P(G, \lambda)$$

is the unique polynomial in the variable λ such that

$$P(G, k) = \text{ number of proper } k\text{-colorings of } G$$

for every integer $k \geq 0$.

A small example.

From now we will be interested in computing this polynomial.

Deleting/Contracting an edge. Let $G = (V, E)$ be a graph and $e = \{u, v\} \in E$ be an edge of G.

The graph $G - e$ is the graph $(V, E - \{e\})$ obtained by removing the edge e from G.

The graph G/e is the graph obtained from G by removing the edge and identifying the vertices u and v (replacing u and v by a single vertex).

Recursive algorithm to compute $P(G, \lambda)$. Let $G = (V, E)$ be a connected graph and $e \in E$ be an arbitrary edge of G. Then

$$P(G, \lambda) = P(G - e, \lambda) - P(G/e, \lambda)$$

Examples. Let us compute the chromatic polynomial of a few small graphs with this technique.

Proof of the recursive formula to compute $P(G, \lambda)$.

Reminder:

$$P(G, \lambda) = P(G - e, \lambda) - P(G/e, \lambda)$$

Let $e = \{u, v\}$.

The number of λ-colorings of $G - e$ in which v and u have different colors:

The number of λ-colorings of $G - e$ in which v and u have the same color:

The total number of λ-colorings of $G - e$ is: