Lecture outline: Homogeneous second-order recurrences

We consider now recurrences of the form

$$a_0 = A,\ \ a_1 = B,\ \ f(a_n, a_{n-1}, a_{n-2}) = g(n)$$

where A and B are given and f is a linear function with constant coefficients.

We will first look at the homogeneous case where $g(n) = 0$ and see that it can be solved quite easily by solving a quadratic equation, although the case of complex roots requires a little more care.

The characteristic equation

Definition. A homogeneous second-order linear recurrence relation with constant coefficients for a sequence $a_k, a_{k+1}, a_{k+2} \ldots$ (often $k = 0$ when we start from a_0) is defined by

- Giving the values of the first 2 terms of the sequence, a_k, a_{k+1} (often a_0, a_1),
- Giving a relation $f(a_{n-2}, a_{n-1}, a_n) = 0$, which is a polynomial of degree 1 in the variables a_{n-2}, a_{n-1}, a_n with constant coefficients.

The given values for a_k and a_{k+1} are called the **initial conditions** of the recurrence.

Example. The most famous example of such recurrence are Fibonacci numbers.

$$\begin{cases} f_0 &= 0 \\ f_1 &= 1 \\ f_n &= f_{n-2} + f_{n-1} \quad n \geq 2 \end{cases}$$

The rule $f_n = f_{n-2} + f_{n-1},\ n \geq 2$ is equivalent to

$$f_n - f_{n-2} - f_{n-1} = 0,\ n \geq 2.$$

We will see how to solve this recurrence in order to find out that

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Idea 1. The solution may be an exponential function.

Take the following recurrence:

$$\begin{cases} a_0 = -1 \\ a_1 = 8 \\ a_n + a_{n-1} - 6a_{n-2} = 0, \ n \geq 2 \end{cases}$$

Assume that $a_n = Cr^n, \ n \geq 0$, where r and C are non-zero real numbers we want to find.

First let's substitute a_n by Cr^n in the relation $a_n + a_{n-1} - 6a_{n-2} = 0$.

So, if our assumption that $a_n = Cr^n, \ n \geq 0$ for some non-zero real r and C, r satisfies $r^2 + r - 6 = 0$.

To find r, we then only need to solve the quadratic equation $r^2 + r - 6 = 0$. This equation is called the **characteristic equation** of the recurrence relation.

This is easy: $r^2 + r - 6 = (r - 2)(r + 3)$, so $r = 2$ or $r = -3$.

Moreover, if $a_n = Cr_n$, by considering $n = 0$ we have $C = a_0 = -1$.

If $r = 2$ and so $a_n = -2^n$, then $a_1 = -2 \neq 8$ so we can not chose $r = 2$.

If $r = -3$ and so $a_n = -(-3)^n$, then $a_1 = 3 \neq 8$ so we can not chose $r = -3$.

So our assumption that $a_n = Cr^n$ is wrong, which is normal in fact as otherwise the sequence $a_0, a_1, a_2 \ldots$ would be defined by a first-order recurrence relation.

We need to explore another avenue.

Idea 2. Solution may be the sum of exponential functions.

Assume now that

$$a_n = X(2)^n + Y(-3)^n,\ n \geq 0$$

i.e. that a_n is defined as the sum of two exponential functions defined by the roots of the characteristic equation.

Let's try to see if this assumption is compatible with both initial conditions.

$$\begin{cases} a_0 &= X + Y \\ a_1 &= 2X - 3Y \end{cases}$$

This is a system of two equations in two unknowns that we can easily solve to obtain that $Y = -2$ and $X = 1$

So if the assumption that $a_n = X(2)^n + Y(-3)^n,\ n \geq 0$ is true, we have in fact

$$a_n = (2)^n + (-2)(-3)^n,\ n \geq 0$$

The characteristic equation.

1. The characteristic equation of a second-order linear homogeneous recurrence relation with constant coefficients defined by the given values of a_0, a_1 and the polynomial

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$$

is the equation

$$C_0 r^2 + C_1 r + C_2 = 0.$$

2. The roots of this quadratic equation are

$$\begin{cases} \alpha &= \frac{-C_1 + \sqrt{C_1^2 - 4C_0C_2}}{2C_0} \\ \beta &= \frac{-C_1 - \sqrt{C_1^2 - 4C_0C_2}}{2C_0} \end{cases}$$

There are three possible cases:

A. [Distinct real roots]
If $C_1^2 - 4C_0C_2 > 0$, then α and β are distinct real numbers.

B. [Distinct complex roots]
If $C_1^2 - 4C_0C_2 < 0$, then α and β are distinct complex numbers.

C. [Double root]
If $C_1^2 - 4C_0C_2 = 0$, then $\alpha = \beta$ and it is a real number.

We will now see how to find a solution for a recurrence using the roots of the characteristic equation. We will need a specific method for each of the three cases described above. We will prove this works later using GF.

Case A. Distinct real roots

Algorithm.
Consider a second-order linear homogeneous recurrence relation with constant coefficients defined by the given values of $a_0,\ a_1$ and the polynomial

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0.$$

Assume that $C_1^2 - 4C_0C_2 > 0$, i.e. the roots α and β of the characteristic equation are distinct real numbers.

Then a_n satisfies

$$a_n = X\alpha^n + Y\beta^n,\ \ n \geq 0$$

for some real numbers X and Y.

The numbers X and Y are the **unique** solution of the system of two equations

$$\begin{cases} X + Y & = & a_0 \\ X\alpha + Y\beta & = & a_1 \end{cases}$$

Remark. The same algorithm applies almost as is if the sequence does not start at a_0 but later: if the given initial condition are the values of a_k and a_{k+1} for some k, then

$$a_n = X\alpha^{n-k} + Y(n-k)\beta^{n-k}n,\ \ n \geq k$$

and X and Y are defined by the system

$$\begin{cases} X & = & a_k \\ X\alpha + Y\alpha & = & a_{k+1} \end{cases}$$

Example. Fibonacci numbers.

Example. Tiling a rectangle.

Usually, we introduce recurrences as a way to solve a counting problem. For example: how many ways are-there to tile a rectangle of dimension $2 \times n$ by tiles that are 1×2 and 1×2 rectangles? Let b_n be such a number for $n \geq 1$.

So here the work to do is to **find** a recurrence relation that defines the sequence $b_1, b_2, \ldots$.

Example. Counting palindromic compositions.

A palindromic composition of size n is an integer composition of n that can be read the same left to right than right to left.

What is the number p_n of palindromic compositions of n?

TO DO. Read Example 10.16, page 461-2 from your textbook.

Case C. Repeated real roots

Let's start with an example.

$$\begin{cases} a_0 = 1 \\ a_1 = 3 \\ a_n - 4a_{n-1} + 4a_{n-2} = 0, \ n \geq 2 \end{cases}$$

The characteristic equation is $r^2 - 4r + 4 = 0$ and has a single solution $\alpha = 2$.

We can not assume that $a_n = X(2)^n, \ n \geq 0$ as $a_0 = 1$ implies that $X = 1$ while $a_1 = 3$ implies that $X = 3/2$.

Here we are going to assume that

$$a_n = X(2)^n + Yn(2)^n, \ n \geq 0$$

This defines the system

$$\begin{cases} X & = & a_0 & = & 1 \\ X\alpha + Y\alpha & = & a_1 & = & 3 \end{cases}$$

that is easy to solve: $X = 1, \ Y = 1/2$.

Algorithm.
Consider a second-order linear homogeneous recurrence relation with constant coefficients defined by the given values of a_0, a_1 and the polynomial

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0.$$

Assume that $C_1^2 - 4C_0C_2 = 0$, i.e. $\alpha = \beta$ is the unique solution of the characteristic equation and is a real number.

Then a_n satisfies

$$a_n = X\alpha^n + Yn\alpha^n, \ n \geq 0$$

for some real numbers X and Y.

The numbers X and Y are the **unique** solution of the system of two equations

$$\left\{ \begin{array}{rcl} X & = & a_0 \\ X\alpha + Y\beta & = & a_1 \end{array} \right.$$

Remark. Again, the same algorithm applies almost as is if the sequence does not start at a_0 but later: if the given initial condition are the values of a_k and a_{k+1} for some k, then

$$a_n = X\alpha^{n-k} + Y(n-k)\beta^{n-k}n, \ n \geq k$$

and X and Y are defined by the system

$$\left\{ \begin{array}{rcl} X & = & a_k \\ X\alpha + Y\alpha & = & a_{k+1} \end{array} \right.$$

Example.

Summary.

Optional: Case B. Distinct complex roots

In this case, we have $C_1^2 - 4C_0C_2 < 0$, so the numbers

$$\begin{cases} \alpha &= \frac{-C_1+\sqrt{C_1^2-4C_0C_2}}{2C_0} \\ \beta &= \frac{-C_1-\sqrt{C_1^2-4C_0C_2}}{2C_0} \end{cases}$$

are complex numbers: $\alpha = a + ib$ and $\beta = a - ib$ for some a and b.

Polar representation of complex numbers. Any complex number $z = x + iy$ can be written as

$$z = r(\cos\theta + i\sin\theta)$$

with

$$r\sqrt{x^2 + y^2}$$

and

$$\tan\theta = \frac{y}{x},\ \ x \neq 0$$

If $x = 0$ and $y > 0$ then

$$z = y(\cos(\pi/2) + i\sin(\pi/2)) = y(\sqrt{2}/2 + i\sqrt{2}/2).$$

If $x = 0$ and $y < 0$ then

$$z = y(\cos(3\pi/2) + i\sin(3\pi/2)) = y(-\sqrt{2}/2 + i\sqrt{2}/2).$$

DeMoivre's theorem.

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta),\ \ n \geq 0$$

So for any complex number z, the number z^n can be written as

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)),\ \text{for some } \theta\ .$$

Now, we can apply this result to the case where

$$\begin{cases} \alpha &= \frac{-C_1+\sqrt{C_1^2-4C_0C_2}}{2C_0} \\ \beta &= \frac{-C_1-\sqrt{C_1^2-4C_0C_2}}{2C_0} \end{cases}$$

We can rewrite this as

$$\begin{cases} \alpha &= \frac{-C_1}{2C_0} + i\frac{\sqrt{-(C_1^2-4C_0C_2)}}{2C_0} \\ \beta &= \frac{-C_1}{2C_0} - i\frac{\sqrt{-(C_1^2-4C_0C_2)}}{2C_0} \end{cases}$$

So we have $\alpha = x + iy$ and $\beta = x - iy$ with $x = -C_1/2C_0$ and $y = \sqrt{-(C_1^2 - 4C_0C_2)}/2C_0$

We can then rewrite them as $\alpha = r(\cos\theta + i\sin\theta)$ and $\beta = r(\cos\theta - i\sin\theta)$

Now if we apply our assumption that

$$a_n = A\alpha^n + B\beta^n,\ \ n \geq 0$$

we obtain that

$$a_n = A(r(\cos\theta + i\sin\theta))^n + B(r(\cos\theta - i\sin\theta))^n,\ \ n \geq 0$$

We can then apply DeMoivre's theorem

$$a_n = Ar^n(\cos(n\theta) + i(\sin(n\theta))) + B(r^n(\cos(n\theta) - i\sin(n\theta))),\ \ n \geq 0$$

We can factor r^n

$$a_n = r^n\left((A+B)\cos(n\theta) + i(A-B)(\sin(n\theta))\right),\ \ n \geq 0$$

which we can rewrite as

$$a_n = r^n\left(X\cos(n\theta) + Y(\sin(n\theta))\right),\ \ n \geq 0$$

with $X = A + B$ and $Y = i(A - B)$.

By considering the cases $n = 0, 1$ we obtain

$$\begin{cases} a_0 &= X \\ a_1 &= r(X\cos\theta + Y\sin\theta) \end{cases}$$

a system that can be easily solved as it is again a system of two equations with two unknowns.

Algorithm.
Consider a second-order linear homogeneous recurrence relation with constant coefficients defined by the given values of a_0, a_1 and the polynomial

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0.$$

Assume that $C_1^2 - 4C_0C_2 < 0$, i.e. the roots α and β of the characteristic equation are distinct complex numbers.

Let r and θ be such that

$$\begin{cases} \alpha & = & r(\cos\theta + i\sin\theta) \\ \beta & = & r(\cos\theta - i\sin\theta) \end{cases}$$

Then a_n satisfies

$$a_n = Xr^n(\cos(n\theta) + Yr^n(\sin(n\theta)), \; n \geq 0$$

for some real numbers X and Y.

The numbers X and Y are the **unique** solution of the system of two equations

$$\begin{cases} X & = & a_0 \\ r(X\cos\theta + Y\sin\theta) & = & a_1 \end{cases}$$

Remark. Again, the same algorithm applies almost as is if the sequence does not start at a_0 but later: if the given initial condition are the values of a_k and a_{k+1} for some k, then '

$$a_n = Xr^{n-k}(\cos((n-k)\theta) + Yr^{n-k}(\sin((n-k)\theta)), \; n \geq k$$

and X and Y are defined by the system

$$\begin{cases} X & = & a_k \\ r(X\cos\theta + Y\sin\theta) & = & a_{k+1} \end{cases}$$

Example.