Generating Functions 3 - Extraction & Recurrences

We continue to work with rational generating functions $A(x)$, so we have

$$A(x) = \frac{p(x)}{q(x)} = \sum_{n=0}^{\infty} a_n x^n$$

Our primary interest will be in going back and forth between expressing $A(x)$ as a quotient of polynomials and in terms of its coefficients.

Coefficient Extraction

Recall: For a generating function $A(x)$ we let $[x^k]A(x)$ denote the coefficient of x^k in $A(x)$.

Coefficient extraction is the process of determining the coefficients of a generating function. The key to coefficient extraction for rational generating functions is **partial fractions**

Example: Find values for A, B, C so that the expression below is true, then use this to determine $[x^n]D(x)$

$$D(x) = \frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

mult by $(x-3)(x-2)^2$

$$\boxed{1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)}$$

3 unknowns
3 eqns

$$= Ax^2 - 4Ax + 4A + Bx^2 - 5Bx + 6B + Cx - 3C$$

$$1 = \underbrace{(A+B)}_{=0}x^2 + \underbrace{(-4A-5B+C)}_{=0}x + \underbrace{(4A+6B-3C)}_{=1}$$

$C = -1 \quad A = 1 \quad B = -1$

$$D(x) = \frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} + \frac{-1}{x-2} + \frac{-1}{(x-2)^2}$$

$$D(x) = \frac{1}{x-3} + \frac{-1}{x-2} + \frac{-1}{(x-2)^2}$$

$$= \left(-\frac{1}{3}\right)\frac{1}{1-\frac{x}{3}} + \left(\frac{1}{2}\right)\frac{1}{1-\frac{x}{2}} + \left(-\frac{1}{4}\right)\frac{1}{\left(1-\frac{x}{2}\right)^2}$$

$$= \left(-\frac{1}{3}\right)\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n x^n + \left(\frac{1}{2}\right)\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n x^n + \left(-\frac{1}{4}\right)\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n (n+1)x^n$$

$$= \sum_{n=0}^{\infty}\left(\left(-\frac{1}{3}\right)\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n + \left(-\frac{1}{4}\right)\left(\frac{1}{2}\right)^n(n+1)\right)x^n$$

This process works in general.

Partial Fractions. Let $q(x)$ be a polynomial which can be factored as $q(x) = (x-r_1)^{d_1}(x-r_2)^{d_2}\dots(x-r_k)^{d_k}$ then there exist constants so that

$$\frac{1}{q(x)} = \frac{A_{1,1}}{x-r_1} + \frac{A_{1,2}}{(x-r_1)^2} + \dots \frac{A_{1,d_1}}{(x-r_1)^{d_1}} + \dots\dots + \dots \frac{A_{k,d_k}}{(x-r_k)^{d_k}}$$

Example: Although we will not solve it, there exist constants A, B, C, D, E, F so that the following expression is valid:

$$\frac{1}{(x-1)(x-2)^2(x-3)^3} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} + \frac{D}{(x-3)} + \frac{E}{(x-3)^2} + \frac{F}{(x-3)^3}$$

Note: We can use this together with the formula

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

To do coefficient extraction whenever we have a rational function and we have factored the denominator.

Recurrences

Generating functions also provide a natural setting to work with and solve recurrence relations.

Example: Consider the recurrence relation

$$a_0 = 0 \qquad \text{and} \qquad a_1 = 1$$

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \quad \text{for } n \geq 2$$

Define the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ What is

$$A(x) - 5xA(x) + 6x^2A(x)?$$

$n \geq 2$

$$[x^n]\left(A(x) - 5xA(x) + 6x^2A(x)\right) = [x^n]A(x) - 5\underbrace{[x^n]xA(x)}_{[x^{n-1}]A(x)} + 6\underbrace{[x^n]x^2A(x)}_{[x^{n-2}]A(x)}$$

$$= a_n - 5a_{n-1} + 6a_{n-2} = 0$$

$$[x]\left(\quad\right) = [x]A(x) - 5[x^0]A(x) = 1$$

$$[x^0]\left(\text{"}\right) = [x^0]A(x) = 0$$

$$\boxed{A(x) - 5xA(x) + 6x^2A(x) = x}$$

Use the above expression to express $A(x)$ as a rational function

$$A(x)\left(1 - 5x + 6x^2\right) = x$$

$$A(x) = \frac{x}{1 - 5x + 6x^2}$$

to extract coeff we use part. frac.

Use coefficient extraction to find the coefficients of $A(x)$.

$$A(x) = \frac{x}{1-5x+6x^2} = x\left(\frac{1}{1-5x+6x^2}\right)$$

Factor

$1-5x+6x^2$

$=(1-2x)(1-3x)$

$$= x\left(\frac{B}{1-2x} + \frac{C}{1-3x}\right)$$

$$\frac{1}{1-5x+6x^2} = \frac{B}{1-2x} + \frac{C}{1-3x}$$

$$1 = B(1-3x) + C(1-2x)$$
$$= (B+C) + (-3B-2C)x$$

$B+C=1$

$-3B-2C=0$

$-B=2$

$B=-2 \quad C=3$

$$A(x) = x\left(\frac{-2}{1-2x} + \frac{3}{1-3x}\right)$$

$$= -2x\sum_{n=0}^{\infty} 2^n x^n + 3x\sum_{n=0}^{\infty} 3^n x^n$$

$$= \sum_{n=0}^{\infty}\left(-2\cdot 2^n + 3\cdot 3^n\right)x^{n+1}$$

$m = n+1$

$$= \sum_{n=0}^{\infty}\left(-2^{n+1} + 3^{n+1}\right)x^{n+1}$$

$$\sum_{m=1}^{\infty}\left(-2^m + 3^m\right)x^m$$