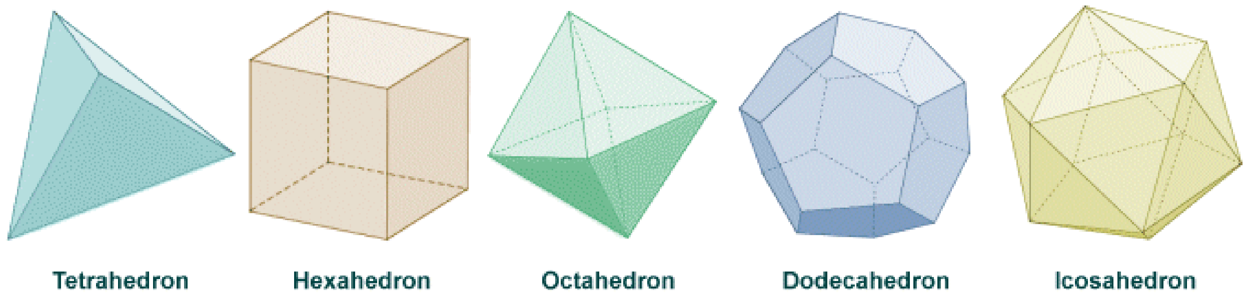
Graphs 3 - Planarity

Definition. A graph G is **planar** if G can be drawn in the plane so that the edges intersect only at the endpoints. Such a drawing is called an **embedding** of G in the plane.

Examples.

Platonic Solids



$K_{3,3}$ and K_5

Property. If G is planar, then every subgraph of G is planar.

Definition. Let $e = \{v_1, v_2\}$ be an edge of G. To **subdivide** e is to delete the edge e, add a new vertex w and then add the edges $\{v_1, w\}$ and $\{v_2, w\}$.

Example.

Definition. We say that a graph H is a **subdivision** of G if H can be obtained from G by a sequence of edge subdivisions.

Example.

Theorem. (Kuratowski-Wagner) A graph G is planar if and only if G does not contain a subdivision of $K_{3,3}$ or a subdivision of K_5.

Definition. Let G be a planar graph embedded in the plane. The embedding breaks the plane into connected regions called **faces**. There is one unbounded face called the **infinite face**, all other faces are **internal faces**.

Example.

Definition. Each face in an embedding of a graph in the plane has vertices and edges on its boundary. They form a closed walk called a **facial walk**

Example.

Theorem (Euler's Formula) If $G = (V, E)$ is a connected graph embedded in the plane and F is the set of faces, then

$$|V| - |E| + |F| = 2.$$

Note: This means that if G is planar, then all embeddings of G in the plane have the same number of faces.

Proof.