

Lecture outline: Inclusion-Exclusion Principle

Inclusion/exclusion is a relatively simple counting technique that allows to count the number of elements of a set that satisfy none from a list of given properties.

It is a very general technique, that has applications in many fields, from theoretical ones (computing the permanent of a matrix) to applied ones (estimating the reliability of a communication network).

The precise topics we will study are:

- The Inclusion/Exclusion: motivation, notation, statement, proof, examples.
- The Inclusion/Exclusion: proof and generalization
- The example of derangements.

Introductory example We want to count the number of integers in $[1, 30]$ that are not divisible by 2 and not divisible by 3 and not divisible by 5.

Think a little bit about it and it is easy to find there are 8 such numbers: $\{1, 7, 11, 13, 17, 19, 23, 29\}$.

What if we had started by subtracting from 30 the number of integers divisible by 2, the number of integers divisible by 3 and finally the number of integers divisible by 5:

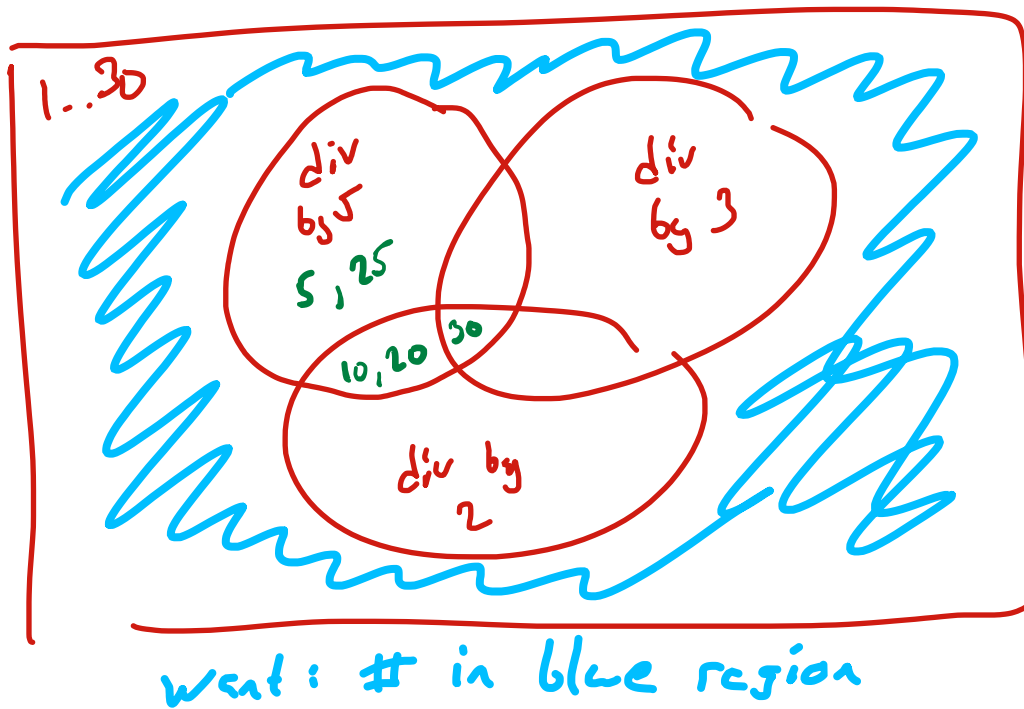
- there are 15 numbers divisible by 2
- there are 10 numbers divisible by 3
- there are 6 numbers divisible by 5

This would give $30 - (15 + 10 + 6) = -1$, which is certainly **not** the correct answer.

What happened is that we **over-counted** (*i.e.* considered more than once) some numbers are divisible by two of $\{2, 3, 5\}$: $\{6, 12, 18, 24, 30\}$ for 2 and 3, $\{10, 20, 30\}$ for 2 and 5, $\{15, 30\}$ for 3 and 5, and we removed each of them twice, so we need to add $5 + 3 + 2 = 10$ to our result, which gives $-1 + 10 = 9$ which is wrong again.

Indeed we counted 30 twice, as it is divisible by 2 and 3 and 5, again some over-counting, that we need to correct by removing 1, to obtain $9 - 1 = 8$ which is correct.

Inclusion-exclusion is only a formalisation of this principle to correct over-counting, which is better visualized on a **Venn Diagram**.



Principle of Inclusion-Exclusion

1. Let \mathcal{S} be a set of N elements (often for us combinatorial objects): $|\mathcal{S}| = N$.

2. Let

$$C_1, \dots, C_t$$

be t properties that the elements of this set could satisfy.

3. We denote the complement of a property C_i by $\overline{C_i}$: an element of \mathcal{S} satisfies property $\overline{C_i}$ if it does not satisfy property C_i .

4. For a given subset $\{i_1, \dots, i_k\}$ of $[t]$, we denote by

$$N(C_{i_1} C_{i_2} \cdots C_{i_k})$$

the number of elements of \mathcal{S} that satisfy **all properties** $C_{i_1}, C_{i_2}, \dots, C_{i_k}$.

5. We denote by \overline{N} the number of elements of \mathcal{S} that satisfy **none of the properties** C_1, C_2, \dots, C_t , i.e.

$$\overline{N} = N(\overline{C_1} \overline{C_2} \cdots \overline{C_t})$$

i.e. the set of all elements

6. For $0 \leq k \leq t$, we define S_k as follows:

$$S_0 = N$$

$$S_k = \sum_{i_1 < i_2 < \cdots < i_k} N(C_{i_1} C_{i_2} \cdots C_{i_k})$$

Be careful: S_k is **not** the number of elements from \mathcal{S} that satisfy **at least** k properties as the elements that satisfy more than k properties are counted more than once.

Inclusion-Exclusion Theorem.

$$\overline{N} = \sum_{k=0}^t (-1)^k S_k$$

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Illustration with our introductory example.

$\mathcal{S} = \{1, 2, \dots, 30\}$, $S_0 = N = 30$ *tot # elts*

C_1 is the property "is divisible by 2"

C_2 is the property "is divisible by 3"

C_3 is the property "is divisible by 5"

Using basic arithmetic we obtain:

$$N(C_1) = \lfloor 30/2 \rfloor = 15$$

$$N(C_2) = \lfloor 30/3 \rfloor = 10$$

$$N(C_3) = \lfloor 30/5 \rfloor = 6$$

$$S_1 = 31$$

$$N(C_1, C_2) = \lfloor 30/6 \rfloor = 5$$

$$N(C_1, C_3) = \lfloor 30/10 \rfloor = 3$$

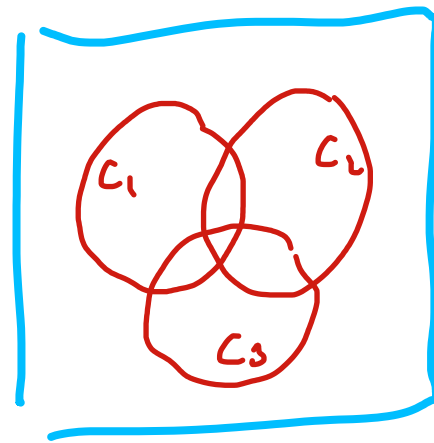
$$N(C_2, C_3) = \lfloor 30/15 \rfloor = 2$$

$$S_2 = 10$$

$$S_3 = N(C_1, C_2, C_3) = \lfloor 30/30 \rfloor = 1$$

So, applying the Inclusion-Exclusion Theorem, we obtain very quickly:

$$\overline{N} = S_0 - S_1 + S_2 - S_3 = 30 - 31 + 10 - 1 = 8$$



Example: rearranging a string with forbidden substrings

How many permutations of the letters of

CTAGCGAAAT

contain neither of the substrings *CAC*, *GAG*, *TAAT* ?

1. \mathcal{S} is the set of all strings with content composed of four symbols *A*, two symbol *C*, two symbols *G* and two symbol *T*. From our study of counting sequences in week 1, we know that

$$N = |\mathcal{S}| = \frac{10!}{4!2!2!2!} =$$

2. We consider three properties of the strings in \mathcal{S} :

C_1 : the string contains *CAC*

C_2 : the string contains *GAG*

C_3 : the string contains *TAAT*

And we want to compute $\overline{N} = N(\overline{C}_1 \overline{C}_2 \overline{C}_3)$

3. $S_0 = N$ is known.

$S_1 = N(C_1) + N(C_2) + N(C_3)$, so we will compute each of the three terms:

$N(C_1)$ is the number of strings that contain the substring *CAC*. So let's consider *CAC* as a single symbol *X*, remove its letters from the pool of symbols and count the number of strings containing *X, A, A, A, G, G, T, T*:

$$N(C_1) = \frac{8!}{1!3!2!2!} =$$

The same idea applies to *GAG* and gives the same result: $N(C_2) = N(C_1) =$).

For C_3 we have

$$N(C_3) = \frac{7!}{1!2!2!2!} =$$

We now turn to $S_2 = N(C_1C_2) + N(C_1C_3) + N(C_2C_3)$.

$N(C_1C_2)$ is the number of strings of content $\{X, Y, A, A, T, T\}$ where X stands for CAC and Y for GAG . So

$$N(C_1C_2) = \frac{6!}{1!1!2!2!}$$

$N(C_1C_3)$ is the number of strings of content $\{X, Y, A, G, G\}$ where X stands for CAC and Y for $TAAT$. So

$$N(C_1C_3) = \frac{5!}{1!1!1!2!}$$

It is obvious that $N(C_1C_3) = N(C_2C_3)$.

Finally, $S_3 = N(C_1C_2C_3)$, is the number of strings that contain the substrings CAC , GAG and $TAAT$. This is the number of strings of $\{X, Y, Z\}$ where X stands for CAC and Y for GAG and Z for $TAAT$, which is $3!$.

So, putting all together, we have that

$$\bar{N} = \frac{10!}{4!2!2!2!} - \left(2 \frac{8!}{1!3!2!2!} + \frac{7!}{1!2!2!2!} \right) + \left(\frac{6!}{1!1!2!2!} + 2 \frac{5!}{1!1!1!2!} \right) - 3!$$

Important remark. The argument above works because the three patterns CAC , GAG and $TAAT$ do not overlap and any forbidden pattern can occur at most once. If we had patterns CAC , GCA and $TAAT$ for example, then we could not simply consider each pattern as a symbol, as a string of the letters that contains the first two patterns could be $GCAC TATAAG$ for example and the calculation would be quite messy.

Example: constrained solutions to an equation

Find the number of solutions of the equation:

$$x_1 + x_2 + x_3 + x_4 = 18$$

such that

$$0 \leq x_i \leq 7, \quad i = 1, 2, 3, 4.$$

First we need to define \mathcal{S} : \mathcal{S} is the set of all **non-negative integer solutions** to $x_1 + x_2 + x_3 + x_4 = 18$, with no constraints on the x_i s.

universe: all non-neg sol to $x_1 + \dots + x_4 = 18$

$$N = \binom{18+3}{3}$$

"binary strings with
18 1's 3 0's"

Next we need to define properties on the elements of \mathcal{S} : C_i is the property that the value of x_i is greater or equal to 8: $x_i \geq 8$.

So now, we have defined all we need to apply the Inclusion/Exclusion, and we want to compute

$$\overline{N} = N(\overline{C}_1 \overline{C}_2 \overline{C}_3 \overline{C}_4).$$

$$N(C_1) = \text{all sol to } x_1 + \dots + x_4 = 18$$

$$\text{where } x_1 \geq 8$$

$$\text{let } y_1 = x_1 - 8$$

$$N(C_1) : \text{all nonneg sol to}$$

$$y_1 + x_2 + x_3 + x_4 = 10$$

same as
 $N(C_2) \quad N(C_3) \quad N(C_4)$

$$= \binom{10+3}{3}$$

$$N(C_1, C_2) = \text{all sol to } x_1 + \dots + x_4 = 18$$

$$\text{where } x_1, x_2 \geq 8$$

$$y_1 = x_1 - 8 \quad y_2 = x_2 - 8$$

$$= \text{all nonneg sol to } y_1 + y_2 + x_3 + x_4 = 2$$

$$= \binom{2+3}{3}$$

$$\bar{N} = N - S_1 + S_2 = \binom{21}{3} - 4 \binom{13}{3} + 6 \binom{5}{3}$$

Summary

When should Inclusion/Exclusion be considered to solve a counting problem?

When we want to count the size of the subset R of a larger set \mathcal{S} composed of all elements of \mathcal{S} that do **NOT** satisfy a list of properties.

To apply the technique of Inclusion/Exclusion to a counting problem, we then need to do the following:

1. Define what is \mathcal{S} , and what is its size.
2. Define what are C_1, \dots, C_t , the properties to avoid.
3. Compute the size of each S_k (this is the hard part).
4. Apply the Inclusion/Exclusion Theorem.

Derangements

Definition. A derangement p of $[n]$ is a permutation of $[n]$ of length n in which $p[i] \neq i$ for all $i \in [n]$.

The question we are interested in is the following: for a given n , how many derangements of n are there? We denote this number by d_n .

Which permutations of $\{1, 2, 3\}$ are derangements?

~~1 2 3~~

~~1 3 2~~

~~2 1 3~~

2 3 1 ←

3 1 2 ←

~~3 2 1~~

We want to count the number of derangements by Inclusion/Exclusion.

1. The set \mathcal{S} is the set of all permutations of $\{1, 2, \dots, n\}$: $|\mathcal{S}| = n!$
2. We define n properties C_1, \dots, C_n as follows: C_i is the property that a permutation P satisfies $P[i] = i$.
3. Counting the numbers S_0, S_1, \dots, S_n is the hard part.

Claim. $S_k = n!/k!$ (to be proved on the next slide).

4. Apply Inclusion/Exclusion.

Theorem.

$$d_n = \sum_{k=0}^n (-1)^k \frac{n!}{k!} \simeq \frac{n!}{e}$$

Claim. $S_k = n!/k!$

by definition, S_k is the sum over all possible sets of k conditions, say

$C_{i_1}, C_{i_2}, \dots, C_{i_k}$ of $N(C_{i_1} \dots C_{i_k})$.

Now $N(C_{i_1} \dots C_{i_k})$ is the number of permutations p satisfying

$$\begin{array}{l} p(i_1) = i_1 \\ \vdots \\ p(i_k) = i_k \end{array}$$

to choose such a permutation, we choose a permutation of the remaining $n-k$ terms

So there are $(n-k)!$ such permutations p .

There are n conditions, so there are $\binom{n}{k}$ ways to choose k of them. Thus

$$S_k = \binom{n}{k} (n-k)! = \frac{n!}{k! (n-k)!} (n-k)! = \frac{n!}{k!}$$

Proof of the Inclusion-Exclusion Theorem

Why is

$$\overline{N} = \sum_{k=0}^t (-1)^k S_k$$

true?

Take an arbitrary element r of \mathcal{S} . As

$$S_k = \sum_{\{i_1, \dots, i_k\} \subseteq [t]} N(C_{i_1} C_{i_2} \cdots C_{i_k})$$

we are going to say that “ S_k counts r p times” if there are exactly p subsets $\{i_1, \dots, i_k\} \subseteq [t]$ such that r satisfies all of the properties $C_{i_1}, C_{i_2}, \dots, C_{i_k}$. We denote this by

$$N(r, k) = p$$

with the convention that

$$N(r, 0) = 1$$

So we want to show that, for every $r \in \mathcal{S}$:

- if r does not satisfy any of the properties C_i , $\sum_{k=0}^t (-1)^k N(r, k) = 1$
- if r satisfies at least one of the properties C_i , $\sum_{k=0}^t (-1)^k N(r, k) = 0$

The first point is immediate: if r does not satisfy any of the properties C_i , then $N(r, k) = 0$ for $k > 0$, so r is counted only by S_0 and $\sum_{k=0}^t (-1)^k N(r, k) = N(r, 0) = 1$.

For the second point, assume that r satisfies exactly c properties, $c > 0$.

$$N(r, 0) = 1$$

$$N(r, 1) = c$$

$$N(r, 2) = \binom{c}{2}$$

$$N(r, 3) = \binom{c}{3}$$

and in general

$$N(r, k) = \binom{c}{k}, \quad k \geq 0$$

So we can apply the Binomial Theorem:

$$\sum_{k=0}^c (-1)^k N(r, k) = \sum_{k=0}^c \binom{c}{k} (-1)^k (1)^{c-k} = (1 - 1)^c = 0$$

Moreover, as

$$N(r, k) = 0, \quad k > c$$

$$\sum_{k=0}^t (-1)^k N(r, k) = \sum_{k=0}^c (-1)^k N(r, k) = 0$$

The Inclusion/Exclusion theorem: A generalization

Now we are interested in a more general question:

How many elements of \mathcal{S} satisfy **exactly** m properties among C_1, C_2, \dots, C_t ?

Let's denote by E_m the number of elements of \mathcal{S} that satisfy exactly m properties.

Theorem.

$$E_m = \sum_{k=0}^{t-m} (-1)^k \binom{m+k}{k} S_{m+k}$$

If now we are interested in counting the number L_m of elements of \mathcal{S} that satisfy **at least** m properties among C_1, C_2, \dots, C_t , we have.

Corollary.

$$L_m = \sum_{k=0}^{t-m} (-1)^k \binom{m+k-1}{m-1} S_{m+k}$$

We will not see the proofs of these results, as they are following very closely the principles of the proof of the Inclusion-Exclusion Theorem, and are only slightly more technical in terms of manipulation of binomial numbers.