# MACM 201 Homework 6 (Quiz Oct. 24)

**Textbook problems:**

| Section | Question |
| --- | --- |
| 9.2 | 1 |
| 9.2 | 2 |
| 9.2 | 4 |
| 9.2 | 6 |
| 9.2 | 8 |
| 10.4 | 1 |

**Instructor question(s):**

1. Define the generating functions $B(x) = \sum_{n=0}^{\infty} 2^n x^n$ and $F(x) = \sum_{n=0}^{\infty} f_n x^n$ where $f_n$ is the Fibonacci sequence determined by the recurrence relation

$$f_0 = 0 \qquad \text{and} \qquad f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2$$

Find the coefficients of the first four terms (constant up to $x^3$) of each GF

(a) $F(x) + B(x)$

(b) $F(x) \times B(x)$

(c) $F(x) \times F(x) \times F(x)$

2. In each problem below you are given an infinite sequence $b_0, b_1, b_2, \ldots$ determined by a recurrence relation. Use this recurrence relation to express the GF for this sequence $B(x) = \sum_{n=0}^{\infty} b_n x^n$ as a rational function.

(a) $b_0 = 2$, $b_1 = 3$, $b_n - 3b_{n-1} + 7b_{n-2} = 0$ for $n \geq 2$

(b) $b_0 = 1$, $b_1 = 2$, $b_n - 5b_{n-1} + 3b_{n-2} = 1$ for $n \geq 2$

(c) $b_0 = 1$, $b_1 = 0$, $b_2 = 3$, $b_n - 2b_{n-1} + b_{n-3} = n$ for $n \geq 3$

3. The goal in this problem is to determine the generating function $W(x)$ for the counting sequence of the family $\mathcal{W}$ of binary strings with the special property that every 1 is followed by an odd number of 0's. (So, if $W(x) = \sum_{n=0}^{\infty} w_n x^n$, then $w_n$ is the number of strings in $\mathcal{W}$ of length $n$).

   (a) Show that the generating function $W_{s,1}$ for the family $\mathcal{W}_{s,1}$ of strings composed of a single 1 followed by an odd number of zeros satisfies

   $$W_{s,1} = \frac{x^2}{1-x^2}.$$

   (b) Show that the generating function $W_s$ for the family $\mathcal{W}_s$ of strings in $\mathcal{W}$ that start with a 1 satisfies

   $$W_s = 1 + W_{s,1} + W_{s_1}^2 + W_{s,1}^3 + \dots$$

   (c) Use the rational expression for $1 + x + x^2 + \dots$ together with (a) and (b) and an infinite substitution to show that

   $$W_s(x) = \frac{1-x^2}{1-2x^2}.$$

   (d) Use (c) to show that

   $$W(x) = \frac{1+x}{1-2x}.$$