MACM 201 Practice Set 1 Solutions — September 28, 2018

Section 10.1

4. $a_{n+1} = a_n + 2.5a_n,\ n \geq 0.$
$a_n = (3.5)^n a_0 = (3.5)^n(1000)$. For $n = 12$, $a_n = (3.5)^{12}(1000) \doteq 3,379,220,508.$

Section 10.2

4. $a_n = a_{n-1} + a_{n-2},\ n \geq 2,\ a_0 = a_1 = 1$
$r^2 - r - 1 = 0,\ r = (1 \pm \sqrt{5})/2$
$a_n = A((1+\sqrt{5})/2)^n + B((1-\sqrt{5})/2)^n$
$a_0 = a_1 = 1 \Longrightarrow A = (1+\sqrt{5})/2\sqrt{5},\ B = (\sqrt{5}-1)/2\sqrt{5}$
$a_n = (1/\sqrt{5})[((1+\sqrt{5})/2)^{n+1} - ((1-\sqrt{5})/2)^{n+1}]$

6. For all three parts, let b_n, $n \geq 0$, count the number of ways to fill the n spaces under the condition(s) specified – including the condition allowing empty spaces.
(a) Here $b_0 = 1$, $b_1 = 3$, and $b_n = 3b_{n-1} + b_{n-2}$, $n \geq 2$. This recurrence relation leads us to the characteristic equation $r^2 - 3r - 1 = 0$, and the characteristic roots $r = (3 \pm \sqrt{13})/2$. Consequently, $b_n = c_1[(3+\sqrt{13})/2]^n + c_2[(3-\sqrt{13})/2]^n$, $n \geq 0$. From $1 = b_0 = c_1 + c_2$ and $3 = b_1 = c_1[(3+\sqrt{13})/2] + c_2[(3-\sqrt{13})/2]$, we find that $c_1 = (3+\sqrt{13})/2\sqrt{13}$ and $c_2 = (-3+\sqrt{13})/2\sqrt{13}$. So $b_n = (1/\sqrt{13})[(3+\sqrt{13})/2]^{n+1} - (1/\sqrt{13})[(3-\sqrt{13})/2]^{n+1}$, $n \geq 0$.
(b) For this part we have $b_n = 2b_{n-1} + 3b_{n-2}$, $n \geq 0$, $b_0 = 1$, $b_1 = 2$. Here the characteristic equation is $r^2 - 2r - 3 = 0$ and the characteristic roots are $r = 3$, $r = -1$. Therefore, $b_n = c_1(3^n) + c_2(-1)^n$, $n \geq 0$. From $1 = b_0 = c_1 + c_2$ and $2 = b_1 = 3c_1 - c_2$, we find that $c_1 = 3/4$, $c_2 = 1/4$. So $b_n = (3/4)(3^n) + (1/4)(-1)^n$, $n \geq 0$.
(c) Here $b_0 = 1$, $b_1 = 3$, and $b_n = 3b_{n-1} + 3b_{n-2}$, $n \geq 2$. The characteristic equation $r^2 = 3r + 3$ gives us the characteristic roots $r = (3 \pm \sqrt{21})/2$. So $b_n = c_1[(3+\sqrt{21})/2]^n + c_2[(3-\sqrt{21})/2]^n$, $n \geq 0$. From $1 = b_0 = c_1 + c_2$ and $3 = b_1 = c_1[(3+\sqrt{21})/2] + c_2[(3-\sqrt{21})/2]$, we have $c_1 = [(3+\sqrt{21})/2\sqrt{21}]$ and $c_2 = [(-3+\sqrt{21})/2\sqrt{21}]$. Consequently, $b_n = (1/\sqrt{21})[((3+\sqrt{21})/2)^{n+1} - ((3-\sqrt{21})/2)^{n+1}]$, $n \geq 0$.

10. Here $a_1 = 1$ and $a_2 = 1$. For $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$, because the strings counted by a_n either end in 1 (and there are a_{n-1} such strings) or they end in 00 (and there are a_{n-2} such strings).

Consequently, $a_n = F_n$, the nth Fibonacci number, for $n \geq 1$.

24. Here $a_1 = 1$, for the case of one vertical domino, and $a_2 = 3$ – use (i) one square tile ; or (ii) two horizontal dominoes; or (iii) two vertical dominoes. For $n \geq 3$ consider the nth column of the chessboard. This column can be covered by
(1) one vertical domino – this accounts for a_{n-1} of the tilings of the $2 \times n$ chessboard;
(2) the right squares of two horizontal dominoes placed in the four squares for the $(n-1)$st and nth columns of the chessboard – this accounts for a_{n-2} of the tilings; and
(3) the right column of a square tile placed on the four squares for the $(n-1)$st and nth columns of the chessboard – this also accounts for a_{n-2} of the tilings.

These three cases account for all the possible tilings and no two cases have anything in common so

$$a_n = a_{n-1} + 2a_{n-2}, \quad n \geq 3, a_1 = 1, a_2 = 3.$$

Here the characteristic equation is $x^2 - x - 2 = 0$ which gives $x = 2$, $x = -1$ as the characteristic roots. Consequently, $a_n = c_1(-1)^n + c_2(2)^2$, $n \geq 1$. From $1 = a_1 = c_1(-1) +$

$c_2(2)$ and $3 = a_2 = c_1(-1)^2 + c_2(2)^2$ we learn that $c_1 = 1/3$, $c_2 = 2/3$. So $a_n = (1/3)[2^{n+1} + (-1)^n]$, $n \geq 1$. [The sequence $1, 3, 5, 11, 21, \ldots$, described here, is known as the *Jacobsthal* sequence.]