Graphs 1 - Degrees, Connectivity, and Proof

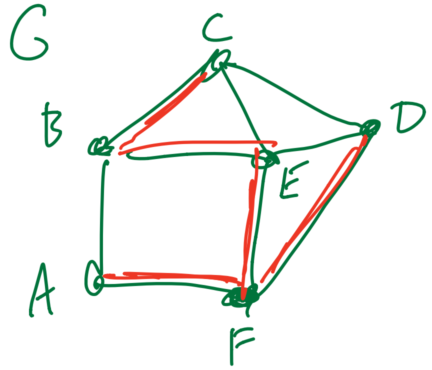
We will continue to explore basic graph theory, but now with an eye toward formally proving theorems.

Definition. If $G = (V, E)$ is a graph, the **degree** of a vertex v is the number of edges incident with v. We denote the degree by

$$\deg_G(v)$$

When G is clear from context, we drop the subscript and write $\deg(v)$.

Example:



$\deg_G(A) = 2$

$\deg_G(E) = 4$

T a span tree of G

$\deg_T(A) = 1$

$\deg_T(E) = 2$

Theorem. For every graph $G = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof 1: let N be the number of pairs (v, e) when $v \in V$ $e \in E$ and e and v are incident

$$N = \sum_{v \in V} \#\text{edges } e \text{ s.t. } e \text{ inc to } v$$

$$= \sum_{v \in V} \deg(v)$$

$$N = \sum_{e \in E} \#\text{vert } v \text{ s.t. } e \text{ inc to } v$$

$$= \sum_{e \in E} 2 = 2|E|$$

$$\sum_{v \in V} \deg(v) = N = 2|E|$$

$1=1^2$ $1+3=2^2$ $1+3+5=9=3^2$
$1+3+5+7=16=4^2$

Proofs by Induction

Let's begin by considering an easy proof by induction.

Theorem A. For every positive integer n, the sum of the first n odd numbers is n^2. So

$$\boxed{\sum_{k=1}^{n}(2k-1)=n^2} \tag{1}$$

To prove this by induction, we imagine giving the proof one number at a time. We first prove it for 1, then 2, 3, 4, This verification splits into two parts:

Base case. Check that the theorem holds for $n=1$

Inductive step. Prove that the formula holds for n under the assumption that it holds for $n-1$.

As you can see, if we can prove the base case and the inductive step, then the statement must hold true for every positive integer.

Let's see how to write out the proof properly.

Proof of Theorem A. We prove this theorem by induction on n.

Base Case. When $n=1$ equation (1) holds because $\sum_{k=1}^{1}(2k-1)=1=1^2$.

Inductive Step. Let $n \geq 2$ and assume that equation (1) is true for all numbers less than n. (We will prove it must then be true for n.) We have

LHS of eqn for n

sum of first n-1 odd int.

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1) &= (2n-1)+\boxed{\sum_{k=1}^{n-1}(2k-1)} \\
&= (2n-1)+(n-1)^2 \qquad \text{by the inductive hypothesis} \\
&= (2n-1)+n^2-2n+1 \\
&= n^2
\end{aligned}$$

Induction can be applied in *much* more general settings. Suppose that you wish to prove that a certain property P holds for an infinite number of things. Here is an inductive way to achieve this:

Size. Suppose that each thing has a certain size. Assume (for simplicity) that the smallest possible size is 1. Instead of trying to prove property P holds for all things at once, we can break things up according to size. Define

$$P(n): \text{ property } P \text{ holds for all things of size } n$$

To prove that P holds by induction, it suffices to show the following:

Base Case. $P(1)$ is true (i.e. property P holds for all things of size 1)

Inductive Step. If $n \geq 2$ and $P(1), P(2), \ldots, P(n-1)$ are true, then $P(n)$ is true. (i.e. if P is true for all things of size $< n$ then P is also true for the things of size n.)

Again it is not difficult to see that if both the base case and inductive step, then P must be true for every thing.

Notes:

- The smallest possible size might be 0 or 2 or something else.
- In general there may be **many** things of size n.
- It is possible that the base case needs to handle more than just the smallest size. In general the base case must take care of all of the instances that are not covered by the inductive step.

Theorem. For every graph $G = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof 2: We proceed by induction on $|E|$

base case $|E| = 0$

$\deg(v) = 0$ for every $v \in V$ so $\sum_{v \in V} \deg(v) = 0 = 2|E|$ ✓

induction step $|E| = m \geq 1$

Since $E \neq \emptyset$ we may choose $e \in E$. Assume $e = \{u, u'\}$

Let G' be the graph obtained from G by deleting e

(show formula holds for G assuming it holds for G')

$$\deg_{G'}(w) = \begin{cases} \deg_G(w) & \text{if } w \neq u, u' \\ \deg_G(w) - 1 & \text{if } w = u \text{ or } w = u' \end{cases}$$

$$\sum_{v \in V} \deg_G(v) = \left(\sum_{v \in V} \deg_{G'}(v)\right) + 2$$

by induction

$$= 2|E(G')| + 2$$

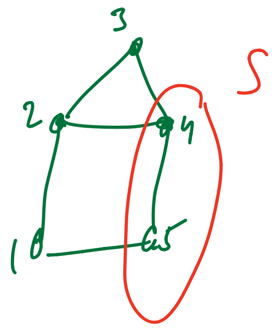
$$= 2|E(G)| \quad ✓$$

Deletion

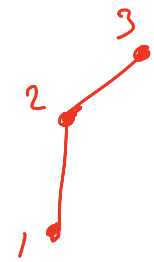
Definition. If $G = (V, E)$ is a graph and $S \subset V$, we define $G - S = G[V \setminus S]$. If $v \in V$ we define $G - v = G - \{v\}$.

Example.

$G =$



$G - S =$



Theorem. Every connected graph $G = (V, E)$ with $|V| \geq 2$ has at least two vertices v_1, v_2 so that $G - v_1$ and $G - v_2$ are both connected.

Proof. Proceed by induction on $|V|$

base case $|V| = 2$

Since G is connected it is isomorphic to K_2

So $G-v$ is connected for both vertices $v \in V$.

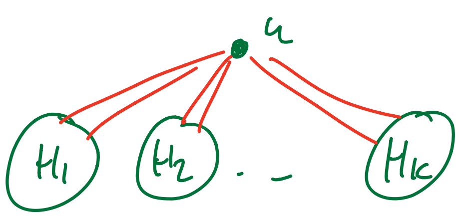
inductive step $|V| = n \geq 3$

If $G-v$ is connected for every $v \in V$

so the thm holds for G ✓

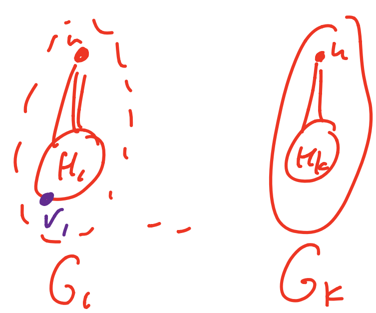
Otherwise we may choose $u \in V$ s.t. $G-u$

has components $H_1, H_2, \ldots H_k$ where $k \geq 2$



for $i = 1..k$

let $G_i = G[V(H_i) \cup \{u\}]$



Every G_i is connected has ≥ 2 vert $|V(G_i)| < |V|$

by induction every G_i has a vertex $v_i \neq u$

st. $G_i - v_i$ is connected.

We claim that $G - v_i$ is connected for every $1 \leq i \leq k$. To see this, note that every vertex in $G_i - v_i$ has a walk to u, and every vertex in $G - V(H_i)$ has a walk to u.

Since $k \geq 2$ this completes the proof.

$\square$