CS 556 - Homework 3

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Problem 1

Write the complete solution of the following linear system:

$$\begin{aligned} x + 2y - z &= 1 \\ 3x + 5y + 2z &= 3 \\ 2x + y + 13z &= 2 \end{aligned}$$

1. Convert equations to matrix/vector form

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ 2 & 1 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

2. Reduce to reduced row-echelon form

   (a) R3 = R3 - 2R1

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ 0 & -3 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

   (b) R2 = R2 - 3R1

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 5 \\ 0 & -3 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   (c) R3 = R3 - 3R2

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   (d) R2 = -1*R2

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   (e) R1 = R1 - 2R2

$$\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

3. Find $x_{\text{particular}}$

   $z = 0$
   $x + 9z = 1 \rightarrow \quad x + 9(0) = 1 \rightarrow \quad x + 0 = 1 \rightarrow \quad x = 1$
   $y - 5z = 0 \rightarrow \quad y - 5(0) = 0 \rightarrow \quad y - 0 = 0 \rightarrow \quad y = 0$
   $x_{\text{particular}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

4. Find x_{null}
   $z = 1$
   $x + 9z = 0 \rightarrow \quad x + 9(1) = 0 \rightarrow \quad x + 9 = 0 \rightarrow \quad x = -9$
   $y - 5z = 0 \rightarrow \quad y - 5(1) = 0 \rightarrow \quad y - 5 = 0 \rightarrow \quad y = 5$

   $x_{\text{null}} = c * \begin{bmatrix} -9 \\ 5 \\ 1 \end{bmatrix}$

$x_{\text{complete}} = x_{\text{particular}} + x_{\text{null}}$

$$x_{\text{complete}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c * \begin{bmatrix} -9 \\ 5 \\ 1 \end{bmatrix}$$

Problem 2

Find the rank of the following matrix: $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$

Rank: number of pivot columns after elimination, when the matrix is in REF.

1. Step 1: Reduce to row-echelon form

   (a) Swap R1 and R2

   $$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$$

   (b) R3 = R3 - 3R1

   $$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix}$$

   (c) R3 = R3 + 5R2

   $$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(d) R3 = R3 + 5R2

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(e) R3 = R3/2

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(f) R2 = R2 - R3

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This is the final matrix in row-echelon form. We have 3 pivots, so the $rank(A) = 3$

Problem 3

Construct a matrix A whose column space contains vectors $\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ and whose null space contains the vector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Since we know that the column space is all linear combinations of the columns of a matrix A, then we know that we can use the given vectors in the column space as the first 2 columns in the matrix A itself. The null space vector given is the solution to $Ax = 0$ so we start with the matrix set up like so:

$$\begin{bmatrix} 3 & 4 & x \\ 6 & 0 & y \\ 2 & 1 & z \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using this matrix equation, we can create equations to solve for variables x, y, z:

$$(3 \cdot 2) + (4 \cdot 2) + (1 \cdot x) \rightarrow 14 + x = 0$$
$$(6 \cdot 2) + (0 \cdot 2) + (1 \cdot y) \rightarrow 12 + y = 0$$
$$(2 \cdot 2) + (1 \cdot 2) + (1 \cdot z) \rightarrow 6 + z = 0$$

This gives us $x = -14, y = -12, z = -6$. Therefore, the final resulting matrix is

$$\begin{bmatrix} 3 & 4 & -14 \\ 6 & 0 & -12 \\ 2 & 1 & -6 \end{bmatrix}$$

Problem 4

Compute the following matrix-vector multiplication as:
a) Linear combination of columns.
b) Dot product of rows.

$$\begin{bmatrix} 2 & 1 & 3 \\ 7 & 1 & 0 \\ 3 & 5 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

a) Linear combination of columns:
Multiply the 3 columns of the matrix given with the 3 items in the vector, and add all together in a linear combination of the form $(c_1 * x) + (c_2 * y) + (c_3 * z)$ to get a final vector of 3x1 size.

$$2 \cdot \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 4 \\ 14 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 20 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 11 \\ 18 \\ 35 \end{bmatrix}$$

b) Dot product of rows: To solve for $Ax = y$ using the dot product method, where A is a 3x3 matrix, x is a 3x1 vector, and y is the resulting 3x1 vector the elements of y are calculated as such:

$$\begin{aligned} y_0 &= a_{0,0}x_0 + a_{0,1}x_1 + a_{0,2}x_2 \\ y_1 &= a_{1,0}x_0 + a_{1,1}x_1 + a_{1,2}x_2 \\ y_2 &= a_{2,0}x_0 + a_{2,1}x_1 + a_{2,2}x_2 \end{aligned}$$

Plugging in :

$$\begin{aligned} y_0 &= 2(2) + 1(4) + 3(1) &&= 11 \\ y_1 &= 7(2) + 1(4) + 0(1) &&= 18 \\ y_2 &= 3(2) + 5(4) + 9(1) &&= 35 \end{aligned}$$

So the final resulting vector y is $\begin{bmatrix} 11 \\ 18 \\ 35 \end{bmatrix}$.

Problem 5

Find the value of k for which the matrix has:
a) Dependent colums
b) Independent columns

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 4 & 8 & k \end{bmatrix}$$

a) To find the value of k for which the columns would be linearly dependent, we must reduce the matrix to row-echelon form

1. R3 = R3 - 2R2

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 0 & 2 & k-2 \end{bmatrix}$$

2. R2 = R2 - 2R1

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 2 & k-2 \end{bmatrix}$$

3. R2 = -1*R2

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & k-2 \end{bmatrix}$$

4. R3 = R3 - 2R2

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & k-8 \end{bmatrix}$$

Now that the matrix is in row-echelon form, we know that if the entire bottom was 0s, the columns would be dependent. So $k - 8 = 0$, or $k = 8$ would make the columns linearly dependent.
b) Therefore, any other values of k that are not $k = 8$ would make the columns of this matrix linearly independent.

Problem 6

Find a basis for the four fundamental subspaces of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The four fundamental subspaces of a matrix are the null space $N(A)$, the column space $C(A)$, the null space of A transposed $N(A^T)$, and the row space $C(A^T)$.

1. Null space $N(A)$ is the solution to $Ax = 0$.
   Reduce to row-echelon form.

   (a) R2 = R2 - R1

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) R3 = R3 - R2

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) R1 = R1 - 3R2

$$\begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here we can see that the pivot columns are columns 2 and 4, and the free columns are columns 1, 3, and 5. Let the free variables $x_1 = r$, $x_3 = s$, and $x_5 = t$. Using these to form equations for the pivots 2 and 4:

$$x_2 + 2s - 2t = 0 \rightarrow x_2 = -2s + 2t$$
$$x_4 + 2t = 0 \rightarrow x_4 = -2t$$

Expressing this in terms of vectors with the free variables s and t:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

So therefore the null space $N(A) = \text{span}\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \}$

2. Column space $C(A)$ are the pivot columns found after reduction from the original matrix A. Using our reduced matrix from the null space calculations, we can see that the pivot columns are columns 2 and 4.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding colmns from the original matrix are

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

3. Null space of A transposed $N(A^T)$
   Start by transposing the matrix and then reduce to row-echelon form.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

   (a) R5 = R5 - 4R2

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

   (b) R4 = R4 - 3R2

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

   (c) R5 = R5 - 2R4

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   (d) R3 = R3 - 2R1

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   (e) R2 = R2 - R4

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   As we can see here, the pivot columns are columns 1 and 2, and column 3 is a free column. Setting $x_3 = s$, the equations for $x_2$andx_4 are as follows:

$$x_1 - x_3 = 0 \rightarrow x_1 = s$$
$$x_2 + x_3 = 0 \rightarrow x_2 = -s$$

   Expressing this in terms of vectors with the free variable s:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

So therefore the null space of A transposed $N(A^T) = \text{span}\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\}$.

4. Row space or column space of A transposed $C(A^T)$ are the pivot columns found after reduction from the original matrix A. Using our the null space calculations, we know that the pivot columns are columns 1 and 2.

   Therefore, the row space $R(A)$ is: $\text{span}\{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix}\}$

Problem 7

Consider the vectors $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

a) In the x-y plane, mark all nine linear combinations $c\vec{v} + d\vec{w}$, with $c = \{-2, 0, 2\}$ and $d = \{0, 1, 2\}$.
b) What shape do all linear combinations $c\vec{v} + d\vec{w}$ fill? A line? The whole plane? Are the vectors $\vec{v}$ and $\vec{w}$ independent?

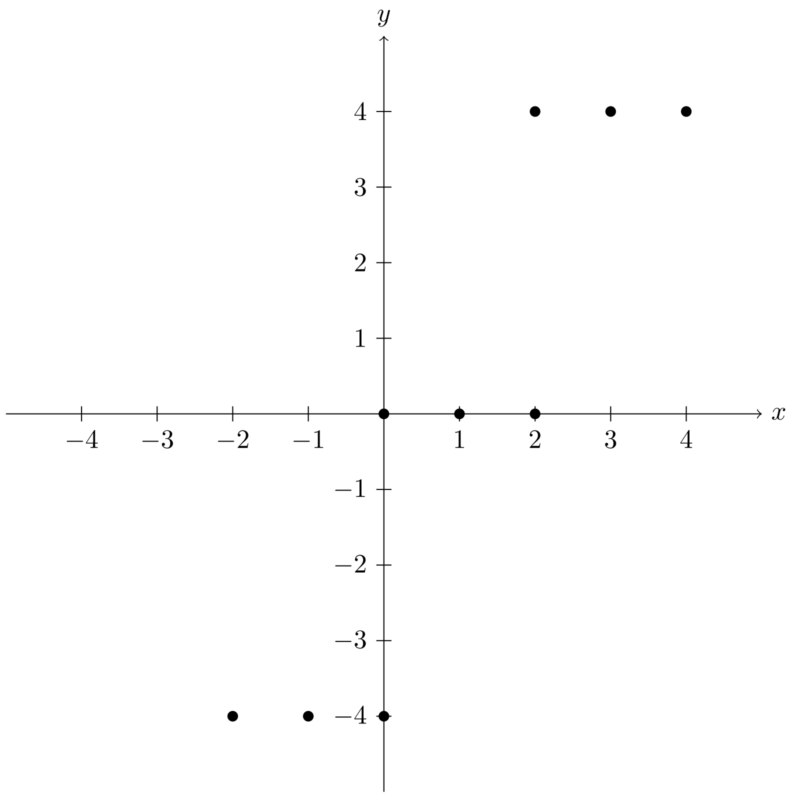
a) To find the points for all 9 linear combinations, we start by setting up the equation as so:

$$c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad \text{This will give us a resulting vector of } \begin{bmatrix} c+d \\ 2c \end{bmatrix}.$$

Now we use that equation to find all 9 points,

$$\begin{array}{llllll} c=-2, d=0 & \rightarrow \begin{bmatrix} -2 \\ -4 \end{bmatrix} & c=-2, d=1 & \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} & c=-2, d=2 & \rightarrow \begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ c=0, d=0 & \rightarrow \begin{bmatrix} -1 \\ -4 \end{bmatrix} & c=0, d=1 & \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & c=0, d=2 & \rightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ c=2, d=0 & \rightarrow \begin{bmatrix} 0 \\ -4 \end{bmatrix} & c=2, d=1 & \rightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix} & c=2, d=2 & \rightarrow \begin{bmatrix} 4 \\ 4 \end{bmatrix} \end{array}$$

Now we can plot all of these on a graph:



b) All linear combinations of $c\vec{v} + d\vec{w}$ would fill the entire x-y plane. The vectors $\vec{v}$ and $\vec{w}$ are linearly independent since no linear combination of one vector would form the other, other than the zero vector.

Problem 8

Consider the vectors $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

a) Can you solve the system $x\vec{u} + y\vec{v} + z\vec{w} = \vec{b}$, if $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

Setting up the equation $x\vec{u} + y\vec{v} + z\vec{w} = \vec{b}$:

$$x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Writing equations to solve for the variables:

$$\begin{aligned} x + 2z = 0 \quad &\rightarrow \quad x = -2z \\ -y + z = 0 \quad &\rightarrow \quad z = y \\ x + y = 1 \quad &\rightarrow \quad x = 1 - y \end{aligned}$$

Using $x+2z=0,\ x+y=1$ and $y=z$, we can combine these into the equation $x+2z-(x+z)=-1$ to solve for z. This gives us $z=y=-1$, which we can plug back into $x=-2z$ to get $x=2$. So now we have the final values $x=2, y=-1, z=-1$.

b) What if $\vec{b}=\begin{bmatrix}0\\0\\0\end{bmatrix}$? How many solutions are there?
Modifying our previous equations from part (a), we now get:

$$\begin{aligned} x+2z=0 \quad &\rightarrow \quad x=-2z \\ -y+z=0 \quad &\rightarrow \quad z=y \\ x+y=0 \quad &\rightarrow \quad x=-y \end{aligned}$$
$$\text{Using } x=-2z \text{ and } x=-y \text{ also gives us } y=2z$$

We now are presented with the contradiction $y=z$ and $y=2z$. The only values of y and z that can make these statements true are if $y=0,\ z=0$. And if $x=-y$, then $x=0$ as well. Therefore, the only solution is $x=0,\ y=0$, and $z=0$.

c) Are the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ dependent or independent?
Since we know from part (b) that the only solution to Ax=0 is the zero vector, this makes the three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ linearly independent.

d) Use parts (a) - (c) to decide if $A=\begin{bmatrix}1&0&2\\0&-1&1\\1&1&0\end{bmatrix}$ is an invertible matrix or not.
Yes, matrix A is invertible using the invertability test that the matrix can have only one solution to Ax=0, where x is equal to the zero vector which we have proven in part (b).

Problem 9

Consider the linear system for some constants b and g:

$$\begin{aligned} x-2y+3z&=3 \\ 2x+y+bz&=-4 \\ x+0y+1z&=g \end{aligned}$$

a) What constant b makes the system singular (missing a pivot)?
b) For the value of b found in Part (a), for which values of g does the system have infinitely many solutions?
c) Find two distinct solutions of the system for that g.

a) To find the value of b that makes the system singular, we must perform row reduction and then find the value for b that would make the entire row 0s after row reduction.
Starting matrix:

$$\begin{bmatrix}1&-2&3\\2&1&b\\1&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}3\\-4\\g\end{bmatrix}$$

1. R3 = R3 - R1

$$\begin{bmatrix}1 & -2 & 3\\2 & 1 & b\\0 & 2 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}3\\-4\\g-3\end{bmatrix}$$

2. R2 = R2 - 2R1

$$\begin{bmatrix}1 & -2 & 3\\0 & 5 & b-6\\0 & 2 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}3\\-10\\g-3\end{bmatrix}$$

3. R1 = R1 + R3

$$\begin{bmatrix}1 & 0 & 1\\0 & 5 & b-6\\0 & 2 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}g\\-10\\g-3\end{bmatrix}$$

4. R3 = R3 / 2

$$\begin{bmatrix}1 & 0 & 1\\0 & 5 & b-6\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}g\\-10\\(g-3)/2\end{bmatrix}$$

5. R2 = R2 - 5R3

$$\begin{bmatrix}1 & 0 & 1\\0 & 0 & b-1\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}g\\-5/2g-17.5\\(g-3)/2\end{bmatrix}$$

For this system to be singular (missing a pivot), the row with b in it needs to contain all 0 values, so $b-1$ must equal 0, therefore $b=1$.

b) The system would have infinitely many solutions when the value for g in the 0 row (where $b=1$), is also equal to 0. We need to find the value for g where $-5/2g-17.5=0$. Solving for g, this gives us $g=7$.

c) Plugging in the values found for b and g, we now have the matrix:

$$\begin{bmatrix}1 & 0 & 1\\0 & 0 & 0\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}7\\0\\2\end{bmatrix}$$

Swap R2 and R3:

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}7\\2\\0\end{bmatrix}$$

The pivots are x and y, and the free variable is z. Making equations:

$$x+z=7\rightarrow x=-z+7$$
$$y-z=2\rightarrow y=z+2$$

In vector form:

$$z\begin{bmatrix}-1\\1\\1\end{bmatrix}+\begin{bmatrix}7\\2\\0\end{bmatrix}$$

To find two distinct solutions, we can use the values of $z = 0$, and $z = 1$:
For $z = 0$:

$$0 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix}$$

For $z = 1$:

$$1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$