CS 556 - Homework Derivates

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Problem 1

Use the limit definition of the derivative to exactly evaluate the derivative:

1. $f(x) = \sqrt{x+4}$

$$\lim_{h \to 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} \to \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} \cdot \frac{\sqrt{x+h+4} + \sqrt{x+4}}{\sqrt{x+h+4} + \sqrt{x+4}}$$
$$= \frac{x+h+4-x-4}{h(\sqrt{x+h+4} + \sqrt{x+4})} = \frac{h}{h(\sqrt{x+h+4} + \sqrt{x+4})}$$

Setting h to 0 and reducing gives you the final answer of:

$$\frac{1}{2\sqrt{x+4}}$$

2. $f(x) = \frac{3}{x}$

$$\lim_{h \to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h} \to \frac{\frac{3x}{x(x+h)} - \frac{3(x+h)}{x(x+h)}}{h} = \frac{\frac{3x-3x-3h}{x(x+h)}}{h}$$
$$= \frac{-3h}{hx(x+h)} = \frac{-3}{x(x+h)}$$

Setting h to 0 and reducing gives you the final answer of:

$$\frac{-3}{x^2}$$

Problem 2

Find the derivatives of the following functions:

1. $f(x) = 3x^3 - \frac{4}{x^2}$

Using difference rule: $\frac{d}{dx} = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$ and power rule: $x^n = nx^{n-1}$

$\frac{d}{dx}(3x^3 = 9x^2)$
$\frac{d}{dx}(\frac{4}{x^2} = 4(\frac{1}{x^2})) = 4(x^{-2}) \to 4(-2x^{-3}) = -8x^{-3} = \frac{-8}{x^3}$
Putting it back together: $9x^2 - \frac{-8}{x^3} = 9x^2 + \frac{8}{x^3}$

2. $f(x) = (4 - x^2)^3$

   Using chain rule: $f(g(x)) = f'(g(x))g'(x)$

   $f(x) = (g(x))^3 \rightarrow f'(x) = 3(g(x))^2 \rightarrow 3(4 - x)^2$
   $g(x) = 4 - x^2 \rightarrow g'(x) = -2x$
   Putting it back together: $3(4 - x^2)^2 \cdot -2x = -6x(4 - x^2)^2$

3. $f(x) = e^{\sin(x)}$

   Using rule for exponential e equations: $f'(e^{g(x)}) = e^{g(x)}g'(x)$

   $g(x) = \sin(x) \rightarrow g'(x) = \cos(x)$
   Putting it back together: $e^{\sin(x)} \cos(x)$

4. $f(x) = ln(x + 2)$

   Using rule for logarithmic equations: $\frac{d}{dx}(ln(g(x))) = \frac{1}{g(x)}g'(x)$

   $$\frac{1}{x+2} \cdot g'(x+2) = \frac{1}{x+2} + 1$$

5. $f(x) = x^2 \cos(x) + x \tan(x)$

   Using sum rule and product rule:

   $$\frac{d}{dx}(x^2 \cos(x)) + \frac{d}{dx}(x \tan(x))$$
   $$\frac{d}{dx}(x^2 \cos(x)) = \cos(x) \cdot \frac{d}{dx}(x^2) + x^2 \cdot \frac{d}{dx}(\cos(x)) = \cos(x)2x - x^2 \sin(x)$$
   $$\frac{d}{dx}(x \tan(x)) = \tan(x) \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\tan(x)) = \tan(x) + x \sec^2(x)$$
   Putting it back together: $2x \cos(x) - x^2(\sin(x)) + \tan(x) + x \sec^2(x)$

6. $f(x) = \sqrt{3x^2 + 4}$

   Using chain rule: $f(g(x)) = f'(g(x))g'(x)$

   $$f(x) = \sqrt{g(x)} \rightarrow f'(x) = \frac{1}{2\sqrt{g(x)}} \rightarrow \frac{1}{2\sqrt{3x^2+2}}$$
   $g(x) = 3x^2 + 2 \rightarrow g'(x) = 6x$
   Putting it back together: $6x \cdot \frac{1}{2\sqrt{3x^2+2}} = \frac{6x}{2\sqrt{3x^2+2}} = \frac{3x}{\sqrt{3x^2+2}}$

7. $f(x) = \frac{x}{4} \sin^{-1}(x)$

Using product rule:

$\sin^{-1}(x) \cdot \dfrac{d}{dx}(\dfrac{x}{4}) + \dfrac{x}{4} \cdot \dfrac{d}{dx}(\sin^{-1}(x))$
$\dfrac{d}{dx}(\dfrac{x}{4}) = \dfrac{1}{4}$
$\dfrac{d}{dx}(\sin^{-1}(x)) = \dfrac{1}{\sqrt{1-x^2}}$
Putting it back together: $\dfrac{1}{4}(\sin^{-1}(x)) + \dfrac{x}{4}(\dfrac{1}{\sqrt{1-x^2}}) = \dfrac{1}{4}(\sin^{-1}(x) + \dfrac{x}{\sqrt{1-x^2}})$

8. $x^2y = (y+2) + xy\sin(x)$

$\dfrac{d}{dx}(x^2y) = y \cdot \dfrac{d}{dx}(x^2) + x^2\dfrac{d}{dx}(y) = y2x + x^2\dfrac{dy}{dx}$
$\dfrac{d}{dx}(y+2) = \dfrac{dy}{dx}$
$\dfrac{d}{dx}(xy\sin(x)) = \sin(x) \cdot \dfrac{d}{dx}(xy) + xy \cdot \dfrac{d}{dx}(\sin(x))$
$\dfrac{d}{dx}(xy) = y \cdot \dfrac{d}{dx}(x) + x \cdot \dfrac{d}{dx}(y) = y + x\dfrac{dy}{dx}$
$\sin(x) cdot (y + x\dfrac{dy}{dx}) + xy \cdot \cos(x) \rightarrow= y\sin(x) + x\sin(x)\dfrac{dy}{dx} + xy\cos(x)$
Putting it back together: $2xy + x^2\dfrac{dy}{dx} = \dfrac{dy}{dx} + y\sin(x) + x\sin(x)\dfrac{dy}{dx} + xy\cos(x) \rightarrow$
$x^2\dfrac{dy}{dx} - x\sin(x)\dfrac{dy}{dx} - \dfrac{dy}{dx} = -2xy + y\sin(x) + xy\cos(x)$
$\dfrac{dy}{dx}(x^2 - 1 - x\sin(x)) = y\sin(x) + xy\cos(x) - 2xy \rightarrow \dfrac{dy}{dx} = \dfrac{y\sin(x) + xy\cos(x) - 2xy}{x^2 - 1 - x\sin(x)}$

Problem 3

1. Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

   Using central difference: $\frac{f(a+h)-f(a-h)}{2h}$

$$\begin{aligned}\frac{f(39+10) - f(39-10)}{20} &= \frac{f(49) - f(29)}{20} \\ &= \frac{145 - 115}{20} = \frac{30}{20} = 1.5\end{aligned}$$

   This is the rate of change of the wind speed at hour 39. This means that the wind speed is increasing at a rate of 1.5mph/hour at this point in time.

2. Using the table, estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Using central difference: $\frac{f(a+h)-f(a-h)}{2h}$

$$\frac{f(83+2)-f(83-2)}{2(2)} = \frac{f(85)-f(81)}{4}$$
$$= \frac{95-125}{4} = \frac{-30}{4} = -7.5$$

This is the rate of change of the wind speed at hour 83. This means that the wind speed is decreasing at a rate of 7.5mph/hour at this point in time.

Problem 4

The famous Regiomontanus' problem for angle maximization was proposed during the 15th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ?
Maximizing the angle is finding where the derivative of the function = 0.

$$\tan(\theta) = \frac{\frac{b}{x} - \frac{a}{x}}{1 + \frac{b}{x} \cdot \frac{a}{x}} \rightarrow$$
$$\tan(\theta) = \frac{b-a}{x} \cdot \frac{x^2}{x^2+ab}$$
$$\frac{d}{dx}\left(\frac{b-a}{x} \cdot \frac{x^2}{x^2+ab}\right) = \frac{(-x^2+ba)(b-a)}{(x^2+ba)^2}$$

Setting equation equal to 0 and solving for x gives:

$$x = \sqrt{ab}$$

Problem 5

An airline sells tickets from Tokyo to Detroit for \$1200. There are 500 seats available and a typical flight books 350 seats. For every \$10 decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?

The equation for the ticket price is: $1200 - 10x$
The equation for the Number of seats sold is: $350 + 5x$

$$(1200-10x) \cdot (350+5x) = -50x^2 + 2500x + 4200$$
$$\frac{d}{dx}(-50x^2 + 2500x + 4200) = -100x + 2500$$
$$100x = 2500 \rightarrow x = 25$$

Plugging back into previous equations: $1200 - 10(25) = 1200 - 250 = 950 \quad \text{(Optimal ticket price)},$

$$350 + 5(25) = 475 \quad \text{(Number of seats sold)}$$