A study of Sylvester Gallai configurations

R&D2 Report

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Abstract

Sylvester's problem states that if given a configuration of points, not all of them collinear, there exists at least one line that passes through exactly two points. We call such a line an ordinary line. We can extend the problem to ask the minimum number of ordinary lines present in a configuration of points. Another extension can be applied to ask the number of 3-rich lines, i.e. the number of lines that passes through exactly three points. The latter problem is called the orchard-planting problem. Coloured extensions of Sylvester's problem are also possible.

This report surveys the proofs of Sylvester's problem and its extension, i.e. the minimum number of ordinary lines. A mathematical object, allowable sequences, is defined which deals with a configuration of points as a sequence of permutations. A proof of Sylvester's problem via allowable sequences is also studied in the report which lays down the future goals for us.

Acknowledgements

I thank Prof Sundar for his guidance, advice and the valuable discussions we had regarding mathematics, research and everything else.

One of the major outcomes of R&D2 and Seminar I have observed is that I am writing better technical reports. This was possible due to Sundar's advice and his conscientious efforts to help me improve my writing. Hopefully, my problem solving abilities have also developed as an indirect effect.

I thank Prof Mrinal, Prof Rohit, Prof Abhiram for their teachings, for their advice whenever I felt stuck, and for helping me grow as a person of academia. Words are not enough to express my gratitude. I am very happy to be a part of the Theory lab at IIT Bombay.

Contents

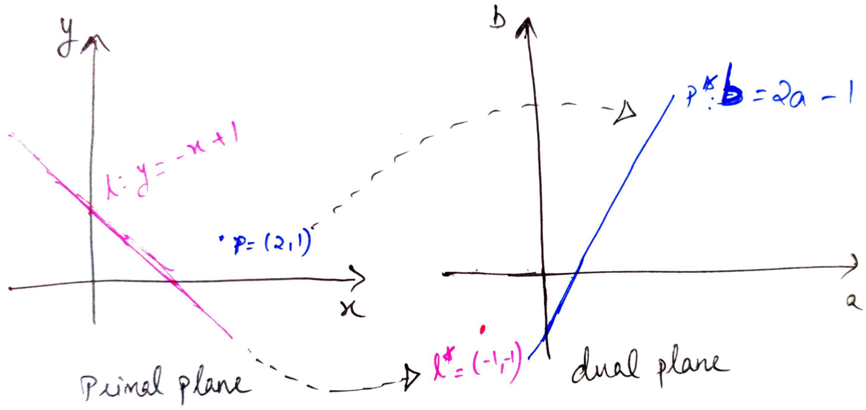
1. **Preliminaries**

References: [Fel20], [Wik21a], [Wik21b], [Tor99]

1.1. **Point-line duality.**

Consider a set of points $P = (p_1, p_2, ..., p_n) \in \mathbb{R}^2$ and a set of lines $\Lambda = (l_1, l_2, ..., l_m)$ in the plane $\mathbb{R}^2$. We can transform points and lines in the original plane, which we call the primal plane, to a new set of lines and points respectively in a different plane. We call this different plane the dual plane and the transformation function as Point-line duality. Point-line duality is defined as

Primal plane		Dual plane
$p : (p_x, p_y)$	$\Longrightarrow$	$p^* : b = p_x a - p_y$
$l : y = mx + c$	$\Longrightarrow$	$l^* : (m, -c)$



Point-line transformation satisfies the following

- $(p^*)^* = p$, $(l^*)^* = l$.
- p lies below/ on/ above l $\iff$ p^* passes above/ through/ below l^*.
- l_1 and l_2 intersect at point p $\iff$ l_1^*, l_2^* lies on p^*.
- p_1, p_2, p_3 are collinear $\iff$ p_1^*, p_2^*, p_3^* intersect at a common point.
- Vertical lines have no dual.

1.2. **Affine geometry.**

Affine geometry is defined as the geometry which remains invariant under affine transformation. In 2-dimension, affine transformations are of the form $Ax + B$ where $A \in GL(2, \mathbb{R})$, B is some 2×1 matrix.

Playfair's axiom is fundamental in affine geometry, i.e. given a line l and point p not on l, exactly one line parallel to l passes through p.
General Linear group
General linear group of degree n is the set of $n \times n$ invertible matrices together with the matrix multiplication operator denoted as $GL(n)$. $GL(n)$ forms a group because it follows the laws of group which are Closure, Associative, Identity and Inverse. Note that matrix multiplication is not commutative.

- $GL(n)$: set of $n \times n$ invertible matrices
- $GL(2, \mathbb{R})$: set of 2×2 invertible matrices over Real numbers

Affine transformation preserves

- Collinearity of points
- Parallelism of lines
- Cross ratio
- Ratio of areas

Cross ratio: Given four collinear points A, B, C, D their cross ratio is defined as $\frac{AC.BD}{BC.AD}$

1.3. **Projective geometry.**

Euclidean geometry with extra points at infinity is Projective Geometry. For any two lines in projective geometry, there is exactly one point that lies on both of them.

Projective transformation preserves

- Incidence structure
- Cross ratio
- Point-line duality

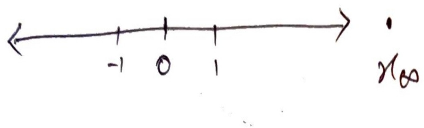
Incidence Structure
A projective plane C can be defined as $C = (P, L, I)$ where P is the set of points, L is the set of lines, and $I \subseteq P \times L$ is the incidence relation. If some $(p, l) \in I$ we can say that p lies on l or equivalently l passes through p.
If we interchange the roles of points and lines, we get $C^* = (L, P, I^*)$.
If C and C^* are isomorphic, we say that C is self dual.
Projective spaces are self dual and C^* is the dual plane to C.

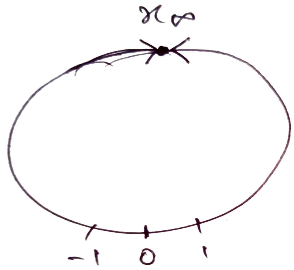
Real Projective line: $\mathbb{RP}^1$
Points of $\mathbb{RP}^1$ are the lines through origin in $\mathbb{R}^2$.

Consider $\mathbb{R}^2$ and a line l': $x_2 = 1$. Then all lines through origin intersect l' at some $(x_1, 1)$, except the line Ox_1, which is parallel to l'. We will assign a special point x_∞, i.e. a point at infinity, where these two lines intersect.

Therefore, $\mathbb{RP}^1$ is the line l' union x_∞.

Two ways to view $\mathbb{RP}^1$ are shown below



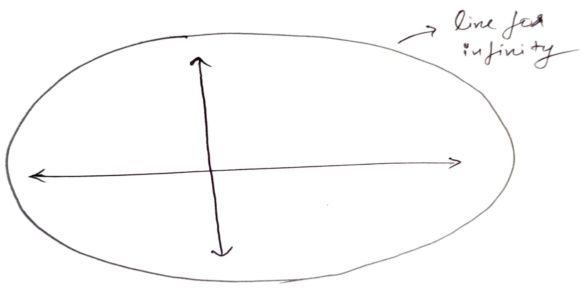


Real Projective plane: $\mathbb{RP}^2$

Points of $\mathbb{RP}^2$ are the lines through origin in $\mathbb{R}^3$.

Consider $\mathbb{R}^3$ and a plane p': $x_3 = 1$. Then all lines through origin intersect p' at some $(x_1, x_2, 1)$, except the lines in plane Ox_1x_2, which is parallel to p'. For all lines in Ox_1x_2, we will assign points at infinity, which signifies points of intersection of Ox_1x_2 and p'.

Therefore, $\mathbb{RP}^2$ is the plane p' union a line for infinity as shown below.



Another way to view $\mathbb{RP}^2$: Spherical view

Consider a hollow sphere S. We will glue all pairs of antipodal points so they essentially represent the same point. Antipodal points are the points on S which are a diameter away. We place the centre of this sphere on the origin. Now, consider a line through origin, which passes through two points in S. These points will clearly be antipodal, and therefore, in our projective space, the line essentially passes through a single point.

2. **Sylvester's Problem**

References: [Nil05], [Bog]

Let P be a finite set of points, not all of them collinear. We call P as a configuration of points.
We define

Definition 2.1 (Connecting line). *A line which contains two or more points from P.*

Definition 2.2 (Ordinary line). *A line which contains exactly two points from P.*

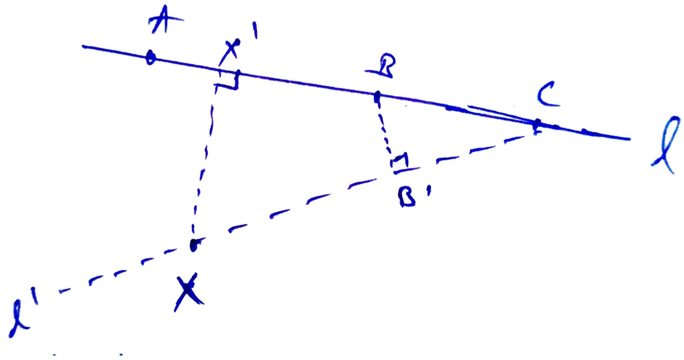
Sylvester conjectured that there exists at least one ordinary line, when given P.

2.1. **Kelly's proof (Euclidean).**

Let Λ be the set of connecting lines formed from the points in P.
Since not all points are collinear, there exists a point and a connecting line such that the perpendicular distance between them is non-zero. We take all possible combinations of such pairs. Let

$$\Gamma = \{(p, l) \mid p \in P, l \in \Lambda, \text{perpendicular distance}(p, l) \neq 0\}$$

Let the pair in Γ with minimum perpendicular distance between them be (X, l). We argue that l is ordinary.
Because, if not, we assume $A, B, C \in P$ lie on l.



We drop a perpendicular from X to l, which intersect l at X'.
At least two of A, B, C lie on the same side of X'. Without loss of generality, let them be B, C.

Note: A can coincide with X'.
We join X and C to form a line and name it l'. Since, $X, C \in P$, therefore, $l' \in \Lambda$. We draw a perpendicular from B to l', intersecting l' at B'.

In $\triangle XX'C$ and $\triangle BB'C$
$\angle B' = \angle X'$ (Right angles)
$\angle C = \angle C$ (Common)
So, by AAA, $\triangle XX'C \sim \triangle BB'C$ and $\frac{BB'}{XX'} = \frac{BC}{XC} = \frac{B'C}{X'C}$

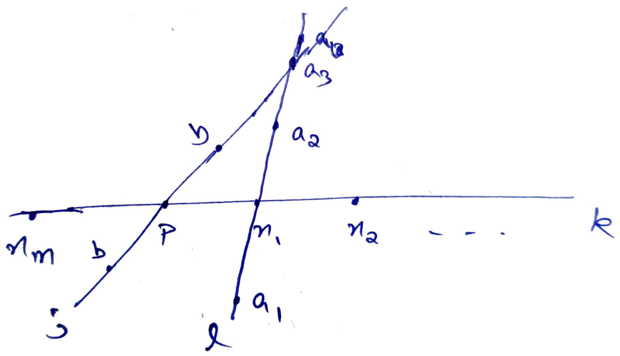
Since, $B'C < X'C$, therefore $BB' < XX'$.

This establishes that (B', l') is a pair with shorter perpendicular distance between them which contradicts our assumption of (X, l) being the pair with the minimum distance.

Hence, l must be ordinary.

2.2. **Steinberg's proof.**

Let P be the configuration of points and Λ be the set of connecting lines. Choose any $p \in P$. If p lies on any ordinary line we are done, so we may assume that p does not lie on any ordinary line. Let k be a line which passes through p but is not a connecting line, or $k \notin \Lambda$. Let k intersect lines in Λ at $x_1, x_2, ..., x_m$ (in cyclic order.)



Let l be the connecting line intersecting k at x_1, such that there is no other intersection point from $k \cap \Lambda$ between p and x_1. (1)

We argue l must be ordinary.

Because, if not, l passes through at least 3 points $a_1, a_2, a_3 \in P$.

Case 1 Any two of a_1, a_2, a_3 lie on the same side of k. Without loss of generality, let them be a_2, a_3.

Note: x_1 can't coincide with a_1, a_2 or a_3 because we assumed k to not be a connecting line.
We join pa_3, let this line be j; $j \in \Lambda$. j can't be ordinary either. So, another point $b \in P$ should lie on it.

(1) b is between p and a_3
then ba_1, a connecting line, intersect k before x_1 which is a contradiction to (1).
(2) b lie outside of pa_3
then ba_2, a connecting line, intersect k before x_1, again a contradiction to (1).

Case 2 All three of a_1, a_2, a_3 lie on the same side of k. Then, we join pa_2 and follow the same argument and conclude either ba_3 or ba_1 intersect k before x_1, contradiction to (1).

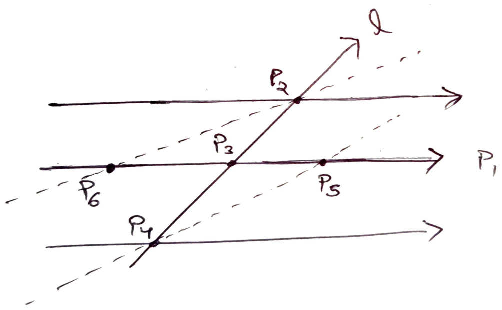
Hence, l must be ordinary.

2.3. **Gallai's proof (Projective).**

Let P be a configuration of points, not all of them collinear. Take a point $p_1 \in P$. If p_1 lies on any ordinary line then we are done. So, let's assume p_1 does not lie on any ordinary line. Let the set of lines passing through p_1 be Ω. Since not all of the points are collinear, Ω will contain at least two lines.

We apply a projective transformation on p_1 such that all lines in Ω after the transformation become parallel. Since all lines in Ω are parallel now and are in a projective space they intersect at the line for infinity. All other connecting lines which do not pass through p_1 intersect Ω at some angle.

Let l be the connecting line which intersects lines in Ω at the smallest angle. We claim that l is ordinary. Because if not, then l intersect lines in Ω at p_2, p_3 and p_4, and some line in Ω contain points p_5, p_3 and p_6 as shown in the figure on the next page. Then, either p_2p_6 or p_5p_4 forms a smaller angle than l which is a contradiction.



3. **Sylvester's Problem Extension**

References: [Nil05], [KM58], [GT13]

With determining the existence of an ordinary line, the natural question to ask is the minimum number of ordinary lines that could be present when given a configuration of points P, and not all of the points are collinear. We define this as $m(P)$ where P contains n points and for all arrangements of n points, we define $m(n)$.

$$m(n) = \min_{|P|=n} m(P)$$

3.1. **Melchoir's proof of the existence of 3 ordinary points.**

We know Projective spaces preserve point-line duality, so we can shift to the dual plane knowing that any valid statement in the dual is also valid in the primal plane. We will particularly use the spherical view of $\mathbb{RP}^2$.

Let the arrangement of lines partition $\mathbb{RP}^2$ into polygonal regions. Let V, E and F denote the number of vertices, edges and faces in this partition respectively.

Let us also suppose v_i be the vertex incident with exactly i lines, f_i be the face with exactly i sides. Since not all lines are concurrent, every face will have 3 sides, therefore, $f_2 = 0$.

Now,

$$2E = \sum_{i \geq 3} i f_i = 2 \sum_{i \geq 2} i v_i$$

Euler's characteristic of Projective space is 1. Therefore,

$$V - E + F = 1$$

$$3V - E + 3F - 2E = 3$$

$$3\sum_{i\geq 2} v_i - \sum_{i\geq 2} iv_i + 3\sum_{i\geq 3} f_i - \sum_{i\geq 3} if_i = 3$$

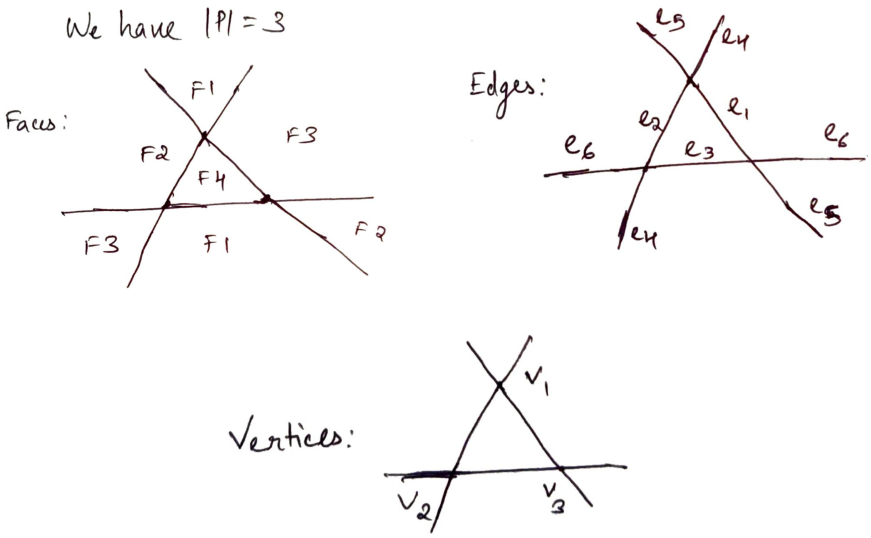
$$\sum_{i\geq 2}(3-i)v_i + \sum_{i\geq 3}(3-i)f_i = 3$$

$$1.v_2 + 0.v_3 + \sum_{i\geq 4}(3-i)v_i + 0.f_3 + \sum_{i\geq 4}(3-i)f_i = 3$$

$$v_2 = 3 + \sum_{i\geq 4}(i-3)v_i + \sum_{i\geq 4}(i-3)f_i$$

Therefore, $v_2 \geq 3$. Hence, any finite set of non-concurrent lines has at least 3 ordinary points. By duality, we can say, any finite set of non-collinear points has at least 3 ordinary lines.

An example is given in the following figure.



3.2. Motzkin's proof of the existence of $O(\sqrt{n})$ ordinary lines.

Consider a point $p \in P$ not lying on any ordinary line. The set of connecting lines not passing through p divide the plane into polygonal regions which we will call a cell. These cells can be open or closed.

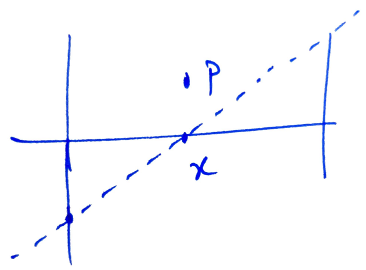
Let the smallest cell containing p be C. No other connecting line passes through C. The edges of cell C are the neighbours of p.

Case 1 If there is no such point p then an ordinary line passes through every point in P. Since an ordinary line passes through exactly two points, there are at least $n/2$ ordinary lines and we are done.

Case 2 Considering such a p exists.

Lemma 3.1. *There can't be a point $x \in P$ lying on the edges of C.*

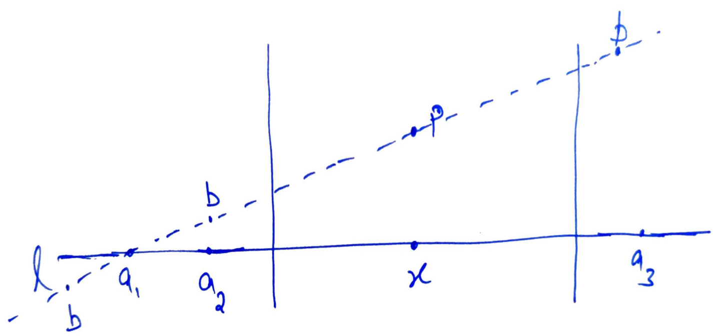
Proof. Let's assume for contradiction such an x exists and it lies on the edge l as shown in the figure below. Then a connecting line passing through x intersects cell C which is a contradiction. □



Note: Points can lie on the intersections of the edges of cell C.

Lemma 3.2. *If p has at least three neighbours then all the neighbours of p are ordinary.*

Or we can say if there are at least three edges in the smallest cell C containing p, then all the edges of C are ordinary.



Proof. Suppose some neighbour l of p is not ordinary. Then l passes through $a_1, a_2, a_3 \in P$ such that a_1, a_2 separate x, a_3 where $x \notin P$ is a point on l on the boundary of cell C, as shown in the figure above. By our hypothesis, lines passing through p are not ordinary. Thus, line a_1p will pass through another point $b \in P$. Then lines either through a_2b or a_3b cuts cell C contradicting the assumption that C was the smallest. □

Theorem 3.3. *A configuration P with n points will have $O(\sqrt{n})$ ordinary lines provided not all of the points are collinear.*

Proof. The set of ordinary lines divides the plane into cells. Each point in P either lie on one of the ordinary lines or in a cell.

A cell can not have more than one point. Suppose p and q both lie in a cell. Then, any connecting line through q cannot be an ordinary line. Now, any line passing through q is a neighbour to p. Since, no ordinary line passes through p, all of its neighbours should be ordinary which contradicts the previous assertion. Thus, at most one point can lie in a cell.

m ordinary lines pass through at most $2m$ points from P and divide the plane into a maximum of $\binom{m}{2} + 1$ regions (can be proved via induction). Each region can contain at most one point from P.

Thus,

$$\begin{aligned} 2m + \binom{m}{2} + 1 &\geq n \\ m^2 + 3m + 2 &\geq 2n \\ m &\geq \sqrt{2n} - 2 \\ \therefore O(\sqrt{n}) &\text{ ordinary lines} \end{aligned}$$

□

3.3. Kelly and Moser's proof of the existence of $3n/7$ ordinary lines.

Let P be the set of points and Λ be the set of connecting lines. We will use p and l to denote some arbitrary point and line respectively.

Connecting lines which do not pass through p divide the plane into polygonal regions which we will call a cell.

We define

Definition 3.4 (Pencil)**.** *Configuration Λ such that all lines in Λ are concurrent.*

Definition 3.5 (Near-pencil)**.** *Configuration Λ such that if Λ contains k lines then exactly $k-1$ of those lines are concurrent.*

Definition 3.6 (Neighbour of p)**.** *Connecting lines which are the edges of the cell containing p.*

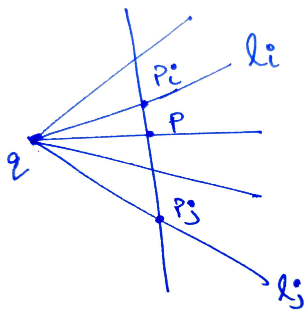
Definition 3.7 (Order of p)**.** *The number of ordinary lines passing through p.*

Definition 3.8 (Rank of p)**.** *The number of neighbours of p which are ordinary lines.*

Definition 3.9 (Index of p)**.** *Summation of the order and the rank of p.*

Lemma 3.10. *If a point q has exactly one neighbour, then Λ is a near-pencil.*

Proof. If q has exactly one neighbour then it implies that out of k connecting lines, $k-1$ lines passes through q. (3.5). □



Lemma 3.11. *If a point p has exactly two neighbours, then Λ is a near-pencil.*

Proof. The lines not passing through p should form a pencil for otherwise p would have at least three neighbours. Let $q \in P$ be the vertex of the pencil. Let l_i and l_j be any two connecting lines passing through q. Let $p_i \in P$ lie on l_i and $p_j \in P$ lie on l_j as shown in the above figure. The connecting line through $p_i p_j$ will not pass through q, therefore, it will pass through p. We can conclude that exactly one line in Λ passes through p and the rest of it forms a pencil configuration with q as vertex. Hence, Λ is a near-pencil. □

Theorem 3.12. *If Λ is not a near pencil, then each point in P has at least three neighbours.*

Proof. Consequence of 3.10 and 3.11. □

Lemma 3.13. *If the order of p is zero then every neighbour of p is an ordinary line.*

Proof. From 3.12, we can say that apart from near-pencil configurations, every point will have at least three neighbours. Therefore, through Motzkin (at 3.2) we can conclude the lemma. □

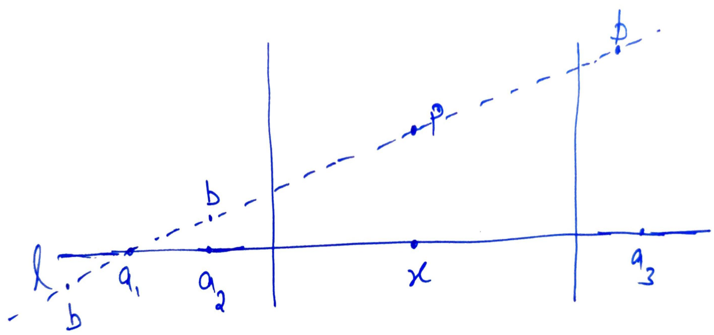
Lemma 3.14. *If the order of p is one then there exists at least one non-ordinary neighbour of p.*

Each non-ordinary neighbour passes through the point a_1 such that pa_1 is the only ordinary line passing through p.

Proof. Since the order of p is one, from 3.13 we can conclude the first part of the lemma.

Through Motzkin at 3.1 we concluded that no point of P can lie on the edges of the smallest cell C containing p. Suppose some neighbour l of p is non-ordinary. Then l passes through points a_1, a_2, x, a_3 such that $a, b, c \in P$ and $x \notin P$ lie on the boundary of the cell C as shown in the figure below.

We claim that pa_1 is ordinary. Because if not, line pa_1 will pass through another point b of P. Then either ba_2 or ba_3 cuts cell C contradicting the assumption that cell C was the smallest.



Therefore, for every non-ordinary neighbour of p, there exists a point a_1 of P such that pa_1 is an ordinary line. But, we stated that the order of p is one. Therefore, every non-ordinary neighbour of point p, whose order is one, passes through the point a_1.

$\square$

Lemma 3.15. *Any point of P not of order two has index at least three.*

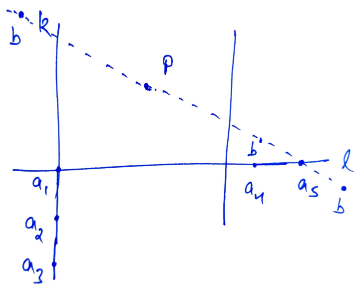
Proof. First, we observe that the lemma is true for near-pencil configuration. Suppose Λ has k lines. For the vertex q of the pencil, the order of q is $k-1$. For any other point p its order is one and rank is two. We will dismiss near-pencil from all future considerations in this lemma.

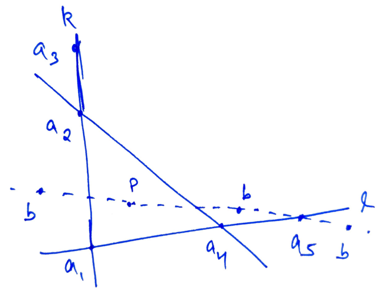
Case 1 Order of p is zero. Then p has at least three neighbours and by 3.13 they are all ordinary. This implies p has a rank of at least three and we are done.

Case 2 Order of p is three then index is trivially at least three.

Case 3 Order of p is one. Let the only ordinary line that passes through p also passes through a_1. We need to show that the rank of p is at least two. From 3.12, there are at least three neighbours of p. If two of them are ordinary we are done.

Suppose two of the neighbours of p are non-ordinary. Let them be k and l. k passes through a_1, a_2, a_3 and l passes through a_1, a_4, a_5 as shown in the figures below. pa_1 is ordinary and no other connecting line through p is ordinary. Therefore, pa_5 passes through another point b of P. Then either ba_4 or ba_1 cuts cell C which is a contradiction.

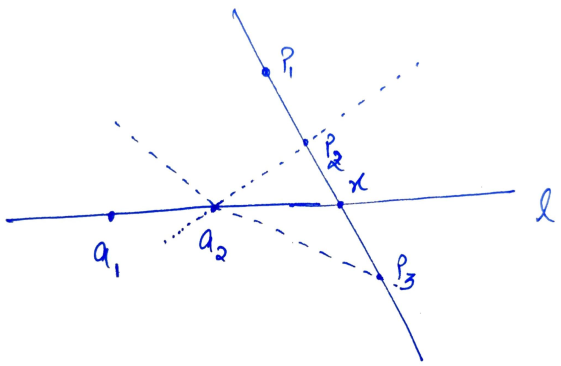




Therefore, p has at least three neighbours and two non-ordinary neighbours is not possible for p if its order is one. This implies that the rank of order one point p is at least two and we are done. $\square$

Lemma 3.16. *Any three points of P that have a common neighbour cannot be collinear.*

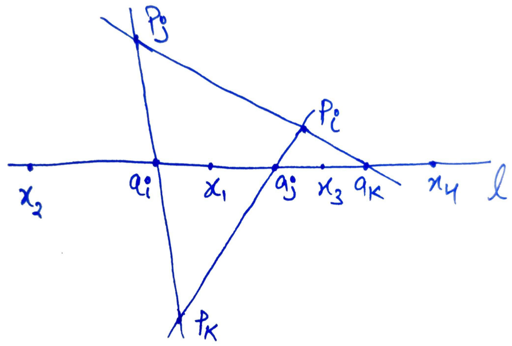
Proof. Suppose for contradiction, p_1, p_2, p_3 of P are collinear and they have a common neighbour l. Let a_1, a_2 be two points of P on l. Suppose p_1p_2 separates xp_3, where x is a point not in P and lies on l as shown



in the figure above. Then, the connecting lines through a_2p_2 and a_2p_3 separates p_1 from l which is a contraction to l being a neighbour of p_1.

We proved the lemma for one case of the configuration. There can be multiple cases possible based on the relative positioning of p_1, p_2, p_3 and a_1, a_2 which all can be proved in the similar fashion. □

Lemma 3.17. *If a line l of Λ is a neighbour of three points p_i, p_j, p_k of P then the points of P which lie on l are on the connecting lines determined by p_i, p_j, p_k.*



Proof. p_i, p_j, p_k can not be collinear 3.16. Suppose connecting lines through p_ip_j, p_jp_k, p_ip_k intersect l at a_k, a_i and a_j respectively as shown in the above figure. Suppose x_1 of P also lie on l. Then the connecting lines through x_1p_j and x_1p_i separates p_k from l which is a contradiction since we assumed l to be a neighbour of p_k. We can prove similarly if another one of x_2, x_3, x_4 lying on l belongs to P. □

Corollary 3.18. *If a line l of Λ is a common neighbour to three points x_1, x_2, x_3 of P then the number of points in P which lie on l can either* *be two or three determined by the set of connecting lines between the points x_1, x_2, x_3.*

Proof. We need at least two points to lie on a line and from 3.17 this quantity cannot exceed three. ☐

Lemma 3.19. *A line l of Λ is a neighbour of at most four points.*

Proof. Suppose l is the neighbour of five points p_1, p_2, p_3, p_4, p_5. Wlog we look at p_1, p_2, p_3 and see that at least two of a_1, a_2, a_3 should be in P. Wlog let them be a_2, a_3.
Neither a_2 nor a_3 can be on the line joining p_1p_4 or p_1p_5 otherwise it will be a contradiction to 3.15.
This means that when we look at p_1, p_4, p_5 the connecting lines through them correspond to a max of one point to lie on l which is a contradiction to 3.18.
This implies that l is not a neighbour of one of the points from p_1, p_4, p_5. ☐

Theorem 3.20. *If I_i is the index of the point p_i, then*

$$m \geq \frac{1}{6}\sum_{i=1}^{n} I_i$$

Proof. We count the number of ordinary lines by counting the index of points in P. Each ordinary line can be counted at most six times, four times as a neighbour and twice for each of the points the ordinary line is incident on. ☐

Theorem 3.21. $m \geq 3n/7$

Proof. Let k be the number of points with order 2. By 3.15 $n-k$ points will have index at least three and k points will have index at least two. Plugging these into 3.20

$$m \geq \frac{3(n-k)+2k}{6} = \frac{3n-k}{6}$$

$$6m \geq 3n - k$$

$$\text{Since, } m \geq k \text{ (trivially)}$$

$$6m \geq 3n - m$$

$$m \geq \frac{3n}{7}$$

☐

4. **Allowable sequences**

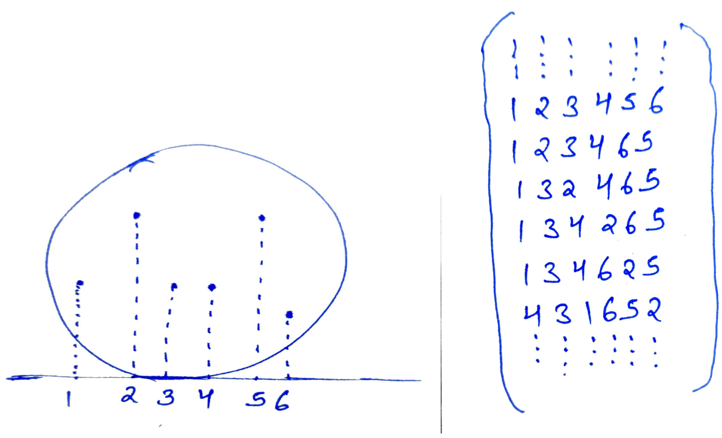
References: [Nil05], [GP93], [Dan20], [GT13]

Allowable sequences is a mathematical object which is used to store combinatorial information about a configuration of points.

Consider a finite configuration of points $P \subseteq \mathbb{Z}^2$. We can assume a big enough circle such that all of the points in P lie inside the circle. Let l be a tangent to the circle such that any connecting line is not perpendicular to l. We orthogonally project the points onto l. In the order of counterclockwise direction, the projection of points determines the permutation $(1, 2, .., n)$ and their corresponding points in P will be made fixed as $(p_1, p_2, ...p_n)$.

We can rotate l, keeping it as a tangent, counterclockwise around the circle. Whenever l passes orthogonally with a connecting line, this will generate a new permutation by reversing the indices of the points on that connecting line. For example, let l pass orthogonally with the connecting line through $(p_i, ..., p_j)$, then the substring $i...j$ will be reversed to $j...i$ to create a new permutation.

Eventually, following these sequences, we would have reversed the entire original sequence and some more sequences later we will be back to the original sequence. These permutations are called the circular sequence of configuration P. An example is shown below.



Properties of Circular sequence

- Circular sequence is double infinite. From a permutation, we can rotate l repeatedly clockwise or counter-clockwise to generate infinite permutations. Generally, we will stick to counter-clockwise rotations only.

- Circular sequence is periodic.
  Its *half-period* will be the number of permutations it takes to reverse the original sequence.
  Similarly, its *period* will be the number of permutations it takes to come back to the original sequence.

Allowable sequences are Circular sequences restricted by the following properties.

- From one permutation to the next, one or more non-overlapping substrings will be reversed.
- If some pair ij gets reversed in a permutation, then they will not be switched again till the entire original sequence is reversed.

We define

Definition 4.1 (Switch)**.** *Reversal of any substring in the sequence.*

Definition 4.2 (Transposition)**.** *Reversal of only two elements. A transposition can also be referred to as a simple switch.*

Properties of Allowable sequence

- $p_i, ..., p_k$ are collinear if and only if $i, ..., k$ all switch simultaneously.
- $p_i p_j$ is parallel to $p_k p_l$ if and only if ij and kl switch simultaneously.

The problem of proving that there exists an ordinary line given P, Sylvester's problem, is equivalent to showing that there exists a simple switch in the allowable sequence of P.

4.1. **Sylvester's problem using allowable sequences.**
Consider an allowable sequence of permutations $1, ..., n$. Each sequence in its half-period will be obtained from the previous sequence by reversal of a substring which is monotonically increasing. We want to show that there exists at least one simple switch.

We claim that the first switch involving a substring either to the right of n or to the left of 1 is a simple switch.

Without loss of originality, we assume n is switched before 1. We can similarly prove the other case by following the arguments given below. Since, we cannot reverse the entire string in one go, because otherwise that will mean all points are collinear, the first switch involving n does not involve 1.

Each switch for the substring involving n will contain the last digit n because it is the largest. When the first substring of this type is reversed, it gets converted into a monotonically decreasing substring that starts from n. This monotonically decreasing substring is "inert" now. It can no longer permute using the elements of its own.

Now, another switch involving n will have to take place because we were not allowed to reverse the entire string at once. This will again form a substring which is monotonically decreasing. If any of these switches is a simple one, we are already done, and so, we assume these switches involve at least three elements.

Thus, to the right of n, there are a bunch of inert substrings each of which is monotonically decreasing but going from the end of one substring to the start of the next substring, the value increases. That is, the length of the largest monotonically increasing substring is two. These can only be resolved through simple switches. Therefore, there exists at least one simple switch.

4.2. **Allowable sequence on Böröczky example X_{12}.**

The Böröczky examples are some of the geometry configurations for which the Dirac-Motzkin conjecture is tight.

Definition 4.3. *(Dirac–Motzkin conjecture) For a configuration of n points not all lying on the same line, there are at least $\lfloor n/2 \rfloor$ ordinary lines.*

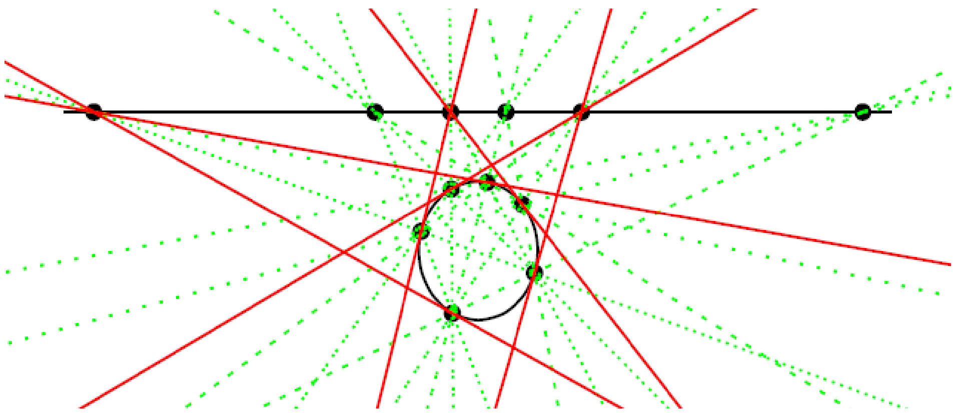
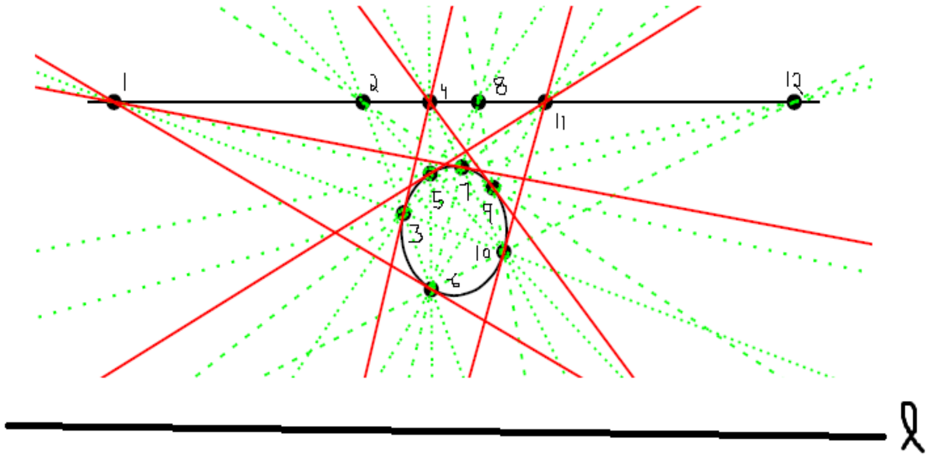


FIGURE 1. The Böröczky example X_{12}

The figure shown above is the Böröczky example X_{12}, i.e. a configuration of $n = 12$ points with 6 ordinary lines that are highlighted in red.

We choose a big enough circle and a tangent l to it as shown in the figure below (only the tangent is shown in bold). We then orthogonally project the points onto l and according to the order, left to right, we label them from 1 to 12.



The set of lines and the points lying on them are given in the table 1 given on the next page.

TABLE 1. Lines and points on it

Line name	Points on it	Line name	Points on it
a	1 2 4 8 11 12	l	4 9
b	1 6	m	8 7 6
c	1 3 10	n	8 5 3
d	1 5 9	o	8 9 10
e	1 7	p	11 9 6
f	2 3 6	q	11 7 3
g	2 5 10	r	11 5
h	2 7 9	s	11 10
i	4 5 6	t	12 10 6
j	4 3	u	12 9 3
k	4 7 10	v	12 7 5

We will now rotate l counterclockwise to create the allowable sequence of X_{12}. In the table 2, the first column represents the line which will pass orthogonally through l and in the second column the allowable sequence after reversal of points.

TABLE 2. Allowable sequence of X_{12}

	1 2 3 4 5 6 7 8 9 10 11 12
i	1 2 3 6 5 4 7 8 9 10 11 12
o	1 2 3 6 5 4 7 10 9 8 11 12
f	1 6 3 2 5 4 7 10 9 8 11 12
k	1 6 3 2 5 10 7 4 9 8 11 12
l	1 6 3 2 5 10 7 9 4 8 11 12
g	1 6 3 10 5 2 7 9 4 8 11 12
h	1 6 3 10 5 9 7 2 4 8 11 12
b	6 1 3 10 5 9 7 2 4 8 11 12
c	6 10 3 1 5 9 7 2 4 8 11 12
d	6 10 3 9 5 1 7 2 4 8 11 12
e	6 10 3 9 5 7 1 2 4 8 11 12
a	6 10 3 9 5 7 12 11 8 4 2 1
v	6 10 3 9 12 7 5 11 8 4 2 1
u	6 10 12 9 3 7 5 11 8 4 2 1
t	12 10 6 9 3 7 5 11 8 4 2 1
r	12 10 6 9 3 7 11 5 8 4 2 1
q	12 10 6 9 11 7 3 5 8 4 2 1
n	12 10 6 9 11 7 8 5 3 4 2 1
p	12 10 11 9 6 7 8 5 3 4 2 1
j, m	12 10 11 9 6 7 8 5 4 3 2 1
s	12 11 10 9 6 7 8 5 4 3 2 1

5. **Conclusions and Future work**

We started with defining Sylvester's problem and looked at some of its proofs by Kelly, Steinberg and Gallai. We then extended the Sylvester's problem to give a lower bound on the number of ordinary lines. We surveyed proofs by Melchoir, Motzkin and Kelly-Moser.

We then defined allowable sequences which view a configuration of points as a sequence of permutations. We solved the Sylvester's problem using allowable sequences and computed the allowable sequence on Böröczky example X_{12}.

A better lower bound than Kelly-Moser was given by Csima-Sawyer [CS93]. They proved there are $6n/13$ ordinary points. Furthermore, Green-Tao proved that the Dirac-Motzkin conjecture holds for all large enough point sets $n > n_0$ [GT13]. Unfortunately, this n_0 is quite large and is double exponential.

Our goal is to prove the Green-Tao theorem, or the Dirac-Motzkin conjecture, using allowable sequences.

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