A study of Sylvester Gallai configurations

for R&D2

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06 May, 2022

Contents

Sylvester's Problem

Consider P, a finite set of points

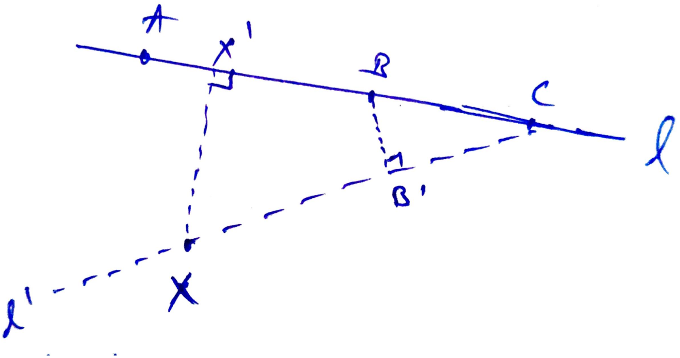
Connecting line: A line which contains two or more points from P.

Ordinary line: A line which contains exactly two points from P.

Sylvester's Conjecture: There exists at least one ordinary line.

Kelly's proof of Sylvester's problem

(X, l) be the pair with minimum perpendicular distance; l must be ordinary



Other proofs for Sylvester's problem

- Kelly (Euclidean)
- Steinberg
- Gallai (Projective)

Sylvester's Problem Extension

Minimum no of ordinary lines

Dirac-Motzkin Conjecture: There are at least $\lfloor n/2 \rfloor$ ordinary lines.

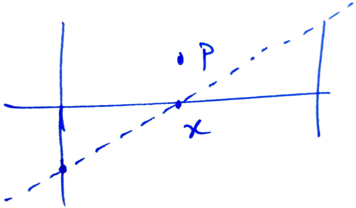
Motzkin's proof I

Consider a point $p \in P$ not lying on any ordinary line.
Let the smallest cell containing p be C.

Case 1 No such p exists

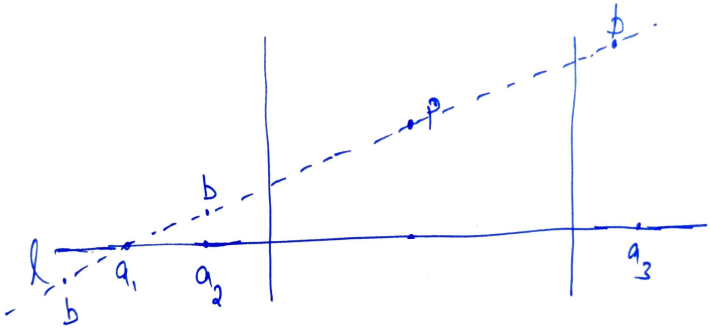
Case 2 Considering such a p exists.

Lemma 1: There can't be a point $x \in P$ lying on the edges of C.



Motzkin's proof II

Lemma 2: If there are at least three edges in the smallest cell C containing p, then all the edges of C are ordinary.



Motzkin's proof III

Consider the set of ordinary lines.

Lemma 3: A cell can not have more than one point.
Suppose p and q both lie in a cell.
No ordinary lines passes through p and q
Connecting line through $q \rightarrow$ neighbour to p
Contradiction to Lemma 2 $\square$

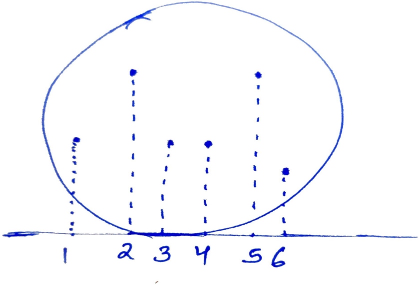
m ordinary lines pass through at most $2m$ points and divide the plane into a maximum of $\binom{m}{2} + 1$ regions

$2m + \binom{m}{2} + 1 \geq n$
$m \geq \sqrt{2n} - 2$
$O(\sqrt{n})$ ordinary lines

Other proofs for Sylvester's problem extension

- Melchoir: 3 ordinary points
- Motzkin: $O(\sqrt{n})$ ordinary lines
- Kelly-Moser: $3n/7$ ordinary lines

Allowable sequence I



$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 6 & 5 \\
1 & 3 & 2 & 4 & 6 & 5 \\
1 & 3 & 4 & 2 & 6 & 5 \\
1 & 3 & 4 & 6 & 2 & 5 \\
4 & 3 & 1 & 6 & 5 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

- Allowable sequence is double infinite and periodic
- Switch: Reversal of any substring , Simple switch: Reversal of only two elements
- Collinear points all switch simultaneously
- Parallel lines switch simultaneously, 1 2 3 4 5 6 $\rightarrow$ 1 3 2 4 6 5

Sylvester's problem using Allowable Sequence I

To show that there exists at least one simple switch

1 2 3 4 ... i ... j ... n

Claim: substring either to the left of 1 or to the right of n $\rightarrow$ simple switch

1 2 3 4 ... i ... n ... j
1 2 3 4 ... n ... i n-1 ... j

Similarly, two simple switches

Future work

- Dirac-Motzkin conjecture using Allowable sequences
- Coloured Extensions
- Csima-Sawyer; $6n/13$ ordinary points
- Green-Tao theorem

References I

[1] Ben Green and Terence Tao. "On sets defining few ordinary lines". In: *Discrete & Computational Geometry* 50.2 (2013), pp. 409–468.

[2] Leroy M Kelly and William OJ Moser. "On the number of ordinary lines determined by n points". In: *Canadian Journal of Mathematics* 10 (1958), pp. 210–219.

[3] Niranjan Nilakantan. "Extremal problems related to the Sylvester-Gallai theorem". In: *Combinatorial and Computational Geometry* 52 (2005), pp. 479–494.

That's all folks!

Thank You!

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