Math
for geometry

Inner product

- Take two vectors and produces a scalar
- Also called **scalar product** or **dot product**
- A Standard inner product is the Euclidean inner product

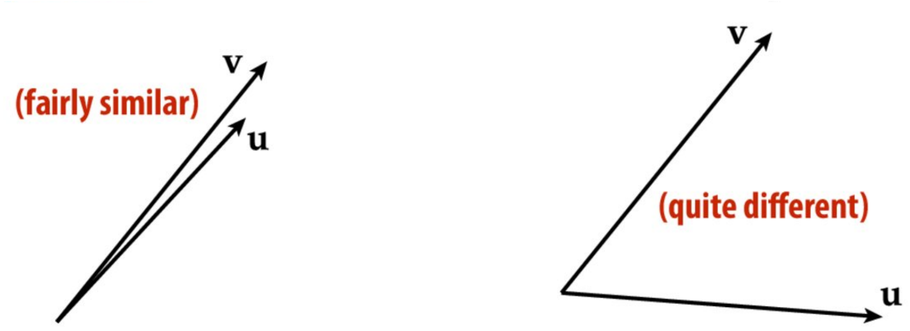
$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle := \sum_{i=1}^{n} u_i v_i$$

- Often convenient to express dot product via matrix product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathsf{T}} \mathbf{v} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i$$

Inner product

- Geometry interpretation: it measures how closely two vectors **align**, in terms of the directions they point

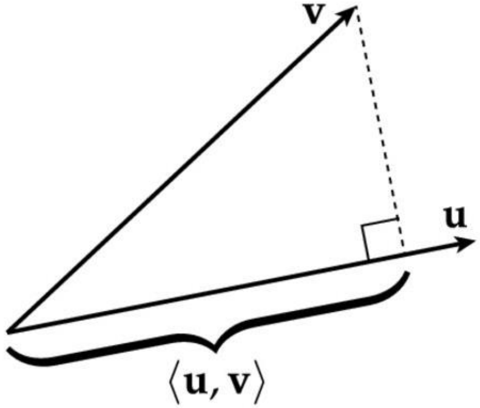


- If u and v are perpendicular, then <u, v> = 0
- If u and v are parallel, then <u, v> = |u||v|
- If u is a unit vector, <u,u> = 1
- In general, <u,u> = $|u|^2$
- |u| is called the **length**, **magnitude** or **norm** of the vector
- Obvious property: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

  u is just as well-aligned with v as v is with u

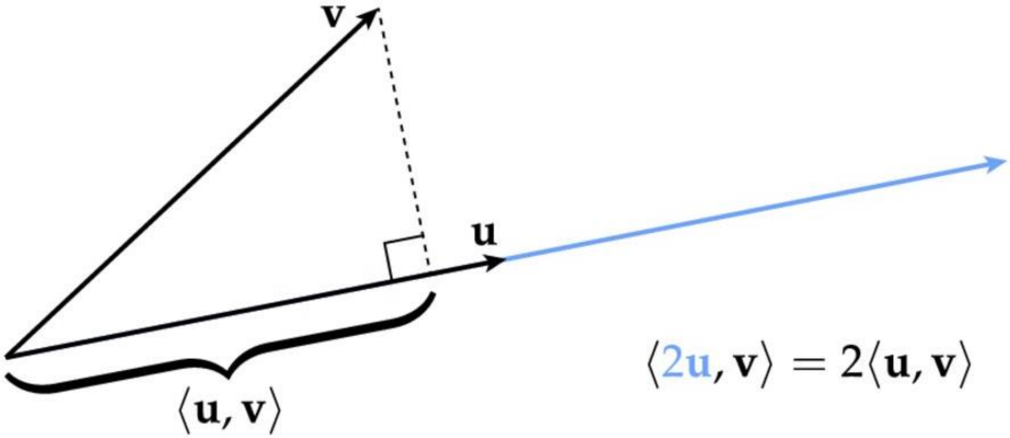
Inner Product – Projection & Scaling

- **For unit vectors $|u|=|v|=1$, an inner product measures the extent of one vector along the direction of the other:**



Q: Is this property symmetric?
I.e., is the length of v along u the same as the length of u along v?

- **If we scale either of the vectors, the inner product also scales:**

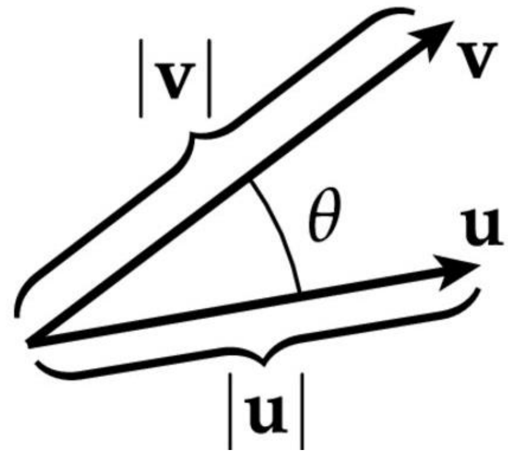


$$\langle 2\mathbf{u}, \mathbf{v} \rangle = 2\langle \mathbf{u}, \mathbf{v} \rangle$$

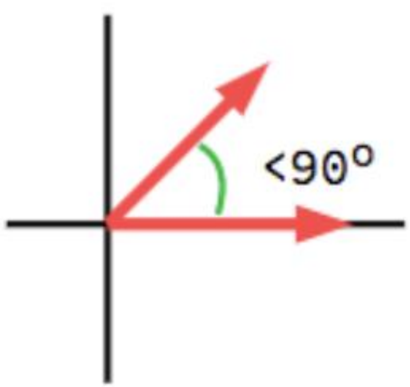
Euclidean inner product

- It relates to the angle between the two vectors

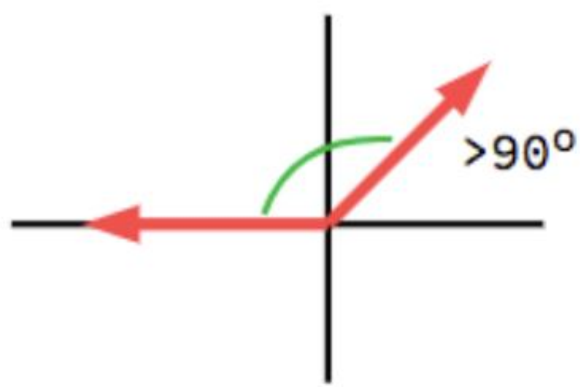
$$\langle \mathbf{u}, \mathbf{v} \rangle := |\mathbf{u}||\mathbf{v}| \cos(\theta)$$



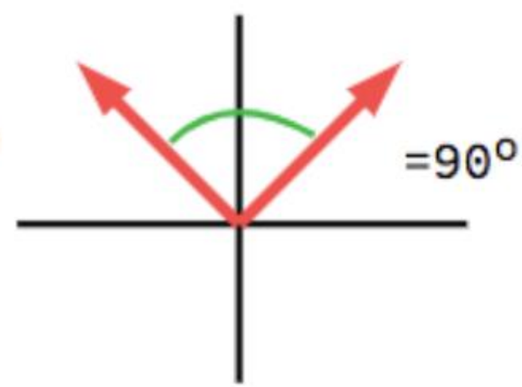
- $\cos\theta = \dfrac{<u,v>}{|u||v|}$



$< u, v > \; > 0$



$< u, v > \; < 0$

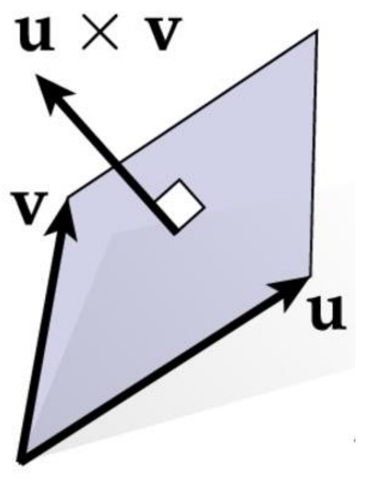


$< u, v > \; = 0$

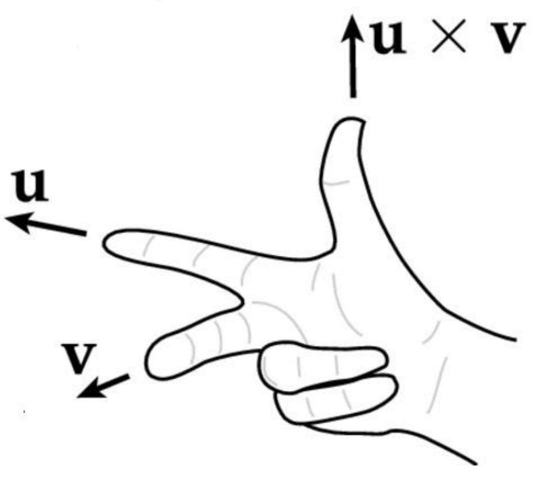
Cross product

- In 3D, cross product takes two vectors and produces a vector

- Geometrically:
  - Magnitude equal to parallelogram area
  - Direction orthogonal to both vectors
  - … but which way?



- Use "right hand rule"



- $u \times v = -(v \times u)$

Cross product

- **Uniquely determines coordinate formula:**

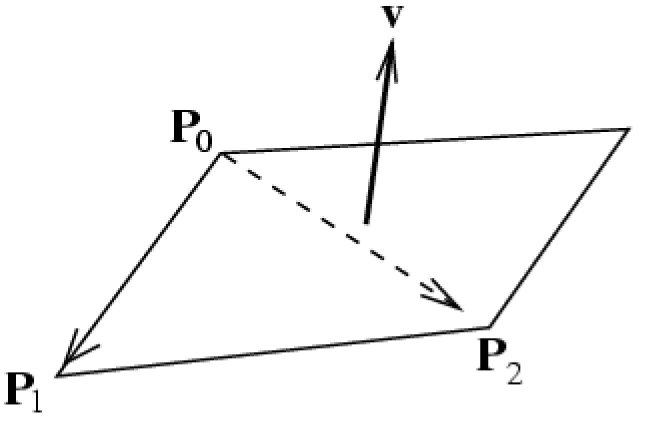
$$\mathbf{u} \times \mathbf{v} := \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

(mnemonic)

Cross product

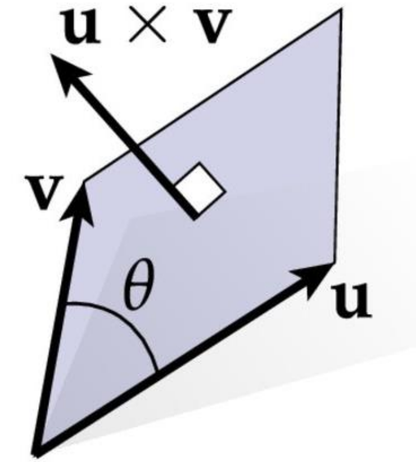
Compute the normal of a triangle



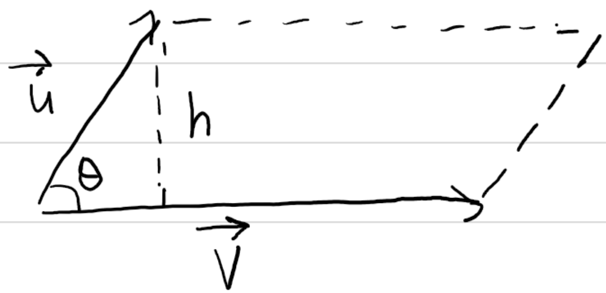
$$\vec{v} = (P_1 - P_0) \times (P_2 - P_0)$$

cross product

Compute area of a triangle



$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$$



$$\text{Area} = b\,h$$
$$= |\vec{v}|\,(|\vec{u}| \sin\theta)$$

Area of a triangle formed by $\vec{u}$ and $\vec{v}$ is

$$\frac{1}{2}\,|\vec{u} \times \vec{v}|$$

Area of a triangle defined by (P_0, P_1, P_2) is

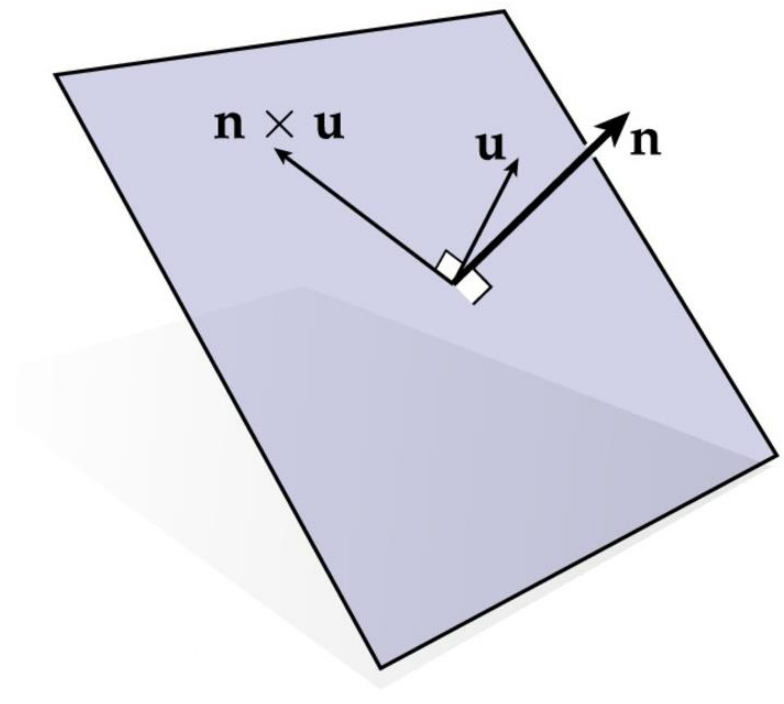
$$\frac{1}{2}\,|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}|$$

counter clockwise ordered, signed area is positive.

Cross product

Compute quarter rotation

- A simple but useful observation for manipulating vectors in 3D: cross product with a unit vector N is equivalent to a quarter-rotation in the plane with normal N



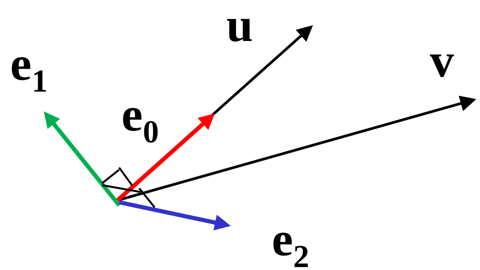
- **Q: What is N x (N x u)?**

Cross product

Compute orthonormal axes

- Given two non-parallel vectors **u** and **v**

- We can define a **3D orthonormal coordinate system**
  - Let the first axis be along **u** ($\mathbf{e}_0$ = normalize(**u**))
  - The second axis be normal to the plane spanned by **u** and **v** ($\mathbf{e}_2$ = normalize(**u** x **v**))
  - The last axis be orthonormal to the first two axes ($\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_0$)



Span

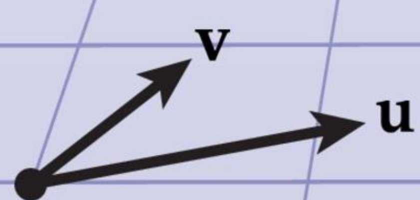
- Q: Geometrically, what is the span of two vectors u, v?

A: The span is the set of all vectors that can be written as a linear combination of u and v, i.e., vectors of the form

$$a\mathbf{u} + b\mathbf{v}$$

for any two numbers a, b.

More generally: $\mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \left\{ \mathbf{x} \in V \,\middle|\, \mathbf{x} = \sum_{i=1}^{k} a_i \mathbf{u}_i,\ a_1, \ldots, a_k \in \mathbb{R} \right\}$



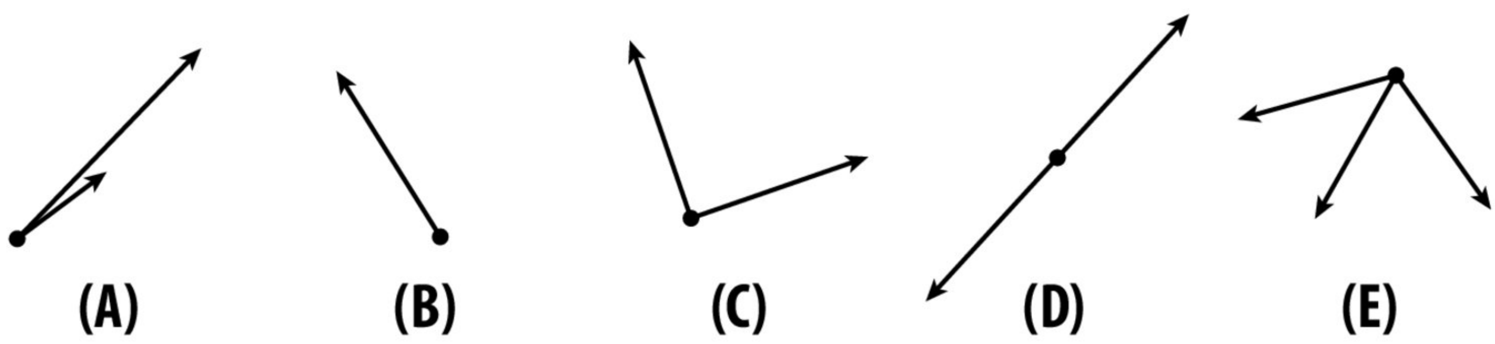
Basis

- Span is also closely related to the idea of a *basis*.
- In particular, if we have exactly n vectors $e_1, \ldots, e_n$ such that

$$\mathrm{span}(\mathbf{e}_1, \ldots, \mathbf{e}_n) = \mathbb{R}^n$$

- Then we say that these vectors are a basis for R^n.
- Note: many different choices of basis!

- Q: Which of the following are bases for the 2D plane (n=2)?

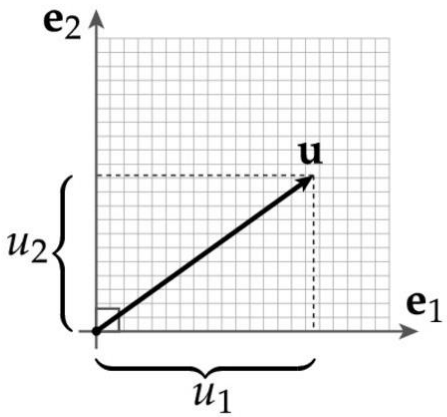


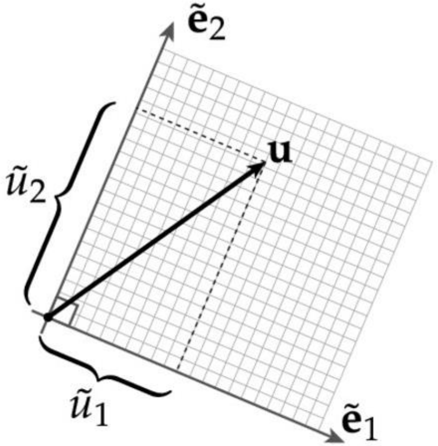
Orthonormal basis

- Most often, it is convenient to have to basis vectors that are (i) unit length and (ii) mutually orthogonal.
- In other words, if $e_1, \ldots, e_n$ are our basis vectors then

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 1, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

- This way, the geometric meaning of the sum $u_1^2 + \ldots + u_n^2$ is maintained: it is the length of the vector u.

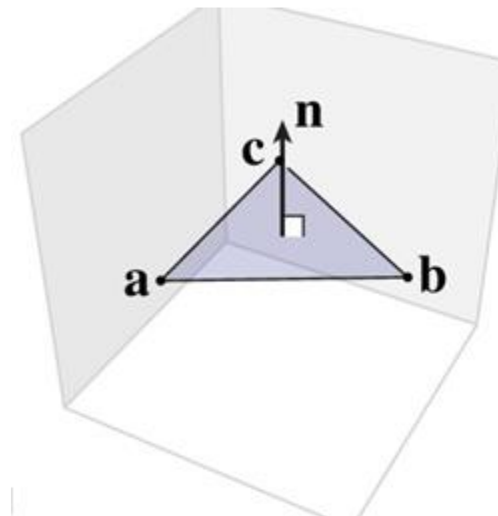




Common bug: projecting a vector onto a basis that is NOT orthonormal while continuing to use standard norm / inner product.

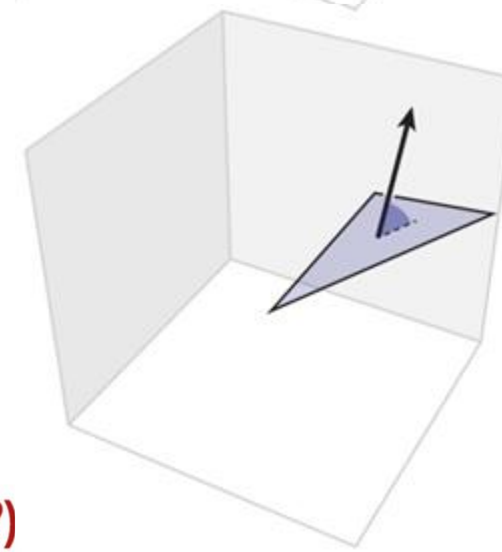
Points vs. Vectors in Homogeneous Coordinates

- **Homogeneous coordinates have another useful feature: distinguish between <u>points</u> and <u>vectors</u>**
- **Consider for instance a triangle with:**
  - **vertices** $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$
  - **normal vector** $\mathbf{n} \in \mathbb{R}^3$



- **Suppose we transform the triangle by appending "1" to** $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{n}$ **and multiplying by this matrix:**

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow$$



Normal is not orthogonal to triangle! (What went wrong?)

Points vs. Vectors in Homogeneous Coordinates

- Let's think about what happens when we multiply the normal vector $\mathbf{n}$ by our matrix:

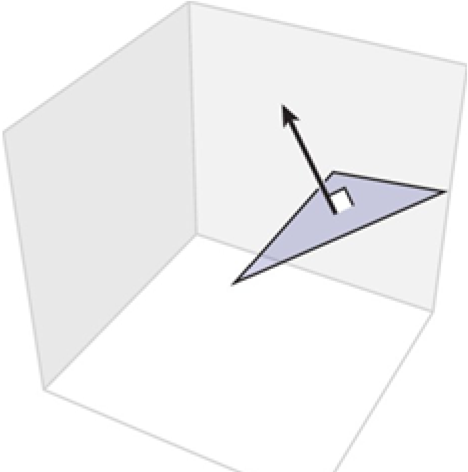
rotate normal around y by θ

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 1 \end{bmatrix}$$

translate normal by (u, v, w)

- But when we rotate/translate a triangle, its normal should <u>just rotate!*</u>
- Solution? Just set homogeneous coordinate to zero!
- Translation now gets ignored; normal is orthogonal to triangle

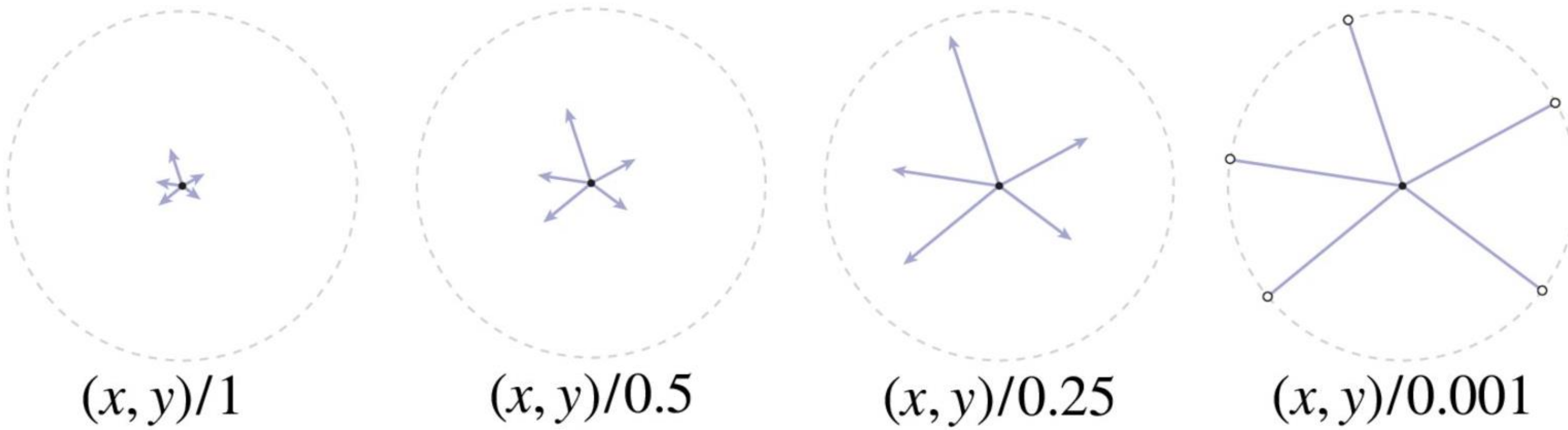
$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 0 \end{bmatrix}$$



*Recall that <u>vectors</u> just have direction and magnitude—they don't have a "basepoint"!

Points vs. Vectors in Homogeneous Coordinates

- What does division by w mean when w=0?
- Consider what happens as w $\rightarrow$ 0



Can think of vectors as "points at infinity" (sometimes called "ideal points")