Parametric Curves and Surfaces

Parametric Polynomial Curves

- The functions are all **polynomials** in the parameter.

$$x(u) = a_0 + a_1 u^1 + a_2 u^2 + \cdots + a_n u^n = \sum_{k=0}^{n} a_k u^k$$

$$y(u) = b_0 + b_1 u^1 + b_2 u^2 + \cdots + b_n u^n = \sum_{k=0}^{n} b_k u^k$$

- We'll assume that u varies from 0 to 1
- Pros:
  - efficient to compute
  - infinitely differentiable
  - Generalization to 3D curves is completely straightforward: add z(u)

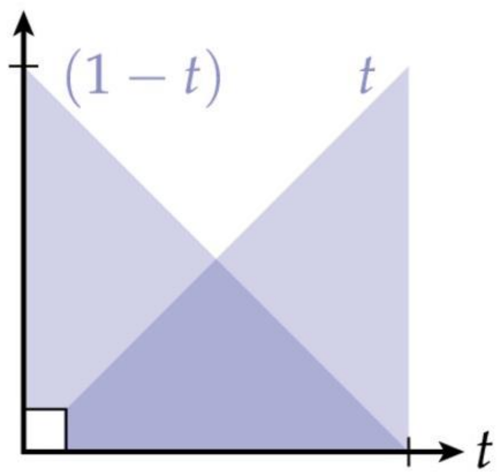
- **Power basis** (1, u, u^2,…) form does not reveal geometry of curves.

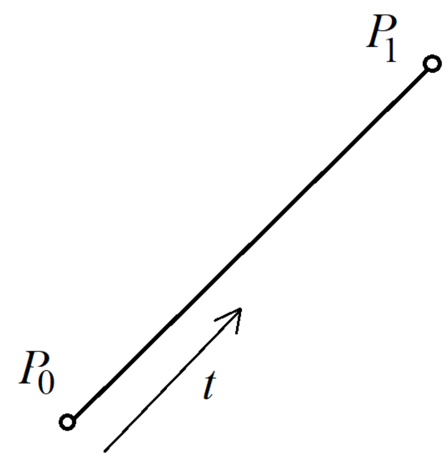
Linear Interpolation (1D)

- Interpolate values using linear interpolation; in 1D:

$$\hat{f}(t) = (1-t)f_i + tf_j$$

- Can think of this as a linear combination of two functions:





- (1-t) and t are two linear basis functions
  - Each gives the contribution of one point while t varies
- Can we use higher-order bases to get more interesting geometry?

Bernstein Basis

- Provide more flexibility by using higher-order polynomials

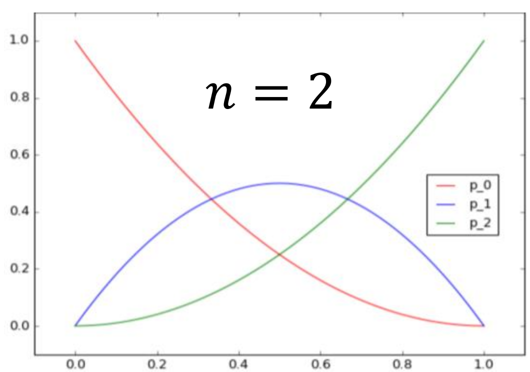
degree

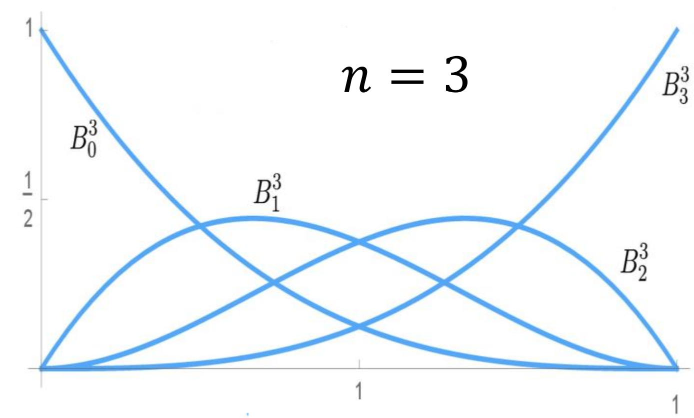
$0 \le x \le 1$

"n choose k"

$$B_k^n(x) := \binom{n}{k} x^k (1-x)^{n-k}$$

$k = 0, \ldots, n$





$$B_0^3(u) = (1-u)^3$$

$$B_1^3(u) = 3u(1-u)^2$$

$$B_2^3(u) = 3u^2(1-u)$$

$$B_3^3(u) = u^3$$

Bezier Curves

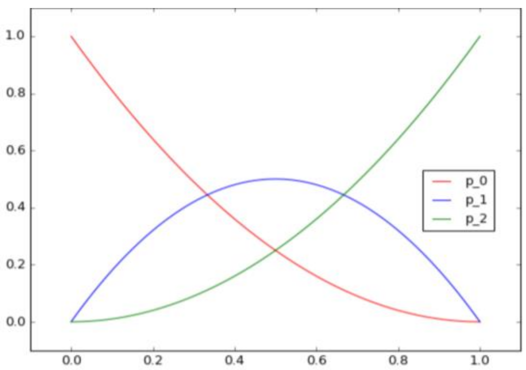
- A Bezier curve is a curve expressed in the Bernstein basis:

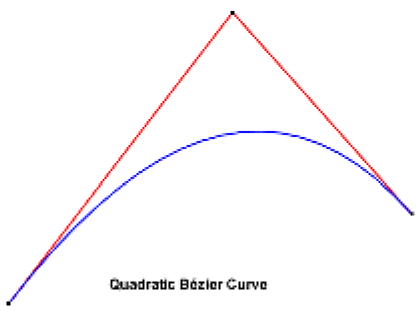
$$C(u) := \sum_{k=0}^{n} B_{n,k}(u) p_k$$

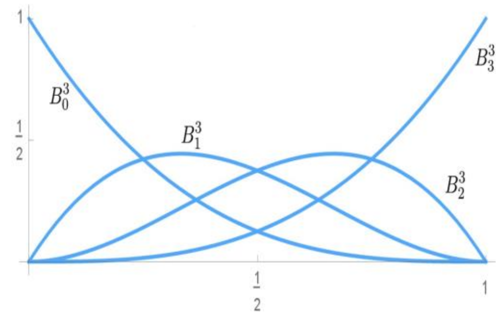
control points

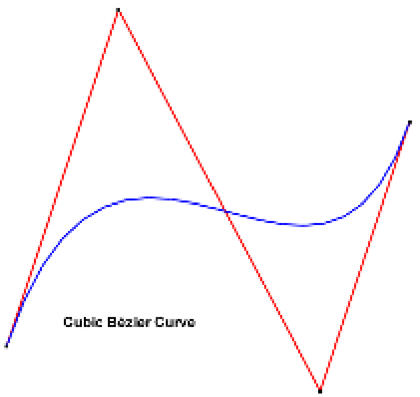
Blending control points using weights computed from basis functions

- For n=1, just get a line segment!
- For n=2, get quadratic Bezier (parabola)
- For n=3, get cubic Bezier









Bezier Curves

- Important features (see next few slides):
  - Interpolates endpoints
  - Tangent to end segments
  - Contained in convex hull
  - Symmetry
  - Affine invariant
  - Variational diminishing
  - …

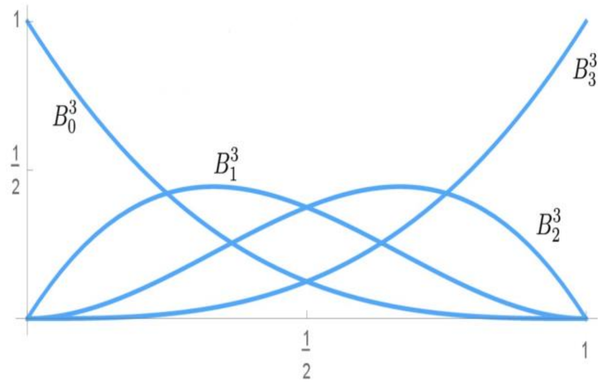
Endpoint interpolations

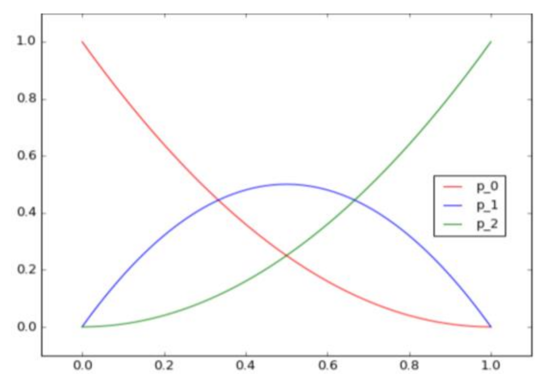
$$B_0^3(u) = (1-u)^3$$

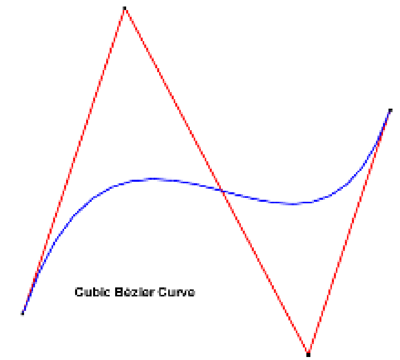
$$B_1^3(u) = 3u(1-u)^2$$

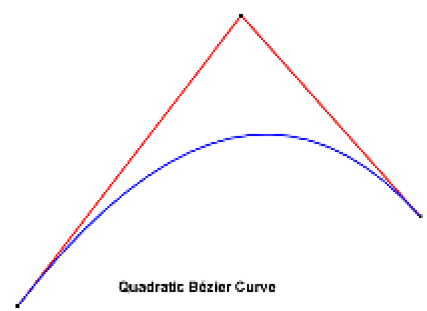
$$B_2^3(u) = 3u^2(1-u)$$

$$B_3^3(u) = u^3$$









At u=0, $B_0(0)=1$, the other functions = 0

At u=1, $B_3(1)=1$, the other functions = 0

Tangent to end segments

- For cubic Bezier curves, we have

$$Q(u) = (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u)V_2 + u^3 V_3$$

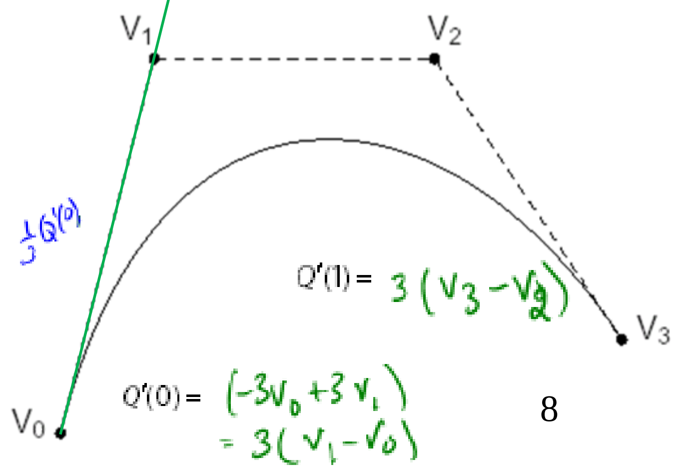
- Expanding the terms in u and rearranging:

$$\begin{aligned} Q(u) = \; & (-V_0 + 3V_1 - 3V_2 + V_3)u^3 \; + \\ & (3V_0 - 6V_1 + 3V_2)u^2 \; + \; (-3V_0 + 3V_1)u \; + \; V_0 \end{aligned}$$

- Differentiating:

$$\begin{aligned} Q'(u) = \; & 3(-V_0 + 3V_1 - 3V_2 + V_3)u^2 + \\ & 2(3V_0 - 6V_1 + 3V_2)u + \\ & (-3V_0 + 3V_1) \end{aligned}$$

- What are the tangents at endpoints?



Contained in Convex hull

- Desirable properties:

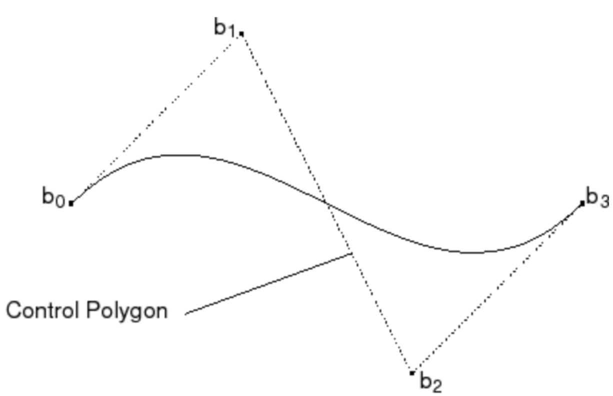
Affine combination

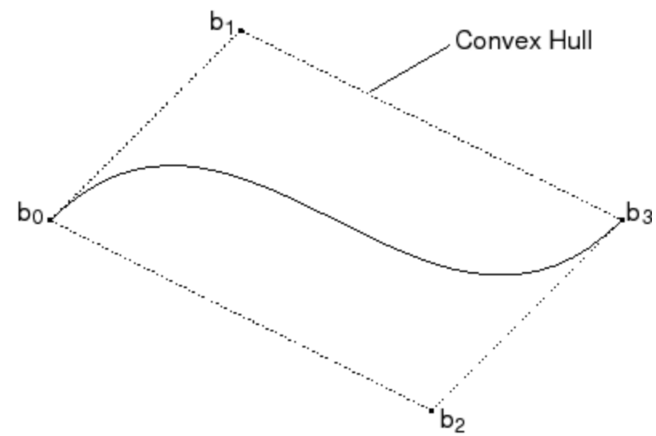
  - **sum** of all coefficients is always **exactly 1** (a.k.a. , a "partition of unity")

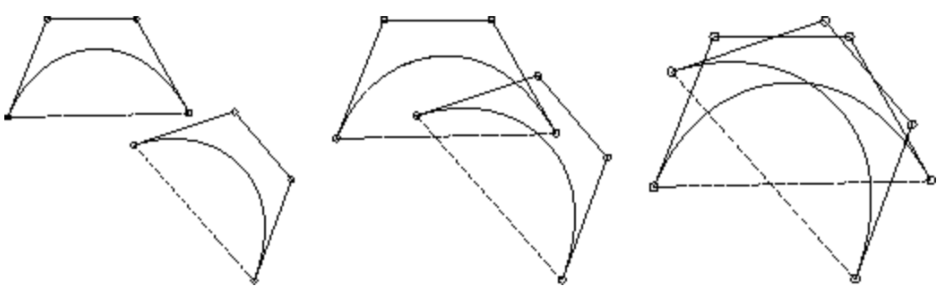
Convex combination

  - each Bernstein coefficient is **positive**

The curve is generated by **convex combinations** of the control points and therefore lies within the **convex hull** of the control points.







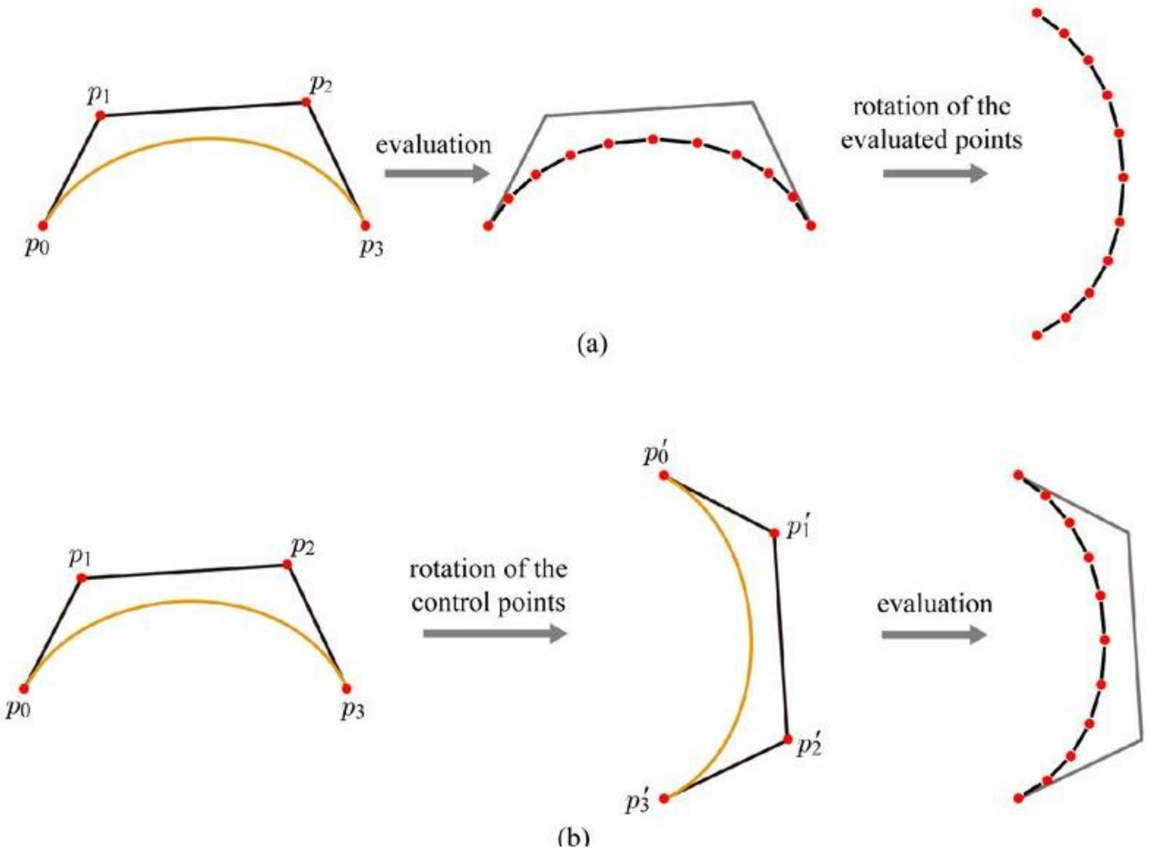
Comparison of convex hull as a mean of detecting intersection

Symmetry

- Relabeling the control points from P_0, P_1,...,P_n to P_n, P_{n-1}..., P_0 and using the symmetry property of Bernstein polynomials, we get the same Bezier curve

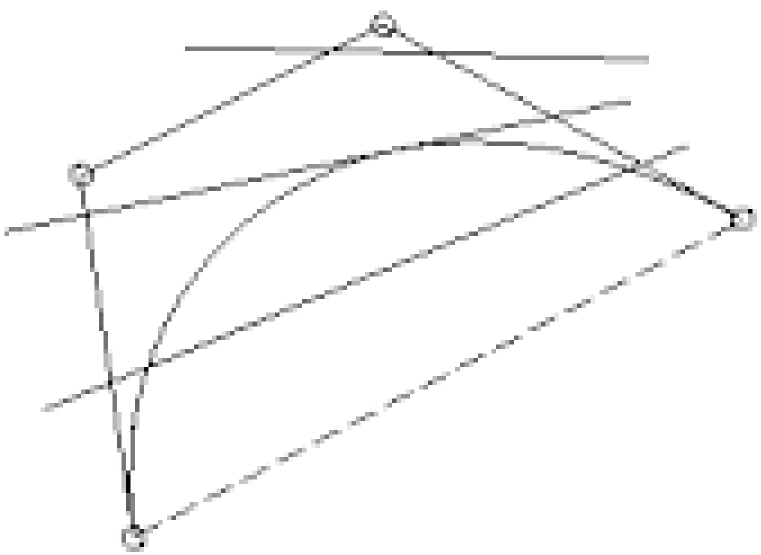
Affine invariance

- Partition of unity and non-negativity also imply **affine invariance**

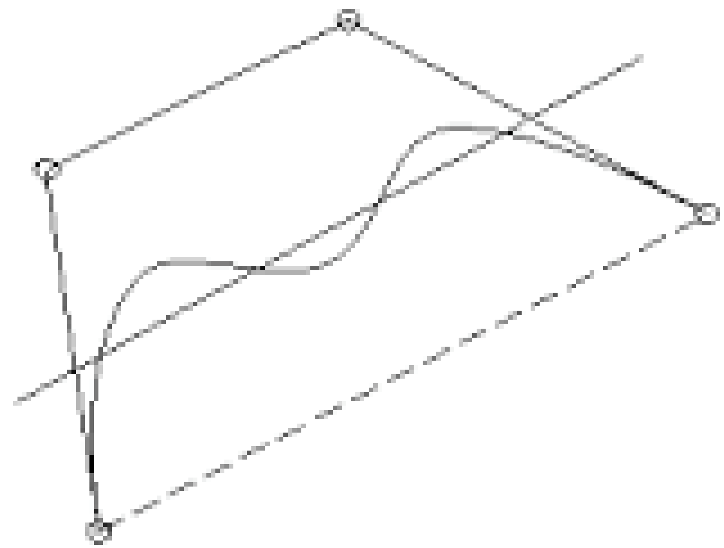


Variational Diminishing

- Bezier curves are smoother than the polygon formed by their control points.
- Any line drawn through the curve has equal or fewer intersections with the curve than with the control polygon.



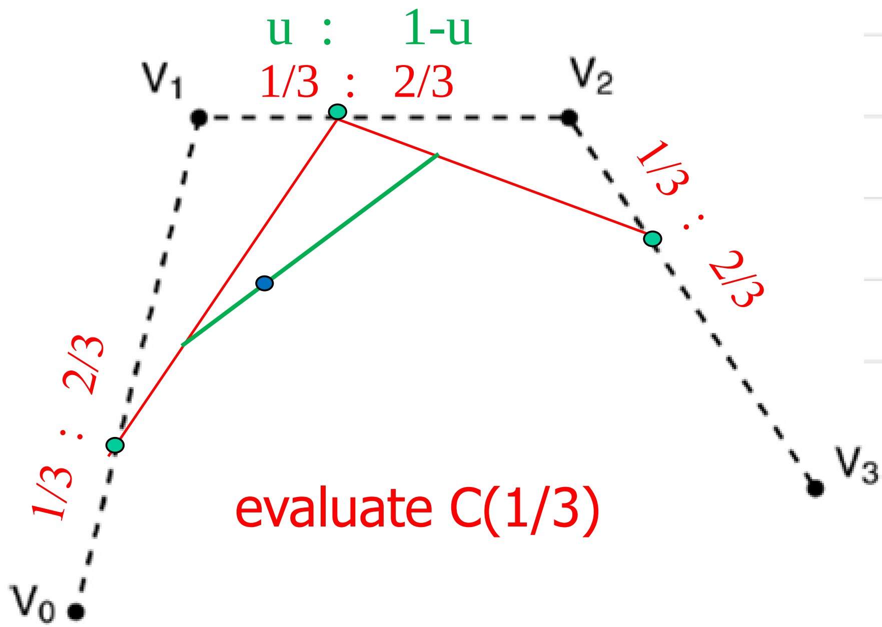
Possible

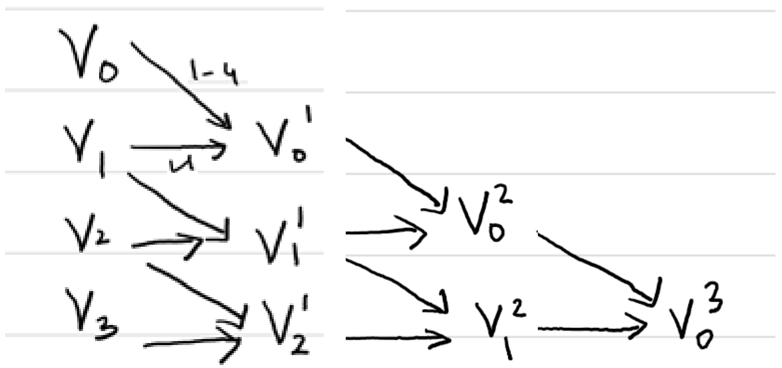


Impossible

Evaluation by recursive interpolations

- A Bezier curve at a particular u can be evaluated using **deCasteljau's algorithm** (recursive linear interpolation)

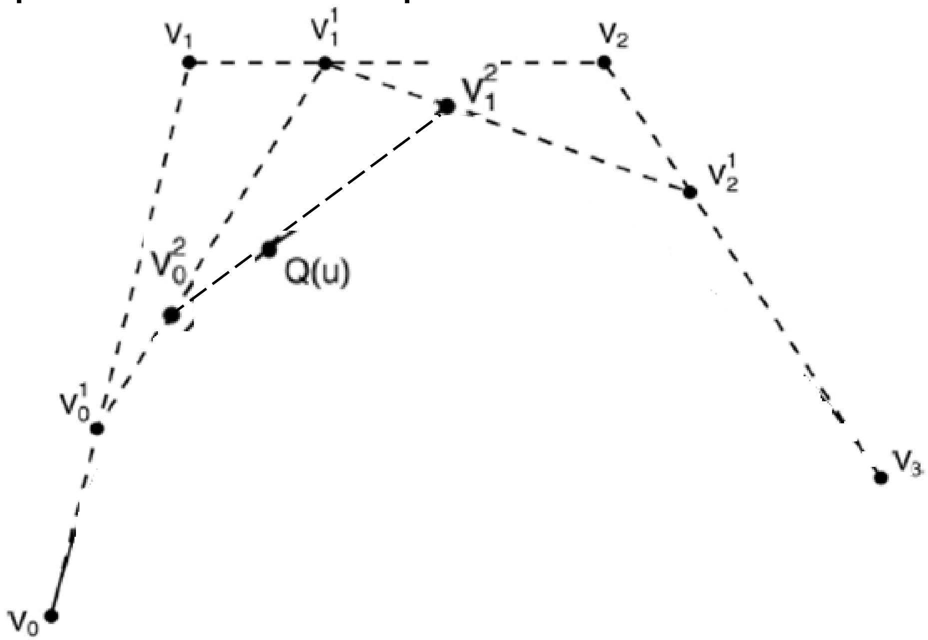




- How many linear interpolations for a degree n Bezier curve?

deCasteljau's algorithm

There are several ways to define Bezier curves.
One way is using Bernstein polynomials.
Here, we show another way: via repeated linear interpolations



$$V_0^1 = (1-u)V_0 + uV_1$$

$$V_1^1 = (1-u)V_1 + uV_2$$

$$V_2^1 = (1-u)V_2 + uV_3$$

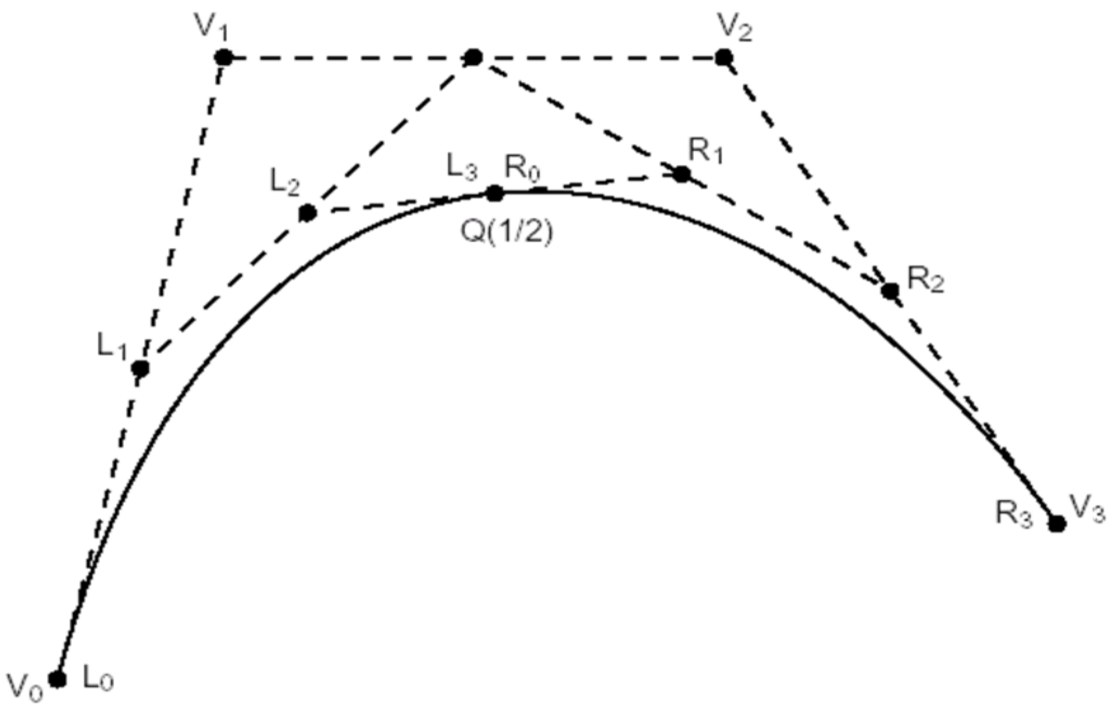
$$V_0^2 = (1-u)V_0^1 + uV_1^1$$

$$V_1^2 = (1-u)V_1^1 + uV_2^1$$

$$\begin{aligned} Q(u) &= (1-u)V_0^2 + uV_1^2 \\ &= (1-u)[(1-u)V_0^1 + uV_1^1] + u[(1-u)V_1^1 + uV_2^1] \\ &= (1-u)[(1-u)\{(1-u)V_0 + uV_1\} + u\{(1-u)V_1 + uV_2\}] + \ldots \\ &= (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u)V_2 + u^3 V_3 \end{aligned}$$

Splitting Bezier curves

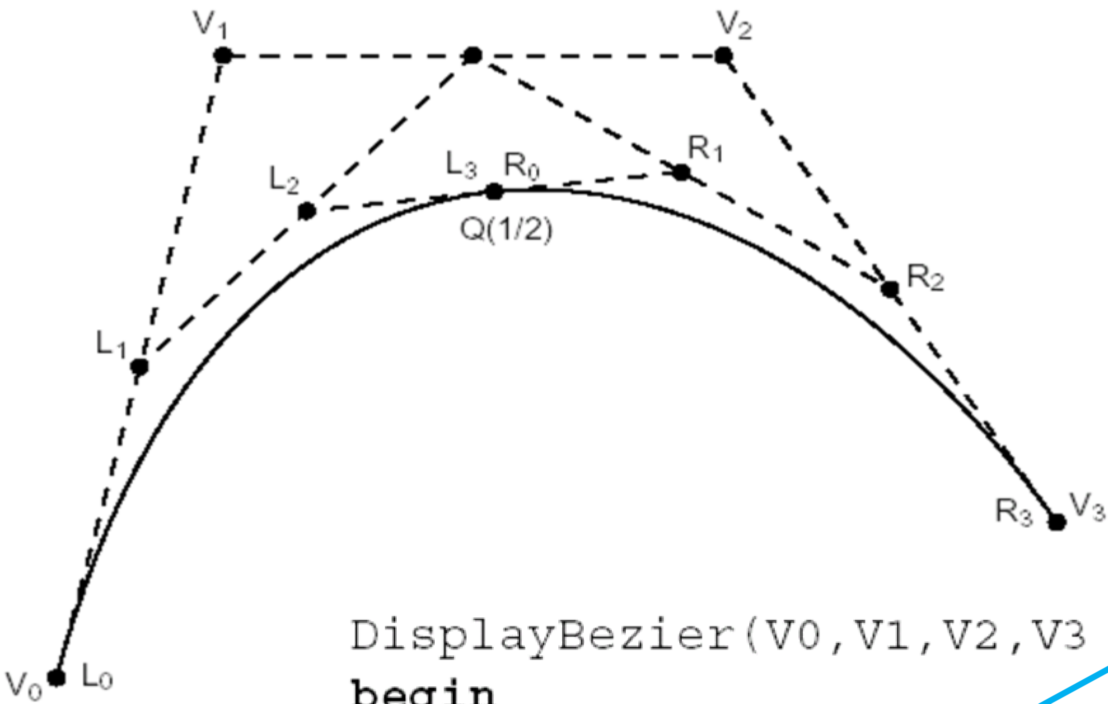
- **deCasteljau's algorithm** is useful for splitting a Bezier curve into two Bezier curves
- Original Bezier curve represented by V_0, V_1 , V_2 , V_3
- Split into two Bezier curves represented by L_0, L_1 , L_2 , L_3 and R_0, R_1 , R_2 , R_3



Useful for

- finding line-Bezier intersection or Bezier-Bezier intersections
- Adaptive display

Subdivide: display



$$\frac{|V_0 - V_1| + |V_1 - V_2| + |V_2 - V_3|}{|V_0 - V_3|} < 1 + \varepsilon$$

```
DisplayBezier(V0,V1,V2,V3 )
begin

   if ( FlatEnough(V0,V1,V2,V3))
       Line(V0,V3);
   else
       Subdivide(V[]) ⇒ L[], R[]

       DisplayBezier(L0,L1,L2,L3);
       DisplayBezier(R0,R1,R2,R3);
end
```

Power-basis form vs Bernstein-basis form

- Mathematically, Bezier curves are equivalent to polynomial curves expressed in power-basis form
- Bezier curves are expressed in terms of meaningful geometric elements (**control points**).

$$Q(u) = (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u) V_2 + u^3 V_3$$

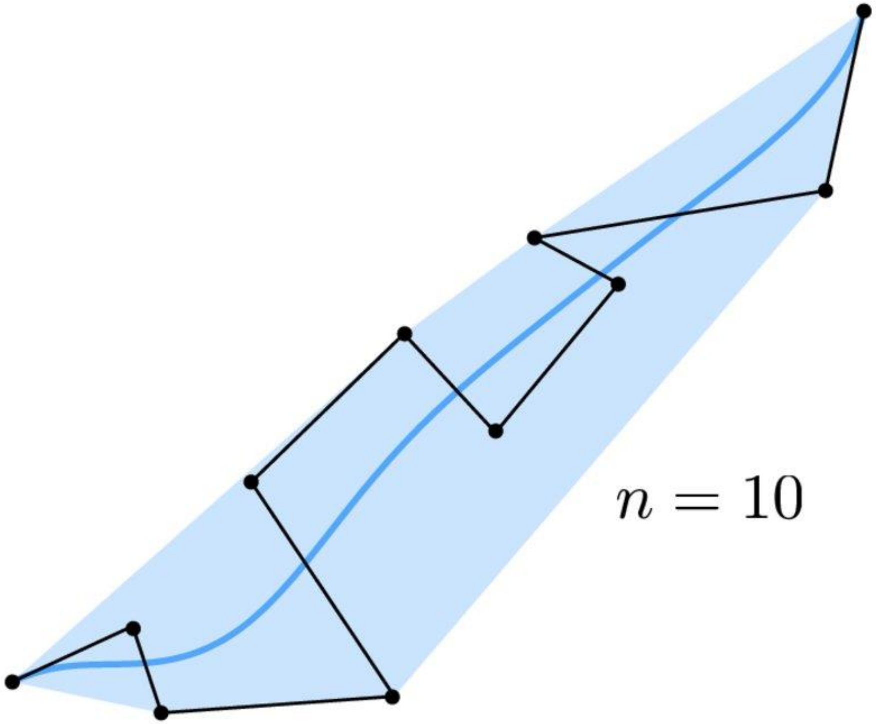
$$= V_0 + (-3V_0 + 3V_1)u + (3V_0 - 6V_1 + 3V_2)u^2 + (-V_0 + 3V_1 - 3V_2 + V_3)u^3$$

$$= \begin{bmatrix} V_{0,x} + (-3V_{0,x} + 3V_{1,x})u + (3V_{0,x} - 6V_{1,x} + 3V_{2,x})u^2 + (-V_{0,x} + 3V_{1,x} - 3V_{2,x} + V_{3,x})u^3 \\ V_{0,y} + (-3V_{0,y} + 3V_{1,y})u + (3V_{0,y} - 6V_{1,y} + 3V_{2,y})u^2 + (-V_{0,y} + 3V_{1,y} - 3V_{2,y} + V_{3,y})u^3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_0 + a_1 u + a_2 u^2 + a_3 u^3 \\ b_0 + b_1 u + b_2 u^2 + b_3 u^3 \\ 1 \end{bmatrix} = \begin{bmatrix} x(u) \\ y(u) \\ 1 \end{bmatrix}$$

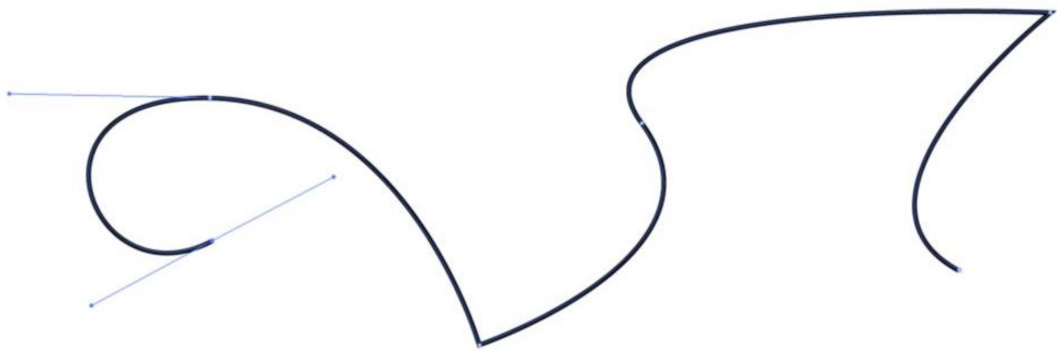
Higher-order polynomials…?

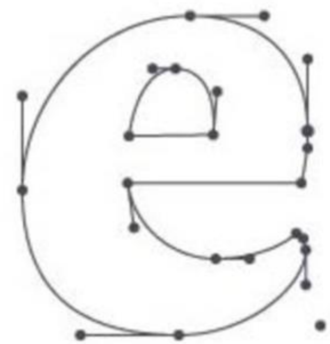
- What if we want more interesting curves than cubic polynomials?
- Use higher-degree polynomials?
  - have more wiggles
  - Very hard to control



Piecewise Bezier curves

- Piece together several Bezier curves to get a Bezier **spline**
- Widely-used (illustrator, fonts, SVG, etc.)





- Formally, **piecewise** Bezier curve:

piecewise Bézier

$$C(u) := C_i\left(\frac{u - u_i}{u_{i+1} - u_i}\right), \qquad u_i \le u < u_{i+1}$$

single Bézier

Bezier splines: continuity

- **First-order continuity** means continuous first derivative

$$Q'(u) = \frac{dQ(u)}{du}$$

- Q′(u) is called the **tangent**
- If we think of u as "time" and Q(u) as the path of a particle through space, Q′(u) represents the **velocity** (direction and magnitude)

- Second-order continuity means continuous second derivative

$$Q''(u) = \frac{d^2Q(u)}{du^2}$$

- Q″(u) represents **acceleration** if Q(u) represents a motion curve

Parametric Continuity

- In general, C^n continuity is defined as follows:

$$Q(u) \text{ is } C^n \text{ continuous}$$
$$\text{iff}$$
$$Q^{(i)}(u) \text{ is continuous for } 0 \le i \le n$$

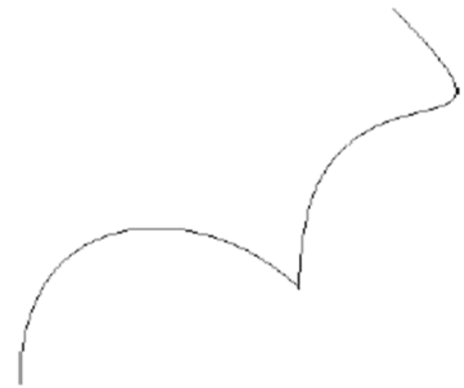
These conditions are *nested*

Two curves $Q_1(u)$ and $Q_2(u)$

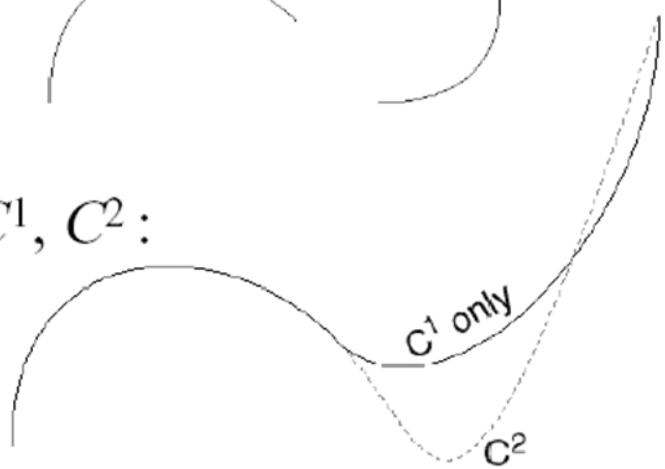
$$Q_1^{(i)}(1) = Q_2^{(i)}(0)$$

C^{-1}:

C^0:



C^1, C^2 :



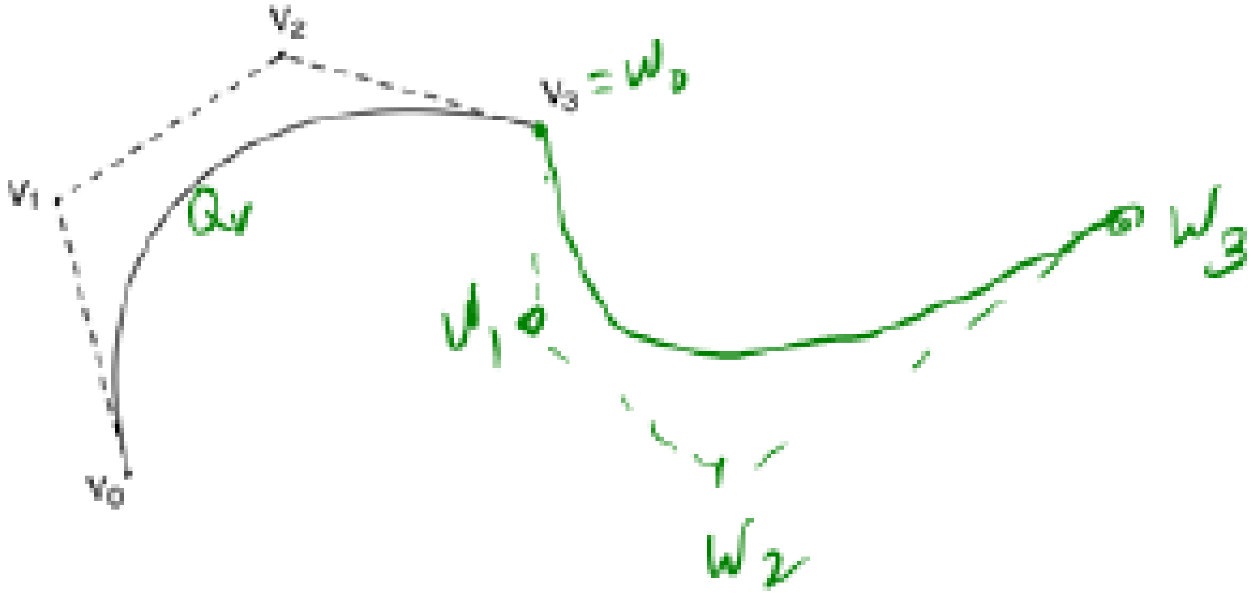
Ensuring C^0 continuity

- $Q_v(u)$ defined by V_0, V_1, V_2, V_3
- $Q_w(u)$ defined by W_0, W_1, W_2, W_3
- Joint is C^0 continuous if

$$C^0 : Q_V(1) = Q_W(0)$$

- What constraint does this place on W_0, W_1, W_2, W_3 ?

$$\boxed{W_0 = V_3}$$



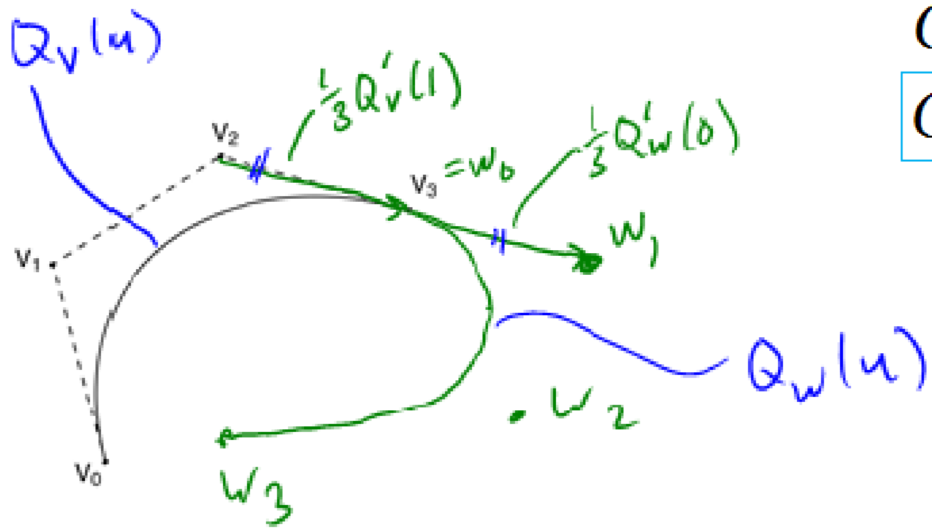
Ensuring C^1 continuity

- Joint is C^1 continuous if

$$C^0 : Q_V(1) = Q_W(0)$$

$$C^1 : Q_V'(1) = Q_W'(0)$$

- What additional constraint does this place on (W_0, W_1, W_2, W_3)?

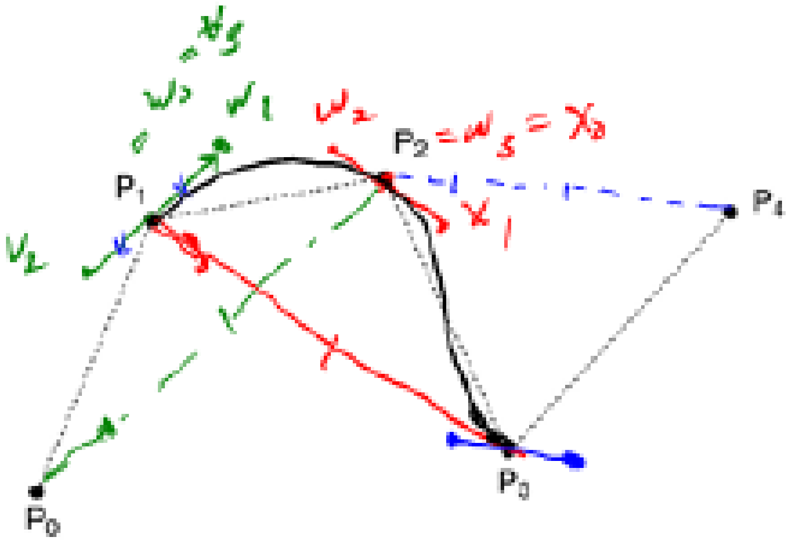


$$C^0 \Rightarrow W_0 = V_3$$

$$\boxed{C^1 \Rightarrow W_1 - W_0 = V_3 - V_2}$$

The C^1 Bezier spline

- How then could we construct a curve passing through a set of points $P_0, \ldots, P_n$ with C^1 continuity?
- We can devise a scheme to place the Bezier control points (consider the interior points; end points need special treatment)



$$V_0 = P_1$$

$$V_1 = P_1 + \tfrac{1}{6}(P_2 - P_0)$$

$$V_2 = P_2 - \tfrac{1}{6}(P_3 - P_1)$$

$$V_3 = P_2$$

Second-order continuity

- To develop C² splines, we need second-order derivatives

$$Q'(u) = 3(-V_0 + 3V_1 - 3V_2 + V_3)u^2 + 2(3V_0 - 6V_1 + 3V_2)u + (-3V_0 + 3V_1)$$

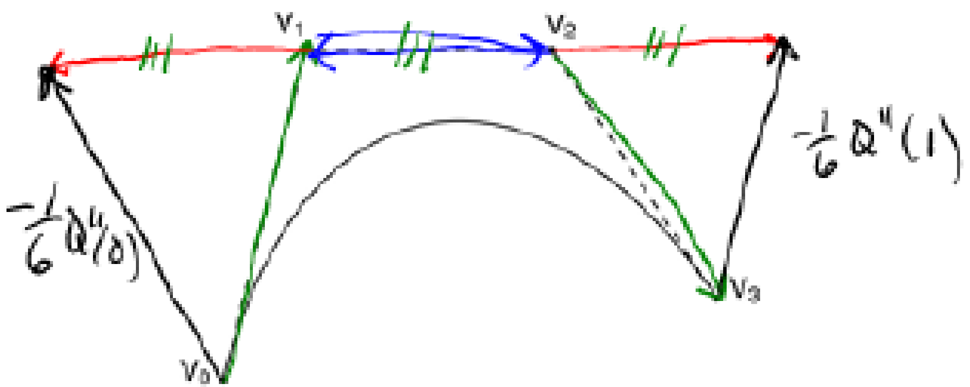
$$Q''(u) = 6(-V_0 + 3V_1 - 3V_2 + V_3)\,u + 2(3V_0 - 6V_1 + 3V_2)$$

At the two endpoints:

$$Q''(0) = 6(V_0 - 2V_1 + V_2) = -6[(V_1 - V_0) + (V_1 - V_2)]$$

$$-\frac{1}{6} Q''(0) = (V_1 - V_0) + (V_1 - V_2)$$

$$Q''(1) = 6(V_1 - 2V_2 + V_3) = -6[(V_2 - V_3) + (V_2 - V_1)]$$

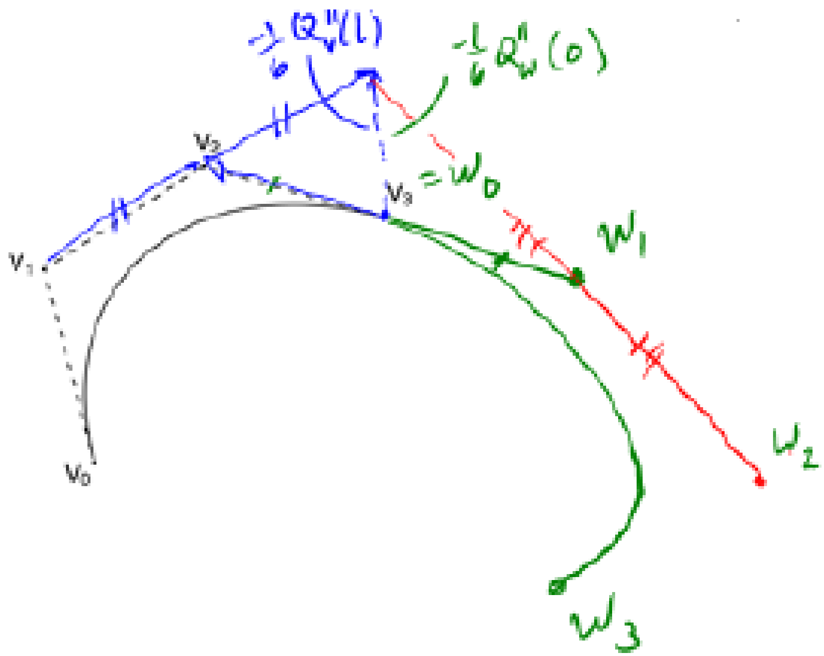


Ensuring C^2 continuity

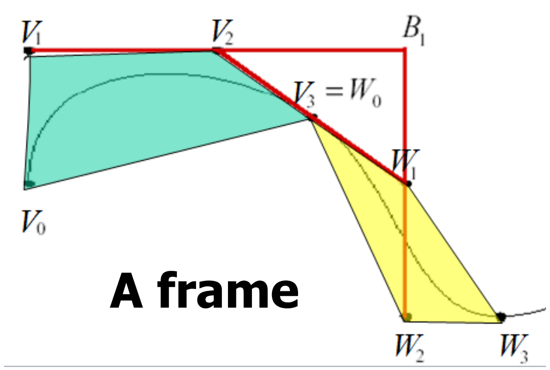
- The joint is C^2 continuous if

$$C^0 : Q_V(1) = Q_W(0)$$
$$C^1 : Q_V'(1) = Q_W'(0)$$
$$C^2 : Q_V''(1) = Q_W''(0)$$

- What additional constraint does this place on (W_0, W_1, W_2, W_3)?

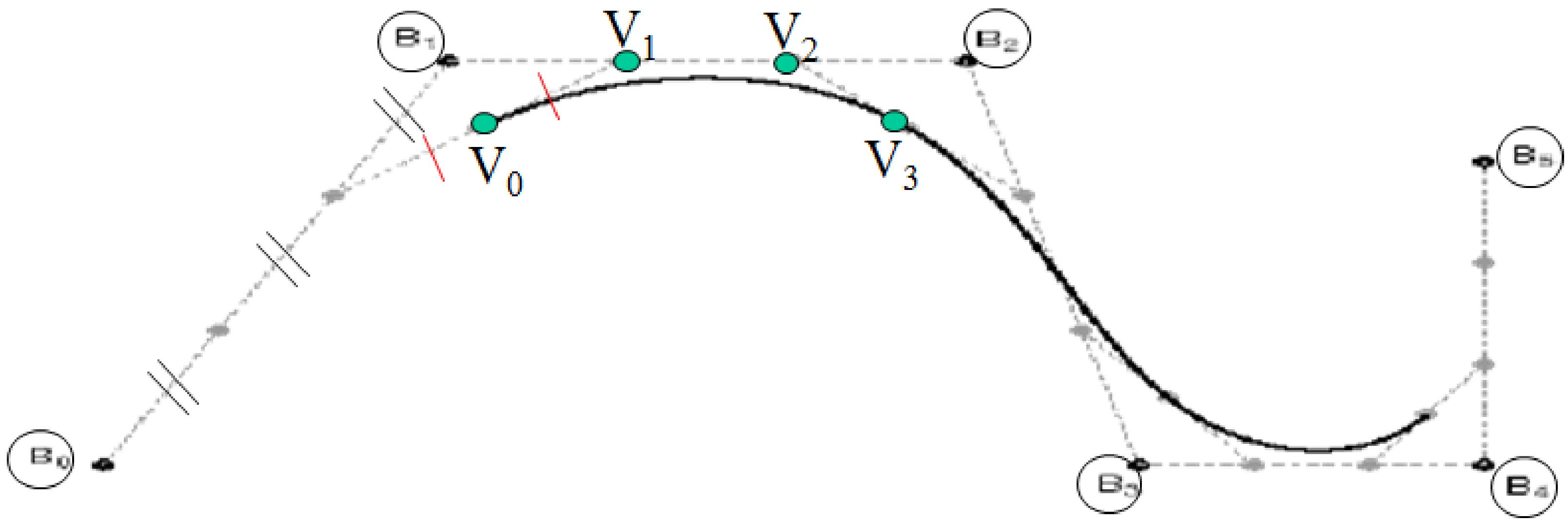


$$\boxed{\begin{aligned}&(W_1 - W_0) + (W_1 - W_2)\\ =\ &(V_2 - V_3) + (V_2 - V_1)\end{aligned}}$$



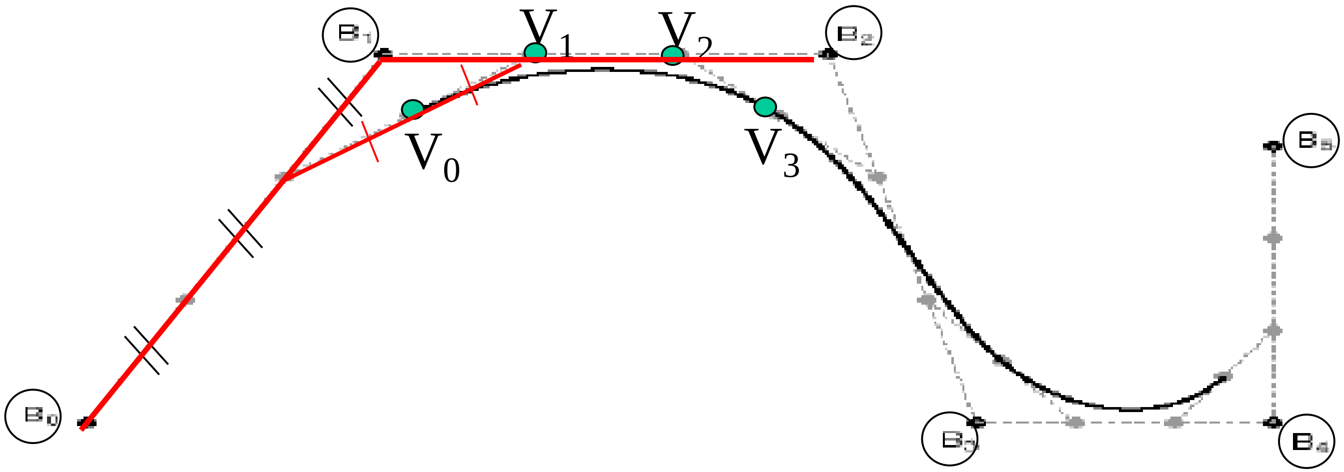
Constructing C^2 Bezier spline

- Given the **corner points** (i.e., $\mathbf{B_i}$) of the **A-frames**
- Let's build a C^2 continuous **approximating** spline



- The $\mathbf{B}_i$ points are called **de Boor points**
- We will see how to define **B-splines** directly using these points later

Constructing C^2 Bezier spline



- Define the Bezier control points (V) in terms of the de Boor points (B):

$$V_1 = \underline{\ 2/3\ }\ B_1 + \underline{\ 1/3\ }\ B_2$$

$$V_2 = \underline{\ 1/3\ }\ B_1 + \underline{\ 2/3\ }\ B_2$$

$$V_0 = \underline{\ 1/2\ }\ [\underline{\ 1/3\ }\ B_0 + \underline{\ 2/3\ }\ B_1] + \underline{\ 1/2\ }\ [\underline{\ 2/3\ }\ B_1 + \underline{\ 1/3\ }\ B_2]$$

$$= \underline{\ 1/6\ }\ B_0 + \underline{\ 4/6\ }\ B_1 + \underline{\ 1/6\ }\ B_2$$

$$V_3 = \underline{\ 1/6\ }\ B_1 + \underline{\ 4/6\ }\ B_2 + \underline{\ 1/6\ }\ B_3$$

Constructing C^2 Bezier spline

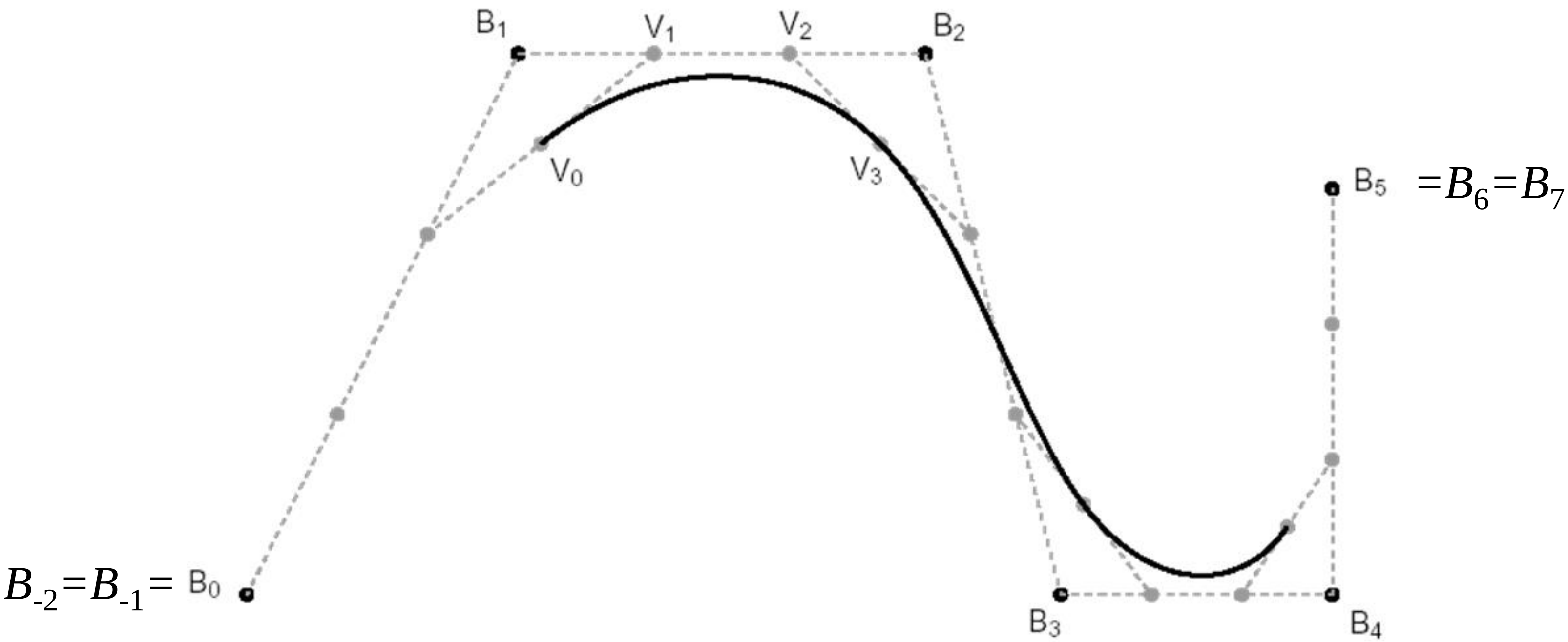
- Express the **Bezier points** in terms of the given **de Boor points:**

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1/6 & 2/3 & 1/6 & 0 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 1/6 & 2/3 & 1/6 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

Endpoint interpolation

- How do we define the first and last curve segments?
- To get endpoint interpolation while still using the same matrix is to simply **repeat the endpoints** (*multiplicity = 3* for *cubics* to create *two curve segments* at each end)



Comparison

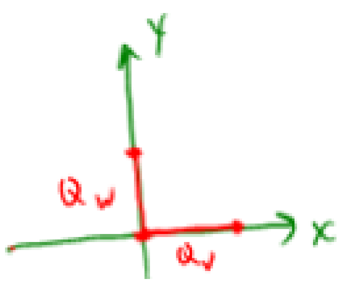
	Local control	interpolatory	C^2
C^1 interpolating Bezier spline	Y	Y	N
C^2 approximating Bezier spline	Y	N	Y
C^2-interpolating spline	N	Y	Y

Note that it is not possible to build a **C^2 continuous interpolating spline** using a *local* procedure. We will need to set up the C^2 continuous constraints for all the V_i points and solve the resulting system of linear equations.

https://courses.cs.washington.edu/courses/csep557/10au/lectures/c2-interp.pdf

Reparameterization

- We have so far consider **parametric continuity**, i.e. continuity of derivatives w.r.t. the parameter u
- This form of continuity makes sense particularly if we really are describing a particle moving over time and want its **motion** (e.g., velocity and acceleration) to be smooth.
- But, what if we're thinking only in terms of the shape of the curve? Is the parameterization actually **intrinsic to the shape**, i.e., is it the case that a shape has only one parameterization?



Piece together 2 curves

C^0

$$\begin{cases} Q_v(u) = \begin{bmatrix} 1-u \\ 0 \end{bmatrix} \\ Q_w(u) = \begin{bmatrix} 0 \\ u \end{bmatrix} \\ Q'_v(u) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ Q'_w(u) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

$\xrightarrow{\qquad}$ reparameterize

C^1

$$\begin{cases} \tilde{Q}_v(u) = \begin{bmatrix} (1-u)^2 \\ 0 \end{bmatrix} \\ \tilde{Q}_w(u) = \begin{bmatrix} 0 \\ u^2 \end{bmatrix} \\ \tilde{Q}'_v(u) = \begin{bmatrix} 2(1-u) \\ 0 \end{bmatrix} \\ \tilde{Q}'_w(u) = \begin{bmatrix} 0 \\ 2u \end{bmatrix} \end{cases}$$

G^n (Geometric) Continuity

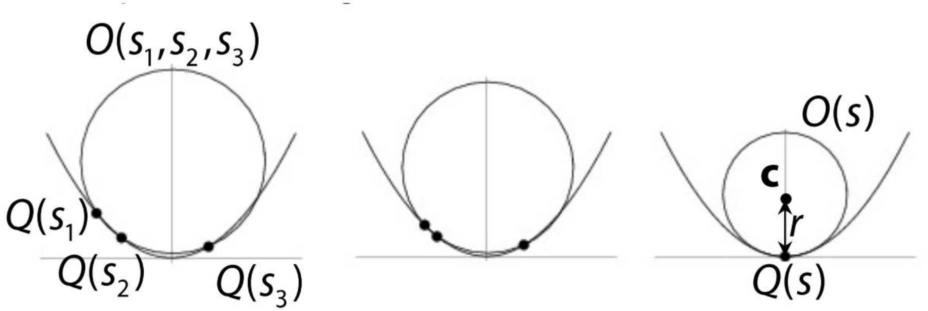
- Geometric continuity is defined in terms of **arc-length parametrization**:

$$Q(s) \text{ is } G^n \text{ continuous} \quad \text{iff}$$

$$Q^{(i)}(s) = \frac{d^i Q(s)}{ds^i} \text{ is continuous for } 0 \le i \le n$$

where Q(s) is **parameterized by arc length** s.

- Then the **first derivative** (tangent) is of unit length
- And the **second derivative** points to center of the osculating circle



$$O(s) = \lim_{s_1,s_2,s_3 \to s} O(s_1,s_2,s_3)$$

curvature

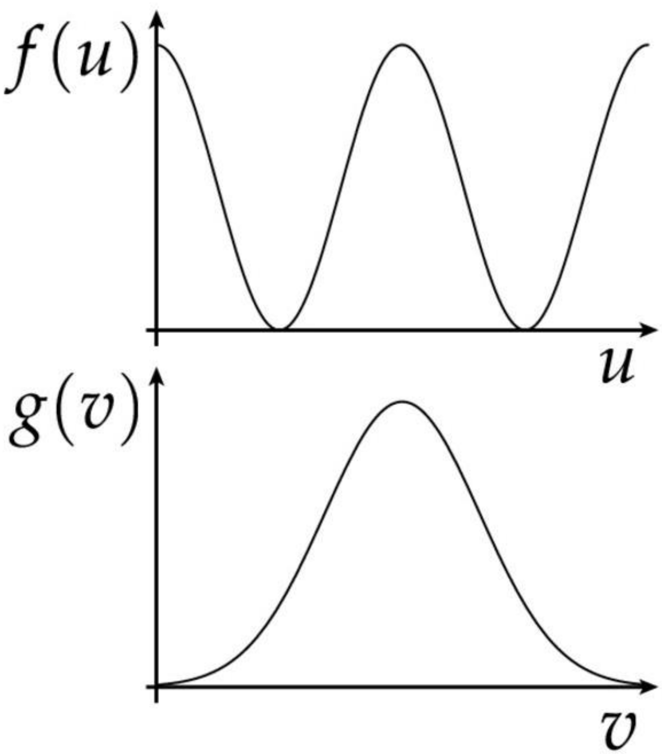
$$\|Q''(s)\| = \kappa(s) = \frac{1}{r(s)}$$

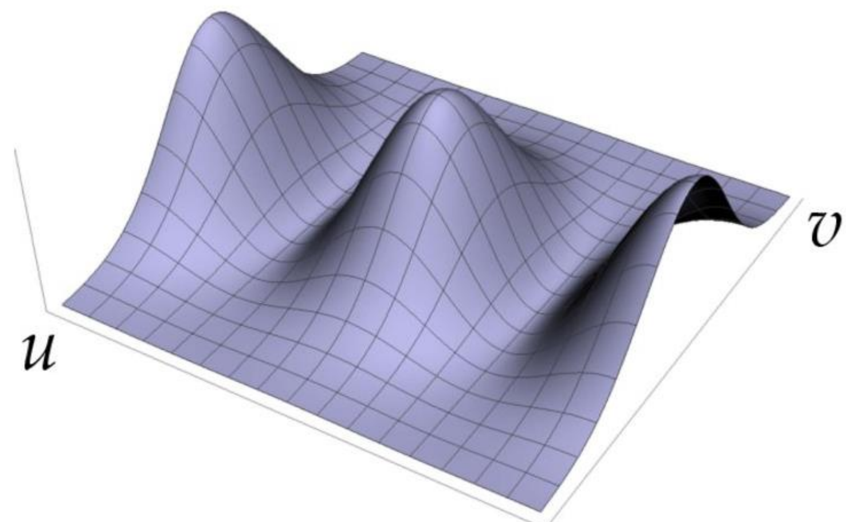
G^n continuity is a weaker constraint than C^n continuity

Parametric Surfaces

Tensor Product surfaces

- Can use a pair of curves to get a surface
- Value at any point (u,v) is given by product of a curve f at u and a curve g at v (sometimes called the "**tensor product**"):

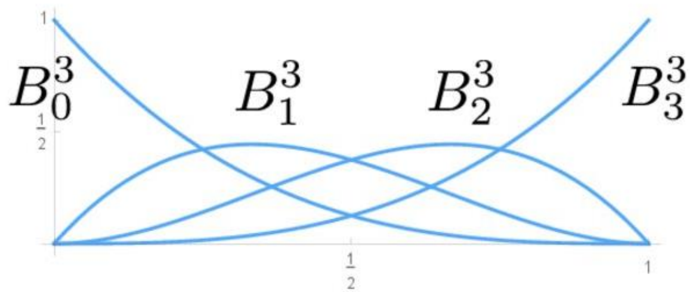




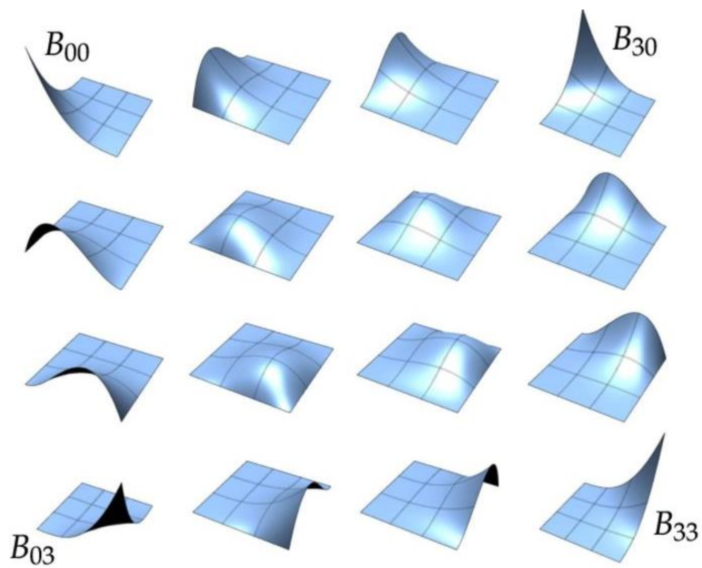
$$(f \otimes g)(u,v) := f(u)g(v)$$

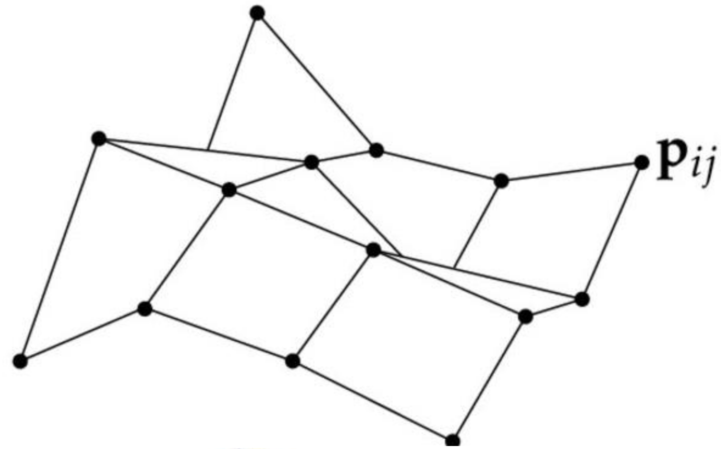
Bezier Patches

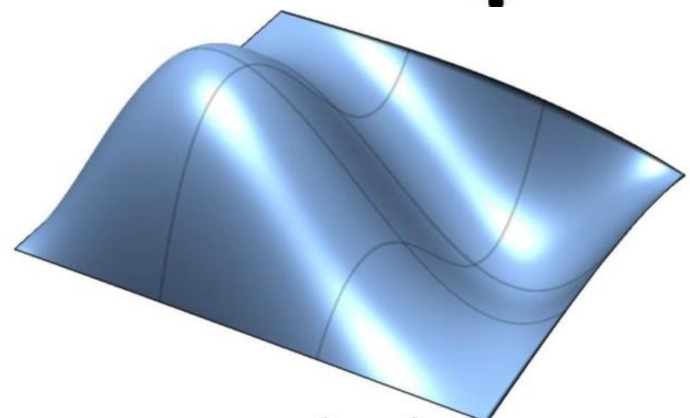
- Bezier patch is sum of (tensor) products of Bernstein bases



$$B_{i,j}^3(u,v) := B_i^3(u)B_j^3(v)$$



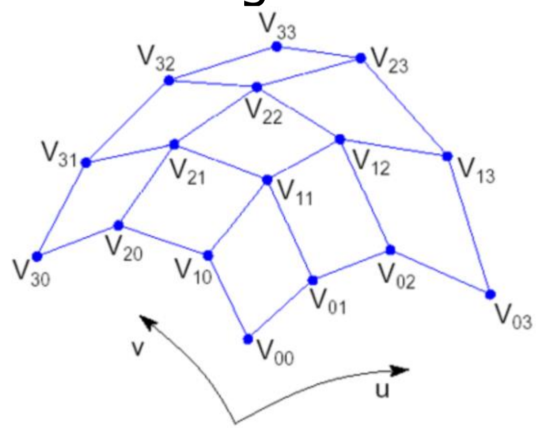


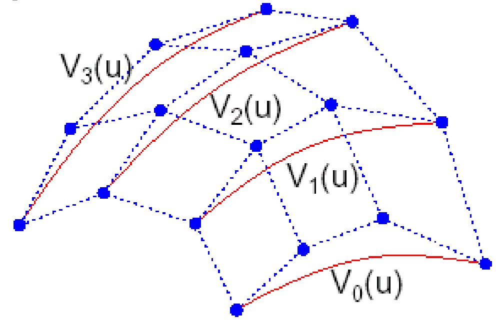


$$S(u,v) := \sum_{i=0}^{3}\sum_{j=0}^{3} B_{i,j}^3(u,v)\mathbf{p}_{ij}$$

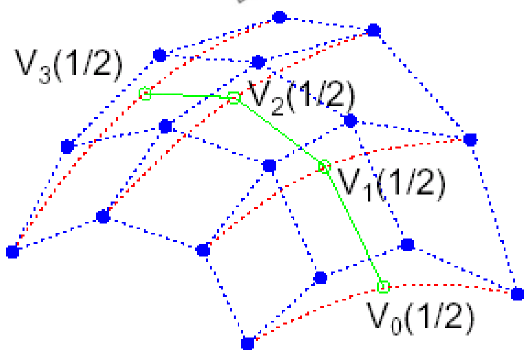
Tensor product Bezier surfaces

- Let's walk through the construction steps:

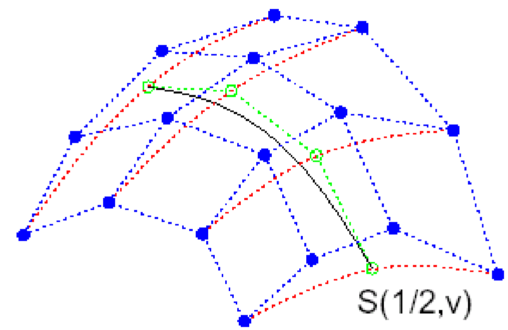




Control curves in *u*



Control polygon at *u*=1/2



Curve at *S*(1/2,*v*)

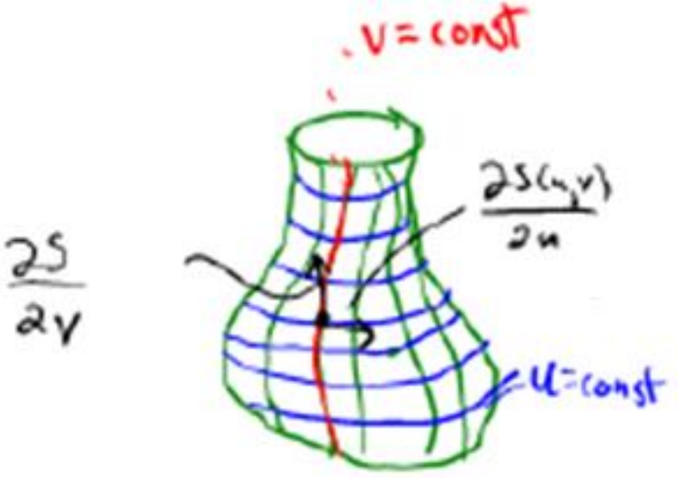
- The surface interpolates the four corner control points
- The boundary curves of a Bézier patch are themselves Bézier curves. E.g.,

$$S(0,v) = \sum_{j=0}^{3} p_{0j} B_{j,3}(v)$$

Tangents and normal

- We can compute tangents to the surface at any point by looking at (infinitesimally) nearby points.
  - Holding one parameter constant and finding two nearby points by varying the other parameter. Thus, we can get **two tangents**:

$$\mathbf{t}_u = \frac{\partial S(u,v)}{\partial u} \qquad \mathbf{t}_v = \frac{\partial S(u,v)}{\partial v}$$



- How do we compute the **normal**?
  - Take cross product of the two tangents

Bézier Patch

- Differentiation at the corners:

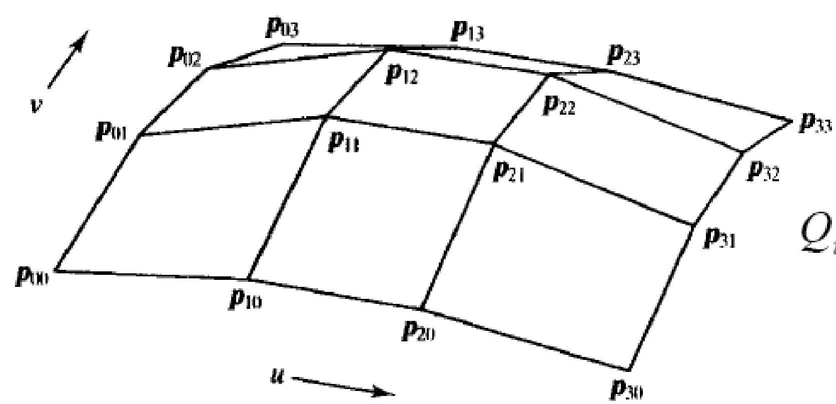
$$Q_u(0,0) = 3(p_{10} - p_{00})$$

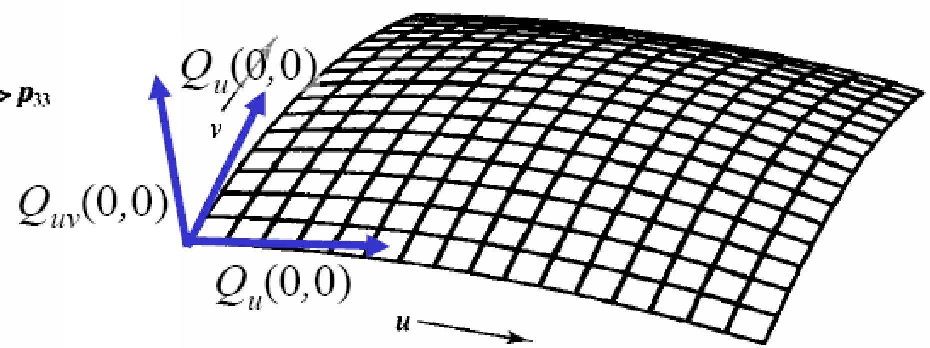
$$Q_v(0,0) = 3(p_{01} - p_{00})$$

tangent vectors

$$Q_{uv}(0,0) = 9(p_{00} - p_{01} - p_{10} + p_{11})$$

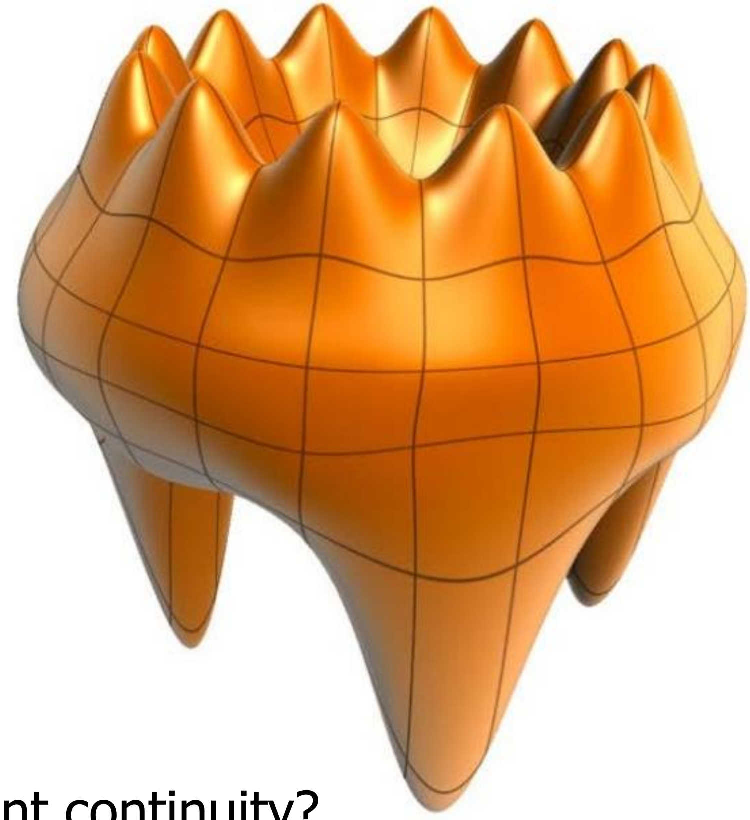
twist vector





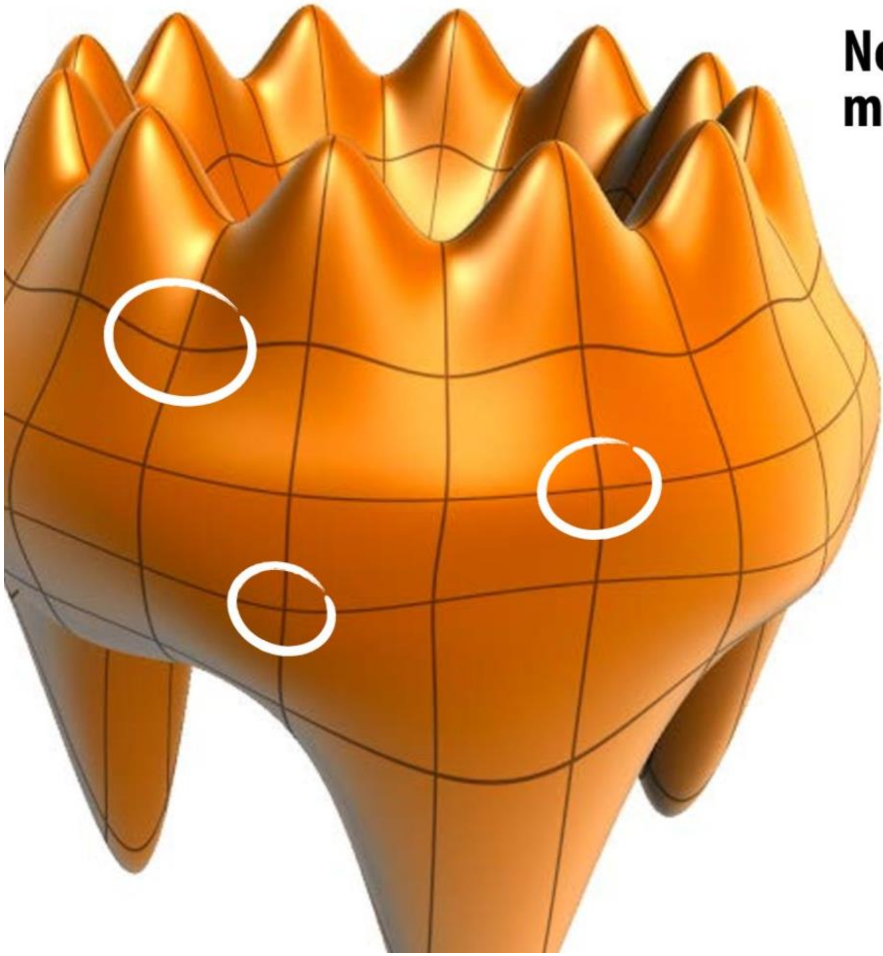
Bezier surface

- Just as we connected Bezier *curves*, we can connect Bezier *patches* to get a surface:



- Can we always get tangent continuity?

Bezier Patches: limited connectivity



Notice that exactly four patches meet around *every* vertex!

In practice, far too constrained.

To make interesting shapes (with good continuity), we need patches that allow more interesting connectivity...

Other parametric schemes

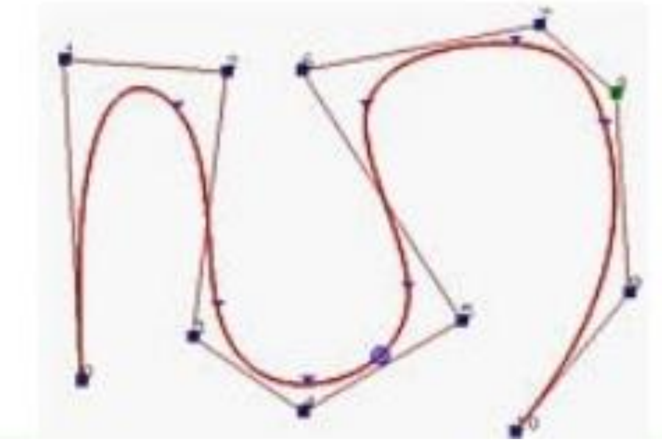
- There are many other parametric curve schemes
  - B-spline
  - Rational Bezier
  - NURBS
  - Hermite
  - Gregory
  - …
- Can form corresponding tensor product surface schemes

Uniform Cubic B-spline

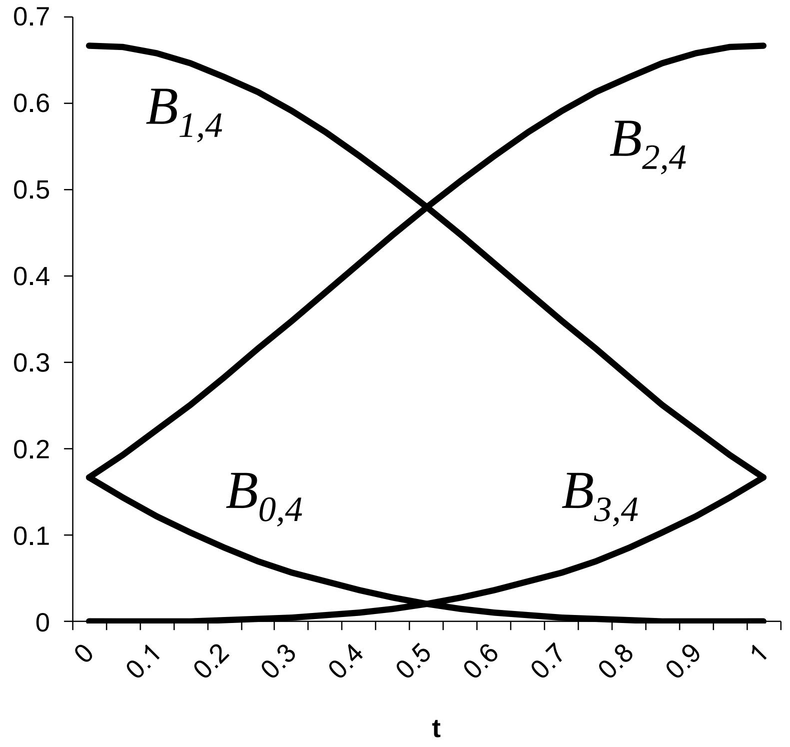
$$C(t) = \sum_{i=0}^{3} P_i B_{i,4}(t)$$

$$= P_0 \frac{1}{6}\left(1 - 3t + 3t^2 - t^3\right) + P_1 \frac{1}{6}\left(4 - 6t^2 + 3t^3\right) + P_2 \frac{1}{6}\left(1 + 3t + 3t^2 - 3t^3\right) + P_3 \frac{1}{6}\left(t^3\right)$$

uniform B-spline basis functions



Uniform cubic B-spline



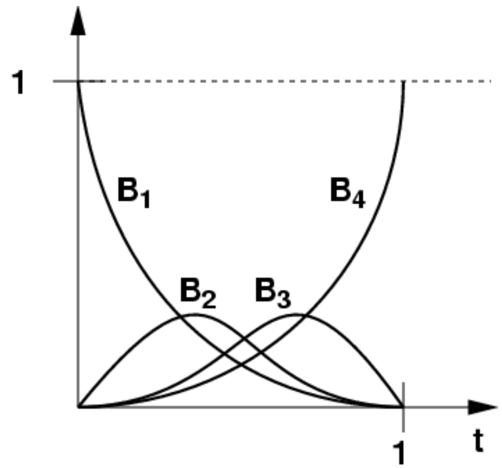
$$
\begin{aligned}
x(t) = {} & P_0 \frac{1}{6}\left(1 - 3t + 3t^2 - t^3\right) \\
& + P_1 \frac{1}{6}\left(4 - 6t^2 + 3t^3\right) \\
& + P_2 \frac{1}{6}\left(1 + 3t + 3t^2 - 3t^3\right) \\
& + P_3 \frac{1}{6}\left(t^3\right)
\end{aligned}
$$

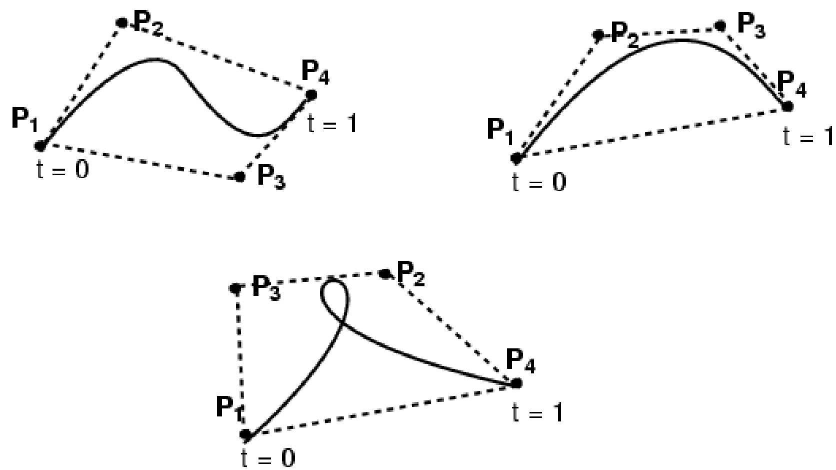
- Does the curve interpolate its endpoints?
- Does it lie inside its convex hull?

Bézier versus BSpline

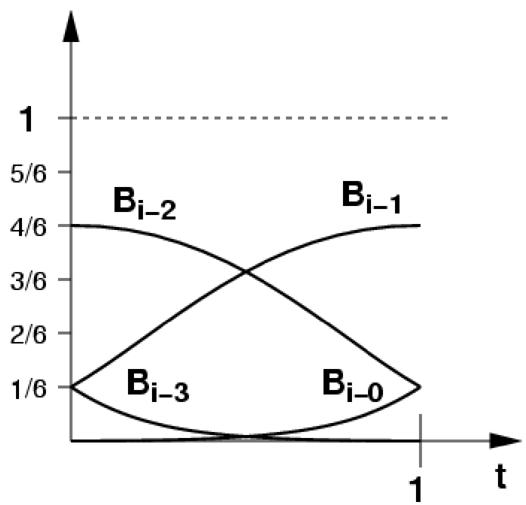
- Relationship to the control points is different

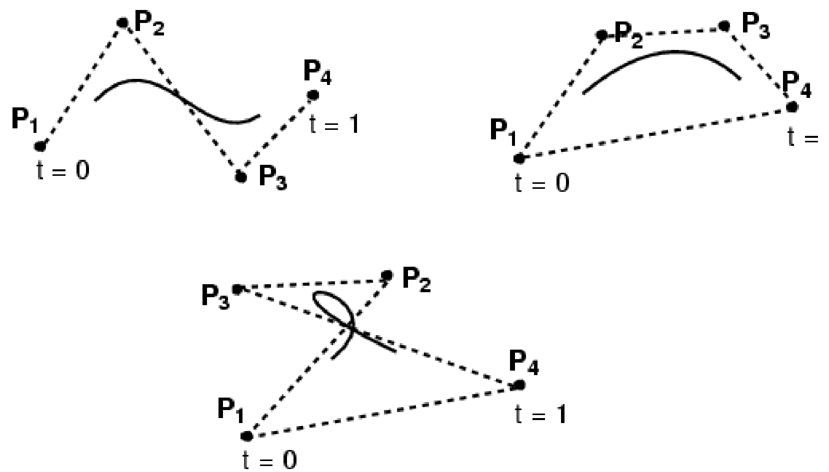
Bézier





BSpline

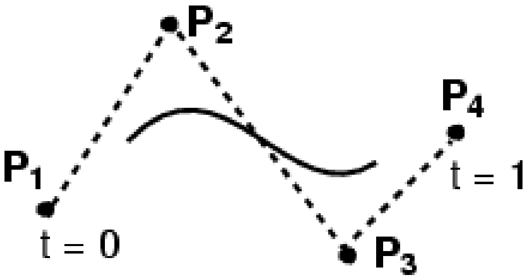


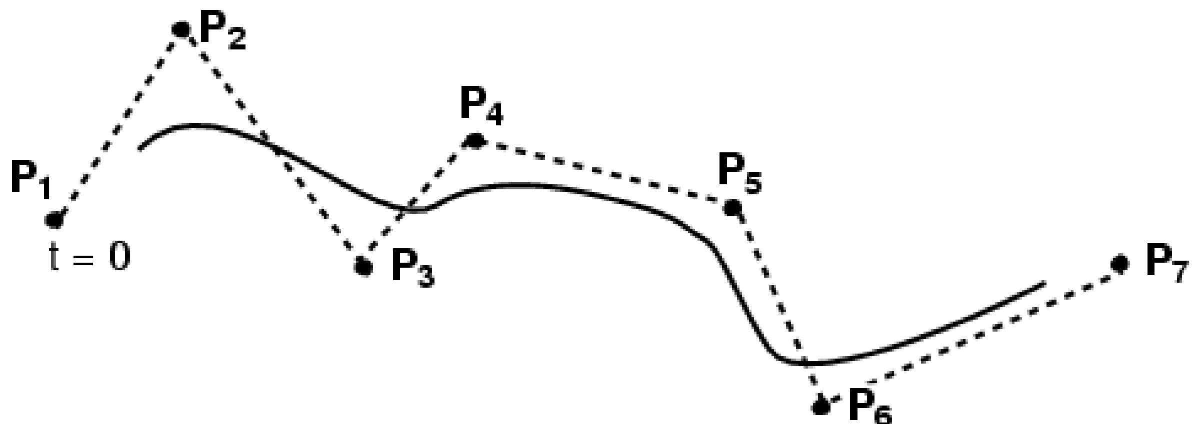


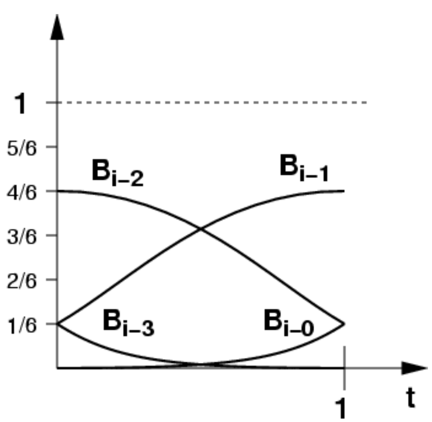
Uniform Cubic B-Splines

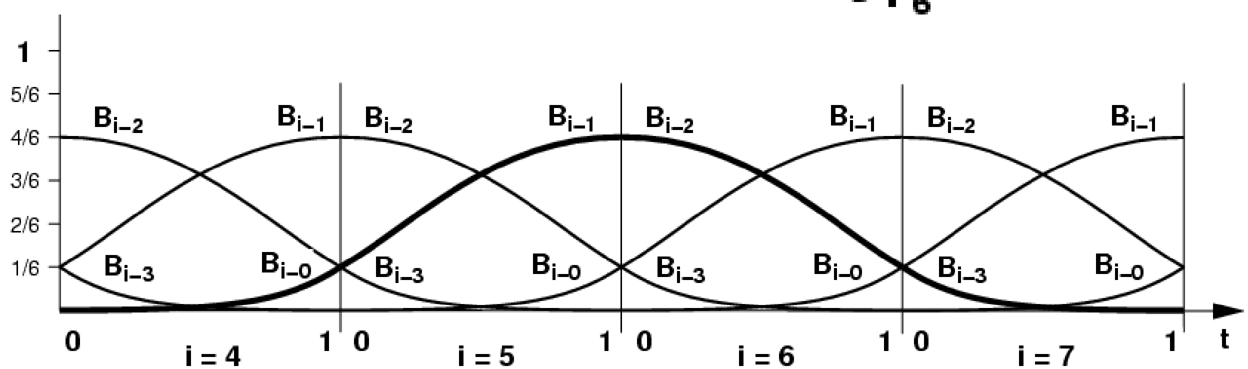
- Basis functions can be chained together
- Local control

Applet









Basis functions are themselves C^2 continuous at knots

Rational Curves

- It is known from classical mathematics that all the **conic curves** (circle, ellipse, hyperbola, parabola) can be represented using **rational functions**, which is defined as **ratio of two polynomials**.

$$x(u) = \frac{X(u)}{W(u)} \qquad y(u) = \frac{Y(u)}{W(u)}$$

where X(u), Y(u) and W(u) are polynomials.

Rational Curves

- Example 1:

Circle of radius 1, centered at the origin

$$x(u) = \frac{1-u^2}{1+u^2} \qquad y(u) = \frac{2u}{1+u^2}$$

- Easy to verify

$$\begin{aligned}(x(u))^2 + (y(u))^2 &= \left(\frac{1-u^2}{1+u^2}\right)^2 + \left(\frac{2u}{1+u^2}\right)^2 \\ &= \frac{1-2u^2+u^4+4u^2}{(1+u^2)^2} = \frac{(1+u^2)^2}{(1+u^2)^2} = 1\end{aligned}$$

- Example 2:

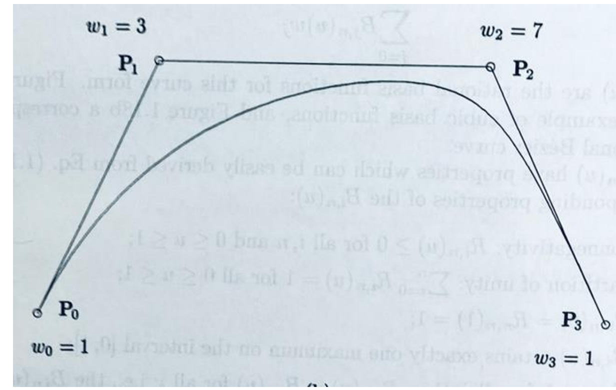
Ellipse, centered at the origin; the y-axis is the major axis, the x-axis is the minor axis, and the major and minor radii are 2 and 1, respectively

$$x(u) = \frac{1-u^2}{1+u^2} \qquad y(u) = \frac{4u}{1+u^2}$$

Rational Bezier

- An n-degree rational Bezier curve is defined as:

$$\mathbf{C}(u) = \frac{\sum_{i=0}^{n} B_{i,n}(u) w_i \mathbf{P}_i}{\sum_{i=0}^{n} B_{i,n}(u) w_i} \qquad 0 \le u \le 1$$

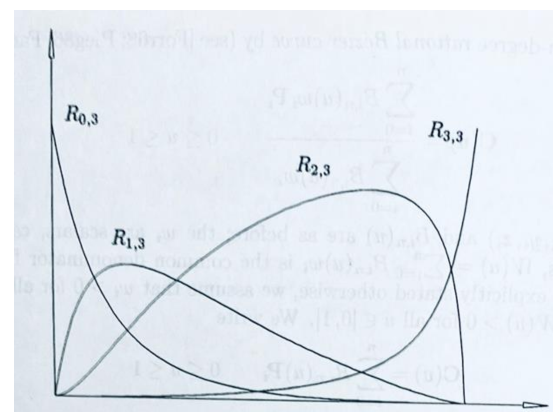


$w_i>0$ are the weights controlling the 'strength' of control points. When $w_i=1$ for all i, we get polynomial Bezier curves.

- Re-writing

$$\mathbf{C}(u) = \sum_{i=0}^{n} R_{i,n}(u) \mathbf{P}_i \qquad 0 \le u \le 1$$

where $$R_{i,n}(u) = \frac{B_{i,n}(u) w_i}{\sum_{j=0}^{n} B_{j,n}(u) w_j}$$



Rational Bezier

- Rational curves in n-dimensional space can be represented as a polynomial curve in (n+1)-dimensional space using homogeneous coordinates.

Now for a given set of control points, $\{\mathbf{P}_i\}$, and weights, $\{w_i\}$, construct the weighted control points, $\mathbf{P}_i^w = (w_i x_i, w_i y_i, w_i z_i, w_i)$. Then define the *nonrational* (polynomial) Bézier curve in four-dimensional space

$$\mathbf{C}^w(u) = \sum_{i=0}^{n} B_{i,n}(u)\,\mathbf{P}_i^w$$

Writing out the coordinate functions:

$$X(u) = \sum_{i=0}^{n} B_{i,n}(u) w_i x_i \qquad Y(u) = \sum_{i=0}^{n} B_{i,n}(u) w_i y_i$$

$$Z(u) = \sum_{i=0}^{n} B_{i,n}(u) w_i z_i \qquad W(u) = \sum_{i=0}^{n} B_{i,n}(u) w_i$$

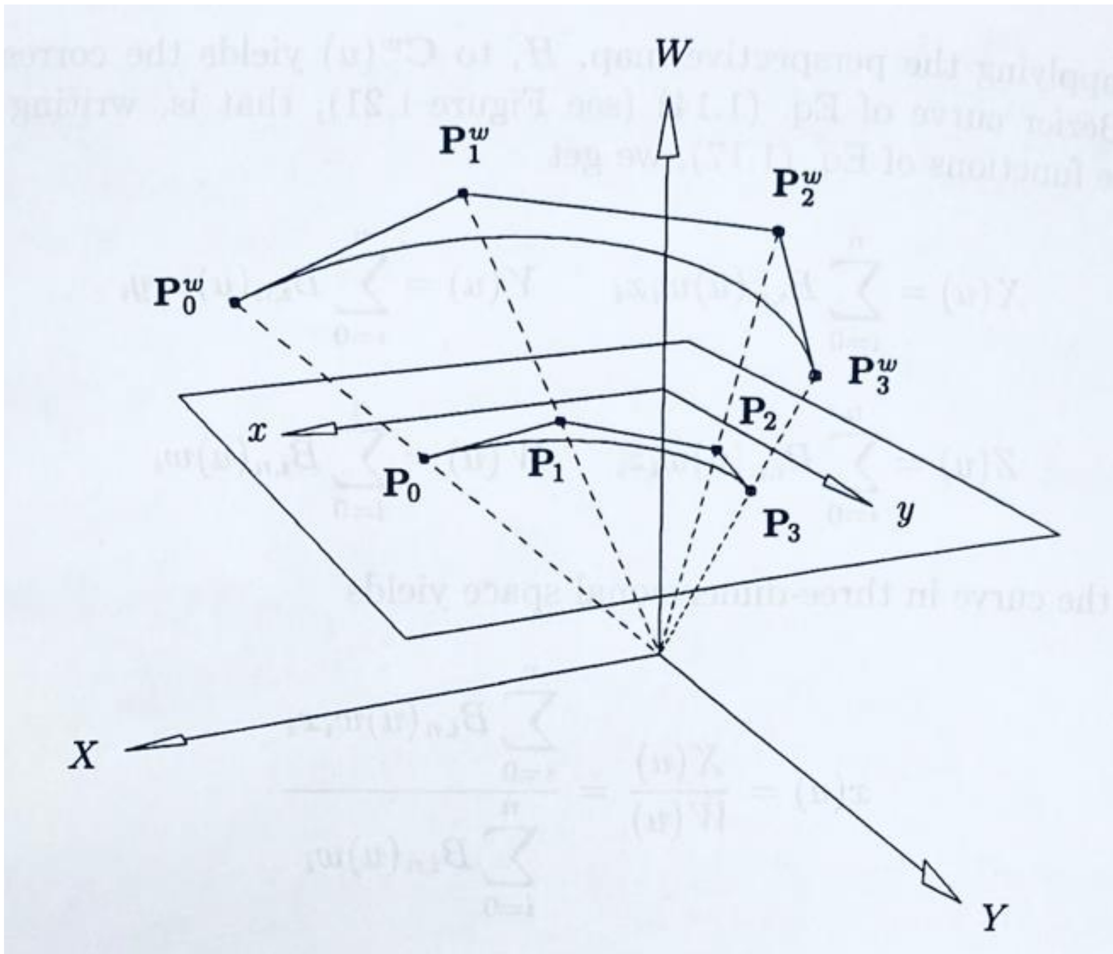
Locating the curve in three-dimensional space yields

$$x(u) = \frac{X(u)}{W(u)} = \frac{\sum_{i=0}^{n} B_{i,n}(u) w_i x_i}{\sum_{i=0}^{n} B_{i,n}(u) w_i}$$

$$y(u) = \frac{Y(u)}{W(u)} = \frac{\sum_{i=0}^{n} B_{i,n}(u) w_i y_i}{\sum_{i=0}^{n} B_{i,n}(u) w_i}$$

$$z(u) = \frac{Z(u)}{W(u)} = \frac{\sum_{i=0}^{n} B_{i,n}(u) w_i z_i}{\sum_{i=0}^{n} B_{i,n}(u) w_i}$$

Rational Bezier



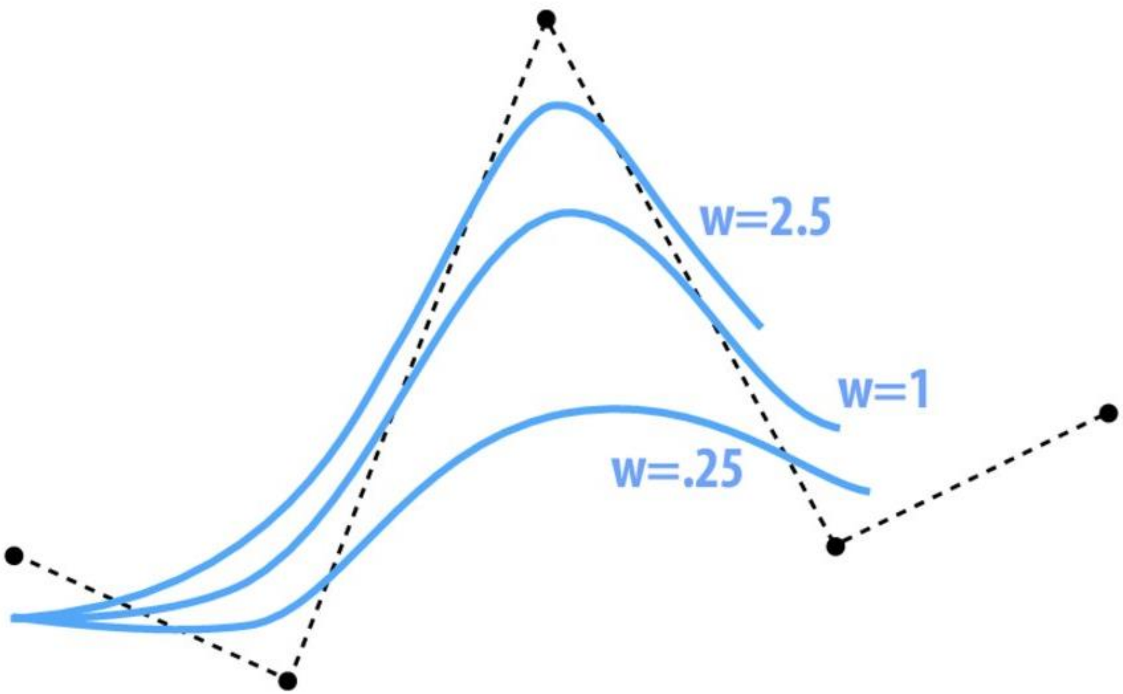
A geometric construction of a rational Bezier curve

Properties of Bezier rational curves

- The rational Bezier basis functions have all the following properties like the polynomial Bezier basis functions:
  - Nonnegativity
  - Partition of unity
  - $R_{0,n}(0) = R_{n,n}(1)=1$,
  - …
- Therefore rational Bezier curves have all the following properties
  - Endpoint interpolation
  - Convex hull property
  - Transformation invariance
  - Variation diminishing
  - The kth derivative at u=0 (u=1) depends on the first (last) k+1 control points and weights

NURBS

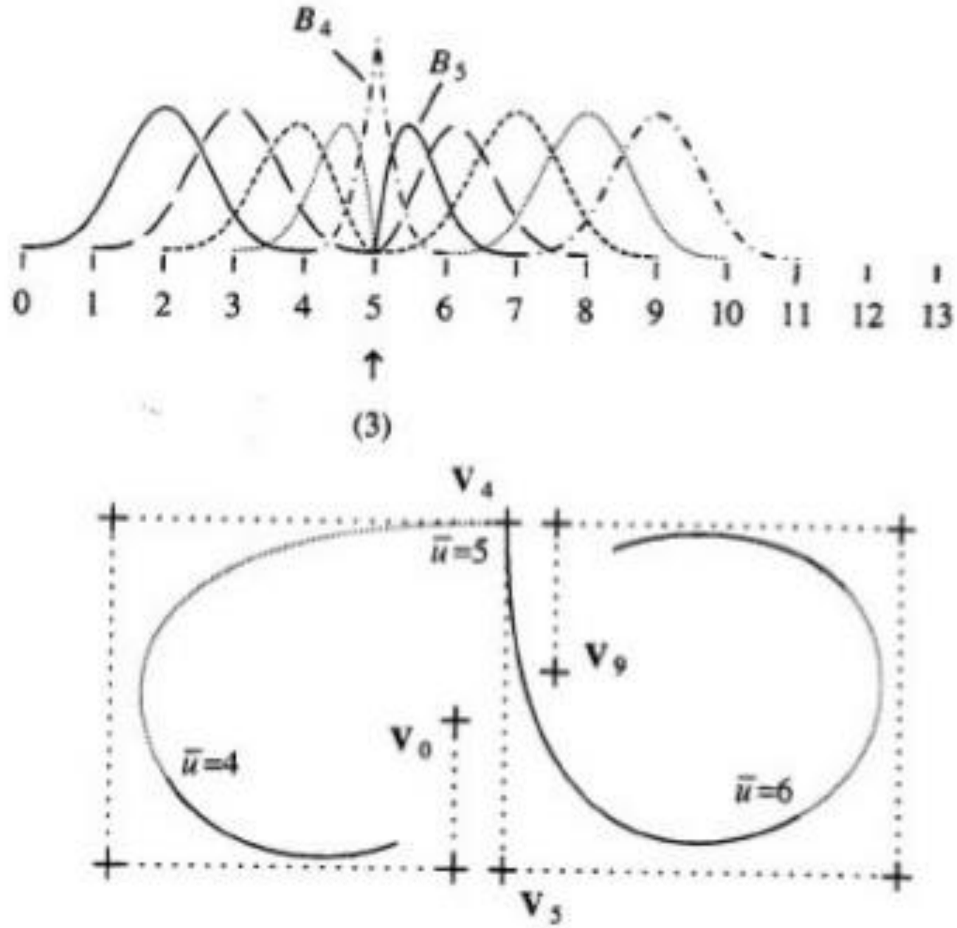
- (N)on-(U)niform (R)ational (B)-(S)pline
  - Knots at arbitrary locations (non-uniform)
  - Ratio of polynomials (rational)
  - Piecewise B-spline curve



NURBS

Non-uniform knots: different spacing between the basis functions

- Knot vector = {0,1,2,3,4,5,5,5,6,7,8,9,10,11}
- Triple knots at 5
- 5 segments



Matrix representation

$$Q(u) = (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u)V_2 + u^3 V_3$$
$$= (-V_0 + 3V_1 - 3V_2 + V_3)u^3 + (3V_0 - 6V_1 + 3V_2)u^2 + (-3V_0 + 3V_1)u + V_3$$

$$= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Bezier basis matrix

$$Q(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

B-spline basis matrix

Matrix Representation

$$Q(u) = UMP$$

where $U = [u^3 \;\; u^2 \;\; u^1 \;\; 1]$

$$Q'(u) = [3u^2 \;\; 2u \;\; 1 \;\; 0\;]MP$$

Pros:

- Compact representation
- Convenient implementation in either hardware or software with available matrix facilities

Conversion between representations

$$UM_iP_i = UM_jP_j$$

$$M_iP_i = M_jP_j$$

$$P_i = M_i^{-1}M_jP_j$$

where M_* is the **characteristic matrix** of a particular curve scheme

$$M_{BS} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$M_B = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

E.g. Convert a cubic uniform B-spline curve to Bézier curve:

$$P_B = M_B^{-1}M_{BS}P_{BS}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} M_{BS}P_{BS} = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix} P_{BS}$$

Bicubic Surfaces

- Matrix equation of a bicubic surface:

$$Q(u,v) = UMPM^{\mathrm{T}}V^{\mathrm{T}}$$

where

$$\begin{aligned} U &= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \\ V &= \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix} \end{aligned}$$

P is a 4×4 matrix containing the 16 control points

Bézier Patch

$$Q(u,v) = UMPM^{\mathrm{T}}V^{\mathrm{T}}$$

- Differentiating

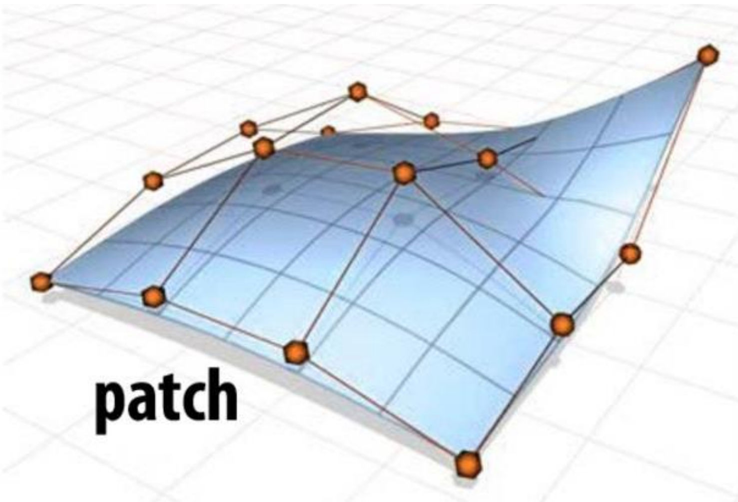
$$Q_u(u,v) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} M_{\mathrm{B}} P M_{\mathrm{B}}^{\mathrm{T}} V^{\mathrm{T}}$$

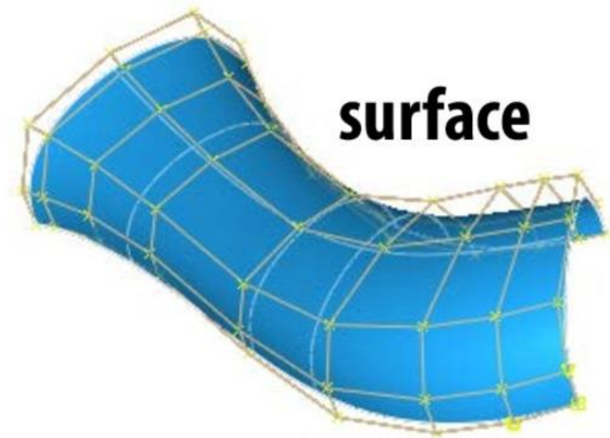
$$Q_v(u,v) = U M_{\mathrm{B}} P M_{\mathrm{B}}^{\mathrm{T}} \begin{bmatrix} 3v^2 \\ 2v \\ 1 \\ 0 \end{bmatrix}$$

$$Q_{uv}(u,v) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} M_{\mathrm{B}} P M_{\mathrm{B}}^{\mathrm{T}} \begin{bmatrix} 3v^2 \\ 2v \\ 1 \\ 0 \end{bmatrix}$$

NURBS surface

- Multiple NURBS patches form a surface





- Pros: easy to evaluate, exact conics, high degree of continuity
- Cons: rectangular underlying topology; difficult to maintain continuity for arbitrary topology