The Problem Formulation

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0-1-2-3-4-S-6-7-8-9-10
P1                  P2
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Two players are at the opposite end of the straight line as shown in the picture above.
An expensive scotch (S) is kept in the middle at position \#5. The players start the
game with $100 in hand. The first player makes a secret bid followed by a secret bid
by the second player. The bottle moves one position closer to the winning bidder. In
the event of a draw, the winner is the player with the draw advantage. Draw advantage
initially starts with the first player, and it alternates every time a draw is encountered
i.e., The first draw is won by the first player. The second draw if it occurs is won by
the second player. And it alternates henceforth. The winning bid is deducted from the
players hand. This deduction happens in case of a draw as well but only from the players
hand who has won the draw. Each bid must be greater than 0$. But a player can bid 0$ only
when there is no money left in his hand. Integer bids are allowed and float bids are
considered as incorrect outputs.

When does the game end?
The bottle reaches a player or both the players run out of money

The function calculate_bid takes in 4 parameters - an integer player, the position of
the scotch pos, an array first_moves that contains the previous bids made by player 1
and an array second_moves that contains the previous bids made by player 2. Complete
the function to return an integer which is your next bid amount.
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The Solution

I will denote player 1 by A and player 2 by B, and their respective money amounts by $\phi(A), \phi(B)$. A subscript $\phi_i(A)$ will denote the amount of money A has at the board location i. This derivation of the game theoretic optimum solution will be from the point of view of A. Also, the derivation will assume that the bidding amounts are real-valued, as opposed to integers.

Now first we consider the only board position where it is immediately obvious what B's bet is. That position is 1, and B must bet as much money as A has plus change for tiebreaking, or else A instantly wins. Let us start with

$$\alpha\phi_1(A) = \phi_1(B)$$

where $\alpha \in \mathbb{R}$ is just any parameter. This says that at position 1, the amount of money B has is a factor α times the amount of money A has. Now suppose B wins the next round by betting $\phi_1(A)$; then

$$\phi_2(B) = (\alpha - 1)\phi_1(A)$$

Now how does A make the next bet? Since B bet the minimum amount of money he needed to get to position 2, A should not expect to get to position 1 with an advantage, if A wins the next bet. Hence, A will bet such that, if he wins the next round of betting, then the ratio of A's money to B's money in position 1 is still α. We can solve for this: let A bet x. Then:

$$(\alpha - 1)\phi_1(A) = (\phi_1(A) - x)\alpha$$

$$x = \frac{\phi_1(A)}{\alpha}$$

Great, now we know what to bet as A in position 2 (the α value will be revealed soon). Now what if A lost that, and we are now in position 3? Assuming B bet the optimum amount to win, we have

$$\phi_3(B) = \left(\alpha - 1 - \frac{1}{\alpha}\right)\phi_1(A)$$

We solve this the same way: let A bet an amount x such that, if A wins the bidding round, then the ratio of $\phi_2(A)$ to $\phi_2(B)$ will be $(\alpha - 1)$.

$$\left(\alpha - 1 - \frac{1}{\alpha}\right)\phi_1(A) = (\phi_1(A) - x)(\alpha - 1)$$

$$x = \frac{\phi_1(A)}{\alpha(\alpha - 1)}$$

If A loses this bidding round, then

$$\phi_4(B) = \left(\alpha - 1 - \frac{1}{\alpha} - \frac{1}{\alpha(\alpha - 1)}\right)\phi_1(A)$$

You can probably see the pattern emerging in these bids. A will bid on location 4:

$$\frac{\phi_1(A)}{\alpha(\alpha - 1)\left(\alpha - 1 - \frac{1}{\alpha}\right)}$$

and if he loses

$$\phi_5(B) = \left(\alpha - 1 - \frac{1}{\alpha} - \frac{1}{\alpha(\alpha - 1)} - \frac{1}{\alpha(\alpha - 1)(\alpha - 1 - \frac{1}{\alpha})}\right)\phi_1(A)$$

And on position 5, A will bid

$$\frac{\phi_1(A)}{\alpha(\alpha - 1)(\alpha - 1 - \frac{1}{\alpha})(\alpha - 1 - \frac{1}{\alpha})(\alpha - 1 - \frac{1}{\alpha} - \frac{1}{\alpha(\alpha - 1)})}$$

Ok but what is the ratio of B's money to A's money in board position 5 (the middle)? It's just 1! So we set ugly thing in parenthesis equal to 1, solve using Wolfram Alpha, and get $\alpha = 1 + \sqrt{3}$. We plug this α into the expressions above and get the optimum fraction of money to bid at each of the board locations 1 through 5.

Now what about board positions 6 through 9? Well, we just reverse the analysis. Compute the optimum bid of B using the above formulas, and bid that.

Now comes the ugly part: dealing with rounding. I admit I didn't spend too much time optimizing this part. Basically, if I was going to win on tie, I bid the floor of by optimum bid. Otherwise, I bid the ceil of it. That is why my bot alternates between opening with \$13 or \$14.