

Direct Proof strategy:

1. Identify the hypothesis and consequence in the compound statement.
2. Replace (if necessary) the hypothesis with a more usable equivalent statement.
3. Replace (if necessary) the consequence with a more usable equivalent statement.
4. Develop a chain of statement, each deducible from its predecessors or other known results, that leads from the hypothesis.

A direct proof for $A \Rightarrow B$ fits the form:

Assume A .

\vdots

Therefore, B .

Thus A implies B .

With in a proof you regularly use properties of the mathematical system, axioms, and logical equivalences to advance through the proof.

Example 1. The product of two odd numbers equals an odd number.

A : (Hypothesis:) Consider any two odd numbers and multiply them. (Since odd numbers are integers we may assume the axioms and basic properties of integers.)

B : (Conclusion:) Their product is an odd number.

Notice that we are multiplying an two numbers that happen to be odd. Thus we must work with symbols representing the odd numbers rather than specific numbers.

Let a and b be two odd numbers. It can be assumed that an odd number is written as

$$a = 2s + 1 \quad b = 2t + 1$$

where s and t are integers. Thus

$$\begin{aligned} ab &= (2s + 1)(2t + 1) \\ &= 4s^2 + 2t + 2s + 4t^2 + 1 \quad \text{by distribution} \\ &= 2(2s^2 + s + t + 2t^2) + 1 \quad \text{by factoring} \end{aligned}$$

The number $p = 2s^2 + s + t + 2t^2$ by the closure properties of integers under multiplication, addition and division. This last implication gives us the result that we want. We are ready to write out our proof.

Proof. Let a and b be two odd numbers (by hypothesis.) We may write

$$a = 2s + 1 \quad b = 2t + 1$$

where s and t are integers. Therefore,

$$ab = (2s + 1)(2t + 1)$$

$$\begin{aligned}
&= 4s^2 + 2t + 2s + 4t^2 + 1 && \text{by distribution} \\
&= 2(2s^2 + s + t + 2t^2) + 1 && \text{by factoring}
\end{aligned}$$

The number $p = 2s^2 + s + t + 2t^2$ is an integer since s , t , and 2 are integers. Thus

$$ab = 2p + 1,$$

with p an integer. Thus, ab is an odd number (conclusion thus the proof is complete). □

Another example:

Example 2. Let x , a , and b be three integers. If x is a divisor of a and b , then x is a divisor of $(a^2 - b^2)$.

A: (Hypothesis:) Let x , a , and b be three integers. Assume x is a divisor of a and b . More explicitly, assume $a = xs$, and $b = xt$ with s and t integer numbers.

B: (Conclusion:) The number x is a divisor of $(a^2 - b^2)$. More explicitly, $a^2 - b^2 = xn$ for some integer n .

From the definition of a divisor both $x \neq 0$.

Proof. By hypothesis and definition of divisor, $a = xs$, and $b = xt$ with s and t integer numbers. Then,

$$\begin{aligned}
a^2 - b^2 &= (xs)^2 - (xt)^2 && \text{(by assumption)} \\
&= x^2s^2 - x^2t^2 && \text{(by properties of exponents)} \\
&= x(xs^2 - xt^2) && \text{(distribution property)} \\
&&& xq
\end{aligned}$$

where $q = xs^2 - xt^2$ is an integer. Thus, x is a divisor of $a^2 - b^2$. □

Example 3. If a and c are odd integers and b is an integer, then $ab + bc$ is an even integer.