One more example on proof by contradiction:
Example 1. Prove by contradiction that if n is a natural number then

$$\frac{n}{n+1} > \frac{n}{n+2}.$$

Natural numbers are positive integers.

Proof. (By contradiction:) Assume n is a natural number and assume $\frac{n}{n+1} \leq \frac{n}{n+2}$. Then

$$\frac{1}{n+1} \leq \frac{1}{n+2}$$
$$\Rightarrow n+2 \leq n+1$$
$$\Rightarrow 2 \leq 1$$

which contradicts the order on natural numbers. Thus, if n is a natural number then $\frac{n}{n+1} \leq \frac{n}{n+2}$. □

"If and Only If" or "Equivalence Theorems" Let n be a natural number. n is even if and only if n^2 is even. Here we can denote the statement n is even by A and the statement n^2 is even by B. Then A and B are both a sufficient and necessary condition.

$$A \Leftrightarrow B \qquad \equiv \qquad (A \Rightarrow B) \wedge (B \Rightarrow A).$$

When we are trying to prove a statement link $A \Leftrightarrow B$ it is at times useful to first prove one direction $(A \Rightarrow B)$ and then the other direction $(B \Rightarrow A.)$
Example 2. Assume p is prime. Then p divides if and only if p divides b^2.

Proof. $(\Rightarrow)$ Assume p is prime and assume p divides b, i.e., $b = pm$ for some integers m. Show p divides b^2, i.e., show $b^2 = pk$ for some integer k. Then

$$b^2 = (pm)^2$$
$$= p^2m^2$$
$$= p(pm^2)$$
$$= pk,$$

where $k = pm^2$ is an integer. Thus, p divides b^2.
$(\Leftarrow)$ Assume p is prime and assume that p divides b^2, i.e., $b^2 = pm$ for some integer m. Show p divides b, i.e., show $b = pk$ for some integer k. This will be a direct proof.
Since $b^2 = pm$, and the right hand side has at least one factor of the prime number p , then the left hand side has at least one factor of p (by Fundamental Theorem of Arithmetic.) If b has no factor of p, then b^2 has no factors of p. Thus, b has at least one factor of p, i.e., $b = pk$ for some integer k. Thus, p divides b. □

You should choose an appropriate proof method for each direction:

Example 3. Prove: the number m is odd if an only if m^2 is odd.

Proof. ($\Rightarrow$) assume m is odd, i.e., $m = 2p + 1$ for some integer p. Want to show that m^2 is odd, i.e., $m^2 = 2k + 1$ for some integer k. Then,

$$\begin{aligned} m^2 &= (2p+1)^2 \\ &= 4p^2 + 4p + 1 \\ &= 2(2p^2 + 2p) + 1 \\ &= 2k + 1 \end{aligned}$$

where $k = 2p^2 + 2p$ is an integer. Thus m^2 is odd.
($\Leftarrow$) (proof by contropositive:) If m is even, then m^2 is even.
Proof:
Assume m is even, i.e., $m = 2k$ for some integer k. Show that m^2 is even, i.e, $m^2 = 2n$ for some integer n. Then

$$\begin{aligned} m^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) = 2n \end{aligned}$$

where $n = 2k^2$ is an integer. Thus m^2 is even. □

At times you will have to prove multiple equivalent statements. (For the example below we should know that the reciprocal of a real non-zero number x is just $\frac{1}{x}$.

Example 4. Let x be a real number that is not equal to 0. Then the following statements are equivalent:

(i) $x > 0$

(ii) The sum of x and its reciprocal is greater then or equal to 2.

(iii) $2^{x+\frac{1}{x}} \geq 4$.

Idea (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(i).

(i)$\Rightarrow$(ii). (Direct proof, work algebra backwards.) Assume x is a real number and $x > 0$. Then,

$$\begin{aligned} &(x-1)^2 \geq 0 \\ \Rightarrow\ &x^2 - 2x + 1 \geq 0 \\ \Rightarrow\ &x - 2 + \frac{1}{x} \geq 0 \qquad \text{since } x > 0 \\ \Rightarrow\ &x + \frac{1}{x} \geq 2. \end{aligned}$$

Thus, (i)$\Rightarrow$(ii).

[(ii)⇒(iii)] (Direct proof:) Assume $x + \frac{1}{x} \geq 2$. Then $2^{x+\frac{1}{x}} \geq 4$, since 2^y is an increasing function.

[(iii)⇒(i)] (Contrapositive: if $x < 0$ then $2^{x+\frac{1}{x}} < 4$:)
Assume $x < 0$. Then $\frac{1}{x} < 0$

$$\Rightarrow x + \frac{1}{x} < 0$$

$$\Rightarrow x + \frac{1}{x} < 2$$

$$2^{x+\frac{1}{x}} < 2^2 = 4.$$

Therefore, (iii)⇒(i). $\square$