The natural numbers system can be constructed from five axioms:

(i) 1 is a natural number.

(ii) every natural number has a unique successor, which is also a natural number.

(iii) no two natural numbers have the same successor.

(iv) 1 is not a successor for any natural number.

(v) Induction property

**Principle of Mathematical Induction (PMI)**

**Definition 1.** If $S$ is a subset of $\mathbb{N}$ with these two properties:

(i) $1 \in S$,

(ii) for all $n \in \mathbb{N}$, if $n \in S$, then $n+1 \in S$,

then $S \in \mathbb{N}$.

A set of natural numbers with the property that whenever $n \in S$, then $n+1 \in S$ is called a inductive set.

**Example 1.** The sets $\{5, 6, 7, \dots\}$ and $\{101, 102, 103, \dots\}$ are inductive sets, but $\{1, 3, 5, 7, \dots\}$ is not.

An inductive definition is a means of defining a infinite set of objects that that can be indexed by the natural numbers. You specify the first object, then the second, then the third,...

**Example 2.** The factorial of a natural number may be defined inductively. A noninductive definition:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2c \cdot 1$$

The inductive definition of $n!$ proceeds as follows:

(i) Define $1! = 1$

(ii) For $n \in \mathbb{N}$, define $(n+1)! = (n+1)n!$.

Many times one wants to prove a statement is true for an infinite collection of natural numbers. Mathematical induction is perfect for this:

Structure:**Proof using the principle of mathematical induction:**

Let $S = \{n \in \mathbb{N} : \text{the statement is true for } n\}$.

(i) Show $1 \in S$.

(ii) Show that $S$ is inductive (that is, for all $n$, if $n \in S$ then $n+1 \in S$).

(iii) By the PMI, $S = \mathbb{N}$.

The trick then is to find how to index your statement that by an integer $n$.

**Example 3.** For every natural number $n$, $\sum_{i=1}^{i=n}(2i-1) = n^2$. Symbolically we can condense the statement to: $P(n)$.

*Proof.* Let $S = \{n \in \mathbb{N} : 1 + 3 + 5 + \cdots + (2n - 1) = n^2\}$. (Here we define $S$ to be the set of natural numbers for which the statement is true. Our aim is to use PMI to show that $S = \mathbb{N}$.

(i) (We first show that $1 \in S$ by showing the statement is true for $n = 1$:) $1 = 1^2$, so $1 \in S$. (boom done!)

(ii) (Next we assume that all-up natural numbers up to $n$ are in $S$. Then we show that this fact implies then $n+1$ is in $S$:) Let $n$ be a natural number such that $n \in S$. Then for this $n$, $1+3+5+\cdot+(2n-1) = n^2$. (We still have not assume what is to be proven.) Then,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2n - 1) + [2(n + 1) - 1] &= n^2 + [2(n + 1) - 1] \\ &= n^2 + 2n + 1 = (n + 1)^2. \end{aligned}$$

This shows $n + 1 \in S$.

(iii) By the PMI, $S = \mathbb{N}$. That is, for every natural number $n, 1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

$\square$