And/Or Negation and Quantifiers

Goal: Have the logical tools to find the negation of a statement like:
Every number satisfying the equation $P(x)=Q(x)$ is such that $|x|<5$.

And/Or/Negation

Definition 1. A *statement* is a sentence or mathematical symbol that can be true or false.

You should be able to identify when a sentence is a statement and when it is not.

Definition 2. A *simple statement* is a statement that cannot be broken down.

Definition 3. Let P and Q be statements.

1. The *conjunction* of P and Q is written as $P \wedge Q$ (read as "P and Q",) and the compound statement is true only when both P and Q are true.
2. The *disjunction* of P and Q is written $P \vee Q$ (read as "P or Q",) and the compound statement is true exactly when at least on of P or Q is true.
3. The *negation* of P is written as $\sim P$ (read as "not P",) and is true whenever P is false.

Definition 4. A *compound statement* is a statement that is made up of finitely many simple statements connected by logical operators.

Truth Tables

We can use truth tables to work out the truth of a compounded statement:

Example 1. Determine the truth table of the compound statement $P \wedge (\sim Q)$

P	Q	$\sim Q$	$P \wedge (\sim Q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Two compound statements are equivalent exactly when the have the same truth table.

Example 2. Show that $\sim (P \wedge Q)$ is equivalent to $(\sim P) \vee (\sim Q)$:

P	Q	$\sim (P \wedge Q)$
T	T	F
T	F	T
F	T	T
F	F	T

P	Q	$(\sim P) \vee (\sim Q)$
T	T	F
T	F	T
F	T	T
F	F	T

Open sentence

Recall a sentence containing a variable, such as $x > 10$ is not a statement since it can be true or false depending on what we may choose x to be.
Definition 1. A sentence containing one or more variables is called an *open sentence*.

Example 1. Define $P(x)$ to be the sentence $x^2 - 1 = 0$. Once we choose a value for x, for example $x = 6$, the open sentence becomes a statement: $P(6)$ which is a false statement.

Example 2. Define $P(x, y)$ to be the open sentence $x - y^2 \geq 1$. Then $P(4, 1)$ is a true statement.

Definition 2. The collection of all objects that may be substituted to make an open sentence a true statement is called a *truth set*. The entire set of objects available for consideration is called a *universal*.

Examples

Example 3. Let the universe be the entire set of integers for the open sentence x is divisible by 2. Then the truth set are all even numbers. Let the universe be the set of all positive integers less then 4 for the same open sentence. Then the truth set is ...

With a particular universe in mind, we say two open sentences are equivalent if and only if they have the same truth set.

Example 4. Let the universe be all positive real numbers, then $x^2 - 1 = 0$ and $x = 1$ are equivalent. Now consider the universe as the set of all real numbers; $x^2 - 1 = 0$ and $x = 1$ are not equivalent.

Universal, Existential, Unique

Definition 3. Let $P(x)$ be an open sentence with variable x.

1. The *universal quantifier* is the statement $(\forall x)P(x)$ (" for all x, $P(x)$") and is true exactly when the truth set for $P(x)$ is the entire universe. In other words, all elements in the universe make $P(x)$ true.
2. The *existential quantifier* is the statement $(\exists x)P(x)$ (" there exists x such that $P(x)$") and is true exactly when the truth set for $P(x)$ is not empty. In other words, there exists at least one element x in the universe for which $P(x)$ is true.
3. The *unique existence quantifier* is the statement $(\exists! x)P(x)$ (" there exists a unique x such that $P(x)$") and is true exactly when the truth set for $P(x)$ has exactly one element.

Examples

Example 5. The following example should help you understand the definitions.

a) Let the universe be the set of all real numbers and consider the open sentence $P(x) : x^2+10 \geq 0$. Consider the quantifier sentence $(\forall x)P(x)$. This statement is true since the truth set is the entire universe.

b) Let the universe be the set of all complex numbers and consider the open sentence $P(x) : x^2 + 10 \geq 0$. Consider the quantifier sentence $(\forall x)P(x)$. This statement is false since $10i \in \mathbb{C}$ and $(10i)^2 + 10 = -100 + 10 = -90 < 0$.

c) Let the universe be all complex numbers. $(\exists x)(x^2 + 1 = 0)$ is true because the truth set consists of $\{-i, i\}$.

d) Let the universe be the set of real numbers. $(\exists x)(x^2 + 1 = 0)$ is false because there is no real number for which the expression is true.

Rewrite as Symbolic Logic

e) Let the universal be the set of all real numbers and consider the open sentence $P(x, y) : x - y^2 = 0$. Consider $(\forall y)(\exists! x)P(x, y)$. This sentence is true.
Notice that the order matters: $(\exists! x)(\forall y)P(x, y)$ means: there is a single, unique real number x so that $x - y^2 = 0$ for all real y. Clearly this is not true.

Example 6. The statement "All dogs chase cars" is qualified with $\forall$. It's reasonable to assume the universe is animals. Use $A(x)$ to represent "x is a dog" and $B(x)$ to represent "x chases cars". Then we can rewrite the statement as $(\forall x)$ if x is a dog, then x chances cars, or $(\forall x)A(x) \Rightarrow B(x)$.

Example 7. Consider the statement "Some dogs eat vegetables". Let $A(x)$ denote "x is a dog" and $C(x)$ denote "x eats vegetables."

Example 8. Try rewriting the following statement in symbolic quantifier form: "There are some animals which chase cars but do not eat vegetables".

Applying negation to quantifiers

Theorem 1. *Let $A(x)$ be an open sentence with variable x. Then*

(a) $\sim (\forall)A(x)$ *is equivalent to* $(\exists x)(\sim A(x))$.

(b) $\sim (\exists)A(x)$ *is equivalent to* $(\forall x)(\sim A(x))$.

Example 9. Use the Theorem to find

a. "∼(some dogs eat vegetables)".

b. "∼(all dogs chase cars)"

c. "∼(there are some animals which chase cars but do not eat vegetables.)"