# Quantum Computing 103: Entanglement

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One of the most fascinating and powerful concepts in quantum computing is **quantum entanglement**. It allows two or more qubits to become interconnected in such a way that the state of one qubit instantly influences the state of another, no matter how far apart they are. This unique property enables quantum computers to perform calculations that are impossible for classical computers.

In this blog, we will explore: - What **entanglement** means in the context of quantum computing
- How it is represented mathematically
- How quantum circuits can be used to create and manipulate entangled states

## 1 Superposition

The concept of **superposition** is one of the fundamental principles of quantum computing. In classical computing, a bit can be either 0 or 1. But in quantum computing a qubit can exist in a superposition of both 0 and 1 at the same time! This means that the qubit has some probability of being measured as 0 and some probability of being measured as 1. A general qubit state is a linear combination of basis vectors $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ given as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.
In the standard computational basis, a qubit is represented as a combination of $|0\rangle$ and $|1\rangle$. However, we can choose other orthonormal bases to express quantum states, such as the Hadamard basis. **Note that superposition depends on the chosen basis.** For example, the **Hadamard basis** consists of the states:

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \ |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Applying the Hadamard transformation to $|0\rangle$ and $|1\rangle$ results in the Hadamard basis states, the qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ transforms as:

$$|\psi\rangle = \alpha'|+\rangle + \beta'|-\rangle$$

where the probability amplitudes are given by:

$$\alpha' = \frac{1}{\sqrt{2}}\left(\alpha + \beta\right), \quad \beta' = \frac{1}{\sqrt{2}}\left(\alpha - \beta\right)$$

and they satisfy the normalization condition $|\alpha'|^2 + |\beta'|^2 = 1$.

## 2 Entanglement in a Two-Qubit System

A two-qubit state is **separable** (or factorizable) if it can be written as a tensor product:

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle, \quad \text{where } |\phi_1\rangle \text{ and } |\phi_2\rangle \text{ are single-qubit states.}$$

**Entangled states are those that cannot be written as a simple tensor product.**

## 2.1 Examples of Separable and Entangled States

1. **Separable State:** The state $|01\rangle$ is **separable** because:

$$|01\rangle = |0\rangle \otimes |1\rangle$$

2. **Entangled State:** The Bell state:

$$\left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

is **not separable**. Let's prove this.

## 2.2 Proof that $|\phi^+\rangle$ is Entangled

Assume, for contradiction, that $|\phi^+\rangle$ is separable, meaning it can be expressed as:

$$\left|\phi^+\right\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

But we know:

$$\left|\phi^+\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

By comparing terms:

$$ac = \tfrac{1}{\sqrt{2}}, \quad ad = 0, \quad bc = 0, \quad bd = \tfrac{1}{\sqrt{2}}$$

Since $ad = 0$ and $bc = 0$, we must have either $a = 0$ or $d = 0$, but this contradicts $ac = \frac{1}{\sqrt{2}}$ and $bd = \frac{1}{\sqrt{2}}$. Thus, no valid values of $a, b, c, d$ satisfy all conditions. Since this results in a contradiction, $|\phi^+\rangle$ cannot be factored, proving its entanglement.

## 2.3 The Four Bell States: Maximally Entangled States

In a two-qubit system, the **Bell states** represent the four maximally entangled states. They are given as:

$$\left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$\left|\phi^-\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$\left|\psi^+\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$\left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Each of these states has the property that measuring one qubit immediately determines the state of the other, demonstrating **quantum entanglement**.

The Bell states $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle$ form an an orthonormal basis for the two-qubit Hilbert space. This means that any two-qubit state can be expressed as a linear combination of these four Bell states. Consequently, measuring in the Bell basis provides a complete and unambiguous way to characterize the entanglement of a two-qubit system. Thus,

$$
\begin{aligned}
|00\rangle &= \frac{1}{\sqrt{2}}(\left|\phi^+\right\rangle + \left|\phi^-\right\rangle) \\
|11\rangle &= \frac{1}{\sqrt{2}}(\left|\phi^+\right\rangle - \left|\phi^-\right\rangle) \\
|01\rangle &= \frac{1}{\sqrt{2}}(\left|\psi^+\right\rangle + \left|\psi^-\right\rangle) \\
|10\rangle &= \frac{1}{\sqrt{2}}(\left|\psi^+\right\rangle - \left|\psi^-\right\rangle)
\end{aligned}
$$

## 2.4 Bell State Measurement

Consider $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
Before measurement, the system exists in a superposition of $|00\rangle$ and $|11\rangle$.
Let's say we measure the first qubit:

1. With 50% probability, we get $|0\rangle$ which means the entire state collapses to $|00\rangle$.
2. With 50% probability, we get $|1\rangle$ which means the entire state collapses to $|11\rangle$.

After measuring one qubit: The entire system collapses to either $|00\rangle$ or $|11\rangle$.

Consider $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.
Before measurement, the system exists in a superposition of $|01\rangle$ and $|10\rangle$.
Let's say we measure the first qubit:

1. With 50% probability, we get $|0\rangle$ which means the entire state collapses to $|01\rangle$ which means the second qubit must be $|1\rangle$.
2. With 50% probability, we get $|1\rangle$ which means the entire state collapses to $|10\rangle$ which means the second qubit must be $|0\rangle$.

After measuring one qubit: The entire system collapses to either $|01\rangle$ or $|10\rangle$.

## 2.5 Qiskit Simulation of Measurement in a Two-Qubit System

Listing 1: Qiskit Simulation of Measurement in a Two-Qubit System

```
    ## necessary imports
from qiskit import QuantumCircuit, transpile
from qiskit_aer import Aer
from qiskit.quantum_info import Statevector

# create a quantum circuit with two qubits and 2 classical bits.
qc=QuantumCircuit(2,2)

# Let us see this statevector
state_qiskit = Statevector.from_instruction(qc)
print("Statevector :")
print(state_qiskit)
# this is by default ket{0}
```

```python
# Apply Hadamard gate to create superposition
qc.h(0)
state_after_hadamard_to_first_qubit = Statevector.
   from_instruction(qc)
state_after_hadamard_to_first_qubit =
   state_after_hadamard_to_first_qubit.reverse_qargs()
print("state_after_hadamard_to_first_qubit :")
print(state_after_hadamard_to_first_qubit)

#Apply a CNOT gate to entangle qubits (0,1)
qc.cx(0,1)

state_after_cnot = Statevector.from_instruction(qc)
state_after_cnot  = state_after_cnot .reverse_qargs()
print("state_after_cnot :")
print(state_after_cnot )

# Measure the qubits
qc.measure([0,1],[0,1]) # Measure both qubits

#Qiskit simulation

# Initialize Qiskit Aer simulator
simulator = Aer.get_backend('aer_simulator')

# Transpile circuit for execution
compiled_circuit = transpile(qc, simulator)

# Run the circuit using the Aer simulator
job = simulator.run(compiled_circuit)
result = job.result()

# Get measurement results
counts = result.get_counts()
print("Measurement Results:", counts)
```

The output of the above code is

```
Statevector :
Statevector([1.+0.j, 0.+0.j, 0.+0.j, 0.+0.j],
            dims=(2, 2))
state_after_hadamard_to_first_qubit :
Statevector([0.70710678+0.j, 0.        +0.j, 0.70710678+0.j,
             0.        +0.j],
            dims=(2, 2))
state_after_cnot :
Statevector([0.70710678+0.j, 0.        +0.j, 0.        +0.j,
             0.70710678+0.j],
            dims=(2, 2))
Measurement Results: {'00': 501, '11': 523}
```

**Conclusion**
The simulation ran 1024 shots, and the counts confirm this:

- 523 times $|11\rangle$

- 501 times $|00\rangle$
- no occurrence of $|01\rangle$ or $|10\rangle$, which confirms entanglement.

# 3 Entanglement in Multi-Qubit Systems

A **fully entangled state** is a state of a multi-qubit system that **cannot** be factored into a tensor product of single-qubit states. While two-qubit entanglement is well understood through Bell states, multi-qubit systems exhibit more complex entanglement patterns.

## 3.1 GHZ State: Multi-Qubit Entanglement

One example of a fully entangled state in a three-qubit system is the **Greenberger–Horne–Zeilinger (GHZ) state**:

$$|\text{GHZ}\rangle = \tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

This state is a direct generalization of the Bell state $|\phi^+\rangle$. If we measure any qubit and find it in state $|0\rangle$, we immediately know that the other two qubits must also be in $|0\rangle$. Similarly, if we measure one qubit in $|1\rangle$, the others must be in $|1\rangle$.

While there are infinitely many possible GHZ-like states (from the $\phi$ parameter), the **canonical GHZ states** are just a few and are commonly used in quantum computing protocols. These states exhibit **unique symmetries and properties** that make them especially significant in applications like quantum cryptography, quantum error correction, and distributed quantum computing.

## 3.2 W State: Another Form of Multi-Qubit Entanglement

Another important type of multi-qubit entanglement is the **W state**, which differs from the GHZ state:

$$|\text{W}\rangle = \tfrac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

Unlike the GHZ state, the W state maintains entanglement even if one qubit is lost. This makes it more robust in quantum networks and applications.

## 3.3 Comparing GHZ and W States

- **GHZ states** exhibit stronger correlations but are more fragile under decoherence.
- **W states** are less correlated but more robust if a qubit is lost.

## 3.4 Entanglement in Larger Systems

Beyond three qubits, entanglement patterns become even richer. Some key points:

- **Cluster states** and **graph states** are used in measurement-based quantum computing.
- **Quantum error correction** relies on multi-qubit entanglement to detect and correct errors.

### 3.5 Qiskit Code for GHZ State

Listing 2: Qiskit Code for GHZ State

```python
## necessary imports
from qiskit import QuantumCircuit, transpile
from qiskit_aer import Aer
from qiskit.quantum_info import Statevector
from qiskit.visualization import plot_histogram
# Create a 3-qubit quantum circuit
qc = QuantumCircuit(3)

# Step 1: Apply Hadamard gate to the first qubit
qc.h(0)

# Step 2: Apply CNOT gates to create entanglement
qc.cx(0, 1)
qc.cx(1, 2)

# Display the circuit
print(qc.draw())

# Simulate the statevector to verify GHZ state formation
state = Statevector.from_instruction(qc)
print("Statevector:", state)

# Step 3: Measure all qubits
qc.measure_all()

# Use the Aer simulator to run the circuit
simulator = Aer.get_backend('aer_simulator')
compiled_circuit = transpile(qc, simulator)
job = simulator.run(compiled_circuit, shots=1024)
result = job.result()

# Get and display the measurement results
counts = result.get_counts()
print("Measurement results:", counts)

# Plot histogram
plot_histogram(counts)
```

The output of the above code is

```
Statevector: Statevector([0.70710678+0.j, 0.        +0.j, 0.        +0.j,
             0.        +0.j, 0.        +0.j, 0.        +0.j,
             0.        +0.j, 0.70710678+0.j],
            dims=(2, 2, 2))
Measurement results: {'111': 496, '000': 528}
```

# 4 Quantum Teleportation

Quantum teleportation is a protocol that allows Alice to send a qubit state to Bob **without physically sending the qubit itself.** Quantum entanglement and classical communication are used to transfer the state information.

Quantum teleportation is fundamentally related to two-qubit systems because it relies on Bell states, which are maximally entangled states of two qubits. The process uses a pre-shared entangled pair between Alice and Bob, and Alice performs a measurement on her two qubits, affecting Bob's qubit instantaneously (but requiring classical communication to complete the teleportation).

## 4.1 Initial set up

1. **Alice has a qubit in an unknown state:**

$$|\psi_a\rangle = \alpha\,|0\rangle_a + \beta\,|1\rangle_a$$

   Here $\alpha$ and $\beta$ are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.
   Alice does not measure this qubit before teleportation, so she does not know the exact values of $\alpha$ and $\beta$.
   Alice wants to teleport this qubit to Bob.

2. **Alice and Bob share the entangled pair (Bell state):**

   (a) **both Alice and Bob start with qubit $|0\rangle$.**

      i. Alice has one qubit $q_t = |0\rangle$.
      ii. Bob has one qubit $q_b = |0\rangle$.

   (b) **Apply Hadamard (H) gate to Alice's qubit $q_t$.**
   This creates a superposition

$$H\,|0\rangle = \frac{1}{\sqrt{2}}\,(|0\rangle + |1\rangle)$$

   Now Alice's qubit is

$$q_t = \frac{1}{\sqrt{2}}\,(|0\rangle + |1\rangle).$$

   (c) **Apply a CNOT (CX) gate with Alice's qubit $|q_t\rangle$ as the control and Bob's $|q_b\rangle$ as the target.** This gives

$$|\phi^+\rangle_{t,b} = \frac{1}{\sqrt{2}}\,(|0_t 0_b\rangle + |1_t 1_b\rangle)$$

   Thus Alice and Bob share an Entangled Bell Pair

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle).$$

   One qubit of this pair is with Alice, and the other is with Bob.
   **Note that**

   - The entangled pair is prepared beforehand and distributed.The entangled pair is typically prepared by a third party (e.g., a quantum lab or entanglement distribution network) and sent to Alice and Bob before teleportation starts.
   - Even if Bob's qubit is on a different machine, it does not have an independent identity; it is linked to Alice's qubits through entanglement.
   - Measuring Alice's qubits collapses the entire system, including Bob's qubit, no matter where he is.

3. **Total System State:** Alice prepares the total system and also performs the measurement
   The total three-qubit state (Alice's unknown qubit + entangled pair) is:

$$|\psi_{total}\rangle = |\psi_a\rangle \otimes |\phi^+\rangle_{t,b} = (\alpha\,|0\rangle_a + \beta\,|1\rangle_a) \otimes \left(\frac{1}{\sqrt{2}}\,(|0_t0_b\rangle + |1_t1_b\rangle)\right)$$

   Expanding,

$$|\psi_{total}\rangle = \frac{1}{\sqrt{2}}\,(\alpha\,|0\rangle_a \otimes |0_t0_b\rangle + \alpha\,|0\rangle_a \otimes |1_t1_b\rangle + \beta\,|1\rangle_a \otimes |0_t0_b\rangle + \beta\,|1\rangle_a \otimes |1_t1_b\rangle)$$

   Thus,

$$|\psi_{total}\rangle = \frac{1}{\sqrt{2}}\left(\alpha\,|000\rangle_{a,t,b} + \alpha\,|011\rangle_{a,t,b} + \beta\,|100\rangle_{a,t,b} + \beta\,|111\rangle_{a,t,b}\right)$$

   Let us rewrite the above expression (using the property that tensor product is associative).

$$|\psi_{total}\rangle = \frac{1}{\sqrt{2}}\left(\alpha\,|00\rangle_{a,t} \otimes |0\rangle_b + \alpha\,|01\rangle_{a,t} \otimes |1\rangle_b + \beta\,|10\rangle_{a,t} \otimes |0\rangle_b + \beta\,|11\rangle_{a,t} \otimes |1\rangle_b\right)$$

4. **Alice applies a Bell measurement on her two qubits (first two qubits of the system)**
   We can express the two-qubit states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ in terms of **Bell basis**. So,

$$\begin{aligned}|\psi_{total}\rangle = \frac{1}{2}\Big[&\alpha(\left|\phi^+\right\rangle + \left|\phi^-\right\rangle) \otimes |0\rangle_b + \alpha(\left|\psi^+\right\rangle + \left|\psi^-\right\rangle) \otimes |1\rangle_b \\ &+ \beta(\left|\psi^+\right\rangle - \left|\psi^-\right\rangle) \otimes |0\rangle_b + \beta(\left|\phi^+\right\rangle - \left|\phi^-\right\rangle) \otimes |1\rangle_b\Big]\end{aligned}$$

   Rewriting $|\psi_{total}\rangle$ in the **Bell basis**:

$$\begin{aligned}|\psi_{\text{total}}\rangle = \frac{1}{2}\Big[&\left|\phi^+\right\rangle(\alpha\,|0\rangle + \beta\,|1\rangle) + \left|\phi^-\right\rangle(\alpha\,|0\rangle - \beta\,|1\rangle) \\ &+ \left|\psi^+\right\rangle(\beta\,|0\rangle + \alpha\,|1\rangle) + \left|\psi^-\right\rangle(\beta\,|0\rangle - \alpha\,|1\rangle)\Big]\end{aligned}$$

   Here,

   - The first two qubits are rewritten in the Bell basis.
   - The last qubit (Bob's) contains a transformed version of Alice's original qubit.

   When Alice performs a Bell measurement, the entire system collapses into one of four possible states. This means Bob's qubit is instantly affected, even if he is far away. However, he cannot immediately use this information until he receives Alice's classical bits.

5. **Alice Sends Classical Information** Alice measures her two qubits. The result could be one of the four Bell states:

   - $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle$
   - Depending on her result, Bob's qubit is in a different state.

Alice transmits two classical bits to Bob, informing him which Bell state she measured. **Note that Alice's qubit is destroyed after measurement (No-cloning theorem).**
**Since Alice's qubit is measured, it is irreversibly altered, meaning she no longer has access to it after teleportation.**

6. **Bob Recovers Alice's Qubit**
Bob's qubit is in a state that is related to Alice's original qubit, but it has undergone a transformation depending on Alice's measurement. Using the two classical bits, he can apply the necessary correction to reconstruct the exact state.
Bob now applies a correction based on Alice's message:

   - If Alice obtained the Bell state $|\phi^+\rangle$, Bob's qubit is already
$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$
No correction is needed.
   - If Alice obtained $|\phi^-\rangle$, Bob must apply the **Pauli-Z** gate ($Z$).
   - If Alice obtained $|\psi^+\rangle$, Bob must apply the **Pauli-X** gate ($X$).
   - If Alice obtained $|\psi^-\rangle$, Bob must apply both the **Pauli-X** and **Pauli-Z** gates ($XZ$).

   After applying the correction, Bob's qubit is exactly:
$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$
which successfully teleports Alice's qubit to Bob!
**Note that Bob does not physically receive Alice's qubit—instead, he reconstructs it after receiving two classical bits from Alice.**