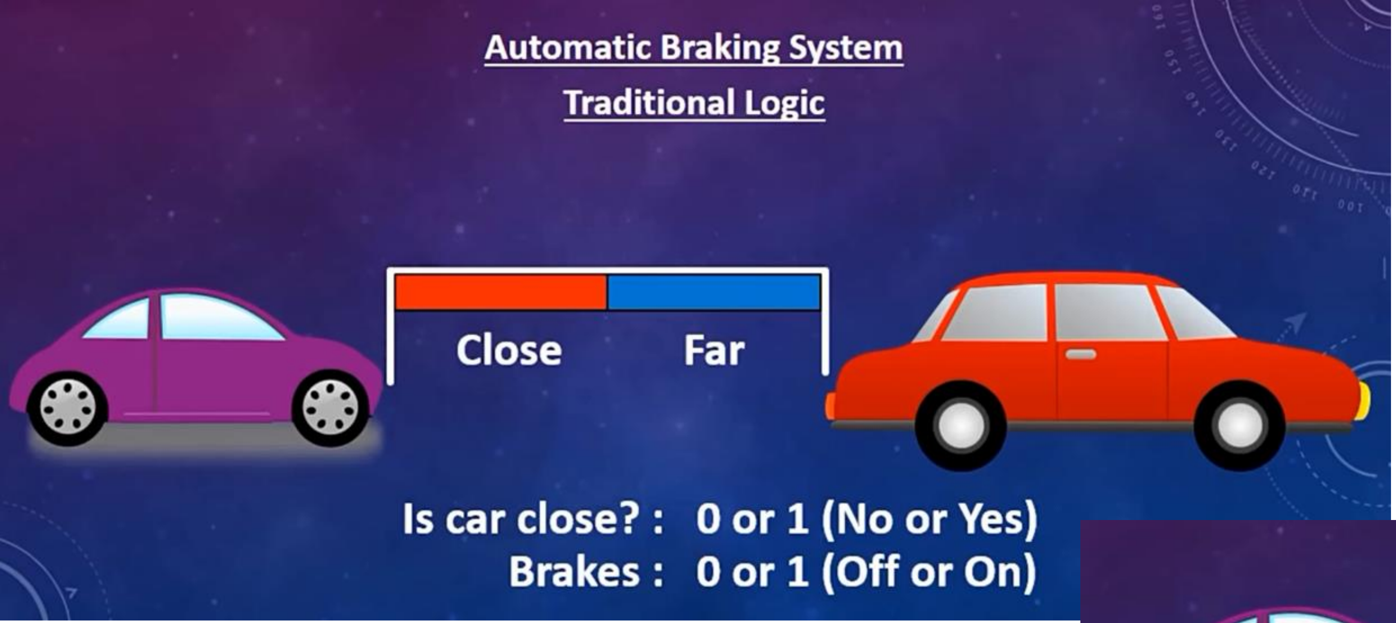
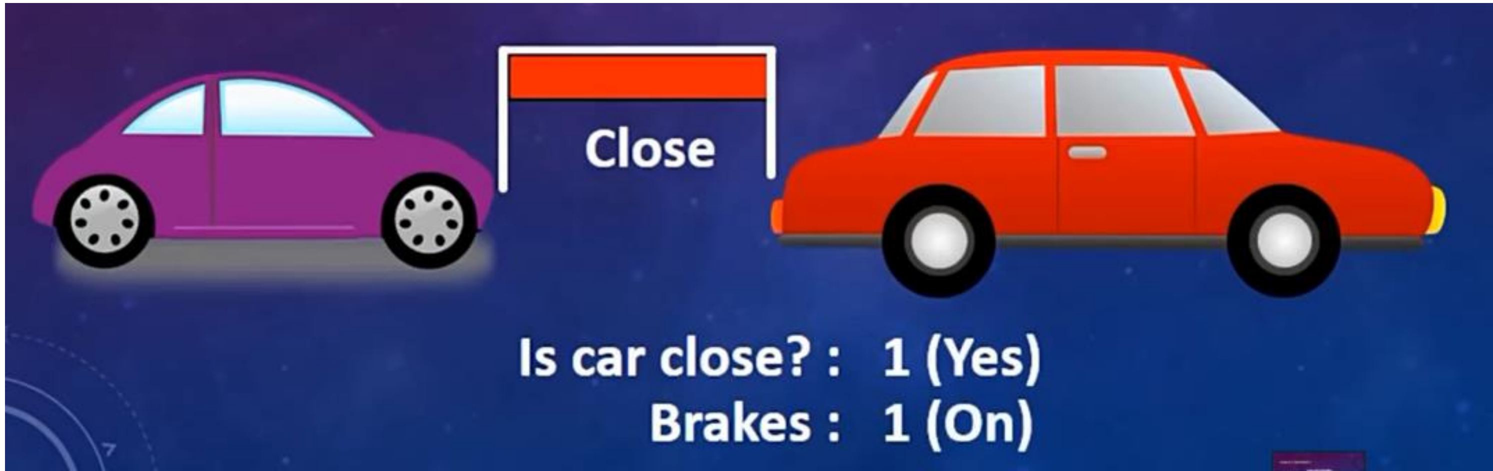
Membership Functions and Operations on Fuzzy Sets

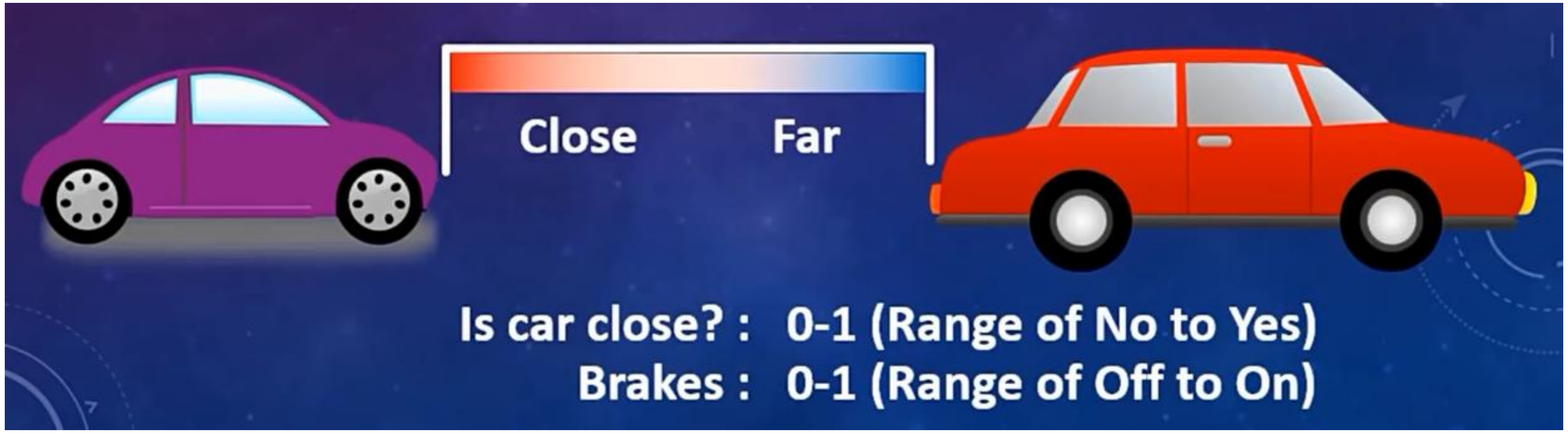
CO542 - Neural Networks and Fuzzy Systems


Automatic Braking System
Traditional Logic
Close
Far
Is car close? : 0 or 1 (No or Yes)
Brakes : 0 or 1 (Off or On)


Close
Far
Is car close? : 0 (No)
Brakes : 0 (Off)


Close
Is car close? : 1 (Yes)
Brakes : 1 (On)

Fuzzy Logic


Close
Far
Is car close? : 0-1 (Range of No to Yes)
Brakes : 0-1 (Range of Off to On)

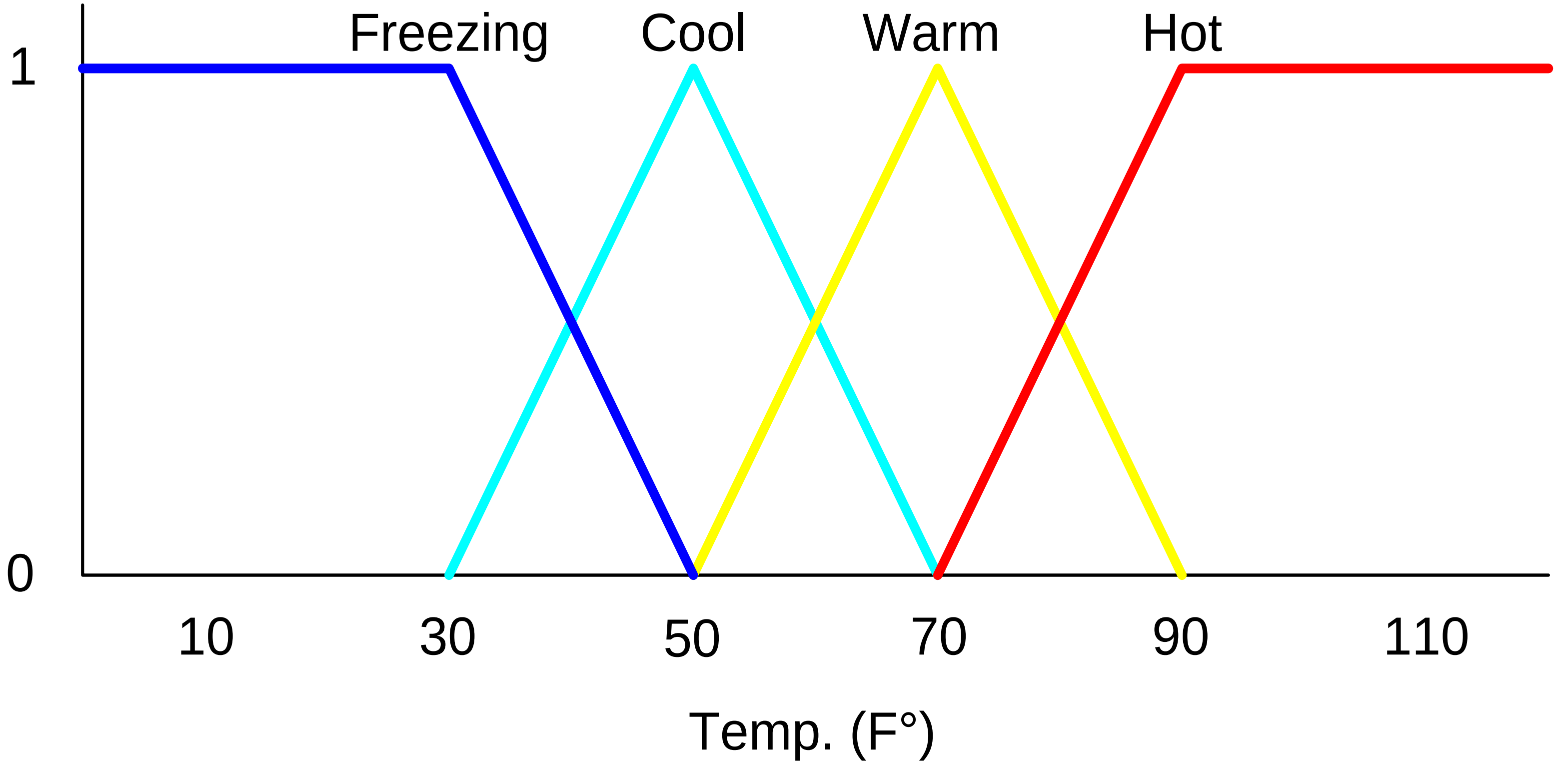

Close
Is car close? : 0.2 (Not very close)
Brakes : 0.2 (Slight pressure)

Fuzzy: Step By Step Example

- Fuzzy Linguistic Variables are used to represent qualities spanning a particular spectrum
- Temp: **{Freezing, Cool, Warm, Hot}**
- Membership Function
  - Question: What is the temperature?
  - Answer: It is warm.
- Question: How warm is it?

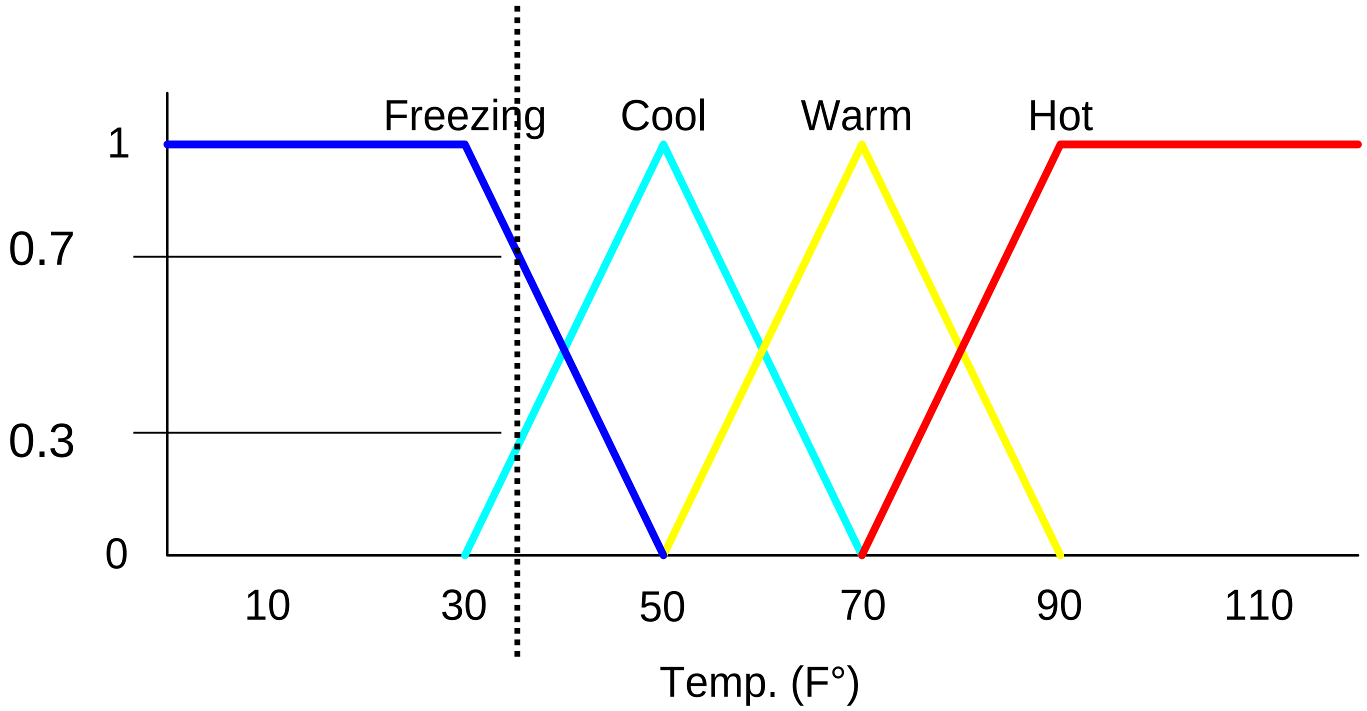
Fuzzy: Step By Step Example

- How cool is 36 F° ?



Fuzzy: Step By Step Example

- It is 30% Cool and 70% Freezing



Fuzzy: Step By Step Example

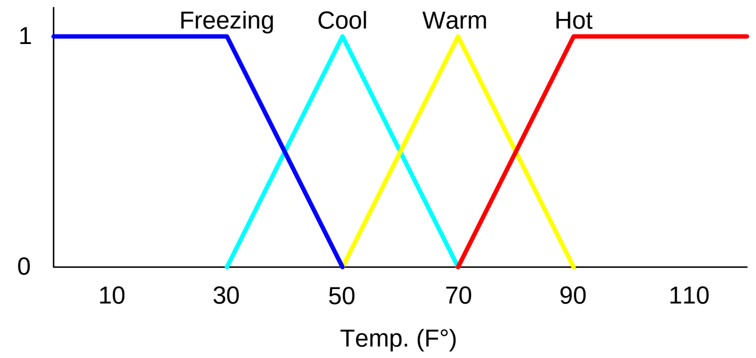
- Fuzzy Control combines the use of fuzzy linguistic variables with fuzzy logic

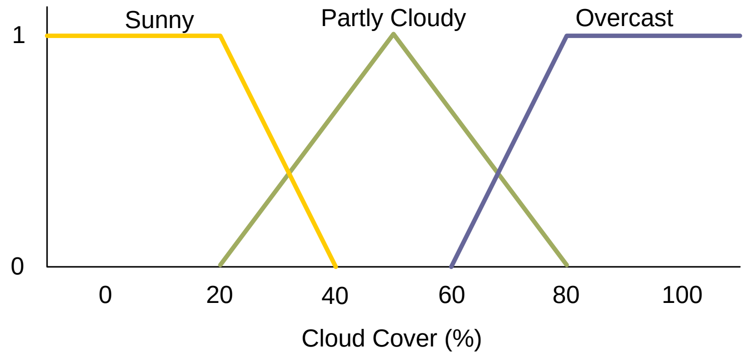
- Example: Speed Control
- How fast am I going to drive today?
  - It depends on the weather.
- Disjunction of Conjunctions

Fuzzy: Step By Step Example

- Inputs: Temperature, Cloud Cover

- Temp: {Freezing, Cool, Warm, Hot}

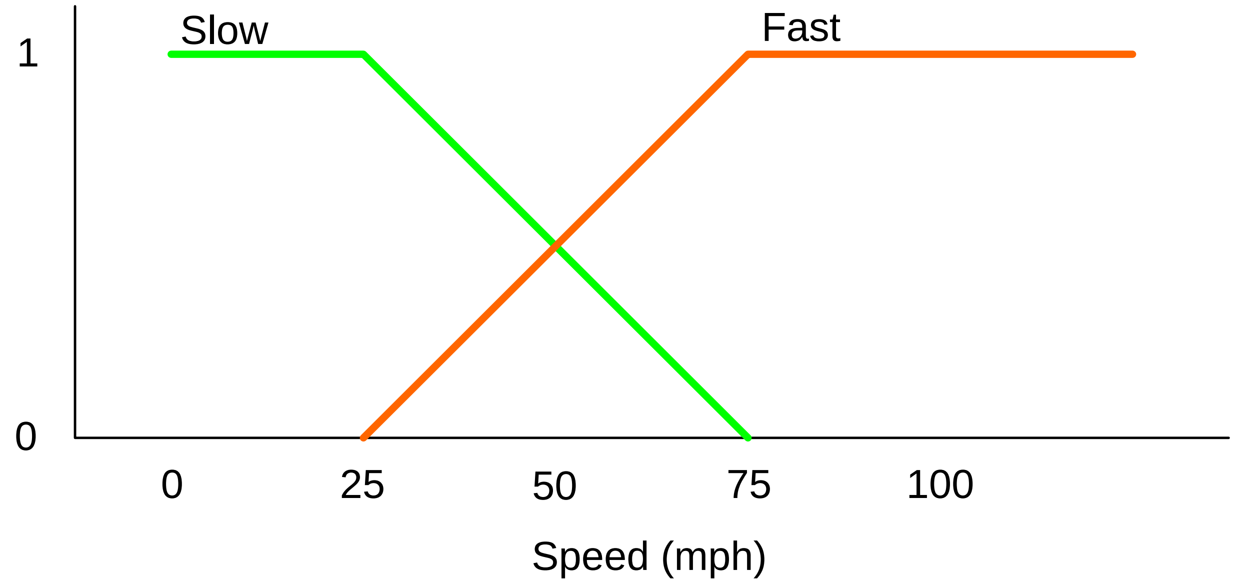




- Cover: {Sunny, Partly, Overcast}

Fuzzy: Step By Step Example

- Speed: {Slow, Fast}



Fuzzy: Step By Step Example

- **Rules**
- If it's Sunny and Warm, drive Fast

  $\text{Sunny(Cover)} \wedge \text{Warm(Temp)} \Rightarrow \text{Fast(Speed)}$

- If it's Cloudy and Cool, drive Slow

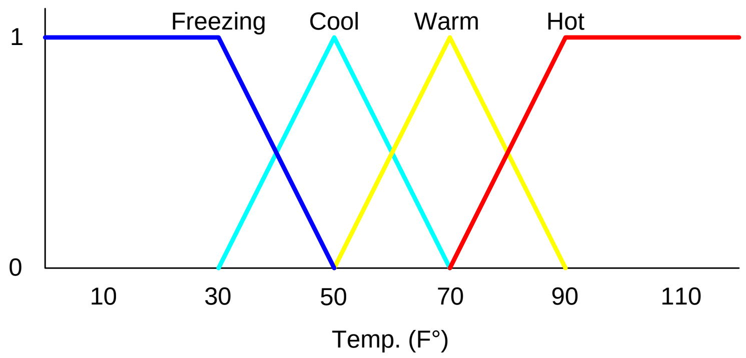
  $\text{Cloudy(Cover)} \wedge \text{Cool(Temp)} \Rightarrow \text{Slow(Speed)}$

- Driving Speed is the combination of output of these rules...

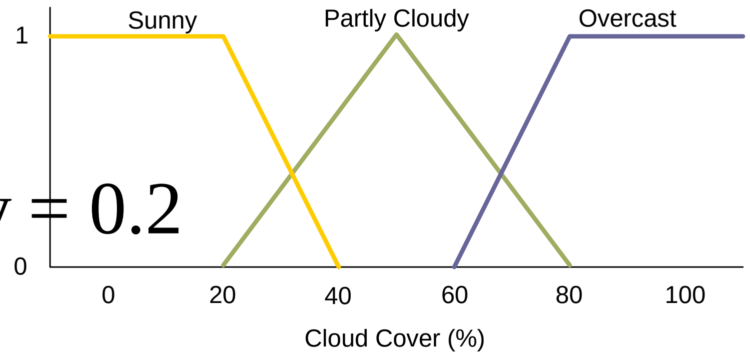
  - How fast will I go if it is 65 F° and 25 % Cloud Cover ?

Fuzzification

- Calculate Input Membership Levels



- 65 F° ⇒ Cool = 0.4, Warm= 0.7



- 25% Cover ⇒Sunny = 0.8, Cloudy = 0.2

Rules

- If it's Sunny and Warm, drive Fast

$$\text{Sunny}(\text{Cover}) \wedge \text{Warm}(\text{Temp}) \Rightarrow \text{Fast}(\text{Speed})$$

$$0.8 \wedge 0.7 = 0.7$$

$$\Rightarrow \ \mathbf{Fast = 0.7}$$

- If it's Cloudy and Cool, drive Slow

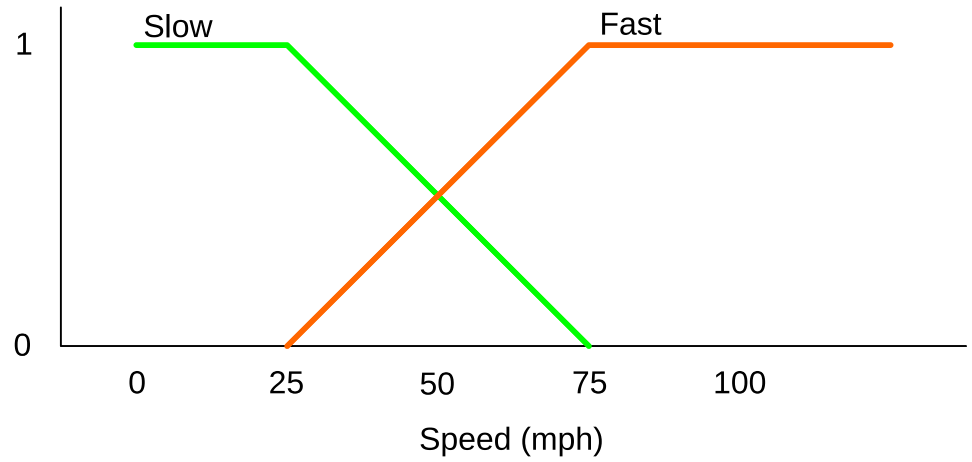
$$\text{Cloudy}(\text{Cover}) \wedge \text{Cool}(\text{Temp}) \Rightarrow \text{Slow}(\text{Speed})$$

$$0.2 \wedge 0.4 = 0.2$$

$$\Rightarrow \ \mathbf{Slow = 0.2}$$

Defuzzification

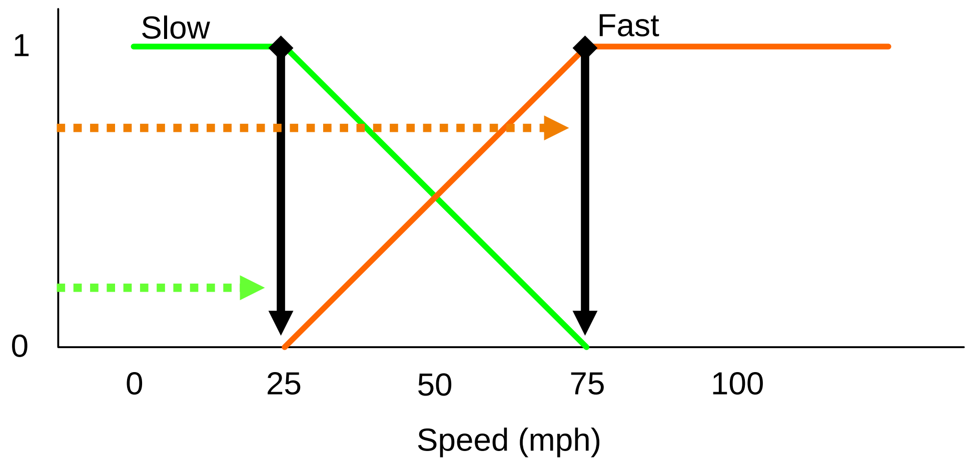
- Constructing the Output
- Speed is 20% Slow and 70% Fast



- Find centroids: Location where membership is 100%

Fuzzy: Step By Step Example

- Speed is 20% Slow and 70% Fast

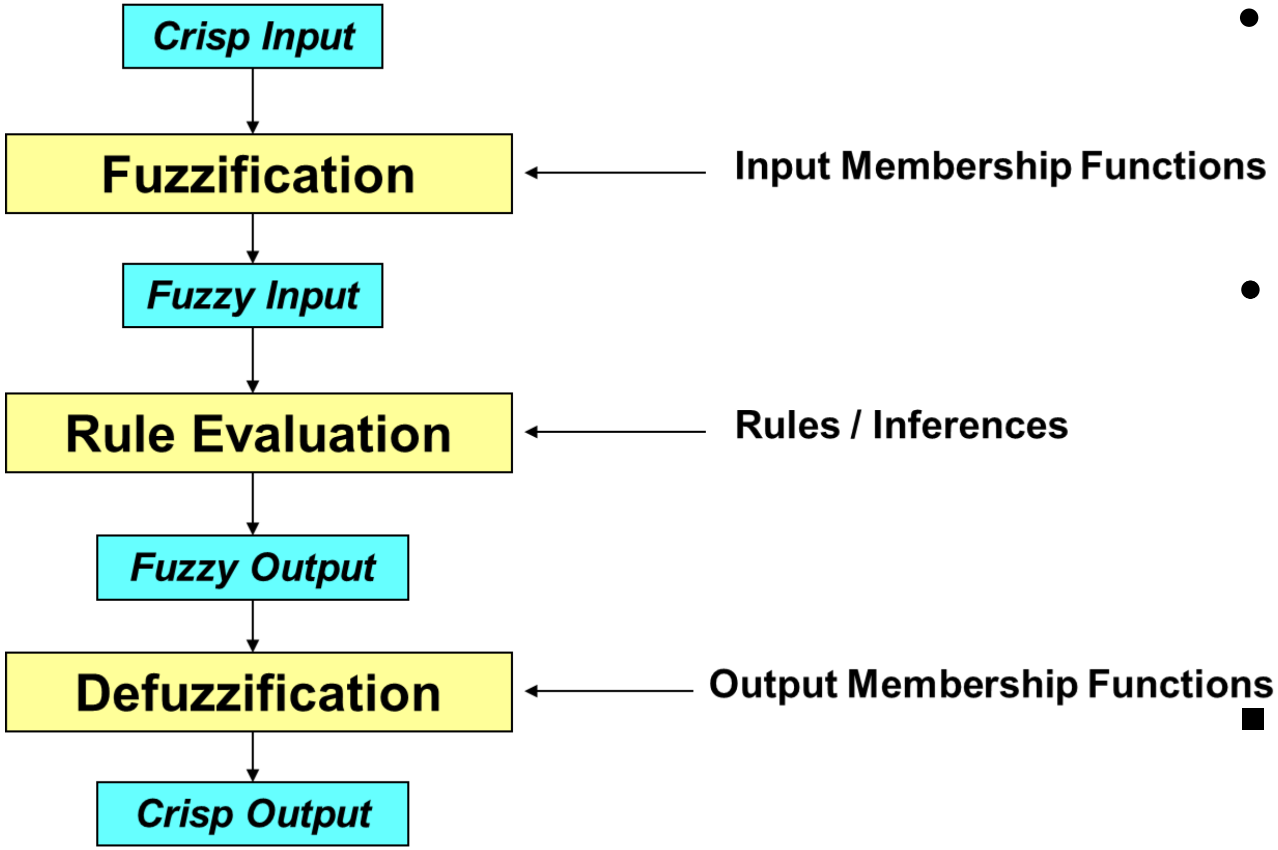


Speed = weighted mean

= (2*25+7*75)/(9)

= 63.8 mph

Fuzzy Control Block Diagram



- Fuzzification:
  - The process of determining the degree to which a value belongs in a fuzzy set
  - The value returned by a fuzzy MF
- Defuzzification
  - process of reducing a fuzzy set into a crisp set or converting a fuzzy member into a crisp member.
  - i.e. producing a quantifiable result in Crisp logic
- Rules
  - infer an output based on input variable
    - Premise: x is A
    - Implication: IF x is A THEN y is B
    - Consequent: y is B

Fuzzy Membership Functions

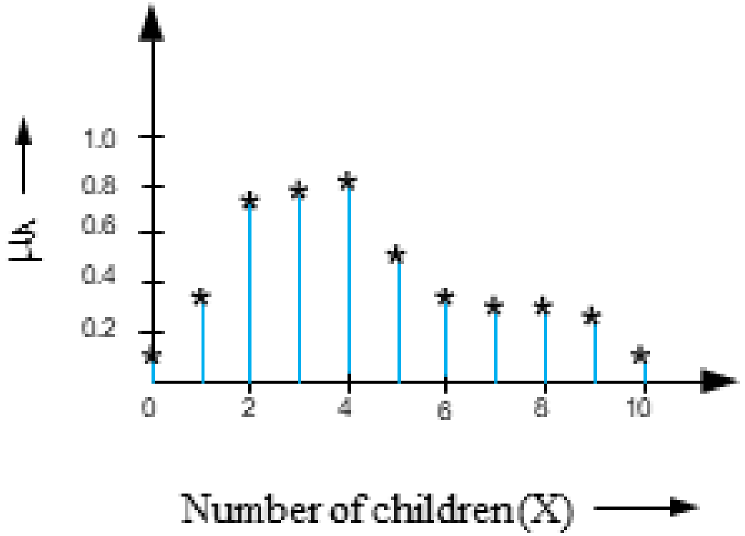
Fuzzy Membership Functions

- A fuzzy set is completely characterized by its membership function (sometimes abbreviated as MF and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

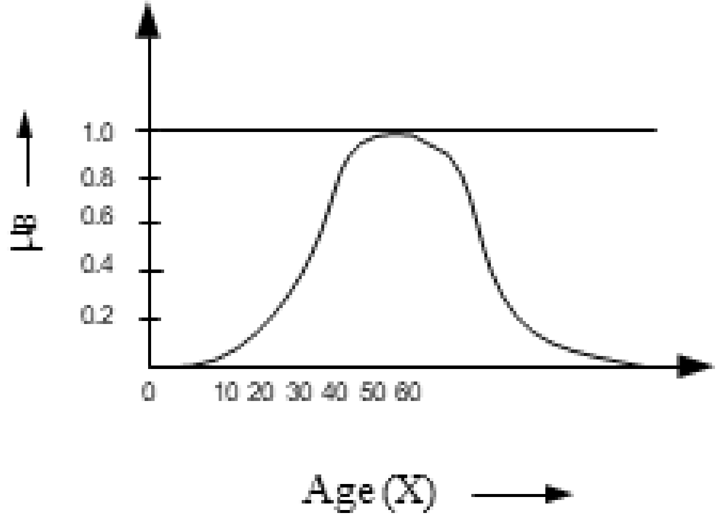
- A membership function can be on
  - a discrete universe of discourse and
  - a continuous universe of discourse.

- Recall: Fuzzy Set

$$A = \{(x, \mu_A(x)) | x \in X\}$$

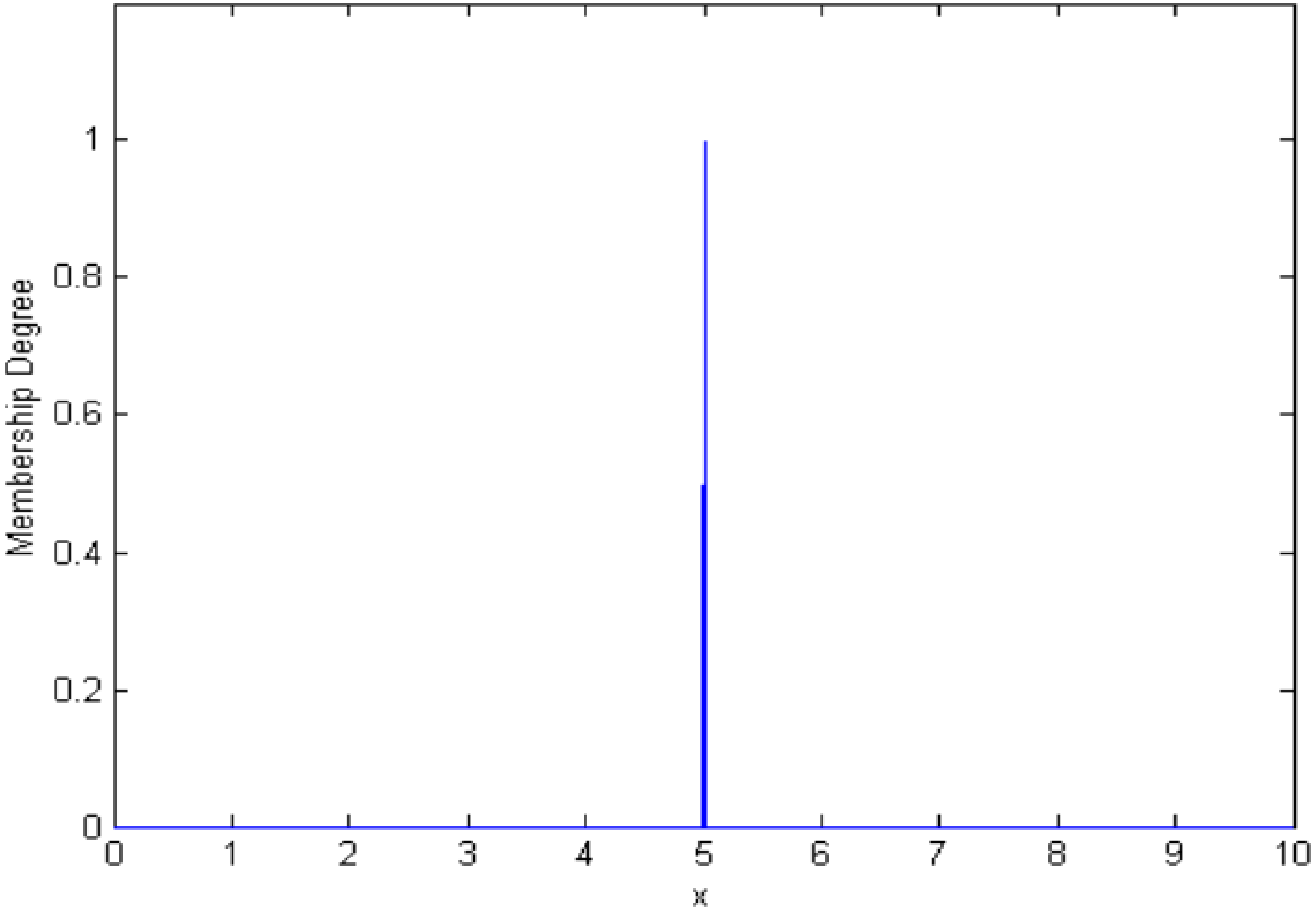


A = Fuzzy set of "Happyfamily"



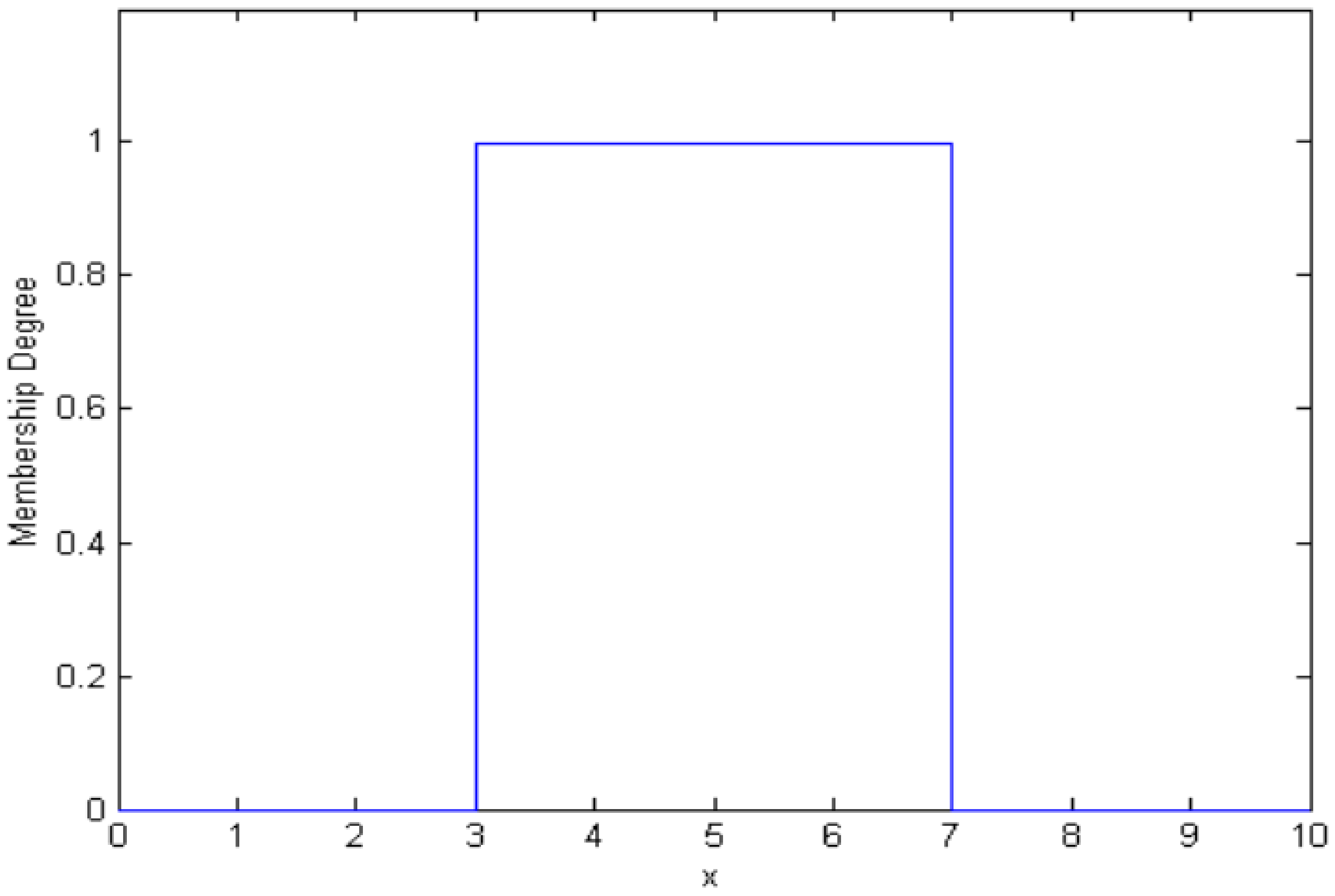
B = "Youngage"

Singleton MF



$$\mu(x) = \begin{cases} 1, & x = c \\ 0, & \text{otherwise} \end{cases}$$

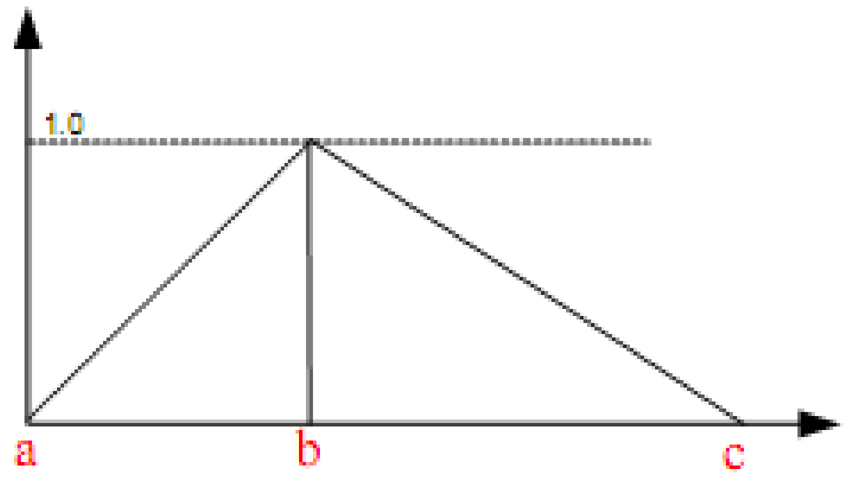
Rectangular MF



$$\mu(x) = \begin{cases} 1, & l \leq x \leq r \\ 0, & \text{otherwise} \end{cases}$$

Triangular MF

- Constantly tend towards zero and one
- A family of MF : Three in the family
  - Left-shouldered
  - Triangular
  - Right-shouldered
- A triangular MF is specified by three parameters {a; b; c} and can be formulated as follows

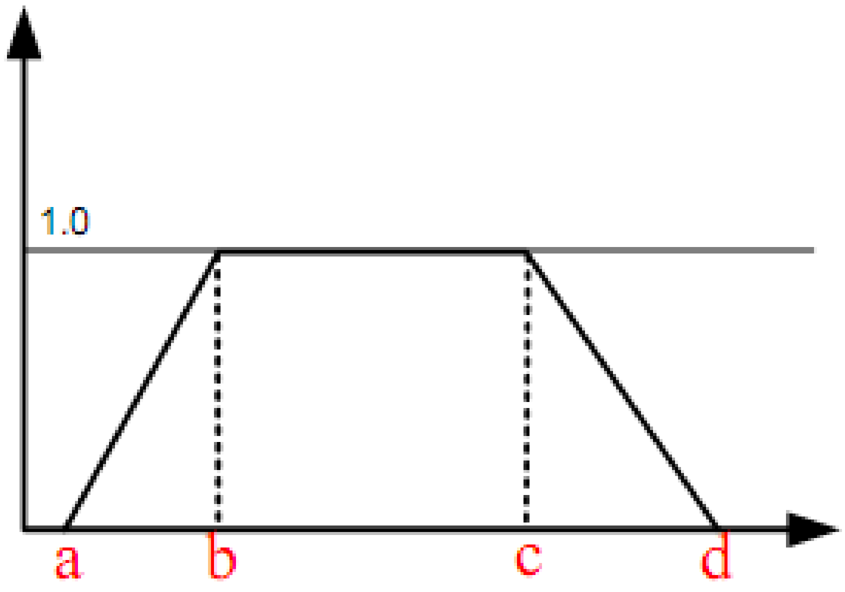


$$triangle(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } c \leq x \end{cases}$$

Trapezoidal MF

- A trapezoidal MF is specified by four parameters {a, b, c, d} and can be defined as follows:

$$trapeziod(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases}$$

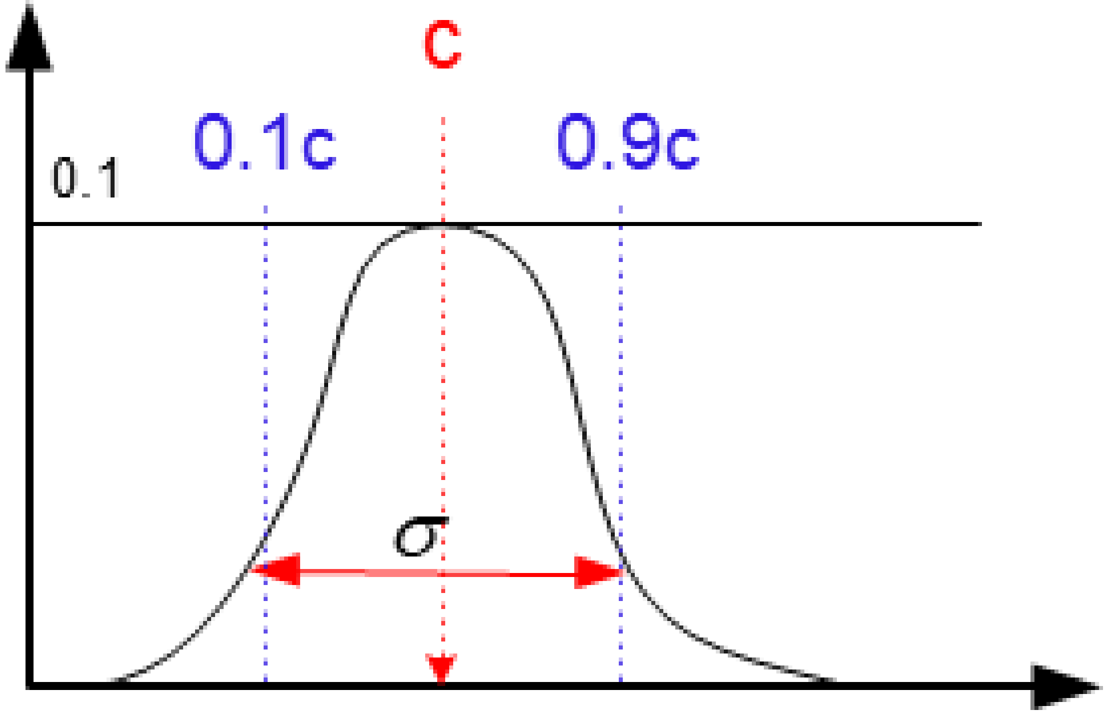


Gaussian MF

- A Gaussian MF is specified by two parameters {c, σ} and can be defined as below

$$\mu_b(x) = exp\left(- \frac{(x-c)^2}{2\sigma^2} \right)$$

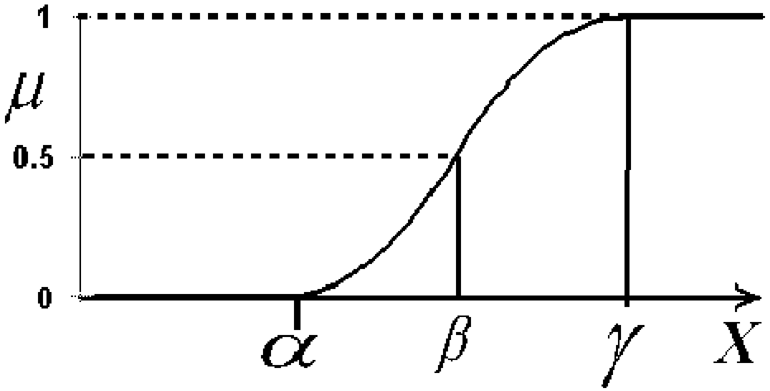
- c is the centre of the MF
- sigma is the width of the MF
- exp is the exponential function



Gaussian MF: S Function

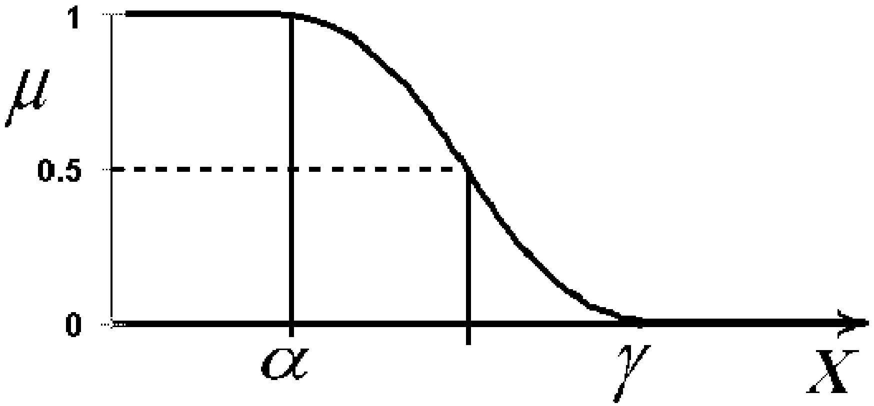
- α is the left hand 'breakpoint' of the MF
- γ is the right hand 'breakpoint' of the MF
- β is the center of the MF

S^+ membership function



$$\mu = \begin{cases} 0 & \textit{for} \quad x < \alpha \\ 2\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 & \textit{for} \quad \alpha \le x \le \beta \\ 1 - 2\left(\dfrac{x-\gamma}{\gamma-\alpha}\right)^2 & \textit{for} \quad \beta \le x \le \gamma \\ 1 & \textit{for} \quad x > \gamma \end{cases}$$

S^- membership function

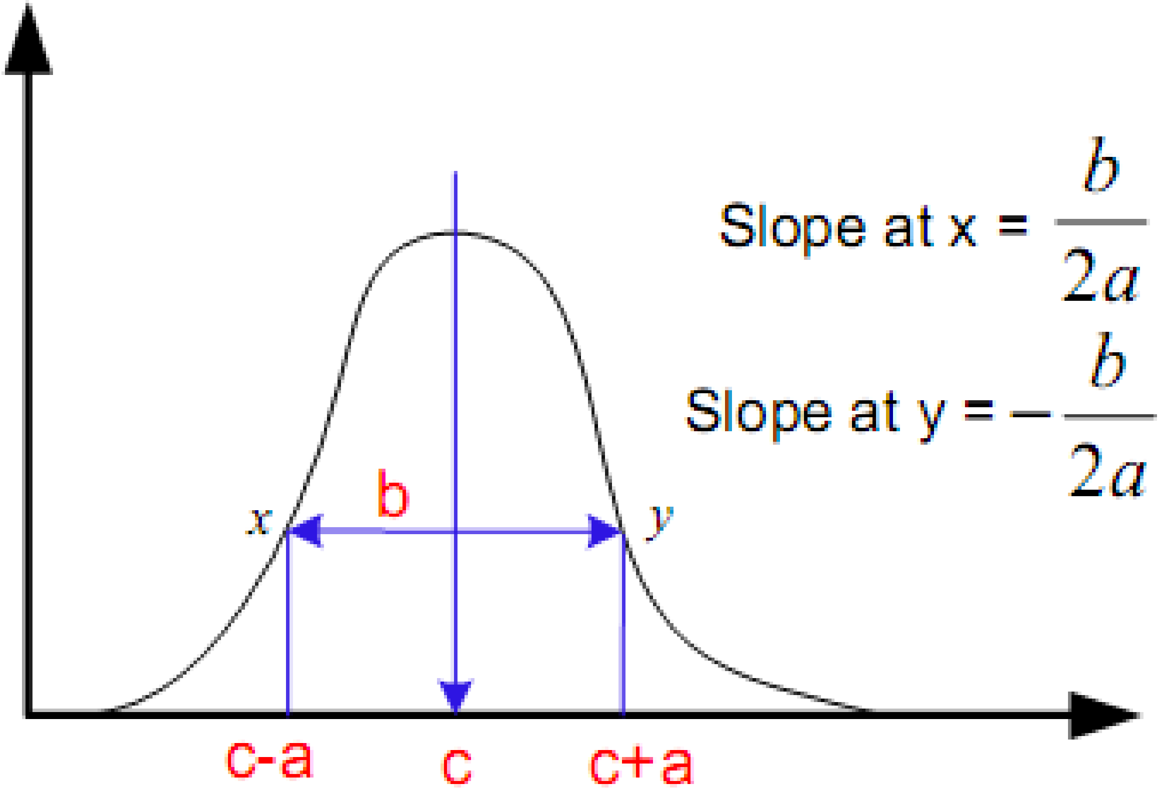


$$\mu = \begin{cases} 1 & \textit{for} \quad x < \alpha \\ 1 - 2\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 & \textit{for} \quad \alpha \le x \le \beta \\ 2\left(\dfrac{x-\gamma}{\gamma-\alpha}\right)^2 & \textit{for} \quad \beta \le x \le \gamma \\ 0 & \textit{for} \quad x > \gamma \end{cases}$$

Generalized bell MF

- It is also called Cauchy MF. A generalized bell MF is specified by three parameters {a, b, c} and is defined as:

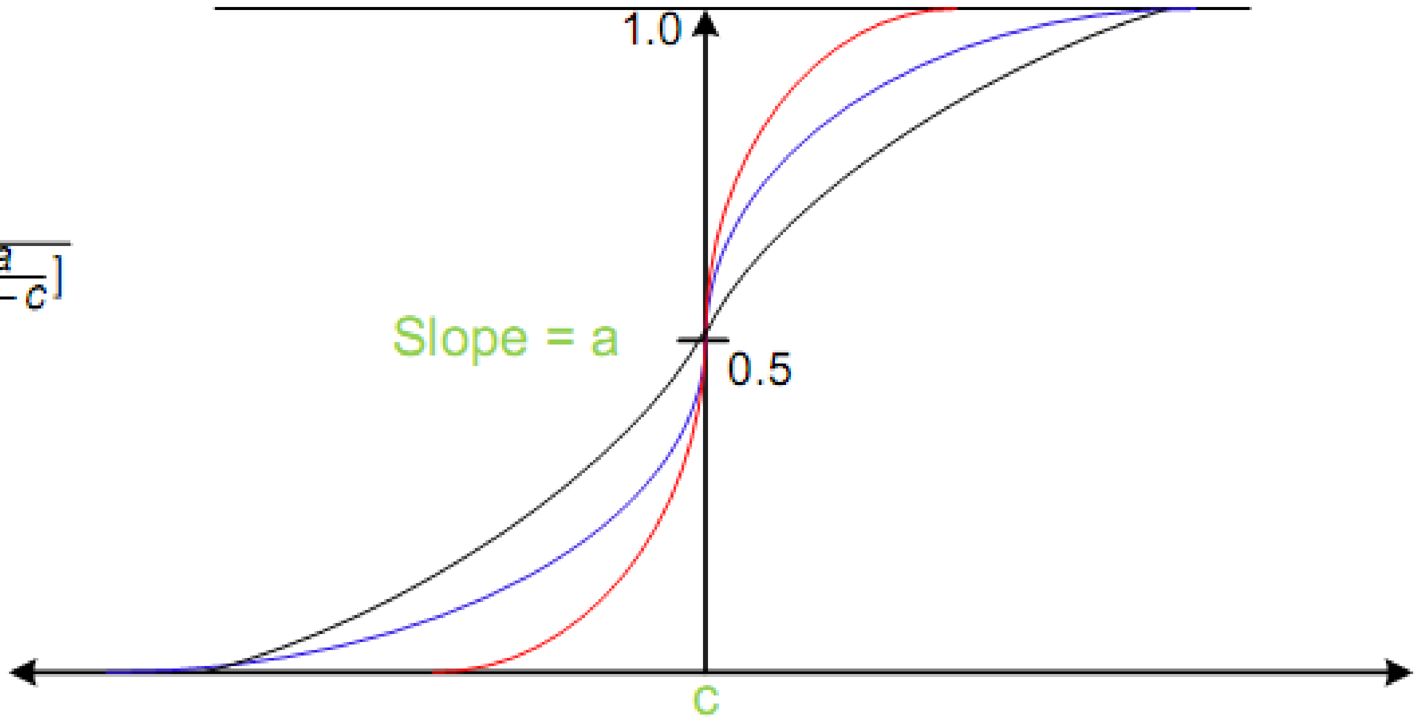
$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$



Sigmoidal MF

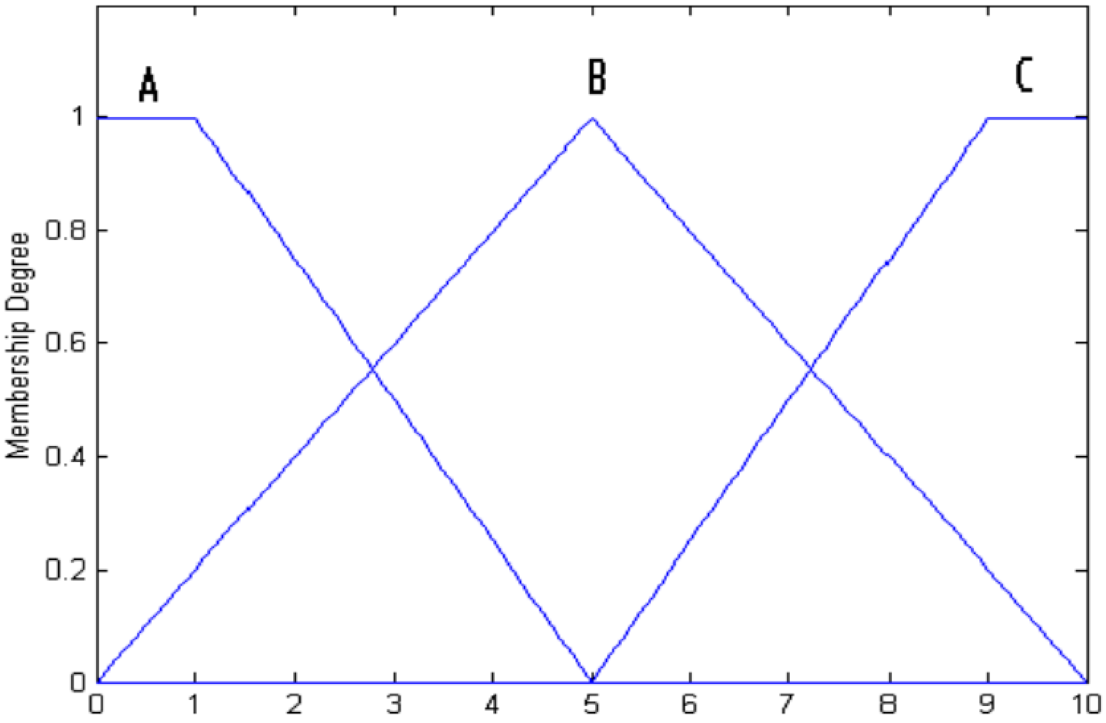
- Parameters: {a,c} ; where c = crossover point and a = slope at c;

$$\mathrm{Sigmoid}(x;a,c)=\frac{1}{1+e^{-\left[\frac{a}{x-c}\right]}}$$



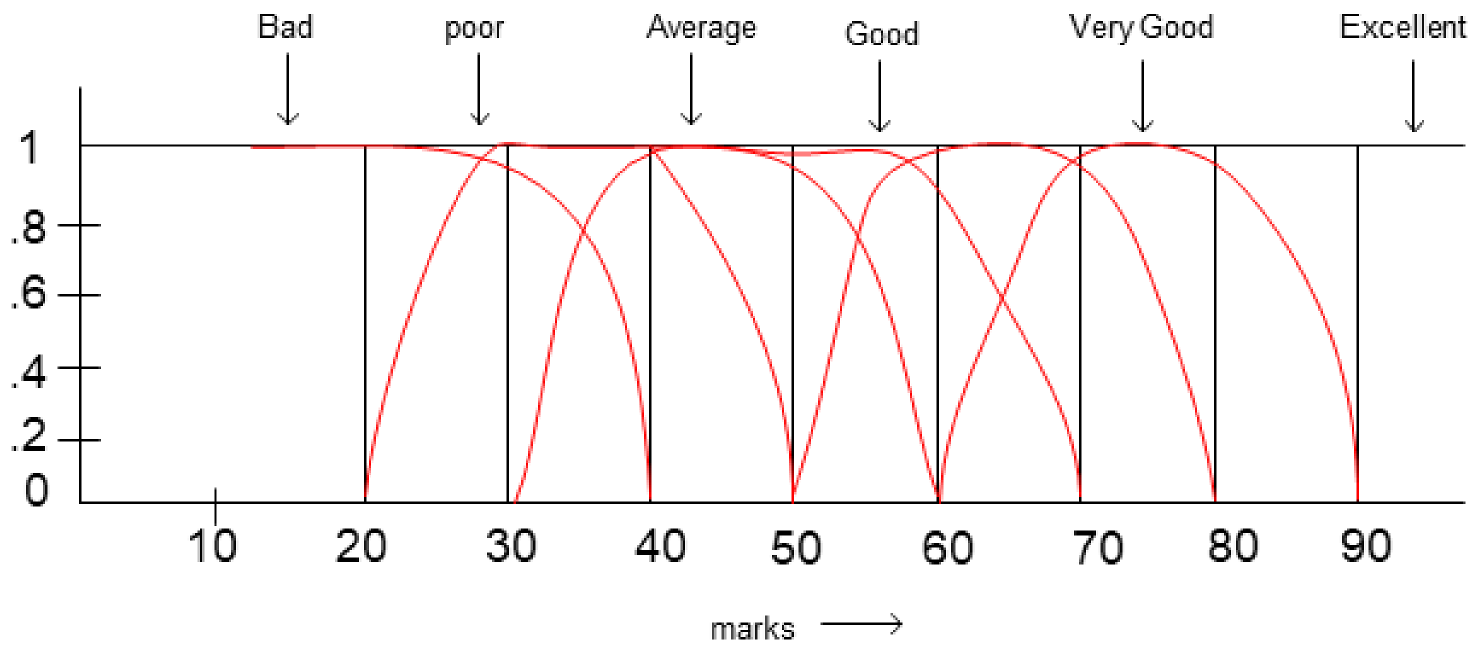
Membership Functions

- MF can also be represented by a set of ordered pairs
- Pairs are crisp-fuzzy values
  - A={(0,1.0),(1,1.0),(2,0.75),(3,0.5),(4,0.25),(5,0.0),(6,0.0),(7,0.0),(8,0.0),(9,0.0),(10,0.0)}
  - B={(0,0.0),(1,0.2),(2,0.4),(3,0.6),(4,0.8),(5,1.0),(6,0.8),(7,0.6),(8,0.4),(9,0.2),(10,0.0)}
  - C={(0,0.0),(1,0.0),(2,0.0),(3,0.0),(4,0.0),(5,0.0)(6,0.25),(7,0.5),(8,0.75),(9,1.0),(10,1.0)}



Example

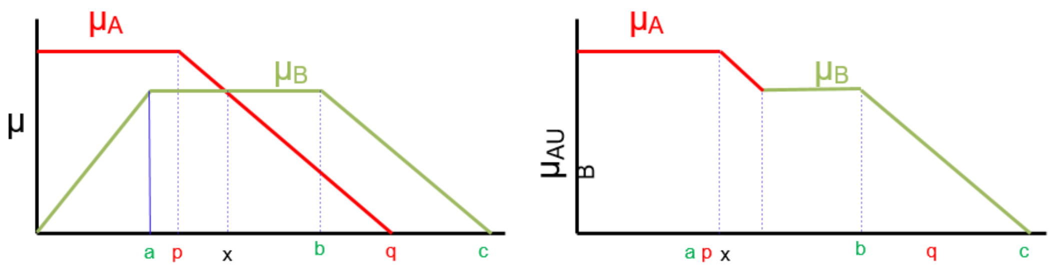
- A fuzzy implementation will look like the following.



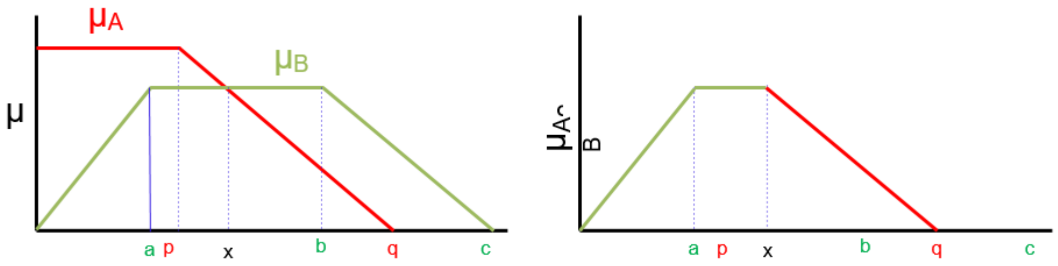
- Fuzzy MF for each of the fuzzy garde?

Properties of Fuzzy Sets (I)

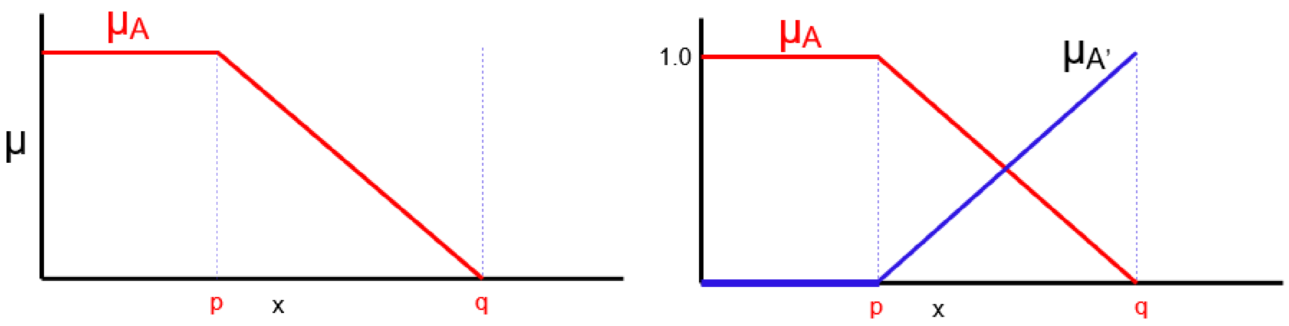
- From last week
  - Union ($A \cup B$):
    - $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$



  - Intersection ($A \cap B$):
    - $\mu A \cap B\ (x) = \min\{\mu A(x), \mu B\ (x)\}$



  - Complement (A^C):
    - $\mu_A^{AC}(x) = 1 - \mu_A(x)$



Operations on Fuzzy Sets

Properties of Fuzzy Sets (II)

- Commutativity :
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$

- Associativity :
  - $A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$

- Distributivity :
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Properties of Fuzzy Sets (III)

- Idempotence
  - $A \cup A = A$
  - $A \cap A = \emptyset$
  - $A \cup \emptyset = A$
  - $A \cap \emptyset = \emptyset$
- Transitivity :
  - If $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$
- Involution :
  - $(A^c)^c = A$
- De Morgan's law :
  - $(A \cap B)^c = A^c \cup B^c$
  - $(A \cup B)^c = A^c \cap B^c$

Operations on Fuzzy Sets (I) : Products

- Algebric product or Vector product (A•B):
  - $\mu_{A \bullet B}(x) = \mu_A(x) \bullet \mu_B(x)$

- Scalar product ($\alpha \times A$):
  - $\mu_{\alpha A}(x) = \alpha \cdot \mu_A(x)$

Operations on Fuzzy Sets (II): Sum and Difference

- Sum (A + B):
  - $\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$
- Difference ($A - B = A \cap B^C$):
  - $\mu_{A-B}(x) = \mu_{A \cap B^C}(x)$
- Disjunctive sum: $A \oplus B = (A^C \cap B) \cup (A \cap B^C)$)
- Bounded Sum: $| A(x) \oplus B(x) |$
  - $\mu_{|A(x) \oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$
- Bounded Difference: $| A(x) \ominus B(x) |$
  - $\mu_{|A(x) \ominus B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$

Operations on Fuzzy Sets (III): Equality and Power

- Equality (A = B):
  - $\mu_A(x) = \mu_B(x)$

- Power of a fuzzy set A^α:
  - $\mu_{A^\alpha}(x) = \{\mu A(x)\}^\alpha$
    - If $\alpha < 1$, then it is called dilation
    - If $\alpha > 1$, then it is called concentration
  - Example : Age = { Young, Middle-aged, Old }
    - Thus, corresponding to Young, we have : Not young, Very young, Not very young and so on.
    - Similarly, with Old we can have : old, very old, very very old, extremely old etc.

    - Thus, Extremely old = $(((old)^2)^2)^2$ and so on
    - Or, More or less old = $A^{0.5} = (old)^{0.5}$

Operations on Fuzzy Sets (III): Composition

- Given R is a relation on X,Y and S is another relation on Y,Z.
- Then R∘S is called a composition of relation on X and Z which is defined as follows.

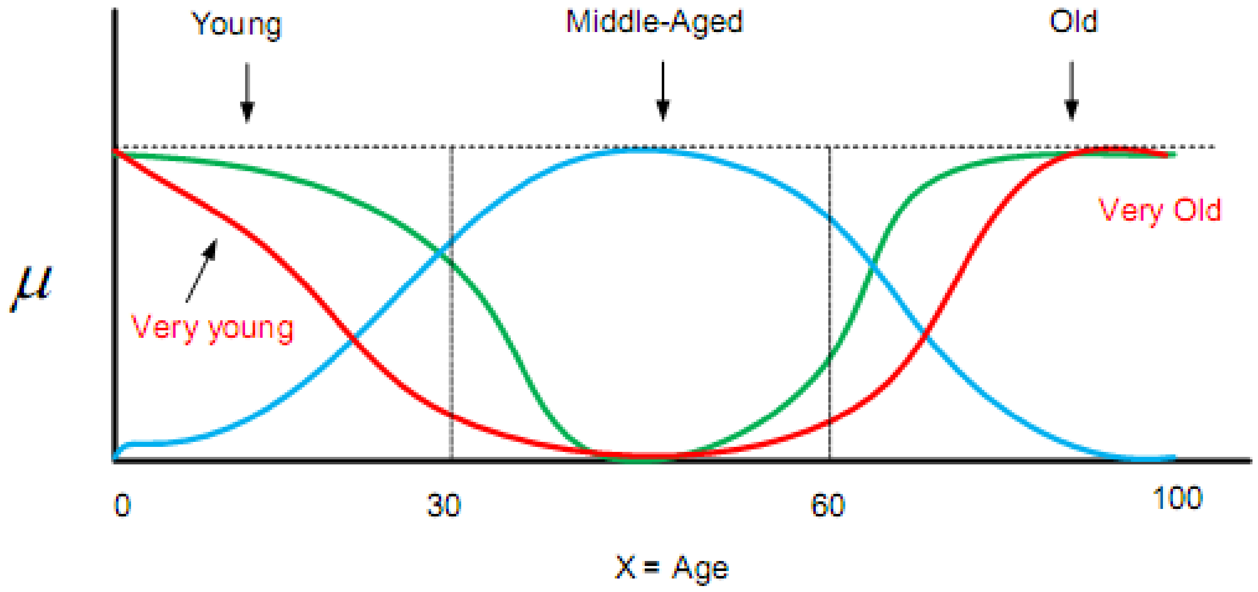
$$R \circ S = \{(x,z) | (x,y) \in R \text{ and } (y,z) \in S \text{ and } \forall y \in Y\}$$

- Max-Min Composition
  - Given the two relation matrices R and S, the max-min composition is defined as T = R ∘ S ;

$$T(x,z) = max\{min\{R(x,y), S(y,z) \text{ and } \forall y \in Y\}\}$$

Composition

- $A = \begin{matrix} 0.2 & 0.3 \\ 0.5 & 0.7 \end{matrix}$
- $B = \begin{matrix} 0.3 & 0.6 & 0.7 \\ 0.1 & 0.8 & 0.6 \end{matrix}$



$$\mu_{young}(x) = bell(x, 20, 2, 0) = \frac{1}{1+(\frac{x}{20})^4}$$

$$\mu_{old}(x) = bell(x, 30, 3, 100) = \frac{1}{1+(\frac{x-100}{30})^6}$$

$$\mu_{middle-aged} = bell(x, 30, 60, 50)$$

Not young = $\overline{\mu_{young}(x)} = 1 - \mu_{young}(x)$

Young but not too young = $\mu_{young}(x) \cap \overline{\mu_{young}(x)}$

Operations on Fuzzy Sets (IV): Cartesian product

- Caretsian Product (A × B):
  - $\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$

- Example 3:
- $A(x) = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$
- $B(y) = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$

$$A \times B = \min\{\mu_A(x), \mu_B(y)\} = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix} \end{array}$$

Example 1

Consider the following two fuzzy sets A and B defined over a universe of discourse [0,5] of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

Determine the membership functions of the following and draw them graphically.

i. $\overline{A}$, $\overline{B}$

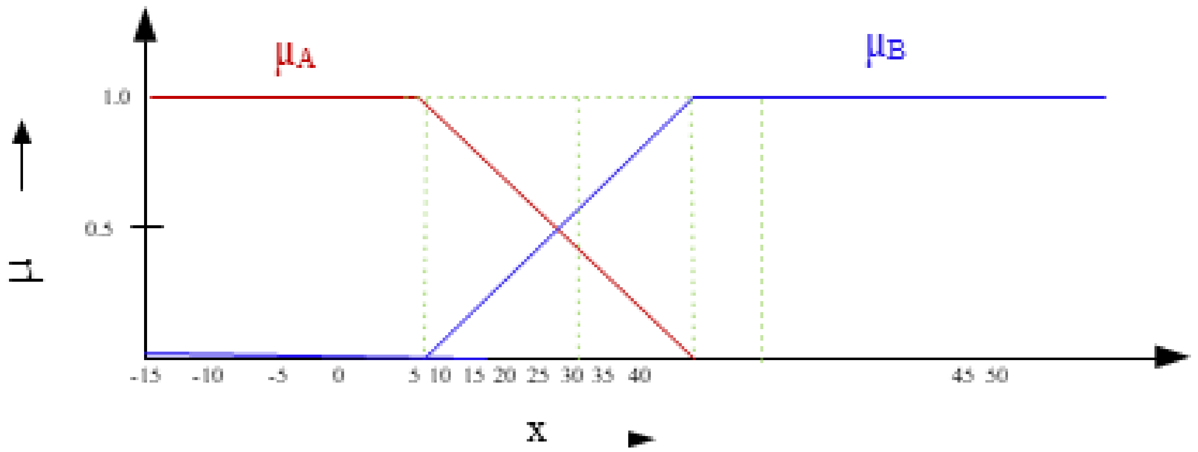
ii. $A \cup B$

iii. $A \cap B$

iv. $(A \cup B)^c$ [Hint: Use De' Morgan law]

Example 2

- Two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.
- A = Cold climate with $\mu_A(x)$ as the MF.
- B = Hot climate with $\mu_B(x)$ as the M.F.



- Here, X being the universe of discourse representing entire range of temperatures.

- What are the fuzzy sets representing the following
  - Not cold climate
  - Not hold climate
  - Extreme climate
  - Pleasant climate

- Note that ”Not cold climate” ≠ ”Hot climate” and vice-versa.

Excersie

Answer would be the following.

1. **Not cold climate**

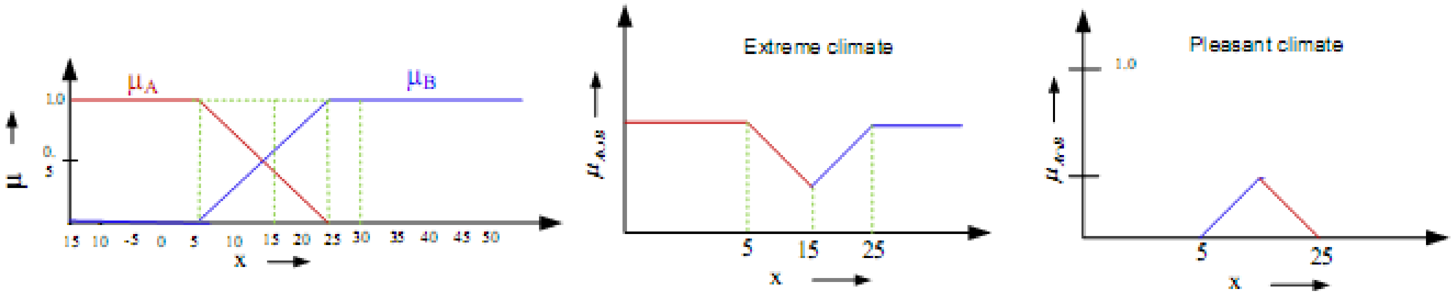
   $\overline{A}$ with $1 - \mu_A(x)$ as the MF.
2. **Not hot climate**

   $\overline{B}$ with $1 - \mu_B(x)$ as the MF.
3. **Extreme climate**

   $A \cup B$ with $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ as the MF.
4. **Pleasant climate**

   $A \cap B$ with $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ as the MF.

The plot of the MFs of $A \cup B$ and $A \cap B$ are shown in the following.



Thank You!

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Comments, Questions, Suggestions