The solutions to Maths Documents 2 are as follows:

1. Given the following matrix:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

We have to compute the singular value decomposition of matrix A.

The singular value decomposition of an $m \times n$ complex matrix M is a factorization of the form $U\Sigma V^*$, where U is an $m \times m$ complex unitary matrix, Σ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal, and V is an $n \times n$ complex unitary matrix.

Specifically:

a. The ordering of the vectors comes from the ordering of the singular values (largest to smallest).
b. The columns of U are the eigenvectors of AA^T.
c. The columns of V are the eigenvectors of A^TA.
d. The diagonal elements of Σ are the singular values, $\sigma_i = \sqrt{\lambda_i}$
e. There are relationships between v_i and u_i (with normalization):

$$Av_i = \sigma_i u_i A \tag{1}$$

$$Tu_i = \sigma_i v_i \tag{2}$$

As computed above:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

So we have:

$$\begin{bmatrix} 3 & 1 & 1 \\ -1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

2. Given the following matrix:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

We have to compute the pseudoinverse of matrix A.
In a nutshell, given the singular decomposition of a matrix A,

$$A = U\Sigma V^* \tag{3}$$

Where U and V are Unitary matrices. The Moore-Penrose pseudoinverse is given by

$$A^+ = V\Sigma^+ U^*. \tag{4}$$

where Σ^+ is formed from Σ by taking the reciprocal of all the non-zero elements, leaving all the zeros alone, and making the matrix the right shape: if is an m by n matrix, then Σ^+ must be an n by m matrix. Interchanging rows and columns, we get the transpose of A as follows:

$$A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$B = AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} * \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

To find out the eigenvalues of matrix B, we write the characteristic equation as follows:

$$\left|B - \lambda \cdot I\right| = \left| \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 11 - \lambda & 1 \\ 1 & 11 - \lambda \end{bmatrix} \right| = 0$$

So the two eigenvalues are $\lambda_1 = 12$ and $\lambda_2 = 10$

Let us find the eigenvector associated first with $\lambda_1 = 12$

We have $B \cdot v_1 = \lambda_1 \cdot \; u_1$ or $(B - \lambda_1) \cdot \; u_1 = 0$

$$or \begin{bmatrix} 11 - 12 & 1 \\ 1 & 11 - 12 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We have: $-u_{1,1} + u_{1,2} = 0$ or $u_{1,1} = u_{1,2}$

Normalizing, we get the following eigenvector associated with eigenvalue $\lambda_1 = 12$ as:

$$\begin{bmatrix} u_{1,1} \\ u_{1,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now let us find the eigenvector associated first with $\lambda_2 = 10$
So we have $B \cdot u_2 = \lambda_2 \cdot \ u_2$ or $(B - \lambda_2) \cdot \ u_2 = 0$

$$\begin{bmatrix} 11 - 10 & 1 \\ 1 & 11 - 10 \end{bmatrix} \begin{bmatrix} u_{2,1} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We have: $u_{2,1} + u_{2,2} = 0$ or $u_{2,1} = -u_{2,2}$

Normalizing, we get the following eigenvector associated with eigenvalue $\lambda_2 = 10$ as:

$$\begin{bmatrix} u_{2,1} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$C = A^T A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} * \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

To find out the eigenvalues of matrix C, we write the characteristic equation as follows:

$$\left| C - \lambda \cdot I \right| = \left| \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 10 - \lambda & 0 & 2 \\ 0 & 10 - \lambda & 4 \\ 2 & 4 & 2 - \lambda \end{bmatrix} \right| = 0$$

Solving we get $(10 - \lambda)[(10 - \lambda)(2 - \lambda) - 16)] + 2[-2(10 - \lambda)] = 0$

So the eigenvalues are $\lambda_1 = 12$, $\lambda_2 = 10$ and $\lambda_3 = 0$

Let us find the eigenvector associated first with $\lambda_1 = 12$
So we have $C \cdot v_1 = \lambda_1 \cdot \ v_1$ or $(C - \lambda_1) \cdot \ v_1 = 0$

$$\begin{bmatrix} 10 - 12 & 0 & 2 \\ 0 & 10 - 12 & 4 \\ 2 & 4 & 2 - 12 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2v_{1,1} + 2v_{1,3} = 0$ or $v_{1,1} = v_{1,3}$

$-2v_{1,2} + 4v_{1,3} = 0$ or $v_{1,2} = 2v_{1,3}$

$2v_{1,1} + 4v_{1,2} - 10v_{1,3} = 0$ or $2v_{1,1} = v_{1,2}$

Normalizing the above values, we get the following eigenvector associated with eigenvalue $\lambda_2 = 12$ as:

$$\begin{bmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Similarly, we get the eigenvector corresponding to $\lambda_2 = 10$ as:

$$\begin{bmatrix} 10-10 & 0 & 2 \\ 0 & 10-10 & 4 \\ 2 & 4 & 2-10 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; or \; \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2v_{2,3} = 0$

$2v_{2,1} + 4v_{2,2} = 0$ or $v_{2,1} = -2v_{2,2}$

Normalizing, we get the following eigenvector associated with eigenvalue $\lambda_2 = 10$ as:

$$\begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

And we get the normalized eigenvector corresponding to $\lambda_3 = 0$ as:

$$\begin{bmatrix} v_{3,1} \\ v_{3,2} \\ v_{3,3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ -\frac{5}{\sqrt{30}} \end{bmatrix}$$

The singular value decomposition of an $m \times n$ complex matrix M is a factorization of the form $U\Sigma V^*$, where U is an $m \times m$ complex unitary matrix, Σ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal, and V is an $n \times n$ complex unitary matrix.

The pseudoinverse of an $m \times n$ complex matrix M is a factorization of the form $V\Sigma^+U^*$, where U is an $m \times m$ complex unitary matrix, Σ^+ is an $n \times m$ rectangular diagonal matrix with non-negative real numbers on the diagonal, formed from Σ by taking the reciprocal of all the non-zero elements, leaving all the zeros alone, and V is an $n \times n$ complex unitary matrix.

Specifically:

a. The ordering of the vectors comes from the ordering of the singular values (largest to smallest).
b. The columns of U are the eigenvectors of AA^T.
c. The columns of V are the eigenvectors of A^TA.
d. The diagonal elements of Σ are the singular values, $\sigma_i = \sqrt{\lambda_i}$
e. There are relationships between v_i and u_i (with normalization):

$$Av_i = \sigma_i u_i A \tag{5}$$

$$Tu_i = \sigma_i v_i \tag{6}$$

As computed above:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

$$V^* = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

So we have the SVD as follows:

$$\begin{bmatrix} 3 & 1 & 1 \\ -1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

So we now have:

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{5}{\sqrt{6}} \end{bmatrix}$$

And

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sqrt{12}} & 0 \\ 0 & \frac{1}{\sqrt{10}} \\ 0 & 0 \end{bmatrix}$$

And we finally have:

$$U^* = U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Now we can write the pseudoinverse of matrix A as follows:

$$\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{5}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{12}} & 0 \\ 0 & \frac{1}{\sqrt{10}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{17}{60} & -\frac{7}{60} \\ \frac{4}{60} & \frac{16}{60} \\ \frac{1}{12} & \frac{1}{12} \end{bmatrix}$$

3a. Rank Definition.
The rank of a matrix is the number of pivots in its reduced row-echelon form. Or the rank of a matrix A, written rank(A), is the dimension of the column space Col(A).

3b. Given the following matrix:

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

In order to determine the rank of the matrix, we transform the given matrix A to the reduced row-echelon form as follows:

Applying $R_1 \to R_1 + R_2$, we get:

$$A = \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Now Applying $R_2 \to R_2 - 3R_1$, we get:

$$A = \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 0 & 2 & -4 & -24 & -32 & 10 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Now Applying $R_3 \to R_3 - 2R_1$, we get:

$$A = \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 0 & 2 & -4 & -24 & -32 & 10 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Now Applying $R_4 \to R_4 - 4R_1$, we get:

$$A = \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 0 & 2 & -4 & -24 & -32 & 10 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 3 & -6 & -36 & -48 & 15 \end{bmatrix}$$

Now Applying $R_2 \rightarrow R_2 - R_3$, we get:

$$A = \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 3 & -6 & -36 & -48 & 15 \end{bmatrix}$$

Now Applying $R_3 \rightarrow R_2 - R_3$, we get:

$$A = \begin{bmatrix} 1 & -3 & 2 & 8 & 11 & -2 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally Applying $R_1 \rightarrow R_1 + 3R_2$, we get:

$$A = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A is now in the row reduced echelon form. If a matrix is in row-echelon form, then the first nonzero entry of each row is called a pivot, and the columns in which pivots appear are called pivot columns. We see that the number of pivot points are 2, or since there are two non zero rows (two leading 1's), the row space and column space are both two-dimensional, hence the rank of A is 2 $[rank(A) = 2]$.

4. To Calculate the 1-norm of

$$B = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$$

We proceed as follows:

Summing down the columns of B we find that

$$||B||_1 = max(5 + |-1| + |-2|,\ |-4| + 2 + 1,\ 2 + 3 + 0)$$

$$= max(8, 7, 5) = 8$$

5. To Calculate the infinity-norm of

$$C = \begin{bmatrix} 3 & 6 & -1 \\ 3 & 1 & 0 \\ 2 & 4 & -7 \end{bmatrix}$$

We proceed as follows:

Summing down the columns of C, we find that

$$||C||_\infty = max(3 + 6 + |-1|,\ 3 + 1 + 0,\ 2 + 4 + |-7|)$$

$$= max(10, 4, 13) = 13$$

6. To calculate the norm of the vector $\vec{u} = (2, -2, 3, -4)$, proceed as follows:

Since $\vec{u} \in \mathbb{R}^4$, we will use the formula

$$||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$$

$$||\vec{u}|| = \sqrt{4 + 4 + 9 + 16} = \sqrt{33}$$

7. Derivative of a Vector-by-Vector (Jacobian matrix)

In vector calculus, the Jacobian matrix of a vector-valued function in several variables is the matrix of all its first-order partial derivatives. When this matrix is square, that is, when the function takes the same number of variables as input as the number of vector components of its output, its determinant is referred to as the Jacobian determinant. Both the matrix and (if applicable) the determinant are often referred to simply as the Jacobian in literature.

Basically, t he first derivative of a vector-valued function f with respect to a vector $x = [x_1\ x_2 \cdots x_n]^T$ is called the Jacobian of f and is defined as

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function such that each of its first-order partial derivatives exist on $\mathbb{R}^n$. This function takes a point $x \in \mathbb{R}^n$ as input and produces the vector $f(x) \in \mathbb{R}^m$ as output. Then the Jacobian matrix of f is defined to be an $m \times n$ matrix, denoted by J, whose $(i, j)th$ entry is $J_{ij} = \frac{\partial f_i}{\partial x_j}$, or explicitly

$$\mathbf{J} = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Given to compute the Jacobian matrix of the function F :$\mathbf{R}^3$ →$\mathbf{R}^4$ with components

$y_1 = x_1$
$y_2 = 5x_3$
$y_3 = 4x_2^2 - 2x_3$
$y_4 = x_3 \sin x_1$

we have

$$\mathbf{J} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \\ \frac{\partial y_4}{\partial x_1} & \frac{\partial y_4}{\partial x_2} & \frac{\partial y_4}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 8x_2 & -2 \\ x_3 cos x_1 & 0 & sin x_1 \end{bmatrix}$$

8. Derivative of a Scalar-by-Vector

We can produce a vector from a scalar (i.e., a function) by differentiation. The derivative of a scalar f by a vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, is written (in numerator layout notation) as:

$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$

In vector calculus, the gradient of a scalar field f in the space $\mathbb{R}^n$ (whose independent coordinates are the components of x) is the transpose of the derivative of a scalar by a vector.

$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = (\frac{\partial f}{\partial \mathbf{x}})^T$

By example, in physics, the electric field is the negative vector gradient of the electric potential.

We know that $r = \sqrt{x^2 + y^2 + z^2}$ is the magnitude of a radial vector, $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\nabla r = (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z})\sqrt{x^2 + y^2 + z^2}$$

$$= \hat{\mathbf{i}}\frac{\partial\sqrt{x^2 + y^2 + z^2}}{\partial x} + \hat{\mathbf{j}}\frac{\partial\sqrt{x^2 + y^2 + z^2}}{\partial y} + \hat{\mathbf{k}}\frac{\partial\sqrt{x^2 + y^2 + z^2}}{\partial z}$$

$$= \frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}}(\hat{\mathbf{i}}\frac{\partial(x^2 + y^2 + z^2)}{\partial x} + \hat{\mathbf{j}}\frac{\partial(x^2 + y^2 + z^2)}{\partial y} + \hat{\mathbf{k}}\frac{\partial(x^2 + y^2 + z^2)}{\partial z})$$

$$= \frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}}(2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}})$$

$$= \frac{1}{r}\mathbf{r}$$

$$= \hat{\mathbf{r}}$$

9. To find the Derivative of a Scalar-by-Scalar with respect to a scalar x

Given function is $f(x) = cosx + sinx + x^2$

$f'(x) = -sinx + cosx + 2x$

10. To Find the derivative of the following matrix with respect to a scalar x

$$A = \begin{bmatrix} x^2 + 2cosx & sinx \\ cosx & x^3 + x^2 + x + 1 \end{bmatrix}$$

We have

$$dA/dx = \begin{bmatrix} 2x - 2sinx & cosx \\ -sinx & 3x^2 + 2x + 1 \end{bmatrix}$$

11. $B = x^T Ax = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$B = x_2^2 + 2x_1x_3$

Since this differentiation of a scalar with respect to a vector, we have:

$\frac{\partial B}{\partial x} = \begin{bmatrix} \frac{\partial B}{\partial x_1} \\ \frac{\partial B}{\partial x_2} \\ \frac{\partial B}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 2x_2 \\ 2x_1 \end{bmatrix}$