1. First write the problem in the standard form required for the application of the KKT theory:

Set

$$L(x, y, z, \mu_1, \mu_2) = f(x, y, z) + \mu_1 h_1(x, y, z) + \mu_2 h_2(x, y, z) \tag{1}$$

$$L(x, y, z, \mu_1, \mu_2) = xyz + \mu_1(x^2 + y^2 - 1 + \mu_2(x + z - 1) \tag{2}$$

The first order conditions are

$$\frac{\partial L}{\partial x} = yz + 2\mu_1 x + \mu_2 = 0 \tag{3}$$

$$\frac{\partial L}{\partial y} = xz + 2\mu_1 y = 0 \tag{4}$$

$$\frac{\partial L}{\partial z} = xy + \mu_2 = 0 \tag{5}$$

$$\frac{\partial L}{\partial \mu_1} = x^2 + y^2 - 1 = 0 \tag{6}$$

$$\frac{\partial L}{\partial \mu_2} = x + z - 1 = 0 \tag{7}$$

In order to solve for μ_1 and μ_2 we divide by y, so we have to assume $y \neq 0$. We will treat the case $y = 0$ later. Assuming $y \neq 0$

$$\mu_1 = -\frac{xz}{2y} \tag{8}$$

$$\mu_2 = -xy \tag{9}$$

Plug these into the eq (10) equation to get

$$yz - 2\frac{x^2 z}{2y} - xy = 0 \tag{10}$$

Multiply both sides by $y \neq= 0$

$$y^2 z - x^2 z - xy^2 = 0 \tag{11}$$

Now we want to solve the two constraints for y and z in terms of x, plug them into this equation, and get one equation in terms of x.

$$3x^3 - 2x^2 - 2x + 1 = 0 \tag{12}$$

$$(1 - x)[1 - 3x^2 - x] = 0 \tag{13}$$

This yields $x = 1, \frac{-1+\sqrt{13}}{6}, \frac{-1-\sqrt{13}}{6}$.

Let's analyze $x = 1$ first. From the second constraint we have that $z = 0$, and from the first constraint we have that $y = 0$. That contradicts our assumption that $y \neq 0$, so this cannot be a solution. Plugging in the other values, we get four candidate solutions $x = 0.4343$, $y = \pm 0.9008$, $z = 0.5657$, $x = -0.7676$, $y = \pm 0.6409$, $z = 1.7676$

Finally, let's look at the case $y = 0$. Then either $x = 1$ and $z = 0$ or $x = -1$ and $z = 2$, from the equality constraints. In the first case, all the other first order conditions hold as well, so $(1, 0, 0)$ is another candidate solution. In the second case, we get $2 = 0$ in the second FOC and therefore this point cannot be a candidate solution

2. This is a problem of equality constrained optimization. By the method of Lagrange multipliers, let us define:

$F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

Or

$F(x, y, \lambda) = 4x^2 + 3y^2 - \lambda(y + 2x - 8)$

To find the stationary points of F, equate the partial derivatives with respect to x, y, λ to 0 to get the KKT conditions as follows:

$$\frac{\partial F}{\partial x} = 8x - 2\lambda = 0 \implies x = \frac{\lambda}{4} \tag{14}$$

$$\frac{\partial F}{\partial y} = 6y - \lambda = 0 \implies y = \frac{\lambda}{6} \tag{15}$$

$$\frac{\partial F}{\partial \lambda} = y + 2x - 8 = 0 \tag{16}$$

Put the values of x, y in Equation (16) to get:

$$\frac{\lambda}{6} + \frac{2\lambda}{4} - 8 = 0 \implies \lambda = 12 \tag{17}$$

So we have $x = 3, y = 2$.

Therefore by the method of Lagrange multipliers, an extrema is obtained at $(3, 2)$ and we get the value of the extrema $f(3, 2) = 48$. Thus $(x^*, y^*) = (3, 2)$ satisfies all the constraints above and so it is a KKT point.

One drawback of the method of Lagrange multipliers is that it does not tell us whether the extrema is a maximum or minimum. In the above example, we have $4x^2 + 3y^2 \geq 0$, $\forall (x, y)$, and so necessarily the function must have a minimum value. Since there was a unique extrema value $(3, 2)$, this must be the minimum value.

IMPORTANT: The KKT condition can be satisfied at a local minimum, a global minimum (solution of the problem) as well as at a saddle point. We can use the KKT condition to characterize all the stationary points of the problem, and then perform some additional testing to the optimal solutions of the problem (global minima of the constrained problem).

Inequality Constrained Optimization

Assume x^* minimizes the following:

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to:} & g_i(x) \geq 0 & \forall i \in I \\ & h_j(x) \geq 0 & \forall j \in J \end{array}$$

The following two cases are possible:

1. $\nabla h_1(x^*) \cdots \nabla h_k(x^*), \nabla g_1(x^*) \cdots \nabla g_m(x^*)$ are linearly dependent.

2. There exists vectors λ^* and μ^* such that:

$$\nabla f(x^*) - \Sigma_{j=1}^{k} \lambda_j \nabla h_j(x^*) - \Sigma_{i=1}^{m} \mu_i \nabla g_i(x^*)$$

$$\mu_i^* g_i(x^*) = 0$$

$$\lambda_j^* h_j(x^*) = 0$$

$$\lambda^* \geq 0$$

$$\mu^* \geq 0$$

- These conditions are known as the Karush-Kuhn-Tucker Conditions.
- We look for candidate solutions x for which we can find λ and μ.
- Solve these equations using complementary slackness.
- At optimality some constraints will be binding and some will be slack.
- Slack constraints will have a corresponding $_i$ of zero.
- Binding constraints can be treated using the Lagrangian.

3. Since this is a problem of an inequality constrained optimization problem, we have $f(x) = x^2$, $g_1(x) = x - 1$ and $g_2(x) = 2 - x$ The Lagrangian of the problem is:

$$L(x, \lambda, \mu) = x^2 - \lambda(x - 1) - \mu(2 - x)$$

(x^*, λ^*, μ^*) is a KKT point if it satisfies KKT conditions.

So we now have the KKT conditions as follows:

$$\frac{\partial L}{\partial x} = 2x - \lambda + \mu = 0 \implies \lambda - \mu = 2x \tag{18}$$

$$\lambda g_1(x) = 0 \implies \lambda(x - 1) = 0 \tag{19}$$

$$\mu g_2(x) = 0 \implies \mu(2 - x) = 0 \tag{20}$$

Apart from the KKT conditions as given by equations (18), (19) and (20), we also have:

$$\lambda \geq 0 \tag{21}$$

$$\mu \geq 0 \tag{22}$$

$$g_1(x) \geq 0 \tag{23}$$

$$g_2(x) \geq 0 \tag{24}$$

From equations (19) and (20), we have the following choices:

1. $\lambda = 0$, $\mu = 0 \implies x = 0$. But this ruled out since we have a constraint $1 \leq x \leq 2$

2. $\lambda = 0$, $(2 - x) = 0 \implies x = 2$. But this is also ruled out. As $x = 2$ and $\lambda = 0$, from equation (18), we have $\mu = -4$. However μ cannot be negative from equation (22).

3. $\mu = 0$, $(x - 1) = 0 \implies x = 1$. Now, if $x = 1, \mu = 0 \implies \lambda = 2$ (from equation (18)). Thus $(x^*, \lambda^*, \mu^*) = (1, 2, 0)$ satisfies all the constraints above and so it is a KKT point.

4. Given:

$$\begin{aligned} \text{minimize} \quad & (x-2)^2 + 2(y-1)^2 \\ \text{subject to:} \quad & x + 4y \leq 3 \\ & x \geq y \end{aligned}$$

We can rearrange the given problem as follows:

$$\begin{aligned} \text{minimize} \quad & (x-2)^2 + 2(y-1)^2 \\ \text{subject to:} \quad & 3 - x - 4y \geq 0 \\ & x - y \geq 0 \end{aligned}$$

Lagrangian of the problem is:

$$L(x, \lambda, \mu) = (x-2)^2 + 2(y-1)^2 - \lambda(3 - x - 4y) - \mu(x - y)$$

(x^*, λ^*, μ^*) is a KKT point if it satisfies KKT conditions.

So we now have the KKT conditions as follows:

$$\frac{\partial L}{\partial x} = 2x - 4 + \lambda - \mu = 0 \tag{25}$$

$$\frac{\partial L}{\partial y} = 4y - 4 + 4\lambda + \mu = 0 \tag{26}$$

$$\lambda g_1(x, y) = 0 \implies \lambda(3 - x - 4y) = 0 \tag{27}$$

$$\mu g_2(x, y) = 0 \implies \mu(x - y) = 0 \tag{28}$$

$$\lambda \geq 0 \tag{29}$$

$$\mu \geq 0 \tag{30}$$

Apart from the KKT conditions as given by equations (25), (26), (27), (28), (29) and (30), we also have:

$$g_1(x, y) \geq 0 \tag{31}$$

$$g_2(x, y) \geq 0 \tag{32}$$

From equations (27) and (28), we have the following choices:

1. $\lambda = 0$, $\mu = 0 \implies x = 2$ [from eq (25)] and $y = 1$ [from eq (26)]. But this ruled out since we have a constraint $g_1(x, y) = (3 - x - 4y)$ at $(2, 1) = -3 \implies < 0$

2. $\lambda = 0$, $(x - y) = 0$. Therefore by solving eq (25) and (26), we get $x = \frac{4}{3}, y = \frac{4}{3}$ and from equation (25) and/or equation (26), we have $\mu = \frac{4}{3}$. But this is also ruled out. Because from equation (27), we have a constraint $g_1(x, y) = (3 - x - 4y)$ at $(\frac{4}{3}, \frac{4}{3}) = -\frac{11}{3} \implies < 0$. However $g_1(x, y) = (3 - x - 4y)$ cannot be negative because we have a constraint that $g_1(x, y) = (3 - x - 4y) \geq 0$.

3. $\mu = 0$, $(3 - x - 4y) = 0$. $(3 - x - 4y) = 0 \implies 4y = 3 - x$. Put $4y = 3 - x$ in eq (26), we get:

$$-x + 4\lambda = 1 \tag{33}$$

Similarly eq (25) can be written as:

$$2x + \lambda = 4 \tag{34}$$

Solving eq (33) and eq (34) we get:

$x = \frac{5}{3}, \lambda = \frac{2}{3}$. From the equation $4y = 3 - x$, we get $y = \frac{1}{3}$. Thus $(x^*, y^*, \lambda^*, \mu^*) = (\frac{5}{3}, \frac{1}{3}, \frac{2}{3}, 0)$ satisfies all the constraints above and so it is a KKT point.

4. $(3 - x - 4y) = 0, (x - y) = 0$. So put $x = y$ in $(3 - x - 4y) = 0$ to get $x = y = \frac{3}{5}$. Put $x = y = \frac{3}{5}$ in equations (25) and (26) to get $\lambda = \frac{22}{25}$ and $\mu = -\frac{48}{25}$. Clearly $\mu = -\frac{48}{25}$ violates the constraint that $\mu \geq 0$ and so $(x^*, y^*, \lambda^*, \mu^*) = (\frac{5}{3}, \frac{1}{3}, \frac{2}{3}, 0)$ satisfies all the constraints above and so it is a KKT point.