The solutions to Assignment 1 are as follows:

1. Given the following matrix:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

a. Interchanging rows and columns, we get the transpose of A as follows:

$$A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

b.

$$B = AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} * \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

c. To find out the eigenvalues of matrix B, we write the characteristic equation as follows:

$$\left|B - \lambda \cdot I\right| = \left|\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{bmatrix}\right| = 0$$

So the two eigenvalues are $\lambda_1 = 12$ and $\lambda_2 = 10$

Let us find the eigenvector associated first with $\lambda_1 = 12$

We have $B \cdot v_1 = \lambda_1 \cdot \ u_1$ or $(B - \lambda_1) \cdot \ u_1 = 0$

$$or \begin{bmatrix} 11-12 & 1 \\ 1 & 11-12 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We have: $-u_{1,1} + u_{1,2} = 0$ or $u_{1,1} = u_{1,2}$

Normalizing, we get the following eigenvector associated with eigenvalue $\lambda_1 = 12$ as:

$$\begin{bmatrix} u_{1,1} \\ u_{1,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now let us find the eigenvector associated first with $\lambda_2 = 10$
So we have $B \cdot u_2 = \lambda_2 \cdot\ u_2$ or $(B - \lambda_2) \cdot\ u_2 = 0$

$$\begin{bmatrix} 11-10 & 1 \\ 1 & 11-10 \end{bmatrix} \begin{bmatrix} u_{2,1} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We have: $u_{2,1} + u_{2,2} = 0$ or $u_{2,1} = -u_{2,2}$

Normalizing, we get the following eigenvector associated with eigenvalue $\lambda_2 = 10$ as:

$$\begin{bmatrix} u_{2,1} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

d.

$$C = A^T A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} * \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

e. To find out the eigenvalues of matrix C, we write the characteristic equation as follows:

$$\left| C - \lambda \cdot I \right| = \left| \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{bmatrix} \right| = 0$$

Solving we get $(10-\lambda)[(10-\lambda)(2-\lambda) - 16)] + 2[-2(10-\lambda)] = 0$

So the eigenvalues are $\lambda_1 = 12$, $\lambda_2 = 10$ and $\lambda_3 = 0$

Let us find the eigenvector associated first with $\lambda_1 = 12$
So we have $C \cdot v_1 = \lambda_1 \cdot\ v_1$ or $(C - \lambda_1) \cdot\ v_1 = 0$

$$\begin{bmatrix} 10-12 & 0 & 2 \\ 0 & 10-12 & 4 \\ 2 & 4 & 2-12 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2v_{1,1} + 2v_{1,3} = 0$ or $v_{1,1} = v_{1,3}$

$-2v_{1,2} + 4v_{1,3} = 0$ or $v_{1,2} = 2v_{1,3}$

$2v_{1,1} + 4v_{1,2} - 10v_{1,3} = 0$ or $2v_{1,1} = v_{1,2}$

Normalizing the above values, we get the following eigenvector associated with eigenvalue $\lambda_2 = 12$ as:

$$\begin{bmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Similarly, we get the eigenvector corresponding to $\lambda_2 = 10$ as:

$$\begin{bmatrix} 10-10 & 0 & 2 \\ 0 & 10-10 & 4 \\ 2 & 4 & 2-10 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ or \ \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2v_{2,3} = 0$

$2v_{2,1} + 4v_{2,2} = 0$ or $v_{2,1} = -2v_{2,2}$

Normalizing, we get the following eigenvector associated with eigenvalue $\lambda_2 = 10$ as:

$$\begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

And we get the normalized eigenvector corresponding to $\lambda_3 = 0$ as:

$$\begin{bmatrix} v_{3,1} \\ v_{3,2} \\ v_{3,3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ -\frac{5}{\sqrt{30}} \end{bmatrix}$$

f. The eigenvalues of matrices $B = AA^T$ and $C = A^T A$ are the same.

2. We use the equation $AX = B$ to solve the system of equation above, where $A \quad B$ are the matrices for values of LHS and RHS of the equations and X consist of x, y and z variables.

So $X = A^{-1}B$, which means you need to find out the inverse of A and compute the dot product of A^{-1} and B (or by multiplying the inverse of the A matrix by the B matrix.) to get the solution.

To find the inverse matrix, augment it with the identity matrix and perform row operations trying to make the identity matrix to the left. Then to the right will be the inverse matrix.

Writing the given equation in the matrix form $AX = B$ as:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & -3 & -1 \\ 1 & -3 & 3 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ -4 \end{bmatrix}$$

Where $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -3 & -1 \\ 1 & -3 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 15 \\ 19 \\ -4 \end{bmatrix}$

The steps to find the inverse of 3×3 matrix are:

a. Compute the determinant of the given matrix
b. Take the transpose of the given matrix
c. Calculate the determinant of 2×2 minor matrices
d. Formulate the matrix of cofactors
e. Finally, divide each term of the adjugate matrix by the determinant Inverse Matrix Formula

First, find the determinant of 3×3 Matrix and then find it's minor, cofactors and adjoint and insert the results in the Inverse Matrix formula given below:

$$A^{-1} = \frac{1}{|A|} Adj(A) \tag{1}$$

$$\left|A\right| = \begin{vmatrix} 2 & 3 & -1 \\ 4 & -3 & -1 \\ 1 & -3 & 3 \end{vmatrix} = -54 \neq 0$$

Therefore A is non-singular

Therefore, the system has the unique solution $X = A^{-1}B$

We have the determinant of 2×2 minor matrices as follows:
$A_{11} = -12$,
$A_{12} = -13$,
$A_{13} = -9$,
$A_{21} = -6$,
$A_{22} = 7$,
$A_{23} = 9$,
$A_{31} = -6$,
$A_{32} = -2$,
$A_{33} = -18$,

Therefore, we now have

$$adj(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -12 & -6 & -6 \\ -13 & 7 & -2 \\ -9 & 9 & -18 \end{bmatrix}$$

And

$$A^{-1} = \frac{adj(A)}{|A|} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{13}{54} & -\frac{7}{54} & \frac{1}{27} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

Now $AX = B \implies X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{13}{54} & -\frac{7}{54} & \frac{1}{27} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 15 \\ 19 \\ -4 \end{bmatrix}$$

And we have:

$x = 5$
$y = 1$
$z = -2$

3. First, find the determinant of 2×2 A Matrix and then find it's minor, cofactors and adjoint and insert the results in the Inverse Matrix formula given below:

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

$$\left|A\right| = \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} = 14 - 15 = -1 \neq 0$$

Therefore A is non-singular

Therefore, the system has the unique solution $X = A^{-1}B$

We have the determinant of 1×1 minor matrices as follows:
$A_{11} = 7$,
$A_{12} = -5$,
$A_{21} = -3$,
$A_{22} = 2$,

Therefore, we now have

$$adj(A) = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}$$

And

$$A^{-1} = \frac{adj(A)}{\left|A\right|} = -\frac{1}{1} \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

4. An orthogonal matrix is a square matrix A if and only its transpose is as same as its inverse. So we have $A^T = A^{-1}$, where A^T is the transpose of A and A^{-1} is the inverse of A.

So we have:

$$A^T = A^{-1} \tag{2}$$

Multiply LHS and RHS of equation (2) by A as follows:

$$A \cdot A^T = A \cdot A^{-1} \tag{3}$$

Since $A \cdot A^{-1} = I$ we have:

$$A \cdot A^T = I \tag{4}$$

Now multiply LHS and RHS of equation (2) by A as follows:

$$A^T \cdot A = A^{-1} \cdot A \tag{5}$$

Since $A^{-1} \cdot A = I$ we have:

$$A^{-1} \cdot A = I \tag{6}$$

We now define that a square matrix A is said to be "orthogonal" if the following conditions are satisfied:

$$A^T = A^{-1} \tag{7}$$

or

$$A \cdot A^T = A^T \cdot A = I \tag{8}$$

Given the matrix:

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

We have:

$$A^T = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

And so:

$$A \cdot A^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \\ 0 & 1 & 0 \\ \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly:

$$A^T \cdot A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \\ 0 & 1 & 0 \\ \\ 0 & 0 & 1 \end{bmatrix}$$

As we have $A \cdot A^T = A^T \cdot A = I$, the matrix A is orthogonal.

In linear algebra, "Orthogonal" means "perpendicular". Two vectors are said to be orthogonal to each other if and only their dot product is zero. In an orthogonal matrix, every two rows and every two columns are orthogonal and the length of every row (vector) or column (vector) is 1.

So in the given matrix A above, Let us find the dot product of the first two columns.

$(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}) \cdot (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$

Similarly, you can check the dot product of every two rows and every two columns. You will get each dot product to be zero.

5a. Given:

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and

$$v = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$u^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

So we have:

$$u^T u = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 = 1 + 4 + 9 = 14$$

5b. Similarly

$$u^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

So we have:

$$u^T v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 1 \cdot -1 + 2 \cdot 2 + 3 \cdot 1 = -1 + 4 + 3 = 6$$