Contents

Nested non-empty compact sets in $\mathbb{C}$ with vanishing diameter contain a unique point

Proof idea

Create a sequence by picking a point in every set. They're nested so each set contains the entire tail of the sequence. And the diameters are vanishing so the sequence is Cauchy.

Proof

Definition 1: If $\Omega \subseteq \mathbb{C}$ is non-empty then

$$\operatorname{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y|.$$

Theorem 1: If $\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_n \supset \dots$ is a sequence of non-empty compact sets in $\mathbb{C}$ with the property that

$$\operatorname{diam}(\Omega_n) \to 0 \text{ as } n \to \infty,$$

then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

Proof: Choose a point z_n in each Ω_n.

Since $\operatorname{diam}(\Omega_n) \to 0$, the sequence is Cauchy: for any given ε, we pick $n \in \mathbb{N}$ such that $\operatorname{diam}(\Omega_n) < \varepsilon$. Then for any $k, m \geq n$,

$$z_k, z_m \in \Omega_n$$

so

$$|z_k - z_m| \leq \sup_{x,y \in \Omega_n} |x - y| = \operatorname{diam}(\Omega_n).$$

Therefore, since $\mathbb{C}$ is complete, (z_k) converges to a limit that we call w.

Now, for any $k, m \in \mathbb{N}$ with $k \geq m$, we know that $\Omega_k \subset \Omega_m$, so $z_k \in \Omega_m$. Therefore for each fixed $n \in \mathbb{N}$ the points z_k s.t. $k \geq n$ form a sequence in Ω_n that converges to w, because it's the tail of a convergent sequence, so it converges to the same limit.

Since the Ω_n are compact, they are sequentially closed. Therefore, looking at the convergent sequence in each Ω_n, its limit is also contained in Ω_n. That is, $w \in \Omega_n$ for all n. This proves existence.

For uniqueness, suppose that w' and w were two distinct points such that $w \in \Omega_n$ and $w' \in \Omega_n$ for all n. Then set $\varepsilon = |w - w'|$ and pick some $N \in \mathbb{N}$ such that

$$\operatorname{diam}(\Omega_N) < \varepsilon.$$

Then $w, w' \in \Omega_N$, so

$$\varepsilon = |w - w'| \leq \sup_{x,y \in \Omega_N} |x - y| = \operatorname{diam}(\Omega_N) < \varepsilon,$$

a contradiction. ■

Remark 1: This proof only uses the fact that the sets are sequentially closed. Is the hypothesis that they are compact really necessary?

Well, the vanishing diameters force the sets to (eventually) become compact. So the sets that the theorem applies to are the same whether we require them to be compact or merely closed. The theorem fails (not just for uniqueness but also existence) if the sets are allowed to be unbounded: take

$$\Omega_n = [n, \infty),$$

a sequence of nested closed sets with empty intersection.

One generalization of this theorem, in analogy with the Nested Intervals Property from Abbott's book, would be to drop the vanishing diameters requirement and just require boundedness. Then it would state that a sequence of nested non-empty compact sets has non-empty intersection. But the intersection could contain more than one point, as the trivial example

$$\Omega_n = [0, 1] \text{ for all } n$$

shows.

My careful reading of the proof of Goursat's theorem in my textbook (Stein and Shakarchi's *Complex Analysis*, Chapter 2, Section 1, page 35) seems to show that it would go through just fine without the uniqueness: they just need *some* point in the interior of every triangle so they can use the fact that it's holomorphic to write it as a constant + linear term + error that goes to 0.