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Small changes I felt that I needed to make in order to make the proofs work

In Problem 5 (c) it seemed to me that I needed to assume that the compact set K is nonempty in order to deduce the existence of a supremum of the image.

In Problem 6 I think the linear map T needs to go from $\mathbb{R}^n$ to itself, not $\mathbb{R}^m \to \mathbb{R}^n$ as written, so that v^*Tv can be defined.

Problem 1: Sequentially compact ⇒ closed and bounded

Proof idea

To prove closed: suppose we have a convergent sequence. By compactness it has a subsequence whose limit is in K. Their limits must be the same, so the parent's limit is in K. So K is closed.

To prove bounded: show not bounded => a sequence exists with no convergent subsequence. That sequence is: for bigger and bigger discs, take a point outside that disc, which must exist if the set is not bounded. Any subsequence of this sequence is also unbounded and so cannot converge.

Proof

Lemma 1: Every convergent sequence in $\mathbb{R}^n$ is bounded.

Proof:

(*Idea: Break sequence into finite prefix + a tail that's within ε of the limit.*)

Let (a_k) be a sequence converging to some $a \in \mathbb{R}^n$. Pick $\varepsilon = 1$. Then there exists some $K \in \mathbb{N}$ such that

$$|a_k - a| < \varepsilon$$

whenever $k \geq K$. Then

$$|a_k| < |a| + \varepsilon$$

by applying the triangle inequality to a and $a_k - a$:

$$|a + a_k - a| \leq |a| + |a_k - a| < |a| + \varepsilon.$$

Let $M = \max(\{|a_k| : k < K\})$, where we may take the maximum because the set is finite. Then the sequence is bounded by $\max(M, |a| + 1)$. ■

Theorem 1: If $K \subseteq \mathbb{R}^n$ is sequentially compact, it is both closed and bounded.

Proof: Let (a_k) be any sequence whose terms are all in K, converging to some limit $a \in \mathbb{R}^n$. By compactness of K, it has a subsequence that converges to some limit $L \in K$.

By the lemma proved in the last problem set (that every subsequence of a convergent sequence converges to the same point) we have $L = a$. So $a \in K$ and so K is closed.

Suppose K is not bounded. Then for every $k \in N$, there must exist a point $a_k \in K$ such that $|a_k| > k$. Let (a_k) be the sequence of these points. Every subsequence $\left(a_{k_i}\right)$ of this sequence is unbounded: given any $M \in \mathbb{R}$, take $i = \lceil M \rceil$. Then

$$\left|a_{k_i}\right| \geq k_i \geq i \geq M.$$

By the lemma proved above, there is no convergent subsequence, so K is not compact. ■

Problem 2: Closed subset of sequentially compact $\Rightarrow$ sequentially compact

Proof idea

Just unwind definitions: any sequence in the subset is also a sequence in the enclosing, so it has a convergent subsequence. But the subset is closed, so the limit is in it.

Proof

Theorem 2: Let K be sequentially compact, and $A \subseteq K$ be closed. Then A is sequentially compact.

Proof: Let (a_k) be any sequence whose terms are all in A. Since $A \subseteq K$, every term $a_k \in K$. By compactness of K, there is a subsequence a_{k_i} converging to a limit $L \in K$. By closedness of A, since $a_{k_i} \in A$, we have $L \in A$. So A is sequentially compact. ■

Problem 3: Bolzano-Weierstrass Theorem: closed and bounded ⇒ sequentially compact

(a) Bounded closed intervals in $\mathbb{R}$ are sequentially compact

Proof idea

Use the concept of a peak term to get a monotone subsequence, then apply the Monotone Convergence Theorem

If the set of peak terms is infinite then they form a monotone subsequence.

Otherwise there's some term a_i after which there are no more peak terms. Then there must exist a bigger term after it, and a bigger one after that, and so on. Because otherwise we'd have another peak term, contradicting the definition of a_i. These form a monotone subsequence.

Proof

Theorem 3: *(Bolzano-Weierstrass)* Bounded closed intervals in $\mathbb{R}$ are sequentially compact.

Proof: Let $K = [a, b]$ be a bounded closed interval in R. Let (x_k) be any sequence whose terms are all in K.

First we show that (x_k) has a monotone subsequence. To that end, let a term x_k of the sequence (x_k) be called a *peak term* if, for each $m \geq k$, we have $x_k \geq x_m$.

Suppose there are infinitely many peak terms of (x_k). Then they form a subsequence; call it $\left(x_{k_i}\right)$. By the definition of peak term, whenever $i \geq j$ we have $x_{k_i} \leq x_{k_j}$. Therefore $\left(x_{k_i}\right)$ is non-increasing, therefore monotone.

Otherwise, if there are not infinitely many peak terms of (x_k), there must be some finite index L (possibly 1) such that for all $j \geq L$, x_j is not a peak term. We form the following subsequence: let x_{k_1} be x_L, the first term that is not a peak term. By the definition of peak term, there must exist some index $k_2 > k_1 = L$ such that $x_{k_2} > x_{k_1}$, otherwise x_{k_1} would be a peak term. Also, by the definition of L as the index after which there are no more peak terms, x_{k_2} is not a peak term, so there must exist some index $k_3 > k_2$ such that $x_{k_3} > x_{k_2}$. Continuing in this way, we form the subsequence $\left(x_{k_i}\right)$ of increasing terms.

Either way we have a monotone subsequence $\left(x_{k_i}\right)$. Since it is contained in K, it is bounded. By the Monotone Convergence Theorem from the last problem set, it converges to some limit $M \in \mathbb{R}$. By Problem 5 in the last problem set, closed intervals in R are closed sets. Therefore $M \in K$, proving that K is sequentially compact. ■

(b) Closed and bounded in $\mathbb{R}^n$ ⇒ sequentially compact

Proof idea

To extend Bolzano-Weierstrass to products of intervals in $\mathbb{R}^n$, find a subsequence whose first coordinate converges, then a subsequence of *that* whose second coordinate converges, and so on. The other obvious strategy – find a convergent subsequence in each dimension, then combine them – seems harder to make work, because the subsequences in each dimension might be mis-aligned, and I couldn't think of a way to fix this. My notation and wording for this proof is very clunky and I couldn't figure out an elegant and precise way to say it.

Then, trap the closed and bounded set inside an enclosing rectange and invoke Problem 2.

Proof

Lemma 2: Closed rectangles in $\mathbb{R}^n$ are sequentially compact.

Proof: In Problem 3 of the last problem set we proved that $v_k \to v$ in $\mathbb{R}^n$ iff, for each i, $v_k^{(i)} \to v^{(i)}$.

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle and let $[a_i, b_i]$ denote the interval in $\mathbb{R}$ corresponding to dimension i of R.

Let (v_k) be any sequence whose terms are all in R. Then $v_k^{(1)}$ lies in a closed and bounded interval of $\mathbb{R}$ and therefore has a convergent subsequence $v_{k_j}^{(1)}$ by Bolzano-Weierstrass. Form the subsequence of $\left(v_{k_j}\right)$ of (v_k) corresponding to the terms k_j of the subsequence converging in the first dimension. This is a subsequence of (v_k) whose first coordinate converges to some limit $v^{(1)}$ in $[a_1, b_1]$.

Now we can find a subsequence of *this* subsequence whose second coordinate converges to some limit in $[a_2, b_2]$. Also, its first coordinate still converges to $v^{(1)} \in [a_1, b_1]$ because every subsequence of a convergent sequence converges to the same limit. Also, a subsequence of a subsequence of (v_k) is still a subsequence of (v_k). So we have constructed a subsequence of (v_k) whose first two coordinates converge to individual limits, both of which lie in the respective intervals of R.

Proceeding in this manner, after a finite number n of steps, we have a subsequence of (v_k), each of whose coordinates converges to some limit in $[a_i, b_i]$. Therefore it converges to some limit in R. So R is sequentially compact. ■

Theorem 4: If $K \subseteq \mathbb{R}^n$ is closed and bounded, then it is sequentially compact.

Proof: Since K is bounded, for each dimension i there must exist bounds a_i, b_i such that $a_i \leq v^{(i)} \leq b_i$ for all $v \in K$, where $v^{(i)}$ denotes the i'th coordinate of v. For, if no such bounds existed, then K would contain points of arbitrarily large norm, contradicting the boundedness of K. (This is so because for the usual Euclidean norm, $\left|v^{(i)}\right| \leq |v|$).

Form the closed rectangle R that is the product of $[a_i, b_i]$ in each dimension. By the lemma proved above, R is sequentially compact. By construction $K \subseteq R$. Therefore by Problem 2, since K is a closed subset of a sequentially compact set, K is sequentially compact. ■

Problem 4: f continuous $\Leftrightarrow$ preimages of closed sets are closed

Proof ideas

Continuous $\Rightarrow$ preimages of closed sets are closed
Just unwind definitions. A convergent sequence in the preimage of B produces, in the B, a convergent sequence, because f is continuous. B is closed, so contains the limit point, so the original limit must then be in the preimage of the set.

Preimages of closed sets are closed $\Rightarrow$ continuous
The main idea is that if f isn't continuous then for some sequence $a_n \to L$, there's a finite bit of room between $f(L)$ and infinitely many of $f(a_n)$. So we can put an open ball around $f(L)$ and take its complement to get a closed set whose preimage isn't closed.

In many more words (still informal):

This side is more interesting. Prove the contrapositive: if f is not continuous, then some closed set has a preimage that is not closed.

f not continuous gives a sequence which converges to L, whose image doesn't converge to $f(L)$. So infinitely many points of $f(a_n)$ stay at least ε away from L.

So there's some room where we can squeeze in an open ball.

Informal mental picture: draw an open ball of radius ε around $f(L)$. It's open, so its complement is closed, and the complement contains infinitely many points of $f(a_n)$. So its preimage contains infinitely many points of a_n but doesn't contain L.

To prove this using the definitions and results we've developed so far, that is, without using open sets or open balls or that their complements are closed, we need a lemma to directly prove that the set of points $\geq \varepsilon$ distance from a fixed point is closed.

We'll also need to use the previously proved fact that any subsequence of a convergent sequence converges to the same limit as the parent. We need this because the set of points whose image under f stays ε away from $f(L)$ form a subsequence, and it is the convergence of this subsequence that shows that the preimage is not closed.

Proof

Lemma 3: Let $a \in R^n$ be a fixed point and $d > 0$ a fixed positive real number. Then the set $K = \{x \in R^n : |x - a| \geq d\}$ is closed.

Proof: Let $(v_k) \in K$ be a sequence converging to some limit $v \in R^n$. Suppose $v \notin K$. Then $|v - a| < d$. Define $r = |v - a|$.

Pick $\varepsilon = d - r > 0$. Then by convergence of (v_k), for some $N \in \mathbb{N}$, $|v_N - v| < d - r$.

Now apply the triangle inequality to $v_N - v$ and $v - a$:

$$\begin{aligned} &|v_N - v + v - a| \leq |v_N - v| + |v - a| \\ \Rightarrow\ &|v_N - a| < \varepsilon + r = d - r + r = d, \end{aligned}$$

a contradiction of $|v_N - a| \geq d$. ■

Theorem 5: If $A \subseteq \mathbb{R}^m$, then a function $f : A \to \mathbb{R}^n$ is continuous iff the preimage of any closed subset of $\mathbb{R}^n$ is closed in A.

Proof: Suppose $f : A \to \mathbb{R}^n$ is continuous and $B \subseteq \mathbb{R}^n$ is a closed set. Let C denote the preimage of B under f. Then let (c_k) be a sequence whose terms are all in C converging to some limit $c \in A$. By continuity of f, the sequence $b_k := f(c_k)$ converges to $b := f(c)$. Since C is the preimage of all of B, $f(C) \subseteq B$: every element of C maps to an element in B. So each $b_k \in B$. Since $b_k \to b$ and B is closed, $b \in B$. But then, since $f(c) = b \in B$, we have $c \in f^{-}1(B) = C$, so C is closed in A. That proves the easy direction.

Now suppose that $f : A \to \mathbb{R}^n$ is not continuous. Then there must exist some sequence $a_n \to a$ with $a_n \in A$ and $a \in A$ for which $f(a_n) \nrightarrow f(a)$. Then there must exist some ε such that for no $K \in \mathbb{N}$ do all terms $f(a_k)$ with $k \geq K$ come within ε of $f(a)$. Then the number of terms at least ε away from $f(a)$ must be infinite, for, if there were only finitely many such terms, then we could pick K to be index of the last of them.

Let $\left(a_{n_i}\right)$ be the sequence such that $f\left(a_{n_i}\right)$ is at least ε away from $f(a)$. Then $\left(a_{n_i}\right)$ is a subsequence of (a_n), but (a_n) is a convergent sequence, so by the lemma in the last problem set about subsequences of convergent sequences, $\left(a_{n_i}\right)$ must also converge to a.

Let $K = \{x \in \mathbb{R}^n : |x - f(a)| \geq \varepsilon\}$. By the lemma proved above, K is closed. Since all $f\left(a_{n_i}\right) \in K$, we have all $a_{n_i} \in f^{-}1(K)$. But $a \notin f^{-}1(K)$, because $f(a) \notin K$. So $f^{-}1(K)$ contains a convergent sequence whose limit is not contained within it. We have found a closed set whose preimage is not closed in A. ∎

Problem 5: the continuous image of a compact set is compact; Extreme Value Theorem

Proof ideas

Continuous image of a compact set is compact

Just unwind definitions: if there's a sequence in the image, then there's a corresponding sequence in the domain, which must have a convergent subsequence, whose convergence is preserved by f, so the image is compact.

Extreme Value Theorem

The image of K under f is a compact subset of the real line, so it's closed and bounded. It contains its supremum. Since it is the image under f of K, some point in K must map to it.

Proofs

(a)

Theorem 6: Suppose $K \subseteq \mathbb{R}^m$ is sequentially compact and $f : K \to \mathbb{R}^n$ is continuous. Then the image of f is sequentially compact.

Proof: Let $B = f(K)$ denote the image of K under f. Let (b_n) be a sequence whose terms are all in B. Then there must be corresponding points $a_n \in K$ such that $f(a_n) = b_n$ for each $n \in \mathbb{N}$; these form a sequence. By sequential compactness of K, there must be a convergent subsequence $\left(a_{n_i}\right)$ whose limit a is contained in K. By continuity of f, the subsequence $f\left(a_{n_i}\right)$ converges to $f(a)$. Since B is the image of K and $a \in K$, we have $f(a) \in B$. We have found a convergent subsequence whose limit is in B, so B is sequentially compact. ■

(b)

Take $K = \mathbb{R}$, the entire real line. It is closed since any convergent sequence in $\mathbb{R}$ converges to some number in $\mathbb{R}$. Take $f(x) = x$, the identity function. It is a rational function, therefore continuous. Then the image of K under f is $\mathbb{R}$ which is unbounded, therefore by Problem 1, not compact.

(c)

Theorem 7: If $K \subseteq \mathbb{R}^m$ is nonempty, closed and bounded and $f : K \to \mathbb{R}$ is continuous, then f achieves its supremum on K, that is, there exists some $v \in K$ such that $f(v) \geq f(w)$ for all $w \in K$.

Proof: By Problem 3, K is compact. By the theorem proved above, the image of K under f is some compact subset B of $\mathbb{R}$. By Problem 1, B is closed and bounded.

Since K is nonempty, its image is also nonempty. So B is bounded and nonempty, therefore it has a supremum. Call it M.

For every $n \in \mathbb{N}$, let $\varepsilon = \frac{1}{n}$. By the lemma proved in the last problem set as part of the Monotone Convergence Theorem, for every ε there exists an element $a \in B$ with $a > M - \varepsilon$. So for every n, form the sequence of these a_n. By its very construction (since $a_n \leq M =$

$\sup(B)$) this sequence converges to M, and B is closed, therefore $M \in B$. But B is the image under f of K, so some element $v \in K$ must map to M. ■

Problem 6: quadratic forms v^*Tv achieve their supremum on the unit sphere

Proof idea

S is the preimage of a closed set (the singleton $\{1\}$) under a continuous function ($v \mapsto v^*v$), so it is closed. Also, it's bounded. So it's compact. So v^*Tv, a rational (therefore continuous) function, must attain its supremum on it.

Proof

Theorem 8: Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a linear map, and let $S = \{v \in \mathbb{R}^m : \|v\| = 1\}$ be the unit sphere. Then the function $v \mapsto v^*Tv$ achieves its supremum on S.

Proof: The function $v \mapsto v^*Tv$ is a quadratic polynomial in the entries v_i of the vector v, therefore rational, therefore continuous.

The condition $\|v\| = 1$ is equivalent to $v^*v = 1$, which is a special case of the above with $T = I$.

So S is the preimage under a continuous function of the closed set $\{1\}$ (since all singleton sets are closed), therefore by Problem 4, S is closed in $\mathbb{R}^m$. Also, S is bounded since every element has a norm of 1. Therefore by Problem 3, S is compact.

Therefore $v \mapsto v^*Tv$, a function from a compact set to the real numbers, must achieve a maximum on S. ■