Contents

Problem 1

I claim that the sequence $(v_k) = 0, 1, 0, 1, \ldots$, i.e., where

$$v_k = \begin{cases} 0 \text{ if } k \text{ is even} \\ 1 \text{ if } k \text{ is odd} \end{cases}$$

is a sequence in $\mathbb{R}^1$ that doesn't converge to anything.

For, suppose that it did converge to some real number v. Then we may pick any $\varepsilon > 0$, and convergence assures that there exists a $K \in \mathbb{N}$ (which may depend on ε) such that for every $k \geq K$, $|v_k - v| < \varepsilon$. Let us pick $\varepsilon = \frac{1}{3}$. For every K, we can find an even $k_e \geq K$ and an odd $k_o = k_e + 1 \geq K$ such that $v_{k_e} = 0$ and $v_{k_o} = 1$.

Then, we have

$$|0 - v| < \varepsilon \text{ and } |1 - v| < \varepsilon$$

so

$$|0 - v| + |1 - v| < 2\varepsilon = \frac{2}{3}.$$

But by the triangle inequality, for every $a, b, c \in \mathbb{R}$,

$$|a - c| \leq |a - b| + |b - c|.$$

Setting $a = 0$ and $c = 1$, we have for every $b \in \mathbb{R}$,

$$|0 - 1| = 1 \leq |0 - b| + |b - 1|.$$

So the triangle inequality requires that the RHS must be greater-than-or-equal to 1 for every $b \in \mathbb{R}$, but convergence requires the existence of a $b = v$ for which the RHS is smaller than $\frac{2}{3}$, a contradiction.

Problem 2

Let (v_k) be a sequence with $(v_k) \to v$ and $(v_k) \to v'$.

Suppose that $v \neq v'$. Then $v - v' \neq 0$, so $|v - v'| > 0$. Define $\delta = |v - v'|$.

By the triangle inequality, for any v_k,

$$|v - v'| \leq |v - v_k| + |v_k - v'|$$

so

$$\delta \leq |v - v_k| + |v_k - v'|. \tag{1}$$

Let us pick $\varepsilon = \frac{\delta}{3}$. Then, because $(v_k) \to v$, there exists a $K_v \in \mathbb{N}$ such that for every $k \geq K_v$, $|v_k - v| < \varepsilon$. Also, because $(v_k) \to v'$, there exists a $K_{v'} \in \mathbb{N}$ such that for every $k \geq K_{v'}$, $|v_k - v'| < \varepsilon$. Let $K = \max(K_v, K_{v'})$. Then for every $k \geq K$ we have both

$$|v_k - v| < \varepsilon$$

and

$$|v_k - v'| < \varepsilon.$$

Adding the two inequalities,

$$|v_k - v| + |v_k - v'| < 2\varepsilon = \frac{2\delta}{3}.$$

Comparing this with Equation 1, we get that

$$\delta \leq |v - v_k| + |v_k - v'| < \frac{2\delta}{3}.$$

Since $\delta > 0$ we may divide both sides by δ to arrive at

$$1 < \frac{2}{3},$$

a contradiction.

Problem 3

($\Rightarrow$) Suppose that $v_k \to v$ in $\mathbb{R}^n$. To prove convergence of each $\left(v_k^{(i)}\right)$ to $v^{(i)}$, we must be able, given an $\varepsilon > 0$, to produce a $K \in \mathbb{N}$ such $\left|v_k^{(i)} - v^{(i)}\right| < \varepsilon$ whenever $k \geq K$.

By convergence of (v_k), given any $\varepsilon > 0$ (which we here take to be the same ε we're trying to bound each coordinate by) there exists a $K \in \mathbb{N}$ such that for every $k \geq K$,

$$\begin{aligned} &|v_k - v| < \varepsilon \\ \Rightarrow &\sqrt{\sum_{j=1}^{n} \left|v_k^{(j)} - v^{(j)}\right|^2} < \varepsilon. \end{aligned}$$

Since both sides are non-negative, we may square them to obtain

$$\sum_{j=1}^{n} \left|v_k^{(j)} - v^{(j)}\right|^2 < \varepsilon^2.$$

Since each term is non-negative, we have for each i that its term in this sum is less-than-or-equal to the sum:

$$\left|v_k^{(i)} - v^{(i)}\right|^2 \leq \sum_{j=1}^{n} \left|v_k^{(j)} - v^{(j)}\right|^2 < \varepsilon^2$$

so

$$\begin{aligned} &\left|v_k^{(i)} - v^{(i)}\right|^2 < \varepsilon^2 \\ \Rightarrow &\left|v_k^{(i)} - v^{(i)}\right| < \varepsilon \end{aligned}$$

where taking square roots is legitimate because $f(x) = \sqrt{x}$ is a strictly increasing function.

(Another way to justify the last step without invoking the $\sqrt{\cdot}$ function: suppose for non-negative a and b that $a^2 > b^2$ but $a \leq b$. Then we may square the last inequality to obtain $a^2 \leq b^2$, a contradiction. (But if I'm being this pedantic then maybe I ought to justify why we can square inequalities when both sides are non-negative… Some level of pedantry seems appropriate when proving statements like these, so close to the axioms.))

($\Leftarrow$) Suppose that each $v_k^{(i)} \to v^{(i)}$. Given $\varepsilon > 0$, define

$$\varepsilon_c = \frac{\varepsilon}{\sqrt{n}}.$$

Then for each $\left(v_k^{(i)}\right)$ there exists a K_i such that for every $k \geq K_i$,

$$\left|v_k^{(i)} - v^{(i)}\right| < \varepsilon_c.$$

Let $K = \max\{K_i : 1 \leq i \leq n\}$; the definition is justified because n is finite. Then the inequality above holds for every i whenever $k \geq K$.

Squaring both sides (legitimate because both sides are non-negative) and summing over all i, we get

$$\sum_{i=1}^{n} \left|v_k^{(i)} - v^{(i)}\right|^2 < n\varepsilon_c^2.$$

The LHS is $|v_k - v|^2$; the RHS is

$$n\frac{\varepsilon^2}{n} = \varepsilon^2$$

so the inequality becomes

$$|v_k - v|^2 < \varepsilon^2$$

and we take square roots to get

$$|v_k - v| < \varepsilon.$$

Problem 4: Monotone Convergence Theorem

Lemma 1: If A is a nonempty bounded subset of $\mathbb{R}$, then for any $\varepsilon > 0$, there is some $a \in A$ with $a > \sup A - \varepsilon$.

Proof: Suppose for some $\varepsilon > 0$ there were no $a \in A$ with $a > \sup A - \varepsilon$. Define $M = \sup A - \varepsilon$. Since no element exceeds M, then every element is $\leq M$: for all $a \in A$ we have $a \leq M$. In other words, M is an upper bound of A. Also, $M < \sup A$ since $\varepsilon > 0$.

So M is an upper bound that is less than $\sup A$, which contradicts the definition of $\sup A$ as the *least* upper bound of A. ■

Theorem 1: Any bounded, monotonic sequence in $\mathbb{R}$ converges.

Proof: Let (a_k) be any bounded, monotonically increasing sequence in $\mathbb{R}$. Let A be the set $\{a_k : k \in \mathbb{N}\}$. Since the sequence (a_k) is bounded, A is also bounded. Since A contains a_1, it is nonempty. Therefore by the least upper bound property of real numbers, A has a supremum.

Fix any $\varepsilon > 0$. By the lemma proved above, there is some $a \in A$ with $a > \sup A - \varepsilon$. Since the set A consists of all the values in the sequence (a_k), there must be at least one $K \in \mathbb{N}$ with $a_K = a$.

Since the sequence is monotonically increasing, for every $k \geq K$, $a_k \geq a_K$. Combining this with the inequality for a_K, we obtain

$$a_k \geq a_K > \sup A - \varepsilon$$

so

$$a_k > \sup A - \varepsilon.$$

Rearranging, we get that

$$\sup A - a_k < \varepsilon.$$

By the definition of supremum, $\sup A \geq a_k$, so both sides of the inequality above are non-negative. Thus we may take the absolute value to obtain that whenever $k \geq K$,

$$|\sup A - a_k| < \varepsilon.$$

Since the choice of ε was arbitrary, this proves that the sequence converges (to $\sup A$).

For a monotonically decreasing sequence (a_k) we may give an analogous proof with infimum in place of supremum. Alternatively, we may consider the sequence $(-a_k)$ which is then monotonically increasing. ■

Problem 5

(a)

Let (x_k) be a sequence in $\mathbb{R}$ converging to some $x \in \mathbb{R}$, and, for some $M \in \mathbb{R}$, we have that each $x_k \leq M$.

Suppose that $x > M$. Let $\varepsilon = x - M > 0$. Then there exists some $K \in \mathbb{N}$ such that

$$|x_k - x| < \varepsilon$$

whenever $k \geq K$. Let $k = K$.

Using a basic identity of $|\cdot|$ in one dimension,

$$-\varepsilon < x_k - x < \varepsilon.$$

(To justify this, we can show it holds for both cases of the absolute value function $f(y) = |y|$: when $y \geq 0$ and when $y < 0$.)

Adding x to both sides, we have

$$x - \varepsilon < x_k < x + \varepsilon.$$

We're interested in the first of these inequalities. Substituting our chosen value of ε, we have

$$\begin{aligned} &x - (x - M) < x_k \\ \Rightarrow\ &M < x_k, \end{aligned}$$

a contradiction, arising from the assumption that $x > M$. So $x \leq M$.

(b)

Let (x_k) be any sequence in $\mathbb{R}$ converging to some $x \in \mathbb{R}$ such that $x_k \in [a, b]$ for every $k \in \mathbb{N}$. That is,

$$a \leq x_k \leq b$$

for every $k \in \mathbb{N}$. By part (a) and the corresponding statement with the inequality in the other direction, we have that

$$a \leq x \leq b.$$

So $x \in [a, b]$.

So, for every convergent sequence whose terms are all in $[a, b]$, the limit is also in $[a, b]$. By the definition of closed set, the interval $[a, b]$ is a closed set. Since the choice of interval was arbitrary, the conclusion holds for all closed intervals in $\mathbb{R}$.

(c)

Let (x_k) be any sequence in $\mathbb{R}^n$ converging to some $x \in \mathbb{R}^n$ such that $x_k \in R$ where R is a closed rectangle in $\mathbb{R}^n$, the product of closed intervals $[a_1, b_1], ..., [a_n, b_n]$.

By Problem 3, for every $i = 1, ..., n$, the sequence obtained from the i'th coordinate $\left(x_k^{(i)}\right)$ of (x_k) converges to $x^{(i)}$.

By the definition of closed rectangle, if $x_k \in R$ then $x_k^{(i)} \in [a_i, b_i]$ for every $i = 1, ..., n$. Therefore, by part (b) above, $x^{(i)} \in [a_i, b_i]$.

By the definition of closed rectangle, then $x \in R$, proving that closed rectangles are closed sets.

Problem 6

I claim that the interval $(0, 1]$, a subset of $\mathbb{R}$, is not closed.

To show this, we need only exhibit one convergent sequence whose terms are all in $(0, 1]$ but whose limit is not in $(0, 1]$.

Take the sequence $a_k = \frac{1}{k}$. First we prove that it converges to 0. For any $\varepsilon > 0$, take $N = \lceil \frac{1}{\varepsilon} \rceil + 1$. Then for all $k \in \mathbb{N}$ such that $k \geq N$ we have

$$\begin{aligned} & k \geq N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1 > \frac{1}{\varepsilon} \\ \Rightarrow\ & k > \frac{1}{\varepsilon} \\ \Rightarrow\ & \frac{1}{k} < \varepsilon \\ \Rightarrow\ & a_k < \varepsilon \\ \Rightarrow\ & |a_k - 0| < \varepsilon \end{aligned}$$

where in the last step we write $|a_k| = a_k$ because $a_k = \frac{1}{k} > 0$. This shows that $a_k \to 0$.

Also, $0 < \frac{1}{k} \leq 1$ for every $k \in \mathbb{N}$, so $a_k \in (0, 1]$.

But $0 \notin (0, 1]$. Therefore $(0, 1]$ is not closed.

Problem 7

(a)

Lemma 2: Let (a_k) be a sequence that converges to L in $\mathbb{R}$. Then any subsequence of (a_k) is also convergent, and it converges to L.

Proof: Let (s_k) be a subsequence of (a_k). Then there is a strictly increasing (therefore injective) mapping $f : \mathbb{N} \to \mathbb{N}$ such that

$$s_k = a_{f(k)}.$$

Fix any real $\varepsilon > 0$. By convergence of (a_k) to L, there exists a $K \in \mathbb{N}$ such that $|a_k - L| < \varepsilon$ for every $k \geq K$. Let K' be any natural number such that $f(K') \geq K$. Such a number exists because f is strictly increasing, hence unbounded.

Then, because f is strictly increasing, for every $k' \geq K'$, we have

$$f(k') \geq f(K') \geq K,$$

so that

$$\left|a_{f(k')} - L\right| < \varepsilon,$$

so

$$|s_{k'} - L| < \varepsilon,$$

proving that $s_k \to L$. ■

Let $U = F \cup F'$ where F, F' are closed sets. Let (a_k) be any convergent sequence whose terms are all in U. Let L be its limit.

Then at least one of F and F' must have an infinite number of terms of (a_k): since $a_k \in U$ and $U = F \cup F'$, each a_k must belong to at least one of F and F'. But if both F and F' have only a finite number of terms of (a_k), then the total number of terms would be finite, a contradiction.

Without loss of generality let F be a set that has an infinite number of terms of (a_k). Then define (f_k) to be the subsequence consisting of terms of (a_k) that are in F. By the lemma proved above, $f_k \to L$. By the fact that F is a closed set, $L \in F$. Therefore $L \in U$, proving that U is a closed set.

(b)

Let I be an indexing set (potentially uncountably infinite) and let $C = \{F_i : i \in I\}$ be a collection of closed sets F_i. Define

$$V = \bigcap_{i \in I} F_i,$$

their intersection.

Suppose (a_k) is any convergent sequence whose terms are all in V. Let L be its limit. Then, by the definition of set intersection and of V, for each $k \in \mathbb{N}$ and every $i \in I$,

$$a_k \in F_i.$$

So for each i, (a_k) is a convergent sequence whose terms are all in F_i. Therefore, because each F_i is a closed set, $L \in F_i$. Therefore by the definition of V, $L \in V$, proving that V is a closed set.

(c)

I think we can find both countable and uncountable counterexamples. In both cases I use the fact proved in Problem 5 (b), that closed intervals in $\mathbb{R}$ are closed sets.

Uncountable

This is the obvious case.

Let r be a real number in the half-open interval $(0, 1]$. Define $I_r = [r, 1]$. Then the collection

$$C = \{I_r : r \in (0, 1]\}$$

is an infinite collection of closed sets. But its union is $(0, 1]$ (proved below) which, by Problem 6, is not closed.

Lemma 3: The union of the collection C is $(0, 1]$.

Proof: Define

$$U = \bigcup_{r \in (0,1]} I_r,$$

i.e., the union of all sets in the collection C.

Let $a \in (0, 1]$. Then $0 < a \leq 1$. The set $I_a = [a, 1]$ is in the collection C, and $a \in I_a$. Therefore $a \in U$, showing that $(0, 1] \subseteq U$.

Let $a \in U$. Then by the definition of U, $a \in [r, 1]$ for some $r \in (0, 1]$. Since $[r, 1] \subset (0, 1]$ for every $r \in (0, 1]$, we have $a \in (0, 1]$, showing that $U \subseteq (0, 1]$. ■

Countable

For any $k \in \mathbb{N}$ define $I_k = \left[\frac{1}{k}, 1\right]$. Then the collection

$$C = \{I_k : k \in \mathbb{N}\}$$

is an infinite collection of closed sets. But its union is $(0, 1]$ (proved below) which, by Problem 6, is not closed.

Lemma 4: The union of the collection C is $(0, 1]$.

Proof: Define

$$U = \bigcup_{k \in \mathbb{N}} I_k,$$

i.e., the union of all sets in the collection C.

Let $a \in (0, 1]$. Then $0 < a \leq 1$. Let $k = \left\lceil \frac{1}{a} \right\rceil + 1$. Then

$$\frac{1}{k} < a \leq 1$$

so $a \in \left[\frac{1}{k}, 1\right]$, which is in the collection C. Therefore $a \in U$, showing that $(0, 1] \subseteq U$.

Let $a \in U$. Then by the definition of U, $a \in \left[\frac{1}{k}, 1\right]$ for some $k \in \mathbb{N}$. Since $\left[\frac{1}{k}, 1\right] \subset (0, 1]$ for every $k \in \mathbb{N}$, we have $a \in (0, 1]$, showing that $U \subseteq (0, 1]$. ■