Contents

Artin 1.1.7 Formula for a matrix power

Theorem 1: For $n \in \mathbb{N}$,

$$\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$

Proof: For $n = 1$ the theorem is true by inspection.

Using the inductive hypothesis, for $n + 1$ we have

$$\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & n+1 & \frac{n(n+1)}{2} + n + 1 \\ & 1 & n+1 \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & n+1 & \frac{(n+1)(n+2)}{2} \\ & 1 & n+1 \\ & & 1 \end{bmatrix}$$

where I did the matrix multiplication in the second step column-by-column: matrix-vector multiplication is taking linear combinations of the columns of a matrix. $\blacksquare$

Artin 1.1.16 If A is nilpotent, then $I + A$ is invertible

Theorem 2: If A is nilpotent, then $I + A$ is invertible.

We give one proof that comes from noticing an analogy to the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

Proof: Notice that if $A^k = 0$ for some $k > 0$ then

$$(I + A)\left(I - A + A^2 - A^3 + \dots + (-1)^{k-1} A^{k-1}\right)$$
$$= I - A + A^2 - A^3 \dots + (-1)^{k-1} A^{k-1})$$
$$+ A - A^2 + A^3 \dots + (-1)^{k-1} A^k$$
$$= I$$

because all the terms involving A cancel out, the series terminates after k terms, and the leftover term with A^k is zero. ∎

We give another proof via eigenvalues:

Proof: If $A^k = 0$ for some $k > 0$ then its eigenvalues must all be 0, because suppose $v \neq 0$ is an eigenvector of A with eigenvalue λ. Then

$$Av = \lambda v$$

so

$$A^k v = \lambda^k v = 0$$

by multiplying k times on the left by A.

Since $v \neq 0$, it must be that $\lambda = 0$.

Also, if λ is an eigenvalue of A with eigenvector v, then $\lambda + \mu$ is an eigenvalue of $A + \mu I$ with the same eigenvector, since

$$(A + \mu I)v = Av + \mu I v = \lambda v + \mu v = (\lambda + \mu)v.$$

So the eigenvalues of $I + A$ are all one, so its determinant is one (the determinant is the product of eigenvalues with algebraic multiplicity). Therefore it is invertible. ∎

Artin 1.1.17 Solving matrix equations

(a)

By doing the algebra in

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we get

$$y_1 = 5 - 4x_1$$
$$z_1 = \frac{-3x_1 - 2y_1}{5}$$
$$y_2 = -2 - 4x_2$$
$$z_2 = \frac{-2x_2 - y_2}{2}$$

for every choice of x_1 and x_2. So there's an infinite 2D space of solutions.

(b)

No C can make

$$AC = I_3$$

hold, because the rank of A is at most 2, so the rank of AC is at most 2, while the rank of I_3 is 3.

Intuitively, the columns of AC live inside the column space of A; every column of AC is a linear combination of the columns of A; the column space of AC is a subspace of the column space of A. Any time you multiply by a matrix, at best you keep all the original directions, but potentially you kill some. If some get killed, they can never be revived; they have already been sent to 0.