Contents

Problem 1: Every convergent sequence is Cauchy

Proof idea

Get both x_n and x_m within $\frac{\varepsilon}{2}$ of the limit x, then apply the triangle inequality.

Proof

Theorem 1: Every convergent sequence is a Cauchy sequence.

Proof: Let (x_n) be a sequence of real numbers converging to a limit $x \in \mathbb{R}$. To prove that it's Cauchy, for any given $\varepsilon > 0$ we must find an $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ whenever $m, n \geq N$.

By convergence of (x_n) there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for every $n \geq N$. Then apply the triangle inequality to $x_n - x_m = x_n - x + x - x_m$:

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $m, n \geq N$. ∎

Problem 2: Give example or prove non-existence of sequences with certain properties

(a) A Cauchy sequence that is not monotone

A cheap way to do this is to take a monotone sequence and just stick a number in front that makes it not monotone:

$$a_n = \begin{cases} 0 & \text{if } n = 1 \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

But we can also give a sequence that is not monotone even after dropping any finite prefix:

$$a_n = (-1)^n \frac{1}{n}$$

This sequence converges to 0 because

$$|a_n - 0| = \left|(-1)^n \frac{1}{n}\right| = \frac{1}{n}$$

so by taking $N = 1 + \left\lceil \frac{1}{\varepsilon} \right\rceil$ in the definition of convergence, it converges to 0. And every convergent sequence is Cauchy, by Problem 1.

(b) A Cauchy sequence with an unbounded subsequence

No such sequence exists.

Theorem 2: Every subsequence of a Cauchy sequence is bounded.

Proof: Every Cauchy sequence is convergent, by the Cauchy criterion. Every subsequence of a convergent sequence converges to the same limit, proved two problem sets ago. And every convergent sequence is bounded, proved last problem set. Therefore every subsequence of a Cauchy sequence is bounded. ■

Actually we can avoid appealing to the difficult/deep Cauchy criterion:

Proof: Every Cauchy sequence is bounded (finite prefix + infinite tail that's within ε of x_N). And every subsequence of a bounded sequence is bounded, by definitions of bounded and subsequence. ■

(c) A divergent monotone sequence with a Cauchy subsequence

No such sequence exists.

Theorem 3: Every monotone sequence with a Cauchy subsequence converges.

Proof: We will prove that the parent sequence is bounded. Then by the Monotone Convergence Theorem, it converges.

Let (x_n) be a monotone sequence with a Cauchy subsequence $\left(x_{n_k}\right)$. Without loss of generality, let (x_n) be increasing. The subsequence is therefore bounded. Let $M = \sup\left(\left\{x_{n_k} : k \in \mathbb{N}\right\}\right)$.

Every term in the original sequence has a term in the subsequence that comes after it. More precisely, for every $n \in \mathbb{N}$, we can find a $n_k \geq n$ by taking $k = n$. Since the sequence is monotone increasing,

$$x_n \leq x_{n_n}$$

for every $n \in \mathbb{N}$. But the subsequence is bounded, so

$$x_n \leq x_{n_n} \leq M.$$

So (x_n) is bounded, and therefore converges. ■

(d) An unbounded sequence containing a subsequence that is Cauchy

Take

$$a_n = \begin{cases} n \text{ if } n \text{ is odd} \\ \frac{1}{n} \text{ if } n \text{ is even} \end{cases}$$

so that the even indices form a Cauchy subsequence, while the odd indices are unbounded, making the sequence itself unbounded.

Problem 3: Direct proofs that the sum and product of two Cauchy sequences are Cauchy

(a) Sum

Proof idea

Get both sequences within $\frac{\varepsilon}{2}$ by taking the maximum of N_x and N_y, then apply the triangle inequality.

Proof

Theorem 4: Let (x_n) and (y_n) be two Cauchy sequences. Then $(x_n + y_n)$ is Cauchy.

Proof: We are given $\varepsilon > 0$ and must produce $N \in \mathbb{N}$ such that $|x_n + y_n - (x_m + y_m)| < \varepsilon$ whenever $m, n \geq N$.

Let $N_x \in \mathbb{N}$ be the index such that $|x_n - x_m| < \frac{\varepsilon}{2}$ whenever $m, n \geq N_x$, and similarly N_y. Let $N = \max(N_x, N_y)$. Then, whenever $n, m \geq N$, we have

$$\begin{aligned}
&|x_n + y_n - (x_m + y_m)| \\
&= |(x_n - x_m) + (y_n - y_m)| \\
&\leq |x_n - x_m| + |y_n - y_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

as desired. ■

(b) Product

Proof idea

Add and subtract $x_n y_m$, factor out, then use the fact that Cauchy sequences are bounded to replace variable terms with the bounds.

Proof

Theorem 5: Let (x_n) and (y_n) be two Cauchy sequences. Then $(x_n y_n)$ is Cauchy.

Proof: Cauchy sequences are bounded (finite prefix + infinite tail that's within ε of x_N), so let M_x and M_y be any bounds for (x_n) and (y_n) respectively. We may assume that the bounds are nonzero because otherwise at least one of the sequences would be identically zero and the theorem is trivial.

Let $\varepsilon > 0$ be given. By definition of Cauchy sequence, there exists $N_x \in \mathbb{N}$ such that $|x_n - x_m| < \frac{\varepsilon}{2|M_y|}$ whenever $m, n \geq N_x$. Similarly, take N_y to get terms of (y_n) within $\frac{\varepsilon}{2|M_x|}$ of each other.

Let $N = \max(N_x, N_y)$.

Then, whenever $m, n \geq N$ we have

$$
\begin{aligned}
&|x_n y_n - x_m y_m| \\
&= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\
&\leq |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m| \\
&= |x_n||y_n - y_m| + |y_m||x_n - x_m| \\
&\leq |M_x||y_n - y_m| + |M_y||x_n - x_m| \\
&< |M_x|\frac{\varepsilon}{2|M_x|} + |M_y|\frac{\varepsilon}{2|M_y|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

as desired. ■

Problem 4: Equivalence of different characterizations of completeness

(a) Bolzano-Weierstrass $\Rightarrow$ Monotone Convergence without using Archimedean Property

Proof idea

Use Bolzano-Weierstrass to get a convergent subsequence, then sandwich every term x_n of the sequence between two terms of the subsequence: the first one after which the subsequence is within ε of its limit, and some term of the subsequence that comes after x_n.

Proof

Theorem 6: If a sequence is monotone and bounded, then it converges.

Proof: Let (x_n) be a monotone and bounded sequence. Without loss of generality, let it be an increasing sequence.

By Bolzano-Weierstrass (*every bounded sequence contains a convergent subsequence*), it contains a convergent subsequence $\left(x_{n_k}\right)$, converging to some $x \in \mathbb{R}$.

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $k \geq N$ we have $\left|x_{n_k} - x\right| < \varepsilon$.

Then for every $k \geq N$ we have

$$\begin{aligned} &\left|x_{n_k} - x\right| < \varepsilon \\ &\Rightarrow -\varepsilon < x_{n_k} - x < \varepsilon \\ &\Rightarrow x - \varepsilon < x_{n_k} < x + \varepsilon. \end{aligned}$$

We will use the left and right sides of this bound below, with different terms x_{n_k}. We will bound each term of (x_n) between the first term of $\left(x_{n_k}\right)$ after N, and any term of $\left(x_{n_k}\right)$ that comes after x_n.

Let $M \in \mathbb{N}$ be the index of N in the original sequence, i.e., $M = n_N$ and $x_M = x_{n_N}$. Since (x_n) is monotone increasing, every term of (x_n) after x_M satisfies

$$x - \varepsilon < x_M \leq x_n.$$

But for every term x_n there is some term x_{n_n} of $\left(x_{n_k}\right)$ that comes after it, so, again because the sequence is monotone increasing,

$$x_n \leq x_{n_n} < x + \varepsilon.$$

Putting these together, we get

$$|x_n - x| < \varepsilon$$

so the sequence converges. ∎

(b) Cauchy Criterion $\Rightarrow$ Bolzano-Weierstrass, using Archimedean Property

Proof idea

Recursively split the interval $[-M, M]$ that contains the sequence, always picking a half which has an infinite number of terms. After enough splittings, the size of the interval is less than ε, and all subsequent terms are trapped within it. The Archimedean property is used to justify that there is a natural number bigger than $\log_2\left(\frac{M}{\varepsilon}\right)$.

Proof

Lemma 1: If $x, y \in [a, b]$ then $|x - y| \leq b - a$.

Proof:

$$x \leq b$$

and

$$a \leq y \Rightarrow -y \leq -a$$

so

$$x - y \leq b - a.$$

Similarly,

$$y \leq b$$

and

$$a \leq x \Rightarrow -x \leq -a$$

so

$$y - x \leq b - a$$

so

$$-(b - a) \leq x - y.$$

Putting these together,

$$-(b - a) \leq x - y \leq b - a$$

so

$$|x - y| \leq b - a.$$

■

Theorem 7: Every bounded sequence contains a convergent subsequence.

Proof: Let (a_n) be a bounded sequence so that there exists $M > 0$ satisfying $|a_n| < M$ for all $n \in \mathbb{N}$. Bisect the closed interval $[-M, M]$ into the two closed intervals $[-M, 0]$ and $[0, M]$. (The midpoint is included in both halves.) At least one of these closed intervals contains an infinite number of the terms of the sequence (a_n). Select a half for which this is the case and label that interval as I_1. Then, let a_{n_1} be some term in the sequence (a_n) satisfying $a_{n_1} \in I_1$.

Next, we bisect I_1 into closed intervals of equal length, and let I_2 be a half that again contains an infinite number of terms of the original sequence. Because there are an infinite number of terms from (a_n) to choose from, we can select an a_{n_2} from the original sequence with $n_2 >$

n_1 and $a_{n_2} \in I_2$. In general, we construct the closed interval I_k by taking a half of I_{k-1} containing an infinite number of terms of (a_n) and then select $n_k > n_{k-1} > \ldots > n_2 > n_1$ so that $a_{n_k} \in I_k$.

The length of I_k is $\frac{\text{len}(I_{k-1})}{2}$ for each $k > 1$ and the length of I_1 is M, so

$$\text{len}(I_k) = \frac{M}{2^{k-1}}.$$

Given an $\varepsilon > 0$, choose $t \in \mathbb{N}$ such that

$$t > \frac{M}{\varepsilon}.$$

Such a t exists by the Archimedean Property and because both M and ε are positive real numbers.

Since t is a natural number there is some power of 2 bigger than it; in other words, the sequence of powers of 2 is unbounded (this is a fact about the natural numbers, not real analysis, and I think we can prove by induction that $2^n \geq n$ for all $n \in \mathbb{N}$).

Therefore pick some $N \in \mathbb{N}$ such that

$$2^{N-1} \geq t > \frac{M}{\varepsilon}$$

so that

$$\varepsilon > \frac{M}{2^{N-1}} = \text{len}(I_N).$$

Then for every $i, j \geq N$, the terms $a_{n_i}, a_{n_j} \in I_N$ so by the lemma proved above,

$$\left|a_{n_i} - a_{n_j}\right| < \varepsilon,$$

so by the Cauchy Criterion, the subsequence $\left(a_{n_k}\right)$ converges. ■

(c) Archimedean Property $\not\Rightarrow$ Axiom of Completeness

The field $\mathbb{Q}$ of rational numbers also has the Archimedean property: for every rational $\frac{p}{q}$, by the Euclidean division algorithm we have

$$p = qd + r$$

where $0 \leq r < q$, so $|d| + 1 > \frac{p}{q}$.

Even more trivially, $|p| + 1 > \frac{p}{q}$.

But we know the rationals do not satisfy the Axiom of Completeness, because the set $\{x \in \mathbb{Q} : x^2 < 2\}$ does not have a least upper bound in $\mathbb{Q}$, because $\sqrt{2}$ is irrational. So it cannot be possible to prove the Axiom of Completeness from the Archimedean property.