IME625A–PROJECT REPORT

TITLE:
MCMC Method For Estimating Implied Volatility in Options Pricing and BS Model

Course Title
Introduction to Stochastic Process

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Abstract:

This report introduces the method of estimating Implied volatility using options data and Independence Metropolis Hastings which is an acceptance-rejection criterion for the Monte Carlo Markov Chain Process. For IV estimation, paper describes this process as an Inverse version of Black Scholes formula and assumes that asset prices follow geometric Brownian motion. Apart from this we have mainly focussed on understanding core concepts such as Brownian Motion, Black Scholes formula, its derivation and No Arbitrage principle. We have tried to explain key concepts from scratch that includes qualitative properties of asset prices, BS model. In this way the report tries to highlight MCMC and its application in finance by taking into consideration the stochastic process.

Introduction to Key Concepts

Options: - Options are financial derivatives which give the buyer of the option the right, not the obligation, to buy or sell some underlying asset at some specified price at, or before, an already specified date.

Present Value of an asset: - More simply, present value is the worth of a future stream of payments, given a specified rate of return. If the risk-free interest rate is r and the asset is known to provide a payment of money = S at time t. Then the present value of the asset is given by,

$$PV = Se^{-rt}$$

The Principle of No Arbitrage: - In derivatives markets, arbitrage is the opportunity of creating risk free profit for sure without any initial investment. Suppose we want to determine the price of some asset which ensures that no one could make a sure profit out of it. Assuming that the no arbitrage principle can be generalised, we can infer that there will be no opportunity to create arbitrage if there is a probability measure on the set of all possible outcomes, such that the expectation of getting sure profits without any risks is zero.

More formally, say the price of an asset at a time t is given by X(t). We consider observing the asset till time s, purchasing (or selling) it at a time s and selling (or purchasing) it later at a time t. By no arbitrage principle and good understanding of present value discussed above, we can say that if **P** is a probability measure over set of possible values of X(t) then

$$E_P[e^{-rt}X(t)|X(u),0 \ \leq u \leq s] = e^{-rs}X(s) \qquad (1)$$

Brownian Motion and Finance: - Botanist Robert Brown was observing the motion of pollen grains in water and he came up with the term Brownian motion to describe the random movement of the pollen grains. The French mathematician, Louis Bachelier, made the term widely popular in finance when he applied it to model the movement of asset prices.

In his theory, in very small amounts of time, the increments in the prices of an asset are independent of the current price and past behaviour of the prices. Based on his assumptions, he inferred that the random movement in prices of an asset could be assumed to be normally distributed.

This led to the development of what today is known as the Random Walk Hypothesis in Finance. In the limit of time periods becoming arbitrarily small, random walk becomes Brownian motion.

Brownian Motion: - Brownian Motion is an example of continuous time continuous state stochastic process. It has the following properties: -

1. It is normally distributed with zero mean and non-zero variance
2. It has normally distributed increments.
3. It follows Markov property which states that the state in which the stochastic process is present only depends on the most recent known past state.
4. It follows Martingale Property which states that conditional expectation of future is equal to the present value, given the information about past events.

Standard Brownian Motion: - In the notations of Wiener, classical equation describing the Brownian motion is given by

$$ds \ = \ \mu dt + \sigma dX$$

where,

ds represent the change in prices of the asset in time dt

μ represents the drift or trend in asset price,

dX is normally distributed, i.e., dX ~ N (0,1),

σ is the price volatility (it is assumed to be a constant here)?

When we model the market prices to be following Standard Brownian motion the future market prices follow Normal Distribution.

Geometric Brownian Motion: - The equation describing the geometric distribution is given by,

$$ds = s\mu dt + s\sigma dX$$

When the market prices are assumed to follow Geometric distribution, the future market prices follow lognormal distribution.

Asset Pricing and Geometric Brownian Motion: - In this work we assume that the asset prices follow Geometric Brownian motion.

Implied Volatility: - Implied volatility is a metric used to calculate the market belief about the price fluctuations of an asset. It is generally the standard deviation of the time series and usually calculated by black Scholes model or the binomial model. Implied volatility, however, does not predict the direction in which the price change will happen. For example, a large volatility implies a large swing in price, but the direction in which prices would move is not conveyed by this information. All we can say is that the prices will fluctuate significantly. On the other hand, low volatility means that the prices of the underlying asset will remain stable.

Options Pricing: - Options are priced according to their intrinsic value and time value. Intrinsic value is made up of the difference and strike price and stock’s price. Often the options with higher volatility are desired

Black Scholes Formula: - Black Scholes Formula is one of the most famous formulas in finance. It is used to find the price of a European option given its volatility, price of the underlying asset, time till expiration of the option, and the risk-free interest rate.

The main assumptions of the black Scholes model are that: -

1. The volatility of the underlying is constant
2. The option is European and could only be exercised on expiry date
3. Asset prices follow geometric Brownian motion
4. The risk-free rate is known and constant

The Black-Scholes Model in mathematical notation is:

$$C = S_t N(d_1) - Ke^{-rt} N(d2)$$

where,

$$d_1 = \frac{ln(\frac{S_t}{K}) + (r + \frac{\sigma_v{}^2}{2})t}{\sigma_s \sqrt{t}}$$

$$d_2 = d_1 - \sigma_s \sqrt{t}$$

$$C = Call\ option\ price$$

$$S = Current\ underlying\ price$$

$$K = Strike\ price$$

$$r = Risk\ free\ interest\ rate$$

$$t = Time\ till\ maturity$$

$$N = Normal\ Distribution$$

Derivation of the Black Scholes Formula: -

Let there be an option for which the underlying asset prices are given by X(t). The option at some time t gives the buyer the right to sell the underlying asset at a price K. Let's say the risk-free interest rate is r. Then, the value of this option at time t

$$= X(t) - K \ \ if\ X(t) \geq K$$

$$= \quad 0 \qquad \text{otherwise}$$

Where the value of option in the second case comes out to be zero due to the fact that option may or may not be exercised by the buyer. The buyer will not exercise the option as it is giving him negative returns.

This implies that the value of this option at time t is $(X(t) - K)^{+}$. Using the knowledge of present value, we infer that the price of this option at time t=0 is $e^{-rt}(X(t) - K)^{+}$.

Suppose the cost of buying the option is C, then assuming that there is no arbitrage we can say that,

$$E_P[e^{-rt}(X(t) - K)^{+}] \ = \ C \qquad (3)$$

Now assuming that our prices follow the geometric Brownian motion, we have

$$X[t] \ = \ x_0 e^{y[t]}$$

Where y(t) follows Brownian motion with the drift parameter μ and the variance parameter σ. Now from the properties of Brownian motion,

$$E[X(t)|X(u), 0 \leqslant u \leqslant s] = X(s)e^{(t-s)(\mu+\sigma^2/2)}$$

If we choose mu and sigma such that

$$\mu + \sigma^2/2 = \alpha$$

Then the equation of no arbitrage is satisfied

Going ahead with our equation 3, we have

$$
\begin{aligned}
ce^{\alpha t} &= \int_{-\infty}^{\infty} (x_0 e^y - K)^+ \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-(y-\mu t)^2/2t\sigma^2}\, dy \\
&= \int_{\log(K/x_0)}^{\infty} (x_0 e^y - K) \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-(y-\mu t)^2/2t\sigma^2}\, dy
\end{aligned}
$$

Making a change of variable

$$w = (y - \mu t)/(\sigma t^{1/2})$$

gives us

$$ce^{\alpha t} = x_0 e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{\sigma w \sqrt{t}} e^{-w^2/2}\, dw - K \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-w^2/2}\, dw$$

Where

$$a = \frac{\log(K/x_0) - \mu t}{\sigma \sqrt{t}}$$

Now the integral below evaluates to

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{\sigma w \sqrt{t}} e^{-w^2/2}\, dw &= e^{t\sigma^2/2} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-(w-\sigma\sqrt{t})^2/2}\, dw \\
&= e^{t\sigma^2/2} P\{N(\sigma\sqrt{t}, 1) \geqslant a\} \\
&= e^{t\sigma^2/2} P\{N(0, 1) \geqslant a - \sigma\sqrt{t}\} \\
&= e^{t\sigma^2/2} P\{N(0, 1) \leqslant -(a - \sigma\sqrt{t})\} \\
&= e^{t\sigma^2/2} \phi(\sigma\sqrt{t} - a)
\end{aligned}
$$

Where phi is the standard normal CDF.

Hence, we can see that,

$$ce^{\alpha t} = x_0 e^{\mu t + \sigma^2 t/2} \phi(\sigma\sqrt{t} - a) - K\phi(-a)$$

Using

$$\mu + \sigma^2/2 = \alpha$$

And letting b = -a we can write the above equation as

$$c = x_0\phi\left(\sigma\sqrt{t} + b\right) - Ke^{-\alpha t}\phi(b)$$

$$b = \frac{\alpha t - \sigma^2 t/2 - \log(K/x_0)}{\sigma\sqrt{t}}$$

The formula above is the black Scholes formula as we have described above.

Parameter Estimation and MCMC: -

Parameters for models can be estimated through different samples and it generally requires to take more estimates from different samples in order to reduce biasness, and make the sample better representative of population. Distribution of these estimates is called as Sampling distribution which quantifies the uncertainty in these estimates. In Bayesian literature, Parameters are treated as Random Variables and are described by PDFs and CDFs. We first describe the Prior distribution for the parameter, which is our belief or experience regarding the parameter. Then we go for experiments and define a likelihood function which quantifies the likelihood of a parameter value given the data. Finally, Posterior distribution allows us to update our belief regarding our parameter. Simply, Posterior distribution is proportional to product of prior and likelihood function.

Posterior = Prior * Likelihood

$$P(\theta|y|) = P(\theta) \cdot P(y|\theta|)$$

More Informative prior will have more influence over the Posterior for a fixed sample size whereases increasing sample size will give likelihood function more control over posterior distribution for a less informative prior.

MCMC stands for Markov Chain Monte Carlo methods. Rather than being a single algorithm it comprises many algorithms which can enable one to sample from a random distribution. These methods use Stationary Markov Chains to generate a random sample from a known distribution and Metropolis Hastings Algorithm will allow us to accept or reject the estimates values produced as described below.

Metropolis-Hastings Algorithm: - Metropolis Hastings Sampling comes under the umbrella of methods described by the MCMC methods. The algorithm is used to generate a time reversible Markov Chain whose stationary probabilities could only be specified up to a multiplicative constant.

More formally let there be a vector valued Markov chain $\boldsymbol{X_1}$, $\boldsymbol{X_2}$, $\boldsymbol{X_3}$.... with stationary transition probabilities

$$\pi(i) \;=\; b(i)/B,\; for\, j = 1,2, \ldots.$$

Where $B = \sum_{i=1}^{inf} b(i)$ is finite but unknown (many times is it computationally inefficient or not even possible to calculate this sum). The procedure is described below.

Let there be a Markov chain $\{Y_1, Y_2, Y_3 \ldots..\}$ with transition probabilities given by $q(i,j)$.

We create a new Markov chain $\{X_1, X_2, X_3 \ldots..\}$ with the transition

probabilities given by

$$P(i,j) \;=\; q(i,j) * \alpha(i,j)\; if\; i \neq j$$

$$P(i,i) \;=\; q(i,i) + \sum_{k \neq i}^{n} \big(1 - \alpha(i,j)\big) * q(i,j)$$

We want our Markov chain $\{X_1, X_2, X_3.\ldots.\}$ to be time reversible have the stationary probabilities given by

$$\pi(i) \;=\; b(i)/B, for\, j = 1,2, \ldots.$$

If we ensure that,

$$\pi(i) * P(i,j) = \pi(j) * P(j,i) \quad for\; j \neq i \qquad (2)$$

Then we will have the time reversibility condition trivially satisfied, the stationarity condition will also be satisfied as proved below,

$$\sum_{i\epsilon\Omega} \pi(i) * P(i,j) = \sum_{i\epsilon\Omega} \frac{\pi(j)*P(j,i)}{P(i,j)} * P(i,j) = \pi(j) \sum_{i\epsilon\Omega} P(j,i) = \pi(j)$$

Now, we need to provide the appropriate $\alpha(i,j)$ such that the condition (2) is satisfied, that is we need to have,

$$\pi(i) * \alpha(i,j) * q(i,j) = \pi(j) * \alpha(j,i) * q(j,i) \quad for\ j \neq i$$

If we set $\alpha(i,j) = min(\frac{\pi(j)*q(j,i)}{\pi(i)*q(i,j)}, 1)$ it is easy to verify that the above equation is satisfied. Now, as the ratio $\frac{\pi(j)}{\pi(i)} = \frac{b(j)}{b(i)}$ is independent of B we reach the conclusion that the $\alpha(i,j)$ as described above, would make a time reversal markov chain with stationary probabilities described up to a constant.

Estimation of IV: -

We are trying to use the data on European option prices to estimate the implied volatility. The problem boils down to an inverse problem, where in we are estimating IV with the help Black Scholes formula by employing the option prices data. Using MCMC-IMH algorithm we will try to obtain the posterior distribution of volatility which is dependent on mean value and the option prices. The paper has proposed IMH sampler for the estimation of IV.

IMH: - IMH refers to Independence Metropolis Hastings. It is a special case of Metropolis Hastings where the new proposed states of the Markov chain are independent of the previous state.

Considering the application of this technique to option pricing, say the financial prices are given by X = X_1….X_T, option prices by, N = N_1,….N_T, returns V. We use the combined posterior $p(\mu, \sigma|X, N, V)$ to estimate the posterior.

As the option values are independent of the μ, it leaves with

$$p(\mu|\sigma, X, N, V) = p(\mu|\sigma, V)$$

And it is normal as the prior is assumed to be normal. The option value and returns are affected by volatility and hence updating σ is not straightforward.

$$\pi(\sigma) = p(\sigma|\mu, X, N, V) = p(N|\sigma, X) * p(V|\mu, \sigma) * p(\sigma)$$

Here we consider applying IMH technique to renew the volatility. The new value volatility is accepted or rejected on the basis of the option prices. One expects the IV to be Inverse Gamma distribution, which is also the proposed distribution.

$$r(\sigma) = p(\sigma|\mu, V) \propto p(V|\mu, \sigma) * p(\sigma)$$

Algorithm: -

1. Draw $\phi_t^{(i+1)}$ from $r(\phi)$ ~ Inverse Gamma
2. Accept $\phi_t^{(i+1)}$ with likelihood $\alpha = \min\left(\frac{P\left(N\middle|\phi_t^i\right), S}{\left\{P\left(N\middle|\phi_t^{i+1}\right), S\right\}}, \ 1\right)$

This algorithm gives draws from the combined posterior.

Conclusion and Future Prospectus: -

We have seen the most famous formula in finance, The black Scholes formula and its derivation using the stochastic concepts like Markov Chain and Brownian Motion. We have also introduced the method of estimating IV through MCMC-IMH process, which refreshes volatility at every time instant and hence time varying volatility has come into picture which was not the case with Black Scholes Formula, in which IV is assumed to be constant.

Though the paper describes every Algorithm quite nicely, there hasn’t been clear description regarding the assumption of prior distributions and proposed distributions. We can extend this Idea to various different distributions and then verify the robustness of this process.

References: -

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