Theory of Computation

Problem Set 2

1. Let $S = \{a, b\}$. A set $C \subseteq S^+$ is called a *code* if every word in S^* has at most one factorization into words from C.

   (a) Prove that $\{aa, ba, baa\}$ is a code.

   (b) Prove that $\{a, ab, ba\}$ is not a code.

   (c) Show that a two-element set $\{u, v\}$ is a code if and only if $uv \neq vu$.

2. Construct a finite automaton over $\{0, 1\}$ that recognizes all strings in which the number of 1's in odd positions is odd and the number of 1's in even positions is even.

3. Design a finite automaton recognizing all words over $\{0, 1\}$ in which the number of 1's occurring at prime positions is even.

4. Determine whether the language

   $$\{\, w \in \{a, b\}^* \mid \#a(u) > 2026 \cdot \#b(u) \text{ for every nonempty prefix } u \text{ of } w \,\}$$

   is regular. Justify your answer.

5. Prove that a language L is regular if and only if its reversal

   $$L^R = \{w^R : w \in L\}$$

   is regular.

6. For a nonempty word $w \in \{0, 1\}^+$ define

   $$0.w = \sum_{i=1}^{|w|} \frac{w_i}{2^i}.$$

   Let us define the following two languages -

   $$L_{<\frac{1}{7}} = \{\, w \in \{0, 1\}^+ : 0.w < \tfrac{1}{7} \,\}$$

   and

   $$L_{<\frac{1}{\sqrt{5}}} = \{\, w \in \{0, 1\}^+ : 0.w < \frac{1}{\sqrt{5}} \,\}$$

   One of them is regular and the other is not. Which one is regular? Why?

7. Let $k \geq 1$. Prove that the language

$$\{ w \in \{a, b\}^* : |w|_a \equiv |w|_b \pmod{k} \}$$

is regular.

8. Give an example of two regular languages whose intersection requires a minimal DFA with strictly more states than either of the original minimal DFAs.

9. Let L be a regular language. Prove that the following languages are also regular:

$$\mathrm{Root}(L) = \{w : \exists n \in \mathbb{N}.\ w^n \in L\},$$

$$\mathrm{Sqrt}(L) = \left\{w : \exists u.\ |u| = |w|^2\ \wedge\ wu \in L\right\},$$

$$\mathrm{Log}(L) = \left\{w : \exists u.\ |u| = 2^{|w|}\ \wedge\ wu \in L\right\},$$

$$\mathrm{Fibb}(L) = \left\{w : \exists u.\ |u| = F_{|w|}\ \wedge\ wu \in L\right\},$$

where F_n is the nth Fibonacci number: $F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$.

10. Let L be a language over the alphabet Σ. Let us define:

$$L_{\frac{1}{2}} = \{x \mid \exists y\ \ |y| = |x|\ \wedge\ yx^R \in L\}$$

where x^R is the string obtained when x is read in reverse. Show that $L_{\frac{1}{2}}$ is regular if L is regular.