MTech CSE – 1st Semester
Student Name: SIDDHARTH KUMAR
Student ID: A125021

Question 7

Prove that positive-definite matrices are suitable for LU decomposition and do not require pivoting to avoid division by zero in the recursive strategy.

Answer:

Key Idea

To establish the result, we show that for a positive-definite matrix:

- LU decomposition exists without row exchanges,
- all pivot elements produced during Gaussian elimination are non-zero.

This guarantees that the recursive LU algorithm never encounters division by zero, eliminating the need for pivoting.

Positive-Definite Matrices

A real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called *positive-definite* if

$$x^T Ax > 0 \quad \text{for all non-zero } x \in \mathbb{R}^n.$$

Positive-definite matrices possess strong structural properties that ensure stable factorization.

Leading Principal Minors

A fundamental result from linear algebra (Sylvester's criterion) states that a real symmetric matrix is positive-definite if and only if all its leading principal minors are strictly positive:

$$\det(A_k) > 0, \quad k = 1, 2, \ldots, n,$$

where A_k denotes the $k \times k$ leading principal submatrix of A.

Connection with LU Decomposition

For LU decomposition without pivoting, the pivot elements satisfy:

$$u_{kk} = \frac{\det(A_k)}{\det(A_{k-1})}, \quad k = 1, 2, \ldots, n,$$

with the convention $\det(A_0) = 1$.

A zero pivot occurs if and only if $\det(A_k) = 0$. Therefore, non-zero leading principal minors guarantee non-zero pivots throughout the elimination process.

Implication of Positive-Definiteness

Since A is positive-definite:

- all leading principal minors are strictly positive,
- all pivots u_{kk} are strictly positive.

Hence, every division performed in the recursive LU algorithm is well-defined, and no breakdown occurs.

Absence of Pivoting

Pivoting is typically introduced to:

- avoid division by zero,
- reduce numerical instability.

For positive-definite matrices:

- division by zero cannot occur,
- pivot elements remain positive at every step.

Thus, pivoting is not required for correctness when performing LU decomposition on positive-definite matrices.

Relation to Numerical Stability

Positive-definite matrices naturally control element growth during Gaussian elimination. As a result:

- pivot magnitudes remain bounded away from zero,
- error amplification is limited in floating-point arithmetic.

This makes LU decomposition without pivoting reliable in practice for such matrices.

Additional Insight: Cholesky Decomposition

Every positive-definite matrix admits a Cholesky decomposition:

$$A = LL^T,$$

which is a special case of LU decomposition with $U = L^T$.

Since Cholesky decomposition never requires pivoting, its existence further confirms that positive-definite matrices are inherently suitable for triangular factorizations.

Final Conclusion

Positive-definite matrices admit LU decomposition without pivoting because:

- all leading principal minors are strictly positive,
- all pivots in Gaussian elimination are non-zero,
- division by zero cannot occur in the recursive strategy.

Therefore, LU decomposition can be safely applied to positive-definite matrices without the need for row exchanges.