MTech CSE – 1st Semester

Student Name: SIDDHARTH KUMAR

Student ID: A125021

Question 11

Define the complexity class **Co-NP**. Explain the nature of problems that belong to this class.

Answer:

Background: Decision Problems and the Class NP

In computational complexity theory, problems are typically formulated as *decision problems*, where each input instance has a binary outcome: YES or NO.

A language L is said to be in the class **NP** if, whenever an input $x \in L$ yields a YES answer, there exists a certificate (also called a witness) whose validity can be checked in polynomial time by a deterministic Turing machine.

Thus, **NP** captures problems for which positive instances admit efficiently verifiable proofs.

Definition of the Class Co-NP

Formal Definition

A decision problem L belongs to the class **Co-NP** if the complement of L is in **NP**.

Formally,

$$L \in \textbf{Co-NP} \iff \overline{L} \in \textbf{NP},$$

where

$$\overline{L} = \{\, x \mid x \notin L \,\}.$$

Alternative Characterization

Equivalently, a problem lies in **Co-NP** if:

- for every NO-instance of the problem,
- there exists a certificate that can be verified in polynomial time.

Hence, **Co-NP** consists of problems whose negative answers admit efficient verification.

Conceptual Understanding

The distinction between **NP** and **Co-NP** can be summarized as follows:

- **NP**: efficient verification of YES answers.
- **Co-NP**: efficient verification of NO answers.

Intuitively, **Co-NP** models problems where proving that no solution exists is easier than finding a solution itself.

Relationship Between P, NP, and Co-NP

Every problem that can be solved in polynomial time belongs to both **NP** and **Co-NP**:

$$\mathbf{P} \subseteq \mathbf{NP} \cap \mathbf{Co\text{-}NP}.$$

However, it is an open question whether:

$$\mathbf{NP} = \mathbf{Co\text{-}NP}.$$

The prevailing belief in complexity theory is that these classes are distinct. Resolving this question would have deep implications, including consequences for the **P** versus **NP** problem.

Comparison of Complexity Classes

Class	Verified Certificate	Time Bound	Representative Problem
P	YES / NO	Polynomial	Shortest Path
NP	YES instance	Polynomial	SAT
Co-NP	NO instance	Polynomial	UNSAT

Nature of Problems in Co-NP

Problems in **Co-NP** are often associated with *universal statements*, such as:

- asserting that no valid solution exists,
- checking that all configurations satisfy a given property,
- verifying the absence of a certain structure.

Many such problems arise naturally as complements of well-known **NP** problems.

Representative Examples

UNSAT

Problem: Given a Boolean formula φ, determine whether it has no satisfying assignment.

Since satisfiability (SAT) is **NP**-complete, its complement UNSAT is **Co-NP**-complete.

TAUTOLOGY

Problem: Given a Boolean formula φ, is it true under every possible truth assignment?

This problem is the complement of SAT and hence belongs to **Co-NP**.

Non-Hamiltonian Graph

Problem: Given a graph G, does G fail to contain a Hamiltonian cycle?

The Hamiltonian Cycle problem is **NP**-complete, implying that its complement lies in **Co-NP**.

Primality

Problem: Given an integer n, is n prime?

This problem lies in **Co-NP** as its complement (compositeness) has polynomially verifiable certificates. Notably, primality testing is also known to be solvable in polynomial time.

Co-NP-Complete Problems

A language L is said to be **Co-NP**-complete if:

- $L \in$ **Co-NP**, and
- every problem in **Co-NP** can be reduced to L in polynomial time.

Classical examples include:

- UNSAT,
- TAUTOLOGY.

Structural Properties

- **Co-NP** is closed under complementation.
- Closure under union and intersection is not known.
- If **NP** = **Co-NP**, the polynomial hierarchy collapses.

Conclusion

The class **Co-NP** consists of decision problems for which NO-instances admit polynomial-time verifiable certificates.

In summary:

- **Co-NP** contains complements of **NP** problems,
- it models efficiently verifiable non-existence proofs,
- it includes problems such as UNSAT and TAUTOLOGY,
- whether **Co-NP** equals **NP** remains an open problem.

Remark. Equality of **NP** and **Co-NP** would imply a collapse of the polynomial hierarchy to its first level.