MTech CSE – 1st Semester
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Question 6

Prove that if a matrix A is non-singular, then its Schur complement is also non-singular.

Answer:

Matrix Partition and Schur Complement

Let the square matrix $A \in \mathbb{R}^{n \times n}$ be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{k \times k}$ is assumed to be invertible. Under this assumption, the *Schur complement* of A_{11} in A is defined as

$$S = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

Objective

Given that the full matrix A is non-singular, we aim to show that the Schur complement S is also non-singular.

Block Factorization of A

Using block Gaussian elimination, the matrix A can be factorized as

$$A = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix}.$$

This factorization separates the elimination process into a lower triangular factor and an upper triangular factor involving the Schur complement.

Invertibility Argument

First Factor

The matrix

$$L = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix}$$

is block lower triangular with identity matrices on the diagonal, and is therefore invertible.

Second Factor

The matrix

$$U = \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix}$$

is block upper triangular. Its determinant satisfies

$$\det(U) = \det(A_{11}) \cdot \det(S).$$

Conclusion from Non-Singularity of A

Since

$$A = LU$$

and A is non-singular while L is invertible, it follows that U must also be invertible. Hence,

$$\det(A_{11}) \cdot \det(S) \neq 0.$$

Because A_{11} is invertible, $\det(A_{11}) \neq 0$, which implies

$$\det(S) \neq 0.$$

Therefore, the Schur complement S is non-singular.

Final Result

If a block matrix A is non-singular and the leading block A_{11} is invertible, then its Schur complement

$$S = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

is also non-singular.

Remark

This result plays a crucial role in block LU and LUP decomposition, where the Schur complement represents the reduced system obtained after eliminating a set of variables. The non-singularity of the Schur complement ensures that the elimination process can proceed without breakdown.