MTech CSE – 1st Semester

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Question 10

Prove that every connected component of the symmetric difference of two matchings in a graph G is either a path or an even-length cycle.

Answer:

Basic Definitions

Let $G = (V, E)$ be an undirected graph.

Matching

A matching $M \subseteq E$ is a set of edges such that no two edges share a common endpoint.

Let M_1 and M_2 be two matchings in the graph G.

Symmetric Difference

The symmetric difference of M_1 and M_2 is defined as:

$$M_1 \oplus M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1).$$

Thus, $M_1 \oplus M_2$ consists of edges that appear in exactly one of the two matchings.

Claim

Every connected component of the graph

$$H = (V, M_1 \oplus M_2)$$

is either:

- a path, or
- a cycle of even length.

Degree Property of Vertices

Consider any vertex $v \in V$.

Since M_1 is a matching, at most one edge of M_1 is incident to v. Similarly, since M_2 is a matching, at most one edge of M_2 is incident to v.

Hence, in the symmetric difference graph H:

$$\deg_H(v) \leq 2.$$

Consequence of Degree Bound

Any graph in which all vertices have degree at most 2 must consist of connected components that are:

- simple paths, or
- simple cycles.

Therefore, each connected component of H is either a path or a cycle. It remains to show that all cycles have even length.

Alternating Nature of Edges

Let v be a vertex with degree 2 in H.

Since no matching can contain two edges incident to the same vertex, the two edges incident to v must belong to different matchings—one from M_1 and one from M_2.

As a result, along any connected component of H, the edges alternate between:

$$M_1 \text{ and } M_2.$$

Analysis of Connected Components

Case 1: Path Components

If a connected component has a vertex of degree 1, then it must be a path. Such a path may begin or end at vertices unmatched in one or both matchings.

Paths impose no restriction on length and are therefore valid components of $M_1 \oplus M_2$.

Case 2: Cycle Components

If a connected component is a cycle, then every vertex in that component has degree exactly 2.

Because edges alternate between M_1 and M_2, the cycle must contain the same number of edges from each matching. This is possible only if the total number of edges in the cycle is even.

Thus, every cycle in $M_1 \oplus M_2$ has even length.

Conclusion

We conclude that:

- each vertex in $M_1 \oplus M_2$ has degree at most 2,
- every connected component is either a path or a cycle,
- all cycles must be of even length due to edge alternation.

Hence, every connected component of the symmetric difference of two matchings in a graph is either a path or an even-length cycle.

Intuition

The symmetric difference highlights exactly where two matchings differ. At each vertex, edges can only alternate between the two matchings, which naturally restricts the structure to paths or even cycles—no other configuration is possible.