80-311 Assignment 11

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2.i

Claim:
Let f: a -¿ b and x and y be subsets of a, then:
$f[x \cup y] = f[x] \cup f[y]$

To prove this we show that the LHS is a subset of the RHS, and the RHS is a subset of the LHS.

Part 1: $LHS \subseteq RHS$
First, let an arbitrary element t be a member of $f[x \cup y]$. Then by the defn of image,
we know that there exists u such that $f(u) = t$ and $u \in (x \cup y)$.
By the defn of union, u must be in either x or y.
Then t is in either $f[x]$ or $f[y]$, so it is in $f[x] \cup f[y]$, which is the RHS.
Therefore any element t which is a member of the LHS is also a member of the RHS.
Then $LHS \subseteq RHS$

Part 2: $RHS \subseteq LHS$
Consider element t in $f[x] \cup f[y]$.
Then there is an element u either in x or y such that $f(u) = t$.
Then u is in $x \cup y$. Then t is in $f[x \cup y]$, so every element in the RHS is also in the LHS. Then $RHS \subseteq LHS$

2.ii

We want to show that every element in $f[x \cap y]$ is also in $f[x] \cap f[y]$.
Consider an arbitrary element t in $f[x \cap y]$.
By the definition of image, there exists a u such that $f(u) = t$ and $u \in x \cap y$.
By the defn of binary intersection, u is in both x and y.
Then t is in both $f[x]$ and $f[y]$, so $t \in f[x] \cap f[y]$, proving
that $LHS \subseteq RHS$.

2.iii

I conjecture that the formula is true when f is injective.

If an element t is in $f[x]$ and $f[y]$, then there is an element u such that $f(u) = t$ and $u \in x$.
There is also an element v such that $f(v) = t$ and $v \in y$.

If f is injective, then $f(v) = t \wedge f(u) = t \implies v = u$.
Then we can say $v \in y => u \in y$, so $u \in x \cap y$, and $t \in f[x \cap y]$.

Fitch Proof of (ii):

3.i

Representability allows us to express THM in terms of first order logic ZF. Specifically, it states that

$$THM(\phi) \iff ZF \vdash thm('\phi')$$

3.ii

Let m and n be natural numbers.

Note: In this problem, $'m'$ is the set theoretic numeral for m.

Claims:
1. If $m = n$ then $ZF \vdash m = n$
2. If $\neg m = n$ then $ZF \vdash \neg 'm' =' n'$

Proof of 1:

By induction. It holds true for the base case when m and n are 0, which is represented by the empty set. For the inductive hypothesis, assume it holds true for all naturals up to and including some fixed m and n such that $m = n$.

$'m+1' = 'm','m-1',...,'1','0'$
$'n+1' = 'n','n-1',...,'1','0'$

and since we know $m = n$ from the inductive hypothesis, then we make a simple substitution to obtain:

$'m+1' = 'n','n-1',...,'1','0'$
$'n+1' = 'n','n-1',...,'1','0'$

Since $'m+1'$ and $'n+1'$ are sets with all the same elements, they are in fact the same set theoretic numerals.

Proof of 2:

By the hint, we seek to prove that if $m < n$ then $ZF \vdash \neg m = n$.

Proceed by induction. We assume n is an arbitrary natural number larger than m. (2) holds true for the base case when m is 0, which is represented by the empty set. n will be presented by some set with elements which do not exist in $'m'$. Thus $\neg 'm' =' n'$.

For the inductive hypothesis, assume (2) is true for all naturals up to and including some fixed m such that $m < n$.

$'m+1' = 'm','m-1',...,'1','0'$
$'n+1' = 'n','n-1',...,'1','0'$

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and since we know $m < n$ from the inductive hypothesis, then we make a simple substitution to obtain:

$'m+1' = 'm','m-1'...,'1','0'$
$'n+1' = 'n',...,'m','m-1'...,'1','0'$

Since $'n+1'$ contains elements not in $'m+1'$ the sets are not equal.

For all m and n in the naturals both claims 1 and 2 hold true.

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