21-301A Combinatorics, 2013 Fall
Homework 1

- The due is on Friday, Sep 6, at beginning of the class.
- Collaboration is permitted, however all the writing must be done individually.

1. How many distinct words (including nonsense ones) produced by the letters in MISSISSIPPI have the property that there is no consecutive I's.

Solution. There are 1 M, 2 P's, 4 S's and 4 I's, where all letters of the same kind are identical. We devide into two steps: first, arrange 1 M, 2 P's and 4 S's in an queue; second, insert 4 I's into queues of the 1st step in such way that no two I's are consecutive. For the 1st step, we have $\frac{7!}{1!2!4!}$ ways to arrange them.

For any fixed queue of the 1st step, we label the 7 letters as 1st object, 2nd,..., 7th object, from left to right; now we want to put 4 I's in 8 slots: the slot before the 1st object, the slot between ith object and $(i+1)$th object for $i = 1, 2..., 6$ and the slot after the 7th object, in such way that each slot only can have at most one I; so we have $\binom{8}{4}$ ways to insert 4 I's. Thus, the number of distinct words is $\frac{7!}{1!2!4!}\binom{8}{4}$. ∎

2. (a) Let n, r be positive integers and $n \geq r$. Give a combinatorial proof of

$$\binom{r}{r} + \binom{r+1}{r} + ... + \binom{n}{r} = \binom{n+1}{r+1}.$$

(b) Use (a) to compute $\sum_{i=1}^{n} i^2$.

Proof. We proved (a) in class. We show (b) here. By choosing $r = 2$, we get the following from (a):

$$\binom{n+1}{3} = \sum_{i=2}^{n} \binom{i}{2} = \sum_{i=2}^{n} \frac{i^2 - i}{2} = \sum_{i=1}^{n} \frac{i^2 - i}{2}.$$

Therefore

$$\sum_{i=1}^{n} i^2 = 2\binom{n+1}{3} + \sum_{i=1}^{n} i = \frac{(n+1)n(n-1)}{3} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{6}.$$

∎

3. Let n, r be be positive integers and $n \geq r$. Give a combinatorial proof of

$$\binom{2n}{2r} \equiv \binom{n}{r} (mod\ 2).$$

Proof. Notice that it suffices to show $\binom{2n}{2r} - \binom{n}{r}$ is an even number!

Consider n couples and label them as couple 1, couple 2,..., couple n. Now, we consider the number of ways to select $2r$ person from the n couples, which equals $\binom{2n}{2r}$. There are two kinds of selections: the selected $2r$ person come from r couples or not.

There are $\binom{n}{r}$ selections satisfying the property that the selected $2r$ person come from r couples. Thus, if we define $\mathscr{A}$ to be the set of selections which are not formed by r couples, then we see

$$|\mathscr{A}| = \binom{2n}{2r} - \binom{n}{r}.$$

All we need now is to show: $|\mathscr{A}|$ is even. We prove this by pairing selections of $\mathscr{A}$.

For any selection, say S, of $\mathscr{A}$, there must be a couple such that one person of the couple is in S and his/her spouse is not. Let couple i be such couple with i minimum and let S^* be obtained from S by exchanging wife and husband of the couple i. Then, we see $S^* \in \mathscr{A}$ and moreover $(S^*)^* = S$. Thus, all selections in $\mathscr{A}$ can be paired as $\{S, S^*\}$. This shows that $|\mathscr{A}|$ is even. ∎

4. How many integer solutions of the inequality

$$x_1 + x_2 + x_3 \leq 25$$

satisfy that $2 \leq x_1 \leq 7, x_2 \geq 0, x_3 \geq 0$?

Solution. The answer is equal to the number (say A) of integer solutions (x_1, x_2, x_3, x_4) to

$$x_1 + x_2 + x_3 + x_4 = 25$$

with $2 \leq x_1 \leq 7$ and $x_2, x_3, x_4 \geq 0$.

Note that $2 \leq x_1 \leq 7$ is the same as x_1 satisfies $x_1 \geq 2$ but does not satisfy $x_1 \geq 8$. If we consider the same equation and change the restriction of $2 \leq x_1 \leq 7$ into $x_1 \geq 2$, after substituting $x_1' = x_1 - 2$, we get $x_1' + x_2 + x_3 + x_4 = 23$ with $x_1', x_2, x_3, x_4 \geq 0$, which gives us $\binom{23+4-1}{4-1} = \binom{26}{3}$ solutions. If we change the restriction of x_1 to $x_1 \geq 8$, by the same argument, we get $x_1'' + x_2 + x_3 + x_4 = 17$ with $x_1'' = x_1 - 8, x_2, x_3, x_4 \geq 0$, which gives $\binom{17+4-1}{4-1} = \binom{20}{3}$ solutions. Now, the answer is given by $A = \binom{26}{3} - \binom{20}{3}$. ∎

5. Let n be a positive integer. Prove that

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k,$$

where $S(n,k)$ is the Strirling number of the second kind and $(x)_k = x(x-1)...(x-k+1)$.

Proof. First we assume that x is an postive integer. We consider the set $[x]^{[n]}$, i.e., the set of all functions $f : [n] \to [x]$. For each function $f \in [x]^{[n]}$, let $Im(f)$ be the set of images $f(i)$, i.e.,

$$Im(f) = \{k : \text{there exists some } i \in [n] \text{ such that } f(i) = k\}.$$

For fixed subset A of size k, the set $\{f : Im(f) = A\}$ contains all surjective functions $f : [n] \to A$, so by the Theorem we proved in Lec 2

$$|\{f : Im(f) = A\}| = S(n,k)k!.$$

Notice that $Im(f)$ can be any non-empty subset of $[x]$. Therefore, we can write $[x]^{[n]}$ as a disjoint union

$$[x]^{[n]} = \cup_{k=1}^{n} \cup_{A \in \binom{[x]}{k}} \{f : Im(f) = A\},$$

which implies that

$$x^n = |[x]^{[n]}| = \sum_{k=1}^{n} \sum_{A \in \binom{[x]}{k}} |\{f : Im(f) = A\}| = \sum_{k=1}^{n} \binom{x}{k} S(n,k)k! = \sum_{k=1}^{n} S(n,k)(x)_k$$

holds for all positive integers x.

Let $f(x) = x^n - \sum_{k=1}^{n} S(n,k)(x)_k$ be a polynomial of variable x. Then we see that all positive integers are the roots of $f(x)$, so f has infinity many roots! The fundamental theorem of algebra says that if a polynomial with degree k has more than $k+1$ roots, then it must be a zero function. But $f(x)$ is a polynomial with degree at most n and with infinity many roots, so $f(x)$ must be a zero function, which implies that

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k$$

holds for all reals x. ∎

6. Let p be a prime and n be a positive integer. Find the largest integer k such that $p^k | n!$. Express such k as a function of p and n. (The notation $a|b$ means that a divides b.)

Solution. For any integer m between 1 and n, write m as the unique expression of $m = p^i m'$, where p does not divide m'. For any integer $i \geq 0$, let the set $A_i = \{m : m = p^i m', 1 \leq m \leq n\}$, then any memeber of A_i will provide exactly i p's to the largest exponent k. Using double counting,

$$k = \sum_{i \geq 1} i|A_i| = \sum_{j \geq 1} (|A_j| + |A_{j+1}| + |A_{j+2}| + ...) = \sum_{j \geq 1} \lfloor \frac{n}{p^j} \rfloor.$$

∎

7. How many ways are there to distribute 30 identical balls among 3 boys and 3 girls if each boy should get an odd number of balls and each girl should get at least 2 balls? Express the answer as a coefficient of a suitable power of x in a suitable product of polynomials.

Solution. Let $B(x) = x + x^3 + x^5 + ... + x^{29}$ and $G(x) = x^2 + x^3 + x^4 + ... + x^{30}$. Also let $f(x) = B^3(x)G^3(x)$. The answer is $[x^{30}]f$. ∎