21-301A Combinatorics, Fall 2013, Test 1

Instructor: Jie Ma

Name: ________________________ ID:________________________

Problem	*Points*
1	
2	
3	
4	
5	

Total :________________

- **Use your best judgement to interpret problems.**
- **Please write down necessary intermediate steps when you solve a problem.**
- **Closed book and closed notes. 50 mins only.**
- Maximum possible score is 20.

Problem 1 (4 points).

Let $n > 0$ be an **even** integer. How many subsets of $\{1, 2, 3, ..., n\}$ with at least $n/2$ elements can we find? Express the answer **without using summation**.

Answer of the form (this is just an example!) $3^{n+5} - \frac{1}{5}\binom{n}{\frac{n}{3}}$ will be acceptable.

Solution. We have

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n,$$

where $\binom{n}{i} = \binom{n}{n-i}$. But n is even, so

$$2^n = \binom{n}{n/2} + 2 \sum_{i=n/2+1}^{n} \binom{n}{i},$$

implying that

$$\sum_{i=n/2+1}^{n} \binom{n}{i} = 2^{n-1} - \frac{1}{2}\binom{n}{n/2}.$$

therefore the number of subsets with at least $n/2$ elements equals

$$\binom{n}{n/2} + \sum_{i=n/2+1}^{n} \binom{n}{i} = 2^{n-1} + \frac{1}{2}\binom{n}{n/2}.$$

Problem 2 (4 points).

Prove that the identity

$$\binom{m+n}{r} = \sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i}$$

holds for all integers $m, n, r \geq 0$ satisfying $0 \leq r \leq m+n$.

Proof. Consider a group of $m+n$ students with m girls and n boys. We want to form a team of r students. Then the number of ways to form the team is $\binom{m+n}{r}$. There is another way to count the number of ways: divide into cases according to the number of girls, say i, where $0 \leq i \leq r$. In the case of i girls, we have $\binom{m}{i}\binom{n}{r-i}$ ways to choose the team. Thus, the total number of ways is also eqaul to

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i}.$$

This shows that

$$\binom{m+n}{r} = \sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i}.$$

Problem 3 (4 points).

Let p, q be two positive integers. How many strings, using exactly p A's and q B's, satisfy the property that every two A's are separated by at least two B's?

Solution.

We first put p A's in the queue and then insert q B's. Let the number of B's before the 1st A be x_1. For $i = 1, 2, ..., p-1$, let the number of B's between the ith A and $(i+1)$th A be x_{i+1}. And let the number of B's after the pth A be x_{p+1}. Then we get

$$x_1 + x_2 + ... + x_p + x_{p+1} = q,$$

where $x_1 \geq 0, x_{p+1} \geq 0$ and $x_2, ..., x_p \geq 2$. Let $y_1 = x_1, y_{p+1} = x_{p+1}$ and for $i = 2, ..., p$, let $y_i = x_i - 2$. Then the equation becomes

$$y_1 + y_2 + ... + y_p + y_{p+1} = q - 2(p-1),$$

where $y_1, ..., y_{p+1} \geq 0$. There are

$$\binom{q - 2(p-1) + (p+1) - 1}{p} = \binom{q - p + 2}{p}$$

many distinct solutions to the equation, which results in that many strings.

Problem 4 (4 points).

Let $S(n, r)$ be the Stirling number of the second kind; that is the number of partitions of set $[n]$ into r non-empty parts. Prove that

$$S(n+1, r+1) = \sum_{k=0}^{n} \binom{n}{k} S(k, r).$$

Proof. Note that $S(n+1, r+1)$ is the numbe of partitions of $[n+1]$ into $r+1$ non-empty parts. Let A be the part (in the $r+1$ partition of $[n+1]$) containing the integer $n+1$. Let $|A| = i+1$, where i means the number of other integers in A other than $n+1$. We see $0 \leq i \leq n$.

When $|A| = i+1$, we have $\binom{n}{i}$ ways to choose these i integers of $A - \{n+1\}$; once a choice of i integers is fixed, we need to partition the remaining $n-i$ integers into r non-empty parts, which together with A form a $(r+1)$-partition of $[n+1]$ and results in $S(n-i, r)$ ways.

Therefore, summing all of the cases up, we get that

$$S(n+1, r+1) = \sum_{i=0}^{n} \binom{n}{i} S(n-i, r) = \sum_{k=0}^{n} \binom{n}{k} S(k, r).$$

Problem 5 (4 points).

Let $\mathbb{N}_o$ be the set of odd positive integers, i.e., $\mathbb{N}_o = \{1, 3, 5, 7, ...\}$. For fixed positive integer n, let b_k be the number of integer solutions $(x_1, x_2, ..., x_n)$ to $x_1 + x_2 + ... + x_n = k$ with $x_i \in \mathbb{N}_o$ for all $i = 1, 2, ..., n$. Express

$$f(x) = \sum_{k=0}^{\infty} b_k x^k$$

without using any summation.

Solution. By Fact of the generating function,

$$f(x) = \left(\sum_{i \in \mathbb{N}_o} x^i \right)^n .$$

Sice $\frac{1}{1-x} = \sum_{i \geq 0} x^i$, we get $\frac{1}{1+x} = \sum_{i \geq 0} (-1)^i x^i$, therefore

$$\sum_{i \in \mathbb{N}_o} x^i = \frac{1}{2} \left(\frac{1}{1-x} - \frac{1}{1+x} \right) = \frac{x}{1-x^2}.$$

Therefore,

$$f(x) = \left(\frac{x}{1-x^2} \right)^n .$$