21-301A Combinatorics, 2013 Fall
Homework 2

- The due is on Friday, Sep 20, at beginning of the class.
- Collaboration is permitted, however all the writing must be done individually.

1. Find the value of $\sum_{i=0}^{n}(-1)^i \binom{n}{i}\binom{n}{r-i}$ for all integers r, n satisfying $0 \leq r \leq 2n$.

2. Let $a_n = \frac{n(n+1)}{2}$ for all integers $n \geq 0$. Find the generating function $f(x)$ of the infinity sequence $a_0, a_1, a_2, ...$ and express it without summation.

3. Prove that for any integer $n \geq 1$, $n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$.

4. Prove that

(a) $1 + x \leq e^x$ holds for any real number x. (Calculus is allowed to use)

(b) $n! \geq e\left(\frac{n}{e}\right)^n$ **by induction** on n.

5. Let $\pi(n)$ be the number of primes in $\{1, 2, ..., n\}$.

(a) Prove that the product of all primes p satisfying $m < p \leq 2m$ is at most $\binom{2m}{m}$, where $m \geq 1$ is any integer.

(b) Use (a) to prove the lower bound of the Prime Number Theorem, that is $\pi(n) \leq \frac{Cn}{\log n}$ for any integer $n \geq 2$ and some absolute constant C. (Hint: by induction)

6. How many integer solutions (x_1, x_2, x_3, x_4) to

$$x_1 + x_2 + x_3 + x_4 = 20$$

satisfy that for each i, $x_i \geq 0$ but $x_i \neq 6$?

7. How many ways are there to seat n couples at a round table with $2n$ chairs in such a way that none of the couples sit next to each other? If one seating plan can be obtained from other plan by a rotation, then we will view them as one plan.

Hint: let A_i be the event such that the couple i sit next to each other and use inclusion-exclusion principle.

8. Let $D(n)$ be the number of permutations π of $[n]$ such that $\pi(i) \neq i$ for any $i \in [n]$ (see Section 3.8). Prove that $D(n+1) = n[D(n-1) + D(n)]$ for all $n \geq 2$.

Note: A proof by plugging in the precise formula of $D(n)$ is not accepted.