21-301A Combinatorics, 2013 Fall
Homework 3

- The due is on Friday, Oct 4, at beginning of the class.
- Collaboration is permitted, however all the writing must be done individually.

1. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by $a_0 = 1, a_1 = 2$ and

$$a_n = 4a_{n-1} - 3a_{n-2} \quad \text{for } n \geq 2.$$

(a) Determine the generating function for the sequence $a_0, a_1, a_2, \ldots$

(b) Use the generating function to find a formula for a_n.

2. Find a sequence $a_0, a_1, a_2, \ldots$ such that for all integer $n \geq 0$

(a) $\sum_{k=0}^{n} a_k a_{n-k} = \binom{n+1}{1}$;

(b) $\sum_{k=0}^{n} a_k a_{n-k} = \binom{n+2}{2}$.

3. Let s_n be the number of all strings of length n with entries from the set of $\{a, b, c\}$ which do not contain "ab" (in consecutive positions). Find the general formula for s_n.

4. There are $2n$ distinct points marked on a circle. We want to divide these $2n$ points into n pairs and connect each pair by a segment (chord) in such a way that these n segments do not intersect. Show that the number of ways to do so is the n'th Catalan number.

5. Let $a_0, a_1, a_2, \ldots$ be a sequence of integers with $a_0 = 1$ and let $f(x) = \sum_{n\geq 0} a_n x^n$ be its generating function (note that here we do not assume that the power series $\sum_{n\geq 0} a_n x^n$ converges!). Prove that there exists a unique formal power series $g(x)$ such that

$$f(x)g(x) = 1.$$

Hint: recall the definition of the multiplication of two formal power series.

6. Let a_n be the number of n-digit numbers with digits $1, 2, 3$ where the number of 1's is even, the number of 2's is odd and the number of 3's is at least 2.

Determine the exponential generating function for the sequence of $a_0, a_1, a_2, \ldots$

7. Let G be a graph in which every vertex has degree at least d. Show that G has a path of length at least d (the length of a path is the number of edges in the path).

8. Let $G = (V, E)$ be a d-regular bipartite graph with parts A and B.

(a) Show that $|A| = |B|$;

(b) G has a perfect matching.