21-301A Combinatorics
Week 1

The notation # means the word "number". Let X be a set of size n. Write $[n] = \{1, 2, 3, ..., n\}$.

Lec 1 (M). Binomial coefficients

- Define $2^X := \{A : A \subset X\}$ and show $|2^X| = 2^{|X|} = 2^n$.
- Define $\binom{X}{k} := \{A \subset X : |A| = k\}$.
- **Fact:** $|\binom{X}{k}| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. That is: the binomial coefficient $\binom{n}{k}$ presents the number of selections of size k out of n distinct elements.

  In its proof, we also show that # of ordered k-tuple $(x_1, ..., x_k)$ with $x_i \in X$ is equal to $(n)_k := n(n-1)...(n-k+1)$.
- If $n < k$, then $\binom{n}{k} = 0$.
- **Fact:** the binomial coefficient $\binom{n}{k}$ also equals # of integer solutions $(x_1, x_2, ..., x_n)$ to $x_1 + x_2 + ... + x_n = k$ with each $x_i \in \{0, 1\}$.
- **Fact:** # of integer solutions $(x_1, ..., x_n)$ to equation $x_1 + ... + x_n = k$ with each $x_i \geq 0$ = # of labellings of k identical objects using n distinct labels = $\binom{n+k-1}{n-1}$

  In its proof, we define a bijection from the set of solutions to $\binom{[n+k-1]}{n-1}$. It is suggested to read another proof in the book on page 69.

Lec 2 (W). Some properties of binomial coefficients and counting functions

- **Fact:** $\binom{n}{k} = \binom{n}{n-k}$.
- **Fact:** $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

  We use a combinatorial proof; here we count the same combinatorial object using two ways, which give us two expressions of the same value.
- **Fact:** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
- We mention the **Pascal Triangle**, whose number in the nth row and kth column is the binomial coefficient $\binom{n}{k}$. This can be explained by using the previous fact.
- Define X^Y to be the set of all functions $f : Y \to X$. Let $|X| = n$ and $Y = [r]$.
- We also can view X^Y as the set of all strings $x_1x_2...x_r$ with elements $x_i \in X$, indexed by elements of Y.
- **Fact:** $|X^Y| = n^r = |X|^{|Y|}$.
- **Fact:** The number of **injective** functions $f \in X^Y$ is equal to $(n)_r$.
- **Definition (The Stirling number of the second kind).** Let $S(r, n)$ be the number of partitions of a set of size r into n non-empty parts.

- Exercise. $S(r,2) = \frac{1}{2}\sum_{i=1}^{r-1}\binom{r}{i}$.
- **Theorem.** The number of **surjective** functions $f \in X^Y$ is equal to $S(r,n)n!$.
- Any injection $f : X \to X$ is called a **Permutation** of X (also a bijection).

  We mention that we may view a permutation in two ways: it is a function from X to X; it also can be think of an arrangement of the elements of X.
- The # of permutations of $[n]$ is $n!$.

Lec 3 (F). Multiplying polynomials and the Binomial Theorem

- Define $[x^k]f$ to be the coefficient of term x^k in a polynomial $f(x)$.
- **Fact 1:** For $j = 1, 2, ..., n$, let
$$f_j(x) := \sum_{i \in I_j} x^i$$
  be a polynomial (note the coefficient of each term is either 1 or 0), where I_j is a set containing nonnegative integers (finite many or infinity). Let $f(x) = f_1 f_2 ... f_n$ be the product of polynomials.

  Then $[x^k]f$ is the number of solutions $(i_1, ..., i_n)$ to $i_1 + ... + i_n = k$ with $i_j \in I_j$ for $j = 1, 2, ..., n$.
- **Fact 2:** Let $f = f_1 f_2 ... f_n$ be a product of polynomials. Then
$$[x^k]f = \sum_{i_1+...+i_n=k}\left(\prod_{j=1}^{n}[x^{i_j}]f_j\right).$$
  This is a more general formula than the Fact 1.
- **The Binomial Theorem.** It holds for any real x and any positive integer n that
$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$
  In the proof we show, we use the Fact 1.
- A nice Exercise. Prove $(1+x)_n = \sum_{k=0}^{n}\binom{n}{k}(x)_k$ for any real x and integer $n \geq 1$, where $(x)_k = x(x-1)...(x-k+1)$ denotes a polynomial.
- **Fact.** $\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}^2$.

  In its proof, we use the Fact 2 as well as the binomial theorem.
- **Fact.** $\sum_{\text{all odd } k}\binom{n}{k} = \sum_{\text{all even } k}\binom{n}{k} = 2^{n-1}$.
- **Fact.** $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$.

Some notices.

- In week 1, Sections 3.1-3.3 and 12.1 from textbook are covered.
- Besides homework problems, the following problems from book are also fun to work on: 2,4,6 in Section 3.1; 1,3,5(b),7(b) in Section 3.2; 3,4,16,17,18 in Section 3.3; 5 in Section 12.1.