21-301A Combinatorics
Week 2

- **The first test** will be on Friday in the week 3, 9/13, in class. It covers all materials in the first 6 lectures, including sections 3.1-3.6 and 12.1 in textbook.
- I will be out of town in week 3 and we will have other professors to fill in for me.
- **Office hours.** We will have some extra office hours on this Friday, 9/7, 4pm–6pm, in Wean 7130. The office hours in week 3 will be cancelled.

Lec 4 (W). Introduction of generating function and estimate (I): factorial function

- Definition. The (ordinary) generating function (GF or short) of an infinite sequence $a_0, a_1, a_2, ...$ is defined as $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

  We can think of GF in 2 ways. We view it as a function of x when the power series $\sum_{k=0}^{\infty} a_k x^k$ converges and therefore we can do operations (like integral and derivative) to $f(x)$. When we do not know if it converges, We treat GFs as formal objects which are allowed to do additions and multiplications.
- **Fact:** $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ holds for any real x with $|x| < 1$.
- **Fact:** (An equivalent form of the Fact 1 of Lec 3)

  For $j = 1, 2, ..., n$, let $f_j(x) = \sum_{i \in I_j} x^i$. Define b_k to be the number of solutions to $i_1 + i_2 + ... + i_n = k$ with each $i_j \in I_j$. Then
  $$\prod_{j=1}^{k} f_j(x) = \sum_{k=0}^{\infty} b_k x^k.$$
- There is an basic idea when using GF. In order to find the expression of a_n in general, we work on its GF $f(x)$; once we find the formula of $f(x)$, then we can expanse $f(x)$ into a power series and find a_n by choosing the coefficient of the right term.
- Using GF and the above idea, we show the fact that $b_k :=$ the number of labelling of k identical objects using 3 different labels is $\binom{k+2}{2}$.

  We have seen this in week 1. This new proof uses derivatives of GF.
- **Theorem.** For any integer $n \geq 1$,
  $$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n.$$
  Here $e = 2.71828...$ is the Euler/natural number. In it proof, we use the curve of $y = \log x$ and its integrals.
- Define $f(n) \sim g(n)$ for functions f and g if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.
- **Stirling's Formula.** $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

  We mention this with NO proof.

- Exercise. For any integer $n \geq 1$, $n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$.

  Modify the proof of the upper bound in previous theorem.

Lec 5 (F). Estimate (II): binomial coefficient

- **Fact:** For fixed integer n, view $\binom{n}{k}$ as a function with $k \in \{0, 1, 2, ..., n\}$. It is increasing when $k \leq \lfloor n/2 \rfloor$ and decreasing when $k \geq \lceil n/2 \rceil$.

  In particular, $\binom{n}{k}$ achieves its maximum when $k = \lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$.

- **Fact:** $\frac{2^n}{n+1} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \leq 2^n$.

  It is a corollary of the previous fact.

- **Fact:** $\frac{2^n}{\sqrt{2n}} \leq \binom{n}{\frac{n}{2}} \leq \frac{2^n}{\sqrt{n}}$ holds for even n.

  It is a better estimate than the previous one. See its proof on page 96 of book.

- **Fact:** Using Stirling's formula, we have $\binom{n}{\frac{n}{2}} \sim \sqrt{\frac{2}{\pi}}\frac{2^n}{\sqrt{n}}$

- **Fact:** $\binom{n}{k} \leq \frac{n^k}{k!}$

- Exercise. $1 + x \leq e^x$ holds for any real x.

  It is allowed to use calculus.

- **Theorem.** For $0 \leq k \leq n$,

  $$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

  We give two proofs for its upper bound. One uses the binomial theorem; the other combines facts $\binom{n}{k} \leq \frac{n^k}{k!}$ and $k! \geq e\left(\frac{k}{e}\right)^k$.

Some more notices.

- Lec 6 will be given on Monday in week 3 and discuss an application of the estimation of binomial coefficient: **The Prime Number Theorem**. See page 97 and Exercise No. 2.

- Fun problems in book: 2, 5 in (3.4); 7, 8, 9, 12 in (3.5); 1 in (3.6).

  The No. 8 in (3.5) is nice and a challenge.

- There are more practice problems for the coming test.

  (1). Find all positive integers $a > b > c$ such that $\binom{a}{b}\binom{b}{c} = 2\binom{a}{c}$.

  (2). Use a combinatorial argument to find integers a, b, c such that $m^3 = a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1}$.

  (3). Determine the number of triples (A, B, C) such that $A \subseteq B \subseteq C \subseteq [n]$.

  (4). How many 9-digit numbers are made of digits 1,2,3,...,9 such that no i is in the ith digit and it is not a palindrome? A palindrome is a number that reads the same in either direction, i.e. 34543.

  (5). Let b_k be the number of integer solutions to $x_1 + x_2 + ...x_n = k$ with $x_i \geq 0$ for all $i = 1, 2, ..., n$. Express $f(x) = \sum_{k \geq 0} b_k x^k$ without using summation.