

## 21-301 Assignment 05

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### 1

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We know that a maximal chain of  $[n]$  contains  $n + 1$  sets, and there is exactly one set of every size from 0 to  $n$ .

We know that the sizes of  $A_i$  strictly increase in the order they are given (from 1 to  $k$ ). That is that  $A_1$  is the smallest and  $A_k$  is the largest  $A_i$ .

We can construct all maximal chains of  $[n]$  containing all  $A_i$  by the following method:

Let your chain contain all  $A_i$ . This is a chain given the subset relationship of all the  $A_i$ . Notice that this chain may not be maximal if there is not a set of every size from 0 to  $n$ . We know that we have sets of sizes  $|A_1|, |A_2|, \dots, |A_k|$ .

1. Now, add a maximal chain of  $2^{A_1}$ .
2. Then find  $|A_j| - |A_i|$  sets in  $2^{[x]}$  of all sizes strictly between  $(|A_i|, |A_j|)$  such that they form a chain, and add them to the chain.
3. Finally find  $n - |A_k|$  sets in  $2^{[x]}$  of all sizes strictly between  $(|A_k|, n)$  such that they form a chain, and add them to the chain.

Note that by this method, you can form any maximal chain of  $2^{[x]}$  which contains all the  $A_i$ . Now we count how many distinct outcomes there are of performing the above procedure.

1. There are  $|A_1|!$  ways of doing step 1.
2. There are  $\prod_{i,j|j-i=1} (|A_j| - |A_i|)!$  ways of doing step 2. This is because you can choose from  $(|A_j| - |A_i|)$  elements for the first set you add, one fewer element for the next set, up til there is only one way to form the last set (then you have  $A_j$ ).
3. There are  $(n - |A_k|)!$  ways of doing step 3 by the same logic.

The product of those three quantities gives the answer:

$$|A_1|! \prod_{i,j|j-i=1} (|A_j| - |A_i|)!(n - |A_k|)!$$

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### 2

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We observe that the size of intersection of  $\mathcal{F}$  and any maximal chain of  $2^{[n]}$ , is at most 2, since if there were 3 sets in common with the chain, those 3 sets would form a chain which would break the constraint on  $\mathcal{F}$  given in the problem. With 2, you do not break the constraint.

The total number of sets in  $\mathcal{F}$  that also exist in some maximal chain is at most  $2n!$  (two in a maximal chain, and there are  $n!$ ).

We can also count this quantity by counting the number of maximal chains containing  $A$  for all  $A \in \mathcal{F}$ . This quantity is  $\sum_{A \in \mathcal{F}} |A|!(n - |A|)!$ , as shown in the proof of Sperner's Theorem in lecture.

Then, we do some algebra:

$$1 \geq \sum_{A \in \mathcal{F}} \frac{|A|!(n - |A|)!}{2n!} = \sum_{A \in \mathcal{F}} \frac{1}{2^{\binom{n}{\lfloor \frac{n}{2} \rfloor}}} = \frac{|\mathcal{F}|}{2^{\binom{n}{\lfloor \frac{n}{2} \rfloor}}}$$

Which implies that  $|\mathcal{F}| \geq 2^{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$  and since  $n$  is even,  $|\mathcal{F}| \geq 2^{\binom{n}{\frac{n}{2}}}$ .

### 3

Consider the  $n$ -length sequence where if  $n$  is even, every element is 1, but if  $n$  is odd,  $n - 2$  elements are 1, 1 element is -1, and 1 element is 1.5.

#### Proof:

Case  $n$  is even:

All numbers in the sequence are 1. Choose half of them to get  $\epsilon = -1$ , and naturally the other half get  $\epsilon = 1$ . Then the dot product of the epsilon vector and the sequence will be 0. There are  $\binom{n}{\frac{n}{2}}$  ways to do this.

Case  $n$  is odd:

Put the  $n - 2$  1s and the  $-1$  in a bag. Note that this forms  $n - 1$  elements, which is even since  $n$  is odd. If you choose half of them to get  $\epsilon = -1$  and the other half to get  $\epsilon = 1$ , the sum will either be 1 or -1. If the sum is 1, choose the 1.5 to be multiplied by  $\epsilon = -1$ . Else,  $\epsilon = 1$ . In the first case, the sum will be  $1 - 1.5 = -0.5$ , and in the second case, the sum will be  $-1 + 1.5 = 0.5$ . There are  $\binom{\frac{n-1}{2}}{\frac{n-1}{4}}$  ways to construct this  $\epsilon$  vector, and since  $n$  is odd, this is equal to  $\binom{\frac{n}{2}}{\frac{n}{4}}$  ways.

### 4

Since every set has a nonempty intersection with every other set, we realize that there must be at least 1 element which is common to all sets in  $\mathcal{F}$ . Then to construct an intersecting family, we have at most the freedom to decide for  $n - 1$  elements whether or not they are in a set in  $\mathcal{F}$ . In other words, at the very least, one element's fate is decided (it is in every set by default). Then the upper bound on the size of an intersecting family is:

$$|\mathcal{F}| \leq 1 \prod_{i=2}^n 2 = 2^{n-1}$$

### 5

Consider a bipartite graph  $(A, B)$  where  $A$  is the set of  $n$  elements and  $B$  is the set of  $S_1 \dots S_n$ . Notice that  $|A| + |B| = 2n$ . An edge exists from a vertex in  $A$  to  $B$  if an element in  $A$  is a member

of a subset  $S_i$  in  $B$ . Notice that the degree of a vertex representing  $S_i$  in  $B$  is equal to  $|S_i|$ , since it is the number of incoming edges, or number of elements which are members of  $S_i$ .

The problem states that two distinct  $S_i, S_j$  may not have an intersection of more than one element. In our bipartite graph, if  $S_i, S_j$  in  $B$  had an intersection of their neighbor sets of two vertexes in  $A$ , it would look like  $K_{2,2}$ , and it would violate the problem statement. Therefore, our bipartite graph does not have  $K_{2,2}$ .

We know from lecture that the maximum number of edges in a bipartite graph that does not have  $K_{2,2}$  is  $|V|^{3/2} + |V|$ . We'll write this in terms of  $n$  and label constants for sake of clarity.

$$|E| \leq (2n)^{3/2} + 2n = c_1 n^{3/2} + c_2 n = c_3 n \sqrt{n}$$

We wish to find an upper bound on the degree of the smallest  $S_i$  in  $B$ , because doing so would prove that an  $S_i$  exists such that the claim in the problem is true. To maximize the degree of the smallest  $S_i$ , we consider the graph with the maximum possible number of edges such that  $|E| = c_3 n \sqrt{n}$ , and we distribute the edges evenly accross vertexes in  $B$ , because giving any vertex more edges than another gives the other vertex a smaller degree. In the above scenario, each vertex in  $B$  would have degree  $\frac{c_3 n \sqrt{n}}{n} = c_3 \sqrt{n}$ . Thus we've shown that the maximum degree of the minimum degree vertex is  $c_3 \sqrt{n}$ .

In any scenario such as the one given in the question, there will always exist an  $S_i$  such that  $d(S_i) = |S_i| \leq C \sqrt{n}$ .

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