21-301A Combinatorics, 2013 Fall
Homework 1

- The due is on Friday, Sep 6, at beginning of the class.
- Collaboration is permitted, however all the writing must be done individually.

1. How many distinct words (including nonsense ones) produced by the letters in MISSISSIPPI have the property that there is no consecutive I's.

2. (a) Let n, r be positive integers and $n \geq r$. Give a combinatorial proof of

$$\binom{r}{r} + \binom{r+1}{r} + ... + \binom{n}{r} = \binom{n+1}{r+1}.$$

(b) Use (a) to compute $\sum_{i=1}^{n} i^2$.

3. Let n, r be be positive integers and $n \geq r$. Give a combinatorial proof of

$$\binom{2n}{2r} \equiv \binom{n}{r} (mod\ 2).$$

4. How many integer solutions of the inequality

$$x_1 + x_2 + x_3 \leq 25$$

satisfy that $2 \leq x_1 \leq 7, x_2 \geq 0, x_3 \geq 0$?

5. Let n be a positive integer. Prove that

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k,$$

where $S(n,k)$ is the Strirling number of the second kind and $(x)_k = x(x-1)...(x-k+1)$.

Hint: first consider the case when x is a positive integer and treat this equation as counting mappings in two ways.

6. Let p be a prime and n be a positive integer. Find the largest integer k such that $p^k | n!$. Express such k as a function of p and n. (The notation $a|b$ means that a divides b.)

7. How many ways are there to distribute 30 identical balls among 3 boys and 3 girls if each boy should get an odd number of balls and each girl should get at least 2 balls? Express the answer as a coefficient of a suitable power of x in a suitable product of polynomials.