21-301A Combinatorics
Week 4

Lec 8 (M). Inclusion-Exclusion and its application

Let $A_1, ..., A_n$ be n subsets of the groud set Ω.

- Definition. Let $A_\emptyset = \Omega$; and for any nonempty subset $I \subseteq [n]$, let

$$A_I = \cap_{i \in I} A_i.$$

For any integer $k \geq 0$, write

$$S_k = \sum_{I \in \binom{[n]}{k}} |A_I|$$

to be the sum of the sizes of all k-fold intersections.

- **Inclusion-Exclusion formula.**

$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} S_k.$$

- Definition. Let $\varphi(n)$ be the number of integers $m \in [n]$ which are relatively prime to n. Here, m is relatively prime to n means that the greatest common divisor of m and n is 1.

- **Fact:** If n can be written as $n = p_1^{a_1} p_2^{a_2} ... p_t^{a_t}$, where $p_1, ..., p_t$ are primes in $[n]$, then

$$\varphi(n) = n \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right).$$

We proved this in Week 3 by considering $\Omega = [n]$ and the sets $A_i = \{m \in [n] : p_i | m\}$ for $i = 1, ..., t$. Note that $\varphi(n) = \left|\Omega \backslash \left(\cup_{i=1}^{t} A_i\right)\right|$

- Definition. A permutation $\sigma : X \to X$ is called a **derangement** of X if $\sigma(i) \neq i$ for any $i \in X$. We use D_n to denote the set of all derangements of $[n]$.

- **Fact:** For any integer $n \geq 1$,

$$|D_n| = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

We apply inclusion-exclusion by letting $A_i = \{\sigma | \sigma(i) = i\}$ for $i = 1, ..., n$.

- **Fact:** $|D_n| \sim \frac{n!}{e}$.

It is beacuse $\lim_{n \to \infty} \frac{|D_n|}{(n!/e)} = e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$.

- Recall. (i) $S(n, k)$ is the number of partitions of a set of size n into k nonempty parts.

(ii) $S(n, k)k!$ is the number of surjective functions from Y to X, where $|Y| = n$ and $|X| = k$.

- **Fact:**
$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$
In its proof, we are using inclusion-exclusion (again!) by considering $\Omega = X^Y$ and its subsets $A_i := \{f : Y \to X\backslash\{i\}\}$.

Lec 9 (W). Generating function: recurrence relation

- We review the inclusion-exclusion formula and mention the following peroperty.

  Exercise. If p is odd integer, then
$$|A_1 \cup A_2 \cup ... \cup A_n| \le \sum_{k=1}^{p}(-1)^{k-1}S_k;$$
  if p is even, then
$$|A_1 \cup A_2 \cup ... \cup A_n| \ge \sum_{k=1}^{p}(-1)^{k-1}S_k.$$

- Recall that if $f(x) = \prod_{i=1}^{k} f_i(x)$ for polynomials $f_1, ..., f_k$, then
$$[x^n]f = \sum_{i_1+i_2+...+i_k=n}\prod_{j=1}^{k}\left([x^{i_j}]f_j\right).$$

- **Definition.** The ordinary generating function (or G.F.) for sequence $a_0, a_1, ...$ is
$$f(x) = \sum_{n\ge 0} a_n x^n.$$
  We have two ways to view the generating function.

  (i). When the power series $\sum_{n\ge0} a_n x^n$ converges (i.e., there exists a radius $R > 0$ of convergence), we view G.F. as a function of x and we can apply operations of calculus on it, including differentiation and integration. For example, in this case we know that
$$a_n = \frac{f^{(n)}(0)}{n!}.$$
  Also racall the following sufficient condition on the radius of convergence that if $|a_n| \le K^n$ for some constant $K > 0$, then $\sum_{n\ge0} a_n x^n$ converges in the interval $(-K, K)$.

  (ii). When we are not sure of the convergence, we view G.F. as a formal object with additions and multiplications. Let $a(x) = \sum_{n\ge0} a_n x^n$ and $b(x) = \sum_{n\ge0} b_n x^n$.

  Addition.
$$a(x) + b(x) = \sum_{n\ge0}(a_n + b_n)x^n.$$
  Multiplication. Let $a(x)b(x) = \sum_{n\ge0} c_n x^n$, where
$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

- Problem. Let $a_0 = 1$ and $a_n = 2a_{n-1}$ for $n \geq 1$. Find a_n.

  We use the menthod of generating function. Let $f(x) = \sum a_n x^n$. Then we show $f(x) = 1 + 2xf(x)$, which implies that $f(x) = \sum 2^n x^n$ and therfore $a_n = 2^n$.

- We see the basic idea of using generating function from the previous problem: instead of working on the sequence $\{a_n\}$, we look at its G.F.; after obtaining the expression of G.F., we get a_n by expanding $f(x)$ as $\sum a_n x^n$.

Lec 10 (F). Recurrence relation (II) and the Newton's binomial theorem

- Problem. Let A_n be the set of strings of length n with entries from the set $\{a, b, c\}$ and with NO "aa" occuring (in the consecutive positions). Find $a_n = |A_n|$ for $n \geq 1$.

- We first observe that $a_1 = 3, a_2 = 8$ and for any $n \geq 2$

  $$a_n = 2a_{n-1} + 2a_{n-2},$$

  therefore $a_0 = 1$. Let $f(x) = \sum_{n\geq 0} a_n x^n$. Then we use the recurrence relation to get

  $$f(x) = 1 + x + 2x(f(x) - 1) + 2x^2 f(x),$$

  which implies that

  $$f(x) = \frac{1+x}{1-2x-2x^2}.$$

  By Partial Fraction Decomposition, we calculate that

  $$f(x) = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1+2x} + \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1-2x},$$

  which implies that

  $$a_n = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1} \left(\frac{-2}{\sqrt{3}+1}\right)^n + \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1} \left(\frac{2}{\sqrt{3}-1}\right)^n.$$

  Note that a_n must be an integer but its expression is of a combination of irrational terms! Observe that $\left|\frac{-2}{\sqrt{3}+1}\right| < 1$, so $\left(\frac{-2}{\sqrt{3}+1}\right)^n \to 0$ as $n \to \infty$. Thus, when n is sufficiently large, a_n is about the value of the second term $\frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1} \left(\frac{2}{\sqrt{3}-1}\right)^n$; equivalently a_n will be the nearest integer to that.

- Definition. For any real r and integer $k \geq 0$, let

  $$\binom{r}{k} = \frac{r(r-1)...(r-k+1)}{k!}.$$

- **Newton's Binomial Theorem.** For any real r,

  $$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

  holds for any $x \in (-1, 1)$.

  The proof is using Taylor series which we did not cover. Note that the Binomial Theorem says that for positive integer n, $(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$ holds for any real x!

- Corollary. Let $r = -n$ for integer $n \geq 0$. Then $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$. Therefore

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k.$$

- We count the number of triangulations of n-gon in next lecture.