21-301 Assignment 05

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We know that a maximal chain of $[n]$ contains $n + 1$ sets, and there is exactly one set of every size from 0 to n.

We know that the sizes of A_i strictly increase in the order they are given (from 1 to k). That is that A_1 is the smallest and A_k is the largest A_i.

We can construct all maximal chains of $[n]$ containing all A_i by the following method:

Let your chain contain all A_i. This is a chain given the subset relationship of all the A_i. Notice that this chain may not be maximal if there is not a set of every size from 0 to n. We know that we have sets of sizes $|A_1|, |A_2|, ..., |A_k|$.

1. Now, add a maximal chain of 2^{A_1}.
2. Then find $|A_j| - |A_i|$ sets in $2^{[x]}$ of all sizes strictly between $(|A_i|, |A_j|)$ such that they form a chain, and add them to the chain.
3. Finally find $n - |A_k|$ sets in $2^{[x]}$ of all sizes strictly between $(|A_k|, n)$ such that they form a chain, and add them to the chain.

Note that by this method, you can form any maximal chain of $2^{[x]}$ which contains all the A_i. Now we count how many distinct outcomes there are of performing the above procedure.

1. There are $|A_1|!$ ways of doing step 1.
2. There are $\prod_{i,j|j-i=1}(|A_j| - |A_i|)!$ ways of doing step 2. This is because you can choose from $(|A_j| - |A_i|)$ elements for the first set you add, one fewer element for the next set, up til there is only one way to form the last set (then you have A_j).
3. There are $(n - |A_k|)!$ ways of doing step 3 by the same logic.

The product of those three quantities gives the answer:

$$|A_1|! \prod_{i,j|j-i=1} (|A_j| - |A_i|)!(n - |A_k|)!$$

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We observe that the size of intersection of $\mathcal{F}$ and any maximal chain of $2^{[n]}$, is at most 2, since if there were 3 sets in common with the chain, those 3 sets would form a chain which would break the constraint on $\mathcal{F}$ given in the problem. With 2, you do not break the constraint.

The total number of sets in $\mathcal{F}$ that also exist in some maximal chain is at most $2n!$ (two in a maximal chain, and there are $n!$).

We can also count this quantity by counting the number of maximal chains containing A for all $A \in \mathcal{F}$. This quantity is $\sum_{A\in\mathcal{F}} |A|!(n-|A|)!$, as shown in the proof of Sperner's Theorem in lecture.

Then, we do some algebra:

$$1 \geq \sum_{A\in\mathcal{F}} \frac{|A|!(n-|A|)!}{2n!} = \sum_{A\in\mathcal{F}} \frac{1}{2\binom{n}{\lfloor\frac{n}{2}\rfloor}} = \frac{\mathcal{F}}{2\binom{n}{\lfloor\frac{n}{2}\rfloor}}$$

Which implies that $\mathcal{F} \geq 2\binom{n}{\lfloor\frac{n}{2}\rfloor}$ and since n is even, $\mathcal{F} \geq 2\binom{n}{\frac{n}{2}}$.

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Consider the n-length sequence where if n is even, every element is 1, but if n is odd, $n-2$ elements are 1, 1 element is -1, and 1 element is 1.5.

Proof:

Case n is even:
All numbers in the sequence are 1. Choose half of them to get $\epsilon = -1$, and naturally the other half get $\epsilon = 1$. Then the dot product of the epsilon vector and the sequence will be 0. There are $\binom{n}{\frac{n}{2}}$ ways to do this.

Case n is odd:
Put the $n-2$ 1s and the -1 in a bag. Note that this forms $n-1$ elements, which is even since n is odd. If you choose half of them to get $\epsilon = -1$ and the other half to get $\epsilon = 1$, the sum will either be 1 or -1. If the sum is 1, choose the 1.5 to be multipled by $\epsilon = -1$. Else, $\epsilon = 1$. In the first case, the sum will be $1 - 1.5 = -0.5$, and in the second case, the sum will be $-1 + 1.5 = 0.5$. There are $\binom{n}{\frac{n-1}{2}}$ ways to construct this ϵ vector, and since n is odd, this is equal to $\binom{n}{\lfloor\frac{n}{2}\rfloor}$ ways.

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Since every set has a nonempty intersection with every other set, we realize that there must be at least 1 element which is common to all sets in $\mathcal{F}$. Then to construct an intersecting family, we have at most the freedom to decide for $n-1$ elements whether or not they are in a set in $\mathcal{F}$. In other words, at the very least, one element's fate is decided (it is in every set by default). Then the upper bound on the size of an intersecting family is:

$$|\mathcal{F}| \leq 1 \prod_{i=2}^{n} 2 = 2^{n-1}$$

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Consider a bipartite graph (A, B) where A is the set of n elements and B is the set of $S_1 ... S_n$. Notice that $|A| + |B| = 2n$. An edge exists from a vertex in A to B if an element in A is a member of a subset S_i in B. Notice that the degree of a vertex representing S_i in B is equal to $|S_i|$, since it is the number of incoming edges, or number of elements which are members of S_i.

The problem states that two distinct S_i, S_j may not have an intersection of more than one element. In our bipartite graph, if S_i, S_j in B had an intersection of their neighbor sets of two vertexes in A, it would look like $K_{2,2}$, and it would violate the problem statement. Therefore, our bipartite graph does not have $K_{2,2}$.

We know from lecture that the maximum number of edges in a bipartite graph that does not have $K_{2,2}$ is $|V|^{3/2} + |V|$. We'll write this in terms of n and label constants for sake of clarity.

$$|E| \leq (2n)^{3/2} + 2n = c_1 n^{3/2} + c_2 n = n(c_3\sqrt{n} + c_2)$$

We wish to find an upper bound on the degree of the smallest S_i in B, because doing so would prove that an S_i exists such that the claim in the problem is true. To maximize the degree of the smallest S_i, we consider the graph with the maximum possible number of edges such that $|E| = n(c_3\sqrt{n} + c_2)$, and we distribute the edges evenly accross vertexes in B. In the above scenario, each vertex in B would have degree $\frac{n(c_3\sqrt{n}+c_2)}{n} = c_3\sqrt{n} + c_2$. Thus we've shown that the degree of the minimum-degree vertex is upper-bounded by $c_3\sqrt{n} + c_2$.

Therefore, in any scenario such as the one given in the question, there will always exist an S_i such that $d(S_i) = |S_i| \leq c_3\sqrt{n} + c_2$. Since n is a positive integer, we can do the following algebra:

$$|S_i| \leq c_3\sqrt{n} + c_2 \leq c_3\sqrt{n} + c_2\sqrt{n} = c_4\sqrt{n} = C\sqrt{n}$$

Thus, we have proven the claim.

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