21-301 Assignment 05

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We know that a maximal chain of $[n]$ contains $n+1$ sets, and there is exactly one set of every size from 0 to n.

We know that the sizes of A_i strictly increase in the order they are given (from 1 to k). That is that A_1 is the smallest and A_k is the largest A_i.

We can construct all maximal chains of $[n]$ containing all A_i by the following method:

Let your chain contain all A_i. This is a chain given the subset relationship of all the A_i. Notice that this chain may not be maximal if there is not a set of every size from 0 to n. We know that we have sets of sizes $|A_1|, |A_2|, ..., |A_k|$.

1. Now, add a maximal chain of 2^{A_1}.
2. Then find $|A_j| - |A_i|$ sets in $2^{[x]}$ of all sizes strictly between $(|A_i|, |A_j|)$ such that they form a chain, and add them to the chain.
3. Finally find $n - |A_k|$ sets in $2^{[x]}$ of all sizes strictly between $(|A_k|, n)$ such that they form a chain, and add them to the chain.

Note that by this method, you can form any maximal chain of $2^{[x]}$ which contains all the A_i. Now we count how many distinct outcomes there are of performing the above procedure.

1. There are $|A_1|!$ ways of doing step 1.
2. There are $\prod_{i,j|j-i=1}(|A_j| - |A_i|)!$ ways of doing step 2. This is because you can choose from $(|A_j| - |A_i|)$ elements for the first set you add, one fewer element for the next set, up til there is only one way to form the last set (then you have A_j).
3. There are $(n - |A_k|)!$ ways of doing step 3 by the same logic.

The product of those three quantities gives the answer:

$$|A_1|! \prod_{i,j|j-i=1} (|A_j| - |A_i|)!(n - |A_k|)!$$

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We observe that the size of intersection of $\mathcal{F}$ and any maximal chain of $2^{[n]}$, is at most 2, since if there were 3 sets in common with the chain, those 3 sets would form a chain which would break the constraint on $\mathcal{F}$ given in the problem. With 2, you do not break the constraint.

The total number of sets in $\mathcal{F}$ that also exist in some maximal chain is at most $2n!$ (two in a maximal chain, and there are $n!$).

We can also count this quantity by counting the number of maximal chains containing A for all $A \in \mathcal{F}$. This quantity is $\sum_{A\in\mathcal{F}} |A|!(n-|A|)!$, as shown in the proof of Sperner's Theorem in lecture.

Then, we do some algebra:

$$1 \geq \sum_{A\in\mathcal{F}} \frac{|A|!(n-|A|)!}{2n!} = \sum_{A\in\mathcal{F}} \frac{1}{2\binom{n}{\lfloor\frac{n}{2}\rfloor}} = \frac{\mathcal{F}}{2\binom{n}{\lfloor\frac{n}{2}\rfloor}}$$

Which implies that $\mathcal{F} \geq 2\binom{n}{\lfloor\frac{n}{2}\rfloor}$ and since n is even, $\mathcal{F} \geq 2\binom{n}{\frac{n}{2}}$.

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Since every set has a nonempty intersection with every other set, we realize that there must be at least 1 element which is common to all sets in $\mathcal{F}$. Then to construct an intersecting family, we have at most the freedom to decide for $n-1$ elements whether or not they are in a set in $\mathcal{F}$. In other words, at the very least, one element's fate is decided (it is in every set by default). Then the upper bound on the size of an intersecting family is:

$$|\mathcal{F}| \leq 1\prod_{i=2}^{n} 2 = 2^{n-1}$$

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