21-301A Combinatorics
Week 5

Lec 11 (M). Catalan number: counting the number of triangulations of n-gon

- Let n-gon be a polygon with n corners, labelled as corner 1, corner 2,..., corner n.
- Definition. A triangulation of the n-gon is a way to add lines between corners to make triangles such that these lines do not cross inside of the polygon.
- Let b_{n-1} be the number of triangulations of the n-gon, for $n \geq 3$. It is not hard to see that $b_2 = 1, b_3 = 2, b_4 = 5$.
- We consider the triangle T for each triangluation of n-gon which contains corners 1 and 2. The triangle T should contain a third corner, say i. Since $3 \leq i \leq n$, we can divide the set of triangulations of n-gon into that many cases.

  (1). If $i = 3$ or n, the triangle T divides the n-gon into triangle T itself plus a $(n-1)$-gon, which results in $b_{n-2)}$ triangulations of n-gon.

  (2). For $4 \leq i \leq n-1$, the triangle T divides the n-gon into three regions: a $(n-i+2)$-gon, triangle T and a $(i-1)$-gon, therefore it results in $b_{i-2} \times b_{n-i+1}$ many triangulations of n-gon. Therefore, combining (1) and (2), we get that

$$b_{n-1} = b_{n-2} + \sum_{i=4}^{n-1} b_{i-2}b_{n-i+1} + b_{n-2} = b_{n-2} + \sum_{j=2}^{n-3} b_j b_{n-j-1} + b_{n-2}$$

  By letting $b_0 = 0$ and $b_1 = 1$, we get

$$b_{n-1} = \sum_{j=0}^{n-1} b_j b_{n-1-j} \quad \text{or} \quad b_k = \sum_{j=0}^{k} b_j b_{k-j} \quad \text{for} \quad k \geq 2.$$

- Let $f(x) = \sum_{k\geq 0} b_k x^k$. Note that $f^2(x) = \sum_{k\geq 0} \left(\sum_{j=0}^{k} b_j b_{k-j}\right) x^k$. Therefore

$$f(x) = x + \sum_{k\geq 2} \left(\sum_{j=0}^{k} b_j b_{k-j}\right) x^k = x + \sum_{k\geq 0} \left(\sum_{j=0}^{k} b_j b_{k-j}\right) x^k = x + f^2(x).$$

- Solving $f^2(x) - f(x) + x = 0$, we get that $f(x) = \frac{1+\sqrt{1-4x}}{2}$ or $\frac{1-\sqrt{1-4x}}{2}$. But notice that $f(0) = 0$, so it has to be the case that

$$f(x) = \frac{1-\sqrt{1-4x}}{2}.$$

- Next, we apply the Newton's binomail theorem to get that

$$f(x) = \frac{1}{2} - \frac{1}{2} \sum_{k\geq 0} \binom{\frac{1}{2}}{k} (-4x)^k,$$

  where $\binom{\frac{1}{2}}{0} = 1$ and for $k \geq 1$, $\binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}2}{4^k} \cdot \frac{(2k-2)!}{k!(k-1)!}$.

- **Fact.** For $k \geq 1$, $b_k = \frac{1}{k}\binom{2k-2}{k-1}$.
- **Fact.** The total number of triangulations of the $(n+2)$-gon is $\frac{1}{n+1}\binom{2n}{n}$, which is also called the nth Catalan number.

Lec 9 (W). Exponential generating function

Let $\mathbb{N}, \mathbb{N}_e$ and $\mathbb{N}_o$ be the sets of nonnegative integers, nonnegative even integers and nonnegative odd integers, respectively.

- Recall that the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$ is $f(x) = \sum_{n\geq 0} a_n x^n$.
- Problem 1. Let S_n be the number of selections of n letters chosen from an unlimited supply of a's, b's and c's such that both of the numbers of a's and b's are even. Let

$$f(x) = \left(\sum_{i\in\mathbb{N}_e} x^i\right)^2 \left(\sum_{j\in\mathbb{N}} x^j\right).$$

  We have known that

$$S_n = \sum_{e_1+e_2+e_3=n,\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}} 1 = [x^n]f(x).$$

- **Definition.** The exponential generating function for the sequence $\{a_n\}_{n\geq 0}$ is defined as

$$f(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

- Problem 2. Let T_n be the number of words (or angements) of n letters chosen from an unlimited supply of a's, b's and c's such that both of the numbers of a's and b's are even. What is the value of T_n?
- **Fact.** If we have n letters including x a's, y b's and z c's (i.e. $x+y+z=n$), then we can form $\frac{n!}{x!y!z!}$ distinct words using these n letters.
- **Fact.**

$$T_n = \sum_{e_1+e_2+e_3=n,\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}} \frac{n!}{e_1!e_2!e_3!}.$$

  To see this, we notice the following two claims. Each selection of n letters (say x a's, y b's and z c's) cooresponds to a solution of the eqution

$$e_1+e_2+e_3=n,$$

  where $e_1, e_2 \in \mathbb{N}_e,\ e_3 \in \mathbb{N}$; each solution, say (e_1, e_2, e_3), should contribute $\frac{n!}{e_1!e_2!e_3!}$ to the count of the number of words.

- **Fact.** Let

$$g(x) = \left(\sum_{i\in\mathbb{N}_e} \frac{x^i}{i!}\right)^2 \left(\sum_{j\in\mathbb{N}} \frac{x^j}{j!}\right),$$

then $g(x)$ is the exponential generating function of the sequence $\{T_n\}$, therefore $T_n = n! \cdot ([x^n]g(x))$.

In its proof, we expand $g(x)$ to get its term of x^n, which is

$$\sum_{e_1+e_2+e_3=n,\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}} \frac{x^{e_1}}{e_1!}\cdot\frac{x^{e_2}}{e_2!}\cdot\frac{x^{e_3}}{e_3!} = \left(\sum_{e_1+e_2+e_3=n,\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}} \frac{n!}{e_1!e_2!e_3!}\right)\frac{x^n}{n!} = T_n\cdot\frac{x^n}{n!}$$

- **Fact.** Two Taylor series: $e^x = \sum_{j\geq 0}\frac{x^j}{j!}$ and $e^{-x} = \sum_{j\geq 0}\frac{(-1)^j}{j!}x^j$ imply that

$$\frac{e^x+e^{-x}}{2} = \sum_{j\in\mathbb{N}_e}\frac{x^j}{j!} \quad \text{and} \quad \frac{e^x-e^{-x}}{2} = \sum_{j\in\mathbb{N}_o}\frac{x^j}{j!}.$$

- **Fact.** Using the previous facts, we get

$$T_n = \frac{3^n+2+(-1)^n}{4}.$$

- Ordinary G.F. can be used to find the number of selections; while exponential G.F. can be used to find the number of arragments or some combinatorial objects **involving ordering**, see the next example.

- Problem 3. Find the number a_n of ways to send n (distinct) students to 3 ordered classrooms (R1,R2,R3) such that each room has at least 1 students.

We show that

$$\sum_{n\geq 0} a_n\cdot\frac{x^n}{n!} = \left(\sum_{k\geq 1}\frac{x^k}{k!}\right)^3 = (e^x-1)^3 = \ldots$$

- **A general formula.** Let $f_i(x) = \sum_{n\geq 0} a_n^{(i)}\frac{x^n}{n!}$ for $1\leq i\leq k$ and let $f(x) = \sum_{n\geq 0} A_n\frac{x^n}{n!}$. Then $f(x) = \prod_{j=1}^k f_j(x)$ if and only if for any $n\geq 0$

$$A_n = \sum_{i_1+\ldots+i_k=n}\frac{n!}{i_1!i_2!\ldots i_k!}\cdot\left(\prod_{j=1}^k a_{i_j}^{(j)}\right).$$

- **We finish Sections 12.1-12.4 from textbook.**

Lec 13 (F). Basic of Graphs

- **Definition.** A graph $G = (V, E)$ consists of a finite set V of vertices (V is called the vertex set) and a set E of edges (E is called the edge set) such that $E \subseteq \{$unordered pairs $(u, v) : u, v \in V\} = V \times V$.

- In our class, we only consider the *simple graphs*, which contain NO loops and NO multi-edges.

- We say vertices i and j are *adjacent* in graph G if (i, j) is an edge of G; and express it as $i \sim_G j$. We also say that edge (i, j) is *incident* to vertices i and j

- Let $e(G)$ be the number of edges in graph $G = (V, E)$, i.e., $e(G) = |E|$.

- Definition. If the edges are *ordered pairs* of vertices, then $G = (V, E)$ is called *directed graph or digraph.* To specify the difference, we call the graphs with unordered edges as undirected graphs or simply just graphs.

- **Notice.** In our class, we always assume that the edges of graphs are unordered pairs (unless specified)!

- The *degree* of a vertex v in graph G, denoted as $d(v)$, is the number of edges in G incident to vertex v.

- The *neighborhood* of vertex v of graph G, denoted as $N(v)$ or $N_G(v)$, is the set of vertices which are adjacent to v. So $|N(v)| = d(v)$.

- A *subgraph* $G' = (V', E')$ of a graph $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$.

- A graph with n vertices is called a *complete graph* or a *clique*, denoted as K_n, if all pairs of vertices are adjacent. So $e(K_n) = \binom{n}{2}$.

- A graph with n vertices is called an *independent set*, denoted as I_n, if it contains NO edge at all.

- Given graph $G = (V, E)$, its *complement* is a graph $\overline{G} = (V, \overline{E})$ with the same vertex set V and the complement $\overline{E}$ of E as its edge set, i.e., $\overline{E} = V \times V - E$.

  We see that $e(G) + e(\overline{G}) = \binom{n}{2}$ for any graph G with n vertices.

- **Definition.** Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there exist a one-to-one mapping $f : V \to V'$ such that $i \sim_G j$ if and only if $f(i) \sim_{G'} f(j)$.

  We took two different drawnings of Peterson graph as an example.

- The *degree sequence* of graph $G = (V, E)$ is a sequence of degrees of all vertices in a increasing order.

- Exercise. If two graphs have the same degree sequence, then are they isomorphic? Answer is NO! And there are infinitely many couterexamples. Can you find one?

- **Definition.** A graph is called *bipartite*, if its vertices can be partitioned into two sets V_1 and V_2 such that any edge joints one in V_1 with another in V_2; equivalently, this means that there is no edge inside each V_i for $i = 1, 2$.

- A *path* of length $k-1$ is a sequence of veritces $v_1, v_2, ..., v_k$ such that (v_i, v_{i+1}) is an edge for all $i = 1, 2, ..., k-1$. Note that the length of path means the number of edges it contains.

- A *cycle* of length k is a sequence of veritces $v_1, v_2, ..., v_k$ such that (v_i, v_{i+1}) is an edge for all $i = 1, 2, ..., k-1$ and moreover (v_k, v_1) is an edge.

- **Shirking Hands Lemma.** In any graph $G = (V, E)$,

$$\sum_{v \in V} d(v) = 2e(G).$$

  In its proof, we do the double counting for the size of set

$$F = \{(e, v) : \text{edge } e \text{ is incident to vertex } v\}.$$

  Note that

$$2e(G) = \sum_{e \in E} (\text{\# of pairs (e,v) with edge e}) = |F| = \sum_{v \in V} (\text{\# of pairs (e,v) with vetex v}) = \sum_{v \in V} d(v).$$