15-251: Great Theoretical Ideas In Computer Science

Homework 2 (due Thursday February 2, 11:59pm)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. In particular, results from other courses, including the textbook Mathematical Thinking by D'Angelo and West, may not be cited directly. Unless otherwise specified, all answers are expected to be given in closed form.*

0. Notice we start at Problem Zero (15 points)

(a) **(5 points)** Let A be a sequence defined by $a_0 = 1, a_1 = 4$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Find a closed form for a_n and prove it by induction on n.

(b) **(10 points)** Let S be the sentence in propostional logic "$(((x \wedge y) \to z) \to (\neg y \to \neg z))$". Prove S is satisfiable. Is it valid?

1. Theorem? I hardly knew him! (15 points)

Consider an axiomatic system where the vocabulary is strings using the two characters '|' and '\$'. The only axiom is $|\$$. The deduction rules are

END If A is a theorem, then $A|$ is a theorem

SURROUND If A and B are theorems, then $|AB|$ is a theorem

(a) **(5 points)** Prove or disprove: $|||\$|\$||\$|||$ is a theorem.

(b) **(10 points)** Prove or disprove: $||\$|||\$\$|$ is a theorem.

2. Teaching Assistant Assistance (15 points)

(a) **(7 points)** Let p and q be propositional variables. Let's define a new connective called $?$. It's definition is that $x?y$ is True unless both x and y are True. Adam likes to do research on interesting questions like the Riemann Hypothesis, P vs. NP, and why the Math Department still thinks zero isn't a natural number, so it comes as no surprise he really likes the ? operator. He realizes that $x?x$ is true if x is false, and it's false if x is true, just like the $\neg$ connective. For each of the formulas $x \wedge y$, $x \vee y$, $x \to y$ and $x \leftrightarrow y$, help Adam find an equivalent formula that *only* uses the $?$ connective. For full credit, make your formulas as short as possible.

(b) **(8 points)** Ever since Tim recovered his memory, he's been worrying students won't forgive his paranoia-induced meanness and students will like all of the other TAs more than him. Being an accomplished logician, Tim wants to model his connundrum. Consider first order logic with the following vocabulary: One constant-name **Tim**, no function-names, and three relation-names, IsTA$(\cdot)$, IsStudent$(\cdot)$ and Prefers$(\cdot, \cdot, \cdot)$ where Prefers(x, y, z) means x prefers y to z. Let L be the statement:

$$(\forall x \ \text{IsStudent}(x) \rightarrow (\forall y \ \text{IsTA}(y) \rightarrow \text{Prefers}(x, y, \textbf{Tim})))$$

Cheer up Tim by proving L is *not* valid. Then give a first-order "sentence" in this vocabulary that expresses "There are students who prefer Tim to another TA."

3. Dragon Chess (15 points)

On the first night of the puzzle hunt, Mark was playing chess in Skibo. After handily defeating three Russian grandmasters, he was looking for a new challenge. Sang was bored because no one had found him, so to celebrate the Chinese New Year and the Year of the Dragon, they design a new game.

The game is played on an $n \times n$ chess board and has one piece: the dragon. Dragons are mighty creatures that do not feel threatened unless attacked not only from the north or south, but also, simultaneously, from the east or west. In other words, we say a dragon is "threatened" if and only if there is another dragon in the same row and also a third dragon in the same column.

Let D_n be the greatest number of dragons that can be placed on an $n \times n$ chess board such that no dragon is threatened. For example, $D_2 = 2$: no two dragons can threaten each other, but when three dragons are placed on a 2×2 board, exactly one of them will be threatened. Give upper and lower bounds for D_n. For full credit, your bounds should match.

4. Post Puzzle Pancake Party (15 points)

To celebrate the end of the puzzle hunt, Jasmine and JD went to Resnik Cafe for some pancakes. When their order of n pancakes arrived, there were two sizes of pancakes, large and small. They pulled out their trusty flipping spatula in an effort to get all of the small pancakes on top of all of the big pancakes. The evil Resnik Cafe chefs stacked the pancakes so that Jasmine and JD would need the largest possible number of flips, T_n. Prove that $T_n = n - 1$ for all $n \geq 1$.

5. The Ghost of Puzzle Hunts Past (15 points)

Shashank and Andrew were reminiscing about the puzzle hunts they did when they took 251 many years ago. Shashank was showing Andrew his old shell when they came across a problem called `secret`. Andrew asked Shashank what `secret` did, but Shashank said he'd never figured it out. This is what the program looked like:

```
input x
base = 2
while x > 0:
    print x
    n = x / base #Integer division
    m = x % base
    base = 2 * base
    x = n * base + m - 1
```

To try it out, they entered 3, and the program output $3\ 4\ 7\ 6\ 5\ 4\ 3\ 2\ 1$.

Prove that `secret` terminates on all natural numbers.