# 15-451 Assignment 03

Karan Sikka
ksikka@cmu.edu
Collaborated with Dave Cummings and Sandeep Rao.
Recitation: A
September 18, 2014

---

### 1: Balanced Splay Trees

---

The outline of the proof is as follows:

1. We show a lower bound on the decrease in potential when performing a deep splay, $3lg(n)+1$.
2. We show an upper bound on the increase in potential when performing a splay, $3lg(n)+1$.
3. From these facts, we conclude that since we start with a perfectly balanced tree (min potential), a deep splay can only occur at most once after a splay, due to restrictions on the potential function.

**(1)** Lower bound on decrease in potential during deep splay.

From the access lemma, we know that:

$$\text{a.c. of splay} \leq 3(r(y) - r(x)) + 1$$

Where $y$ is the root node and $x$ is the node being splayed.
We know that $r(y) = lg(n)$ and in the case of a deep splay, $r(x) = 0$ since it's terminal, subtree size is 1, $lg(1) = 0$.

We also know the definition of amortized cost in terms of the potential function. Then in the case of a deep splay,

$$\# \text{ of rots} + \Delta\Phi \leq 3lg(n) + 1$$

In the case of a deep splay, the number of rotations is $d - 1$ since a rotation moves the terminal node 1 level closer to the root.
And a deep splay only occurs when the tree is off-balance which means that $d \geq 6lg(n) + 3$. Combining these facts, we do the following algebra:

$d - 1 + \Delta\Phi \leq 3lg(n) + 1$
$\implies \Delta\Phi \leq 3lg(n) + 1 - (d - 1)$
$\implies \Delta\Phi \leq 3lg(n) + 1 - (6lg(n) + 3 - 1)$
$\implies \Delta\Phi \leq -(3lg(n) - 1)$

**(2)** Upper bound on increase in potential of a splay.

Combine the access lemma with the defn of amortized cost to establish the following truth:

$$\Delta\Phi \leq 3(r(y) - r(x)) + 1 - \# \text{ of rots}$$

We know that $r(y) = lg(n)$. We know $r(x) \geq 0$ and same with the number of rotations, so we can omit them from our upper bound. Thus, we obtain the following.

$$\Delta\Phi \leq 3lg(n) + 1$$

**(3)** We start with a perfectly balanced tree, so that the potential function is at its minimum. From there, a deep splay can only occur after a splay, and it can only occur once per splay, otherwise the potential function would go lower than it was when the tree was perfectly balanced.

## 2: Universal Hashing

**(a.i.)** No it is not universal.

Consider the case when $M < n$ and consider two strings which vary only in their $M^{th}$ character.

When you compute the hash function for these two strings using any hash function in the family, you'll get identical hash values.

This is because the only differing term in the sums formed by the hash computations, representing the $M^{th}$ character, will have a coefficient of $M$ and $M \mod M$ is zero. So the different term can be eliminated and the hash functions are guaranteed to output identical values.

Thus the probability of collision (over all hash functions in the family) on these two keys is 1, and $1 > \frac{1}{M}$.

**(a.ii.)** It is not 2-universal, because if it were, then it would also be universal, but as we showed in part i, it's not.

**(b.i.)** $h_i = \sum_{k=1}^{n} T_i[k][s_i] \mod M$