15-251: Great Theoretical Ideas in Computer Science

Homework 3 (due Thursday, February 09)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. Unless otherwise specified, all answers are expected to be given in closed form.*

0. The World is Ending (14 points)

The earthquakes have placed the world in ruin. It seems as if mother nature is playing a game with us.

The Earth's surface is an $n \times m$ grid of points in a rectangle where adjacent points in the same row or column are at a distance of 1 apart. Each day, an earthquake occurs along a straight fault line between two points whose distance is either 1 or $\sqrt{2}$. In response, the people of Earth construct a massive straight steel beam immediately after to reinforce a line between two points on the Earth's surface. Constructing steel beams is expensive though, so the length of each steel beam is also either 1 or $\sqrt{2}$.

The chasms opened by the earthquakes are so massive that steel beams cannot be placed across them. The steel beams are so strong that no earthquake fault line can cross them. However, steel beams can cross each other and fault lines can intersect, and furthermore, fault lines and steel beams may intersect each other at their endpoints.

We know that if three fault lines ever form a triangle of area $\frac{1}{2}$ the Earth will be destroyed. We also know that if three steel beams can be placed to form a triangle of area $\frac{1}{2}$ the Earth will be stabilized and the earthquakes will stop.

Prove that if mother nature tries as hard as possible to destroy Earth then there is no way the people of Earth can stabilize it. I.e., prove that either the Earth will be destroyed or that it will be so torn up that it becomes impossible for any new earthquakes to occur or new beams to be placed.

1. The Warlords Appear (14 points)

With the world being destroyed, war for supremacy breaks out between the world's former two largest powers. The final battle plays out as follows.

The battleground is an $n \times m$ grid situated between chasms formed by the earthquakes. Each side has one soldier in each row of the grid. During each minute the two generals leading the battle will alternately direct one of their soldiers to advance or retreat any non-zero distance subject to the following rules.

- Each soldier is confined to his row.
- Two enemy soldiers cannot pass each other or be in the same place (they are all very cowardly).

- The battle ends when one of the generals can no longer direct any of his soldiers to move and is thus forced to surrender.

Prove that one of the generals is able to ensure his win and force his opponent to surrender. Find the P and N positions of this battle.

2. Civilian Life (14 points)

Things have settled down in the shelters. Trivialities like games have been largely forgotten; ironically some people use them to survive by betting on combinatorial games.

A particularly popular one is 1,2,5-Nim. The game is played with a stack of n chips like Nim, but players are allowed to take only 1, 2, or 5 chips each turn. A player loses when there are no chips left and he is unable to move.

(a) What are the Nimbers of 1,2,5-Nim?

(b) Suppose there are four stacks of chips containing 12, 6, 13, and 2 chips, respectively. Players alternate turns removing from one stack each turn. 1,2,5-Nim rules are used for the first two stacks, and regular Nim rules are used for the last two stacks. Would you want to go first or second? If you go first, what move do you make? If you go second, what move do you make if your opponent takes 5 chips from the first stack on his turn?

3. Decay of Knowledge (14 points)

With the fall of civilization came a loss of much of modern mathematics. In particular all knowledge of binary numbers were lost.

In an attempt to create a new representation of natural numbers, mathematicians of the post-apocalyptic age decided to represent numbers as sums of Fibonacci numbers instead of as sums of powers of two. Fortunately, Fibonacci numbers have many properties similar to those that make powers of two nice.

(a) Prove that any sum of Fibonacci numbers $\{F_k \mid k \geq 2\}$ containing no neighboring Fibonacci numbers with maximum element F_n is less than F_{n+1} .

(b) Prove that all positive integers can be uniquely represented as the sum of a subset of Fibonacci numbers $\{F_k \mid k \geq 2\}$ containing no neighboring Fibonacci numbers.

4. Crime Wave (14 points)

With local police forces in disarray, two rival crime syndicates have set their sights on stealing a total of $\$1,000,000,000$ from institutions around the country. So as not to alert the authorities, they agree on the following rules.

The syndicates alternate nights on which they steal. Each night a syndicate may only steal up to twice as much as was stolen the previous night but must steal at least $\$1$. Only a whole number of dollars can be stolen each night. On the first night the first syndicate steals $\$1$.

Assuming both syndicates make it their goal to steal the last dollar of the $\$1,000,000,000$, who succeeds, the first or second syndicate? Find the P and N positions for this game.
Hint 1: You'll need to use the result of the previous problem.
Hint 2: You'll probably want to do some programming to help you solve this problem.

5. Endgame (14 points)

That's it, the world really has ended.

In a last ditch effort to save humanity a spaceship was launched. Unfortunately it's headed straight toward the sun.

In their last minutes before oblivion, the last remaining humans decide to play a card game. A total of n cards are laid face up in a row. Two players alternate turns by choosing a face up card, flipping it over, and then optionally flipping the card to its immediate left. The game ends in a loss for the player who cannot make a move.

Find the Nimbers of the positions in this game.