

15-451 Assignment 07

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Recitation: A

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1: A Densely-Knit Community

(a) (b) (c) (d)

2: Large + Dense = Difficult

3: A Well-Separated Problem

(a) The problem is in NP because there exists a poly-time verifier as follows:

Define the proof of the solution as a K -element subset of X . We compute distances between each pair of distinct points and check that they are greater than or equal to Δ . If all pairs satisfy the condition, then we verify that this is a solution. Otherwise it is not.

The Well-Separated problem is in NP-Hard because the independent set decision problem which is NP-hard reduces to it. The reduction is as follows:

Independent set

Given a graph $G = (V, E)$ and integer k ,

we want to output YES

if there exists a set of vertices of size k

such that no two of them are adjacent.

To craft our input to the Well-Separated oracle,

we construct a set X from the vertices of G ,

let $K = k$,

let $\Delta = 1.25$,

for all $i \in V$ let $d(i, i) = 0$,

for all $i, j \in V : i \neq j \wedge (i, j) \in E$ let $d(i, j) = 1$

for all $i, j \in V : i \neq j \wedge (i, j) \notin E$ let $d(i, j) = 1.5$

We pass this input to the well separated problem, which will return YES

if there exists a set of elements of size K where all distances are greater than 1.25.

Observe these elements in X map directly to vertices in V which are not adjacent due to the way we constructed the input to the Well-Separated problem.

Also note that the construction of d correctly obeys the triangle inequality, because two of the shortest distances ($1 + 1$) is still greater than the longest distance (1.5).

(b) First we will present the algorithm, then prove its correctness.

Algorithm:

Call one set with separation at least $\Delta^*/2$ set C .

We will maintain a vector of all points which are potentially in C , initially containing all points.

We will also maintain a vector of points which we know are in C .

1. Pick a point u potentially in C that's not already in C , and examine how far away all other points are from it.
2. Remove all potential points not at least $\Delta^*/2$ from u from the set of points potentially in C .
3. Add u to the set of known points C .

Repeat for a total of K iterations,
resulting in a set C with K points at least $\Delta^*/2$ away from each other
since at each step we eliminate points which could invalidate this invariant.

Now we need a proof that we can perform this K times, or in other words, we never run out of points potentially in C from which to select at each iteration.

Proof:

We were allowed to assume there exists some set of K points with separation Δ^* .

For convenience, let's call these the optimal points.

Lets examine how many optimal points are eliminated from C in one iteration of the algorithm.

If we select optimal point u , we will see that all the other optimal points are at least Δ^* away from u , and they will not be eliminated from C in this iteration.

If we select non-optimal point u , we will see that at most one optimal point is less than $\Delta^*/2$ away from u .

This due to the triangle inequality. Consider a nonoptimal point u , the nearest optimal point v , and any other optimal point w .

$$\Delta^* \leq d(v, w) \leq d(v, u) + d(u, w)$$

Since v no farther from u than w , $d(v, u)$ may be less than $\Delta^*/2$, but $d(u, w)$ will certainly be at least $\Delta^*/2$.

Therefore at most one optimal point is removed from C in each iteration of the algorithm.

Therefore this algorithm can be run at least K iterations before running out of points to select.

(c) Modify the algorithm from B to return NONE if it runs out of points potentially in C that are not already in C .

Observe that the optimum separation delta may only be one of $\binom{|X|}{2}$ values: the distances $d(i, j) : i \neq j$.

Sort the values from highest to lowest, and run the algorithm with these values.

Return the none-NONE answer corresponding to the input with highest Δ^*