AM213B Numerical Methods for the Solution of Differential Equations

Lecture 16
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List of topics in this lecture

- A better entropy fix for the upwind method
- Lax-Wendroff method for conservation laws, two-step Lax-Wendroff
- Godunov method, exact solution of Riemann problem
- Framework of non-oscillating high resolution methods

Review: Numerical solution of conservation laws:

Numerical method in conservation form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right)$$

where numerical flux $F_{i+1/2}$ is constructed based on a few local cells:

$$F_{i+1/2} = F_{num}(u_{i-1}^n, u_i^n, u_{i+1}^n, u_{i+2}^n) \qquad \text{or} \qquad F_{i+1/2} = F_{num}(u_i^n, u_{i+1}^n)$$

To describe a method, we only need to specify its numerical flux.

Theorem (Lax-Wendroff):

Numerical method in conservation form + assumed convergence

==> Converging to a weak solution

FTCS method:

$$F_{i+1/2}^{\text{(FTCS)}} = \frac{1}{2}\left[F(u_{i+1}^n) + F(u_i^n)\right]$$

Lax-Friedrichs method:

$$F_{i+1/2}^{\text{(L-F)}} = \underbrace{\frac{1}{2}\left[F(u_{i+1}^n) + F(u_i^n)\right]}_{\text{Flux of FTCS}} - \underbrace{\frac{\Delta x}{2\Delta t}(u_{i+1}^n - u_i^n)}_{\text{Added viscosity term}}$$

Upwind method

$$F_{i+1/2}^{\text{(Up)}} = \underbrace{\frac{1}{2}\left[F(u_{i+1}^n) + F(u_i^n)\right]}_{\text{Flux of FTCS}} - \underbrace{\frac{1}{2}\left|\alpha(u_i^n, u_{i+1}^n)\right|(u_{i+1}^n - u_i^n)}_{\text{Added viscosity term}}$$

where α is the propagation velocity of local discontinuity/variation $[u_i^n, u_{i+1}^n]$.

$$\alpha(u_i^n, u_{i+1}^n) = \begin{cases} \dfrac{F(u_{i+1}^n) - F(u_i^n)}{u_{i+1}^n - u_i^n}, & u_i^n \neq u_{i+1}^n \\ F'(u_i^n), & u_i^n = u_{i+1}^n \end{cases}$$

This upwind method does not obey the entropy condition.

<u>An ad hoc entropy fix</u> (Harten):

We modify $|\alpha|$ to make the added viscosity strictly positive,

$$\theta(\alpha) \equiv \begin{cases} |\alpha|, & |\alpha| \geq \varepsilon \\ \dfrac{1}{2\varepsilon}(\alpha^2 + \varepsilon^2), & |\alpha| < \varepsilon \end{cases}$$

$$F_{i+1/2}^{(\text{Up})}(\text{new 1}) = \underbrace{\frac{1}{2}\Big[F(u_{i+1}^n) + F(u_i^n)\Big]}_{\text{Flux of FTCS}} - \underbrace{\frac{1}{2}\theta\Big(\alpha(u_i^n, u_{i+1}^n)\Big)(u_{i+1}^n - u_i^n)}_{\text{Added viscosity term}}$$

<u>End of review</u>

<u>A better entropy fix</u> (LeVeque):

We distinguish two cases:

- Characteristics are converging (shock wave)
- Characteristics are diverging (rarefaction wave)

We replace $|\alpha(u_i^n, u_{i+1}^n)|$ with $\psi(u_i^n, u_{i+1}^n)$, denoted concisely as $\psi_{i+1/2}$

$$\psi_{i+1/2} = \max\left\{\left|\alpha(u_i^n, u_{i+1}^n)\right|, -F'(u_i^n), F'(u_{i+1}^n)\right\}$$

- <u>When</u> the discontinuity (u_i^n, u_{i+1}^n) is a shock wave, we have

$$F'(u_i^n) > \alpha(u_i^n, u_{i+1}^n) > F'(u_{i+1}^n)$$

$$\Longrightarrow \quad \begin{cases} \alpha(u_i^n, u_{i+1}^n) > F'(u_{i+1}^n) \\ -\alpha(u_i^n, u_{i+1}^n) > -F'(u_i^n) \end{cases}$$

<u>Note:</u> If $A > B$, then $|A| > B$.

$$\Longrightarrow \quad \begin{cases} \left|\alpha(u_i^n, u_{i+1}^n)\right| > F'(u_{i+1}^n) \\ \left|-\alpha(u_i^n, u_{i+1}^n)\right| > -F'(u_i^n) \end{cases}$$

$$\Longrightarrow \quad \left|\alpha(u_i^n, u_{i+1}^n)\right| > \max\left\{-F'(u_i^n), F'(u_{i+1}^n)\right\}$$

$$==> \quad \psi_{i+1/2} = \max\left\{ \left|\alpha(u_i^n, u_{i+1}^n)\right|, \ -F'(u_i^n), \ F'(u_{i+1}^n) \right\} = \left|\alpha(u_i^n, u_{i+1}^n)\right|$$

For a real shock wave, we don't increase the numerical viscosity.

- When the discontinuity (u_i^n, u_{i+1}^n) is a rarefaction wave, we have

$$F'(u_i^n) < \alpha(u_i^n, u_{i+1}^n) < F'(u_{i+1}^n)$$

$$==> \quad \begin{cases} F'(u_{i+1}^n) > \alpha(u_i^n, u_{i+1}^n) \\ -F'(u_i^n) > -\alpha(u_i^n, u_{i+1}^n) \end{cases}$$

Note: If $B > A$ and $C > -A$, then $\max\{B, C\} > |A|$.

$$==> \quad \max\left\{ -F'(u_i^n), \ F'(u_{i+1}^n) \right\} > \left|\alpha(u_i^n, u_{i+1}^n)\right|$$

$$==> \quad \psi_{i+1/2} = \max\left\{ \left|\alpha(u_i^n, u_{i+1}^n)\right|, \ -F'(u_i^n), \ F'(u_{i+1}^n) \right\} > \left|\alpha(u_i^n, u_{i+1}^n)\right|$$

For a rarefaction wave, we increase the numerical viscosity to spread it out.

The new (version 2) numerical flux is

$$F_{i+1/2}^{(\mathrm{Up})}(\text{new } 2) = \underbrace{\frac{1}{2}\left(F(u_{i+1}^n) + F(u_i^n)\right)}_{\text{Flux of FTCS}} - \underbrace{\frac{1}{2}\psi_{i+1/2}(u_{i+1}^n - u_i^n)}_{\text{Added viscosity term}}$$

Lax-Wendroff method:

We use Taylor expansion to derive Lax-Wendroff method for $u_t + F(u)_x = 0$.

We need to keep everything in the conservation form.

$$u_i^{n+1} = u_i^n + u_t\big|_{(i,n)} \Delta t + u_{tt}\big|_{(i,n)} \frac{(\Delta t)^2}{2} + \cdots$$

$$u_t = -F(u)_x$$

$$u_{tt} = \left(-F(u)_x\right)_t = -\left(F(u)_t\right)_x = -\left(F'(u)u_t\right)_x = \left(F'(u)F(u)_x\right)_x$$

We need to discretize $F(u)_x\big|_{(i,n)}$ and $\left(F'(u)F(u)_x\right)_x\Big|_{(i,n)}$ in the conservation form.

$$F(u)_x\big|_{(i,n)} = \frac{F(u_{i+1}^n) - F(u_{i-1}^n)}{2\Delta x} = \frac{1}{\Delta x}\left[\frac{1}{2}\left(F(u_{i+1}^n) + F(u_i^n)\right) - \frac{1}{2}\left(F(u_i^n) + F(u_{i-1}^n)\right)\right]$$

$$\equiv \frac{1}{\Delta x}\left[\frac{1}{2}\left(F(u_{i+1}^n) + F(u_i^n)\right) - \underbrace{\{\quad\}\big|_{i-1/2}}_{\text{Short notation}}\right]$$

We use $F'(u)\Big|_{(i+1/2,n)} \approx \alpha(u_i^n, u_{i+1}^n)$ to discretize $\left(F'(u)F(u)_x\right)_x\Big|_{(i,n)}$

$$\begin{aligned}\left(F'(u)F(u)_x\right)_x\Big|_{(i,n)} &= \frac{1}{\Delta x}\left[\alpha(u_i^n, u_{i+1}^n)\left(\frac{F(u_{i+1}^n)-F(u_i^n)}{\Delta x}\right)-\alpha(u_{i-1}^n, u_i^n)\left(\frac{F(u_i^n)-F(u_{i-1}^n)}{\Delta x}\right)\right] \\ &\equiv \frac{1}{\Delta x}\left[\alpha(u_i^n, u_{i+1}^n)\left(\frac{F(u_{i+1}^n)-F(u_i^n)}{\Delta x}\right)-\left\{\quad\right\}\Big|_{i-1/2}\right]\end{aligned}$$

Substituting into the Taylor expansion, we obtain

$$\begin{aligned} u_i^{n+1} &= u_i^n - \Delta t\, F(u)_x\Big|_{(i,n)} + \frac{(\Delta t)^2}{2}\left(F'(u)F(u)_x\right)_x\Big|_{(i,n)} \\ &= u_i^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}\left(F(u_{i+1}^n)+F(u_i^n)\right)-\left\{\quad\right\}\Big|_{i-1/2}\right] \\ &\quad -\frac{\Delta t}{\Delta x}\left(\frac{-\Delta t}{2}\right)\left[\alpha(u_i^n, u_{i+1}^n)\left(\frac{F(u_{i+1}^n)-F(u_i^n)}{\Delta x}\right)-\left\{\quad\right\}\Big|_{i-1/2}\right] \\ &= u_i^n - \frac{\Delta t}{\Delta x}\left[\left\{\frac{1}{2}\left(F(u_{i+1}^n)+F(u_i^n)\right)-\frac{\Delta t}{2}\alpha(u_i^n, u_{i+1}^n)\left(\frac{F(u_{i+1}^n)-F(u_i^n)}{\Delta x}\right)\right\}-\left\{\quad\right\}\Big|_{i-1/2}\right]\end{aligned}$$

<u>Lax-Wendroff method</u> for $u_t + F(u)_x = 0$ is

$$F_{i+1/2}^{(\text{L-W})} = \underbrace{\frac{1}{2}\left(F(u_{i+1}^n)+F(u_i^n)\right)}_{\text{Flux of FTCS}} - \underbrace{\frac{\Delta t}{2\Delta x}\alpha(u_i^n, u_{i+1}^n)\left(F(u_{i+1}^n)-F(u_i^n)\right)}_{\text{Added viscosity term}}$$

or $$F_{i+1/2}^{(\text{L-W})} = \underbrace{\frac{1}{2}\left(F(u_{i+1}^n)+F(u_i^n)\right)}_{\text{Flux of FTCS}} - \underbrace{\frac{\Delta t}{2\Delta x}\alpha(u_i^n, u_{i+1}^n)^2(u_{i+1}^n-u_i^n)}_{\text{Added viscosity term}}$$

<u>Note:</u> we only need to specify the numerical flux $F_{i+1/2}$.

Two-step Lax-Wendroff method (for systems of conservation laws)

<u>Clarification:</u>

This is not a multi-step method. $u^{(n+1)}$ depends only on u^n, not $u^{(n-1)}$. A proper name should be two-stage since it refers to two sub-steps within one time step.

<u>Motivation:</u>

Why do we want a two-step version of Lax-Webdroff method?

For a system of conservation laws,

- $F(u)$ is a vector function and $F'(u)$ is the Jacobi matrix.
- $\alpha(u_i^n, u_{i+1}^n)$ is an extended Jacobi matrix satisfying

$$\underbrace{F(u_{i+1}^n)-F(u_i^n)}_{\text{Vector}}=\underbrace{\alpha(u_i^n,u_{i+1}^n)}_{\text{Matrix}}\underbrace{(u_{i+1}^n-u_i^n)}_{\text{Vector}}$$

- The simple expression for scalar conservation laws

$$\underbrace{\alpha(u_i^n,u_{i+1}^n)}_{\text{Matrix}} = \frac{\underbrace{F(u_{i+1}^n)-F(u_i^n)}_{\text{Vector}}}{\underbrace{(u_{i+1}^n-u_i^n)}_{\text{Vector}}}$$ is no longer valid for a system.

- The calculation of matrix $\alpha(u_i^n, u_{i+1}^n)$ is complicated and expensive.

We want to avoid the calculation of matrix $\alpha(u_i^n, u_{i+1}^n)$.

Strategy:

We use $\alpha(u_i^n,u_{i+1}^n)\approx F'(u)$ and reverse the Taylor expansion to get rid of $F'(u)$.

The numerical flux of the Lax-Wendroff method is

$$F_{i+1/2}^{(\text{L-W})} = \underbrace{\frac{1}{2}\left(F(u_{i+1}^n)+F(u_i^n)\right)}_{\text{Flux of FTCS}} - \underbrace{\frac{\Delta t}{2\Delta x}\alpha(u_i^n,u_{i+1}^n)\left(F(u_{i+1}^n)-F(u_i^n)\right)}_{\text{Added viscosity term}}$$

$$\approx \underbrace{F\left(\frac{u_{i+1}^n+u_i^n}{2}\right)}_{F(u)}+\underbrace{F'\left(\frac{u_{i+1}^n+u_i^n}{2}\right)}_{F'(u)}\underbrace{\left[\frac{-\Delta t}{2\Delta x}\left(F(u_{i+1}^n)-F(u_i^n)\right)\right]}_{\Delta u}$$

We approximate the RHS as $F(u)+F'(u)\Delta u\approx F(u+\Delta u)$.

The approximate numerical flux is

$$F_{i+1/2}^{(\text{L-W})}\approx F\left(\underbrace{\frac{u_{i+1}^n+u_i^n}{2}-\frac{\Delta t}{2\Delta x}\left(F(u_{i+1}^n)-F(u_i^n)\right)}_{\text{One step of Lax-Friedrichs}}\right)$$

Version 1 of two-step Lax-Wendroff (Robert Richtmyer):

$$u_{i+1/2}^{*}=\frac{u_{i+1}^n+u_i^n}{2}-\frac{\Delta t}{2\Delta x}\left(F(u_{i+1}^n)-F(u_i^n)\right)$$

This is one step of Lax-Friedrichs with time step $\Delta t/2$.

$$F_{i+1/2}^{(\text{2S–L-W})} = F(u_{i+1/2}^{*})$$

Draw the stencil of this two-step Lax-Wendroff method.

<u>Another approach</u> for a two-step method

Alternatively, we can write the numerical flux of Lax-Wendroff as

$$F_{i+1/2}^{(\text{L-W})} = \underbrace{\frac{1}{2}\Big(F(u_{i+1}^{n}) + F(u_{i}^{n})\Big)}_{\text{Flux of FTCS}} - \underbrace{\frac{\Delta t}{2\Delta x}\alpha(u_i^n, u_{i+1}^n)\Big(F(u_{i+1}^{n}) - F(u_{i}^{n})\Big)}_{\text{Added viscosity term}}$$

$$= \frac{1}{2}F(u_{i+1}^n) + \frac{1}{2}\left\{F(u_i^n) - \frac{\Delta t}{\Delta x}\alpha(u_i^n, u_{i+1}^n)\Big(F(u_{i+1}^{n}) - F(u_{i}^{n})\Big)\right\}$$

$$\approx \frac{1}{2}F(u_{i+1}^n) + \frac{1}{2}\left\{\underbrace{F(u_i^n)}_{F(u)} + \underbrace{F'(u_i^n)}_{F'(u)}\underbrace{\left[-\frac{\Delta t}{\Delta x}\Big(F(u_{i+1}^{n}) - F(u_{i}^{n})\Big)\right]}_{\Delta u}\right\}$$

We approximate the term inside { } as $F(u) + F'(u)\Delta u \approx F(u + \Delta u)$.

The approximate numerical flux becomes

$$F_{i+1/2}^{(\text{L-W})} \approx \frac{1}{2}F(u_{i+1}^n) + \frac{1}{2}F\left(\underbrace{u_i^n - \frac{\Delta t}{\Delta x}\Big(F(u_{i+1}^{n}) - F(u_{i}^{n})\Big)}_{\text{One step of FTFS}}\right)$$

<u>Version 2 of two-step Lax-Wendroff</u> (Robert MacCormack):

$$u_i^{*} = u_i^n - \frac{\Delta t}{\Delta x}\Big(F(u_{i+1}^{n}) - F(u_{i}^{n})\Big)$$

This is one step of FTFS with time step Δt.

$$F_{i+1/2}^{(\text{L-W})} = \frac{1}{2}F(u_{i+1}^n) + \frac{1}{2}F(u_i^{*})$$

Draw the stencil of this two-step Lax-Wendroff method.

Godunov method

(A more sophisticated version of upwind for conservation laws)

We use a more accurate way to calculate the numerical flux $F_{\text{num}}(u_i^n, u_{i+1}^n)$.

We shift the coordinate system to make $x_{i+1/2} = 0$.

Let $u_- = u_i^n$ and $u_+ = u_{i+1}^n$. We consider the Riemann problem

$$\begin{cases} u_t + F(u)_x = 0 \\ u(x,0) = \begin{cases} u_- , & x<0 \\ u_+ , & x>0 \end{cases} \end{cases}$$

The exact solution of the Riemann problem is a self-similar function.

$$u(x,t) = \eta\left(\frac{x}{t}; u_-, u_+\right)$$

The exact flux at $x = 0$ is

$$F\big(u(0,t)\big) = F\big(\eta(0; u_-, u_+)\big)$$

In Godunov method, we use the exact flux of the Riemann problem.

$$\boxed{F_{i+1/2}^{(\text{Godunov})} = F\Big(\eta(0; u_i^n, u_{i+1}^n)\Big)}$$

Accuracy of Godunov method

We consider the linear hyperbolic PDE: $u_t + au_x = 0$, $a > 0$.

We write it as a conservation law:

$$u_t + F(u)_x = 0, \qquad F(u) = au$$

All characteristics are parallel to $x = at$. There is no shock wave or rarefaction wave. The discontinuity travels with velocity a. The exact solution of Riemann problem is

$$\eta\left(\frac{x}{t}; u_-, u_+\right) = \begin{cases} u_- , & \dfrac{x}{t} < a \\ u_+ , & \dfrac{x}{t} > a \end{cases}$$

For $a > 0$, the exact flux at $x = 0$ is $F\Big(\eta(0; u_-, u_+)\Big) = au_-$.

The numerical flux of Godunov method for $u_t + au_x = 0$ is

$$F_{i+1/2}^{(\text{Godunov})} = au_i^n = F_{i+1/2}^{(\text{Up})}$$

For $u_t + au_x = 0$, Godunov method is simply the upwind method, which is first order accurate in time and in space. Therefore, Godunov method is first order accurate in time and in space.

Remark:

Godunov method is the best version of upwind method. It is also the most expensive version since it requires the exact solution of Riemann problem. For systems of conservation laws, the exact solution of Riemann problem is difficult to find.

We studied a good first order method and a good second order method.

- Upwind method captures discontinuities with no numerical oscillation. But it has only first order accuracy and it smooths out discontinuities too much.
- Lax-Wendroff method has second order accuracy. But it has numerical oscillations at discontinuities.

We want a second order method that has no numerical oscillation.

Non-oscillating high resolution methods

We want to achieve two things:

i) non-oscillating and

ii) second order accuracy.

We start with the upwind method and Lax-Wendroff method.

Upwind method:

$$F_{i+1/2}^{(\mathrm{Up})} = \underbrace{\frac{1}{2}\left[F(u_{i+1}^n)+F(u_i^n)\right]}_{\text{Flux of FTCS}} - \underbrace{\frac{1}{2}\psi_{i+1/2}(u_{i+1}^n - u_i^n)}_{\text{Added viscosity term}}$$

where

$$\psi_{i+1/2} = \max\left\{\left|\alpha(u_i^n, u_{i+1}^n)\right|, -F'(u_i^n), F'(u_{i+1}^n)\right\}$$

The upwind method is non-oscillating, but it is only first order.

Lax-Wendroff method:

$$F_{i+1/2}^{(\text{L-W})} = \underbrace{\frac{1}{2}\left(F(u_{i+1}^n)+F(u_i^n)\right)}_{\text{Flux of FTCS}} - \underbrace{\frac{\Delta t}{2\Delta x}\alpha(u_i^n, u_{i+1}^n)^2(u_{i+1}^n - u_i^n)}_{\text{Added viscosity term}}$$

Lax-Wendroff method is second order in regions where solution is smooth, but it has numerical oscillations at discontinuities.

Question:

How to combine these two to get both i) non-oscillating and ii) second order?

Observation:

We write Lax-Wendroff method as upwind method + correction.

$$F_{i+1/2}^{(\mathrm{LW})} = F_{i+1/2}^{(Up)} + \underbrace{\left[F_{i+1/2}^{(\mathrm{LW})} - F_{i+1/2}^{(Up)}\right]}_{\text{Correction}}$$

The correction term leads to second order in regions where solution is smooth, which is good. It also leads to numerical oscillations at discontinuities, which is bad.

Strategy:

We keep the correction term in regions where solution is smooth. At discontinuities, we limit the correction term to suppress the numerical oscillations.

A framework for non-oscillating high resolution methods

We use a coefficient to switch the correction on and off and in-between

$$F_{i+1/2}^{(\mathrm{HR})} = \underbrace{F_{i+1/2}^{(Up)}}_{\text{Upwind}} + \phi_{i+1/2} \underbrace{\left[F_{i+1/2}^{(\mathrm{LW})} - F_{i+1/2}^{(Up)} \right]}_{\text{Correction}}$$

The switching coefficient ϕ satisfies

$\phi_{i+1/2} \approx 1$ in regions where solution is smooth;

$\phi_{i+1/2} \approx 0$ at discontinuities

A non-oscillating high resolution method (Ami Harten)

(This is a simplified version of Harten's method)

$$\begin{aligned} & F_{i+1/2}^{(\mathrm{HR})}(u_{i-1}^n, u_i^n, u_{i+1}^n, u_{i+2}^n) \\ & \quad = \underbrace{F_{i+1/2}^{(Up)}(u_i^n, u_{i+1}^n)}_{\text{Upwind}} + \phi_{i+1/2} \underbrace{\left[F_{i+1/2}^{(\mathrm{LW})}(u_i^n, u_{i+1}^n) - F_{i+1/2}^{(Up)}(u_i^n, u_{i+1}^n) \right]}_{\text{Correction}} \end{aligned}$$

where

$$F_{i+1/2}^{(\mathrm{Up})} = \underbrace{\frac{1}{2}\left[F(u_{i+1}^n) + F(u_i^n) \right]}_{\text{Flux of FTCS}} - \underbrace{\frac{1}{2}\psi_{i+1/2}(u_{i+1}^n - u_i^n)}_{\text{Added viscosity term}}$$

$$\psi_{i+1/2} = \max\left\{ \left| \alpha(u_i^n, u_{i+1}^n) \right|, -F'(u_i^n), F'(u_{i+1}^n) \right\}$$

$$F_{i+1/2}^{(\mathrm{LW})} - F_{i+1/2}^{(Up)} = \frac{1}{2}\left(\psi_{i+1/2} - r\alpha(u_i^n, u_{i+1}^n)^2 \right)(u_{i+1}^n - u_i^n), \qquad r = \frac{\Delta t}{\Delta x}$$

$$\phi_{i+1/2} = \phi\left(\frac{\Delta u_{i-1/2}^n}{\Delta u_{i+1/2}^n}, \frac{\Delta u_{i+3/2}^n}{\Delta u_{i+1/2}^n} \right), \qquad \Delta u_{i+1/2}^n = u_{i+1}^n - u_i^n$$

$$\phi(c_L, c_R) = \max\left(0, \min(1, qc_L, qc_R)\right), \qquad 1 \le q \le 2 \text{ is a parameter}$$

Behavior of $\phi_{i+1/2}$

- In regions where the solution is smooth, we have

$$c_L = \frac{\Delta u_{i-1/2}^n}{\Delta u_{i+1/2}^n} \approx 1, \qquad c_R = \frac{\Delta u_{i+3/2}^n}{\Delta u_{i+1/2}^n} \approx 1$$

==> $qc_L \geq 1, \quad qc_R \geq 1$ (This depends on q)

==> $\phi_{i+1/2} = \max\big(0,\ \min(1,\ qc_L,\ qc_R)\big) \approx 1$

Here $1 \leq q \leq 2$ is a parameter controlling how aggressive we want to be in pursuing sharp transitions at discontinuities by extending the second order accuracy toward discontinuities as much as possible. If we are too aggressive, it may cause possible distortions/oscillations. If we are not aggressive enough, the discontinuities may be smoothed out.

q = 1: least aggressive

q = 2: most aggressive

<u>Example</u>

Suppose near a discontinuity, we have

$$c_L = \frac{\Delta u_{i-1/2}^n}{\Delta u_{i+1/2}^n} = \frac{3}{2}, \qquad c_R = \frac{\Delta u_{i+3/2}^n}{\Delta u_{i+1/2}^n} = \frac{1}{2}$$

The switching coefficient is

$$\phi_{i+1/2} = \max\big(0,\ \min(1,\ qc_L,\ qc_R)\big) = \min\left(1,\ \frac{1}{2}q\right)$$

Depending on the value of q, the correction term is partially switched off.

- <u>At a location</u> where one of $c_L = \frac{\Delta u_{i-1/2}^n}{\Delta u_{i+1/2}^n}$ and $c_R = \frac{\Delta u_{i+3/2}^n}{\Delta u_{i+1/2}^n}$ is negative.

(This situation arises when numerical oscillations start to develop.)

One of qc_L and qc_R is negative

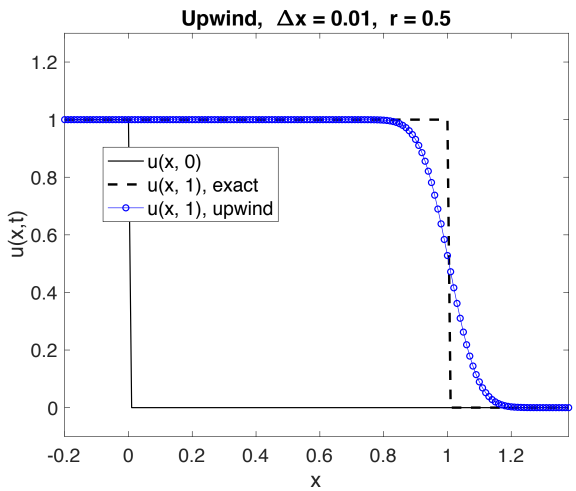
==> $\min(1,\ qc_L,\ qc_R) < 0$

==> $\phi_{i+1/2} = \max\big(0,\ \min(1,\ qc_L,\ qc_R)\big) = 0$

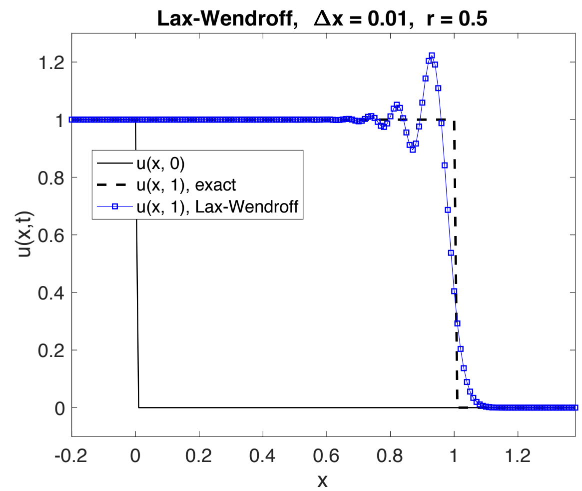
<u>Remark:</u>

In the high-resolution method above, at (i+1/2) both the flux of upwind and the correction term depend only on 2 neighboring cells (u_i^n, u_{i+1}^n). It is the switching coefficient that depends on 4 neighboring cells (u_{i-1}^n, u_i^n, u_{i+1}^n, u_{i+2}^n).

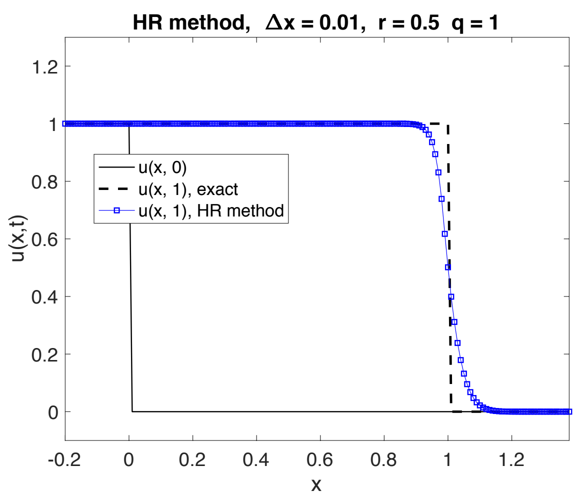
Numerical results of these methods for $u_t + u_x = 0$ are compared below.



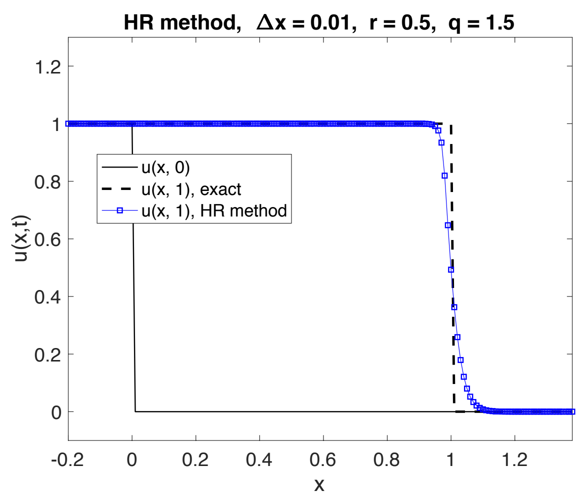
Upwind method



Lax-Wendroff method



High-resolution method (q = 1)



High-resolution method (q = 1.5)

Stability of numerical methods for solving conservation laws

Both the PDE and the numerical method are non-linear.

von Neumann stability analysis is not applicable.

We can only check/enforce CFL condition.

Example:

The numerical domain of dependence of the Lax-Wendroff method is

$D_{\text{num}} = [x_i - \Delta x, x_i + \Delta x]$

CFL condition is

$$\Delta t \cdot \underbrace{\max|f'(u_i)|}_{\substack{\text{Maximum velocity} \\ \text{of characteristics}}} \le \Delta x$$

Note: For a non-linear conservation law, $f'(u_i)$ is no longer a constant.

Example:

The numerical domain of dependence of the upwind method is

$D_{\text{num}} = [x_i - \Delta x, x_i + \Delta x]$

(It is essentially a 3-point method, covering the correct side automatically)

CFL condition is

$$\Delta t \cdot \underbrace{\max|f'(u_i)|}_{\substack{\text{Maximum velocity} \\ \text{of characteristics}}} \le \Delta x$$

Example:

The numerical domain of dependence of the high-resolution method is

$D_{\text{num}} = [x_i - \Delta x, x_i + \Delta x]$

(It is essentially a 3-point method; it switches between two methods;

both the upwind and Lax-wendroff methods are 3-point methods).

CFL condition is

$$\Delta t \cdot \underbrace{\max|f'(u_i)|}_{\substack{\text{Maximum velocity} \\ \text{of characteristics}}} \le \Delta x$$