AM213B Assignment #6

Problem 1 (Theoretical)

Consider the Lax-Friedrichs method for the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Part 1:

Carry out von Neumann stability analysis.

Part 2:

Use Taylor expansion to find coefficients of $O((\Delta t)^2)$, $O((\Delta t)(\Delta x))$ and $O((\Delta x)^2)$ terms in the local truncation error e_i^n. There are the leading terms of e_i^n.

Part 3:

For fixed $\dfrac{\Delta t}{\Delta x} = r$, write $\dfrac{e_i^n}{\Delta t}$ in terms of Δx only. Do we have $\lim\limits_{\Delta x \to 0} \dfrac{e_i^n}{\Delta t} = 0$?

For fixed $\dfrac{\Delta t}{(\Delta x)^2} = c$, write $\dfrac{e_i^n}{\Delta t}$ in terms of Δx only. Do we have $\lim\limits_{\Delta x \to 0} \dfrac{e_i^n}{\Delta t} = 0$?

Problem 2 (Theoretical)

Carry out von Neumann stability analysis on each of the methods below

i) the BTCS method for the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = u_i^n - \frac{ar}{2}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

ii) the implicit upwind method for the case of $a > 0$

$$u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Hint: Examine $|1/\rho|^2$ instead of $|\rho|^2$.

iii) the Crank-Nicolson type method the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = u_i^n - \frac{ar}{4}\left((u_{i+1}^n - u_{i-1}^n) + (u_{i+1}^{n+1} - u_{i-1}^{n+1})\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Problem 3 (Computational)

Use respectively, the first order and the second order numerical differentiations to approximate $\frac{d}{dx}\sin(x)$ at $x = 1.45$

$$q_1(h) = \frac{\sin(x+h) - \sin(x)}{h}$$

$$q_2(h) = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

Use the exact derivative to calculate the total errors

$$E_{T,1}(h) = \left| q_1(h) - \cos(x) \right|$$

$$E_{T,2}(h) = \left| q_2(h) - \cos(x) \right|$$

Calculate the total errors $E_{T,1}(h)$ and $E_{T,2}(h)$ for h = 10.^(−[1:0.2:14]).

Plot the total errors $E_{T,1}(h)$ vs h and $E_{T,2}(h)$ vs h in one figure.

Use logarithmic scales for both h and $E_T(h)$.

Remark:

This numerical exercise clearly demonstrates that the total error attains a minimum at a certain value of step size h. The minimum total error is not zero. A higher order method can achieve a smaller minimum total error.

Problem 4 (Computational)

Consider the 2D IBVP of the heat equation

$$\begin{cases} u_t = u_{xx} + u_{yy}, \quad (x,y) \in (0,8)\times(0,8), \quad t > 0 \\ u(x,y,0) = f(x,y) \\ u(0,y,t) = g_L(y,t), \quad u(8,y,t) = g_R(y,t) \\ u(x,0,t) = g_B(x,t), \quad u(x,8,t) = g_T(x,t) \end{cases}$$

where

$$f(x,y) = 0$$

$$g_L(y,t) = g_R(y,t) = -4\sin(\pi y/8)\cdot\tanh(2t)$$

$$g_B(x,t) = g_T(x,t) = x(1 - x/8)\cdot\tanh(2t)$$

Use the 2D FTCS method to solve the IBVP to $T = 2$.

Use $\Delta x = \Delta y = 0.08$ and $\Delta t = 1.25\times10^{-3}$.

Plot $u(x, 4)$ vs x at t = 0.2, 0.4, 1, and 2 in one figure.

Plot $u(4, y)$ vs y at t = 0.2, 0.4, 1, and 2 in one figure.

Problem 5 (Computational)

Continue with the numerical solution of IBVP in Problem 4.

Use the same (Δx, Δy, Δt) as in problem 3. Solve the IBVP to T = 20.

In each time step, calculate and track the maximum of $| u_t |$.

$$E(t_n) = \max_{(i,j)} \frac{|u_{i,j}^{n+1} - u_{i,j}^{n}|}{\Delta t}$$

Plot $E(t)$ vs t. Use logarithmic scale for $E(t)$ and linear scale for t.

Plot u vs (x, y) as a surface at t = 20.

Hint:

In Matlab, to calculate the maximum of elements of matrix A, we use

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max(abs(A), [ ], 'all');
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