AM213B Assignment #2

Problem 1 (Theoretical)

Recall that for Runge-Kutta methods, we derived

Condition for first order: $\sum_{i=1}^{p} b_i = 1$

Additional condition for second order:

$$\sum_{i=1}^{p}\left(b_i \sum_{j=1}^{p} a_{ij} \right) = \frac{1}{2}, \qquad \sum_{i=1}^{p}(b_i c_i) = \frac{1}{2}$$

Consider the Runge-Kutta method below.

$$k_1 = h f\left(u_n,\ t_n + h\right)$$

$$k_2 = h f\left(u_n + \frac{1}{2} k_1,\ t_n + \frac{1}{2} h \right)$$

$$u_{n+1} = u_n + k_2$$

<u>Verify that this RK method satisfies the conditions above.</u> Thus, it has second order.

<u>Verify that this RK method does not satisfy the internal consistency condition:</u>

Internal consistency condition: $c_i = \sum_{j=1}^{p} a_{ij}$

<u>Remark:</u> This result demonstrates that the internal consistency condition is not a necessary condition for second order.

Problem 2 (Theoretical)

Recall that in lecture, we carried out polynomial interpolation based on 3 points and derived

- 3-step Adams-Bashforth method and
- 2-step Adams-Moulton method

<u>Carry out polynomial interpolation based on 2 points and derive</u>

- 2-step Adams-Bashforth method and
- 1-step Adams-Moulton method

Problem 3 (Theoretical)

Consider a linear multi-step method (LMM) applied to solving $u' = 0$

$$\sum_{j=0}^{r} \alpha_j u_{n+j} = 0, \qquad \alpha_r = 1 \tag{E03}$$

The characteristic polynomial $\rho(\xi)$ is

$$\rho(\xi) \equiv \sum_{j=0}^{r} \alpha_j \xi^j$$

Suppose q_1 is a double root of $\rho(\xi)$. Show that

- $\{u_k = q_1^{\,k},\ k = 0, 1, 2, \ldots\}$ is a solution of (E03).
- $\{u_k = k q_1^{\,k},\ k = 0, 1, 2, \ldots\}$ is also a solution of (E03).

Hint:

"q_1 is a double root" means $\rho(q_1) = 0$ and $\rho'(q_1) = 0$ where

$$\rho(\xi) = \sum_{j=0}^{r} \alpha_j \xi^j, \qquad \rho'(\xi) = \sum_{j=1}^{r} j \alpha_j \xi^{j-1}$$

Problem 4

Use the classic 4th-order Runge-Kutta method (RK4) to solve the IVP below

$$y'' - \mu(1 - y^2) y' + y = 0$$
$$y(0) = y_0, \quad y'(0) = v_0$$

(This is called the van der Pol equation.)

Use $y_0 = 0.5$, $v_0 = 1$, and $h = 0.025$. Solve the IVP to $T = 30$.

Solve the IVP respectively for parameter values

$\mu = 0.5$, $\mu = 1.0$, and $\mu = 2.5$

Plot $y(t)$ vs t and $y'(t)$ vs t.

Plot $y'(t)$ vs $y(t)$ (this is called phase plane plot or phase plot).

Hint:

Look at sample code on how to implement RK methods for solving ODE systems.

Problem 5 (continue with the IVP in Problem 4)

Use $y_0 = 0.5$, $v_0 = 1$, and $\mu = 2.5$. Use RK4 to solve the IVP to $T = 30$.

Run simulations, respectively, with time step h and time step $h/2$.

Use the two results to estimate the error in numerical solution of time step h.

Try time steps

$$h = \frac{1}{2^3},\ \frac{1}{2^4},\ \frac{1}{2^5},\cdots$$

Part 1: From the sequence above, select a time step h_C such that

$$\max_{t_n \in [0,30]} \left(\text{norm of estimated error at } t_n\right) < 5\times 10^{-8}$$

Report the value of h_C.

Plot the norm of estimated error vs t for time step h_C.

Part 2: Treat the numerical solution of time step h_C as the "exact solution". Use the two methods below to calculate the error in numerical solution of time step h = 0.025.

$$\text{Err}_1 = \frac{1}{1-(0.5)^4}\left\|\left(\text{result of step } h\right) - \left(\text{result of step } \frac{h}{2}\right)\right\|$$

$$\text{Err}_2 = \left\|\left(\text{result of step } h\right) - \left(\text{"exact solution"}\right)\right\|$$

Plot Err_1 vs t and Err_2 vs t in ONE figure to compare the two errors.

Plot two versions of the figure: one with linear scale for error, the other with logarithmic scale for error. Use linear scale for the time in both versions.

Hint:

Look at sample code on how to estimate error in numerical solution of ODE system.

Note that (result of step h) and ("exact solution") are represented on *different* numerical grids. Look at the sample code on how to deal with this situation.

Problem 6 (continue with the IVP in Problem 4)

The Fehlberg method is an embedded Runge-Kutta method with orders 5 and 4. Implement the Fehlberg method to solve the IVP to T = 30.

Use y_0 = 0.5, v_0 = 1, μ = 2.5 and time step h =0.025.

Estimate the error as follows:

$$E_{\text{est}}(h) \approx \frac{e_n(h)}{h} \approx \frac{\left\| w_{n+1} - \tilde{w}_{n+1} \right\|}{h}$$

where w_{n+1} and $\tilde{w}_{n+1}$ are respectively the results of the 5th-order component and the 4th-order component in the Fehlberg method. In each time step, both w_{n+1} and $\tilde{w}_{n+1}$ are calculated from w_n in the Fehlberg method:

$$w_{n+1} = w_n + \sum_{i=1}^{p} b_i k_i$$

$$\tilde{w}_{n+1} = w_n + \sum_{i=1}^{p} \tilde{b}_i k_i$$

Calculate the "exact" error using the "exact" solution" from Problem 5:

$$E_{\text{exa}}(h) = \left\| \left(\text{result of 5th order method}\right) - \left(\text{"exact" solution}\right) \right\|$$

Plot $E_{\text{est}}(h)$ vs t and $E_{\text{exa}}(h)$ vs t in ONE figure to compare the two errors.

Use logarithmic scale for the errors. Use linear scale for the time.

Hint:

The coefficients (p, matrix A, vectors c, b5 and b4) of the Fehlberg method are given in the folder of sample code on how to implement RK methods