AM213B Assignment #3

Problem 1 (Theoretical)

Consider the 2-step multi-step method below:

$$u_{n+2} - 2u_{n+1} + u_n = hf(u_{n+1}, t_{n+1}) - hf(u_n, t_n)$$

Part 1:

Use Taylor expansion to calculate the order of its local truncation error.

Part 2:

Find whether or not the method is zero-stable.

Problem 2 (Theoretical)

Consider the Runge-Kutta method described by Butcher tableau

Butcher tableau: $$\begin{array}{c|cc} \alpha & \alpha & 0 \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array}$$

where $\alpha > 0$ is real.

This method is called 2S-DIRK (2-Stage Diagonally Implicit Runge-Kutta).

Part 1:

Use Taylor expansion to show that the method is second order for $\alpha = 1 - \frac{1}{\sqrt{2}}$.

Caution: "second order" means that the local truncation error is $O(h^3)$.

Part 2:

Apply the 2S-DIRK to solving $u' = \gamma u$.

Derive analytically the expressions for k_1, k_2 and the stability function $\phi(z)$.

$$k_1 = \frac{1}{1-\alpha z} \cdot z u_n$$

$$k_2 = \frac{(1-\alpha z) + (1-\alpha) z}{(1-\alpha z)^2} \cdot z u_n$$

$$\phi(z) = \frac{1 + (1-2\alpha) z}{(1-\alpha z)^2}$$

Part 3:

Suppose we know that the 2S-DIRK is A-stable for $\alpha = 1 - \frac{1}{\sqrt{2}}$ (see Problem 3).

Show that it satisfies the second condition in L-stability.

Problem 3 (Theoretical)

We continue with the 2S-DIRK method in Problem 2.

Here we show that the method is A-stable for $\alpha = 1 - \frac{1}{\sqrt{2}}$.

- First we look at the absolute value of the numerator of $\phi(z)$.

  Show that

$$\begin{aligned}\left|1+(1-2\alpha)z\right|^2 &= \left(1+(1-2\alpha)z\right)\left(1+(1-2\alpha)\bar{z}\right)\\ &= 1+(1-2\alpha)(z+\bar{z})+(1-2\alpha)^2|z|^2\end{aligned}$$

- Next we look at the absolute value of the denominator of $\phi(z)$.

  It is straightforward to verify the inequality

  $(1+a_1+a_2)^2 \geq 1+2(a_1+a_2)$ for all a_1 and a_2 real

  Use the inequality above to show that

$$\begin{aligned}(|1-\alpha z|^2)^2 &= \left[(1-\alpha z)(1-\alpha\bar{z})\right]^2 = \left[1-\alpha(z+\bar{z})+\alpha^2|z|^2\right]^2\\ &\geq 1-2\alpha(z+\bar{z})+2\alpha^2|z|^2\end{aligned}$$

- We compare the corresponding terms in $\left|1+(1-2\alpha)z\right|^2$ and $(|1-\alpha z|^2)^2$.

  For $\alpha = 1 - \frac{1}{\sqrt{2}}$ and Re(z) < 0, show that:

  $(1-2\alpha) > -2\alpha$

  $(1-2\alpha)(z+\bar{z}) < -2\alpha(z+\bar{z})$ for Re(z) < 0

  $(1-2\alpha)^2 = 2\alpha^2$

- Combine all results above to show that for $\alpha = 1 - \frac{1}{\sqrt{2}}$, we have

$$\underbrace{\left|1+(1-2\alpha)z\right|^2}_{\text{numerator of } \phi(z)} < \underbrace{(|1-\alpha z|^2)^2}_{\text{denominator of } \phi(z)} \qquad \text{for Re}(z) < 0$$

which leads to $|\phi(z)| < 1$ for $\mathrm{Re}(z) < 0$.

Therefore, for $\alpha = 1 - \frac{1}{\sqrt{2}}$, the 2S-DIRK is A-stable.

Remark: for $\alpha = 1 - \frac{1}{\sqrt{2}}$, the 2S-DIRK is 2nd order, A-stable and L-stable.

Problem 4 (Theoretical and Computational)

Part 1:

Find analytically the stability function $\phi(z)$ for each of

- Heun's method (RK2)
- Classic 4th order Runge-Kutta method (RK4)
- The two-stage DIRK method with $\alpha = 1 - \frac{1}{\sqrt{2}}$.

Part 2:

The region of absolute stability is described by the contour line:

$$\left\{ (x, y) \middle| \left| \phi(x + i\, y) \right| = 1 \right\}.$$

Plot the region of absolute stability for each of the three methods listed above.

Part 3:

Which of RK2 and RK4 has the larger region of absolute stability?

Hint:

See sample code on how to plot contour lines and how to plot contour lines of multiple functions in one figure.

Problem 5 (Computational)

Implement respectively, the backward Euler method and the 2S-DIRK method with $\alpha = 1 - \frac{1}{\sqrt{2}}$, to solve the IVP

$$\begin{cases} u' = -\left(\frac{1}{4} + \exp\left(20\cos(t)\right) \right) \sinh\left(u - \cos(t)\right) \\ u(0) = 0 \end{cases}$$

Solve the IVP to $T = 30$. Try time steps $h = \frac{1}{2^3}, \frac{1}{2^4}, \ldots, \frac{1}{2^8}$.

For each numerical method, carry out numerical error estimation.

Part 1:

Plot the estimated errors vs t for $h = 2^{-5}$. Plot the two curves in ONE figure to compare the two methods. Use logarithmic scale for the errors, linear scale for the time.

Part 2:

In a separate figure, plot errors vs t of the two methods for $h = 2^{-7}$.

Part 3:

Plot the numerical solution $u(t)$ vs t of the 2S-DIRK for $h = 2^{-5}$.

Plot $\cos(t)$ vs t in the same figure for comparison.

Does the solution $u(t)$ always follow the function $\cos(t)$ very closely?

Observe the value of $\cos(t)$ where $u(t)$ follows $\cos(t)$ very closely?