AM213B Assignment #8

Problem 1 (Computational)

Consider the IVP of Burgers' equation

$$\begin{cases} u_t + \left(\frac{1}{2}u^2\right)_x = 0, \quad t > 0 \\ u(x,0) = \begin{cases} \frac{-1}{2}, & x \le 0 \\ 1, & 0 < x \le 1 \\ 0, & x > 1 \end{cases} \end{cases} \qquad \text{(IVP-1)}$$

For $t \le 2$, the exact solution of (IVP-1) is

$$u_{\text{ext}}(x,t) = \begin{cases} \frac{-1}{2}, & x \le \frac{-1}{2}t \\ \frac{x}{t}, & \frac{-1}{2}t < x \le t \\ 1 & t < x \le 1 + \frac{1}{2}t \\ 0 & x > 1 + \frac{1}{2}t \end{cases}$$

We implement two methods below to solve (IVP-1).

- Upwind method

$$F_{i+1/2}^{(\text{Up})} = \frac{1}{2}\left(F(u_{i+1}^n) + F(u_i^n)\right) - \frac{1}{2}\psi_{i+1/2}(u_{i+1}^n - u_i^n)$$

$$\psi_{i+1/2} = \max\left\{\left|\alpha(u_i^n, u_{i+1}^n)\right|, -F'(u_i^n), F'(u_{i+1}^n)\right\}, \qquad \alpha(u_i^n, u_{i+1}^n) = \frac{1}{2}(u_i^n + u_{i+1}^n)$$

- Lax-Wendroff method

$$F_{i+1/2}^{(\text{LW})} = \frac{1}{2}\left(F(u_{i+1}^n) + F(u_i^n)\right) - \frac{\Delta t}{2\Delta x}\alpha(u_i^n, u_{i+1}^n)^2(u_{i+1}^n - u_i^n)$$

We select $[L_1, L_2]$ with $L_1 = -1$ and $L_2 = 2$ as the computational domain.

We construct the numerical grid

$$\Delta x = \frac{L_2 - L_1}{N}, \qquad x_i = L_1 + i\,\Delta x, \qquad x_0 = L_1, \qquad x_N = L_2$$

In each time step, we calculate $\left\{u_i^{n+1}, 1 \le i \le N-1\right\}$. We need u^n at x_0 and x_N. We use artificial boundary conditions: $u_0^n = u_1^n$, $u_N^n = u_{N-1}^n$.

Use $N = 300$ and $r = \Delta t/\Delta x = 0.5$ in simulations.

Part 1:

Plot the exact solution and numerical solutions of the upwind and Lax-Wendroff methods at $t = 1$ in one figure. Observe the behavior of Lax-Wendroff method.

Part 2:

Plot in one figure, numerical solutions of the upwind method at $t = 0$, $t = 1$, $t = 1.5$, $t = 3$, and $t = 6$ to show the time evolution of solution.

Observe the sign of u at the two ends of the computational domain. Based on the sign of u, conclude that none of characteristics is going into the domain, which justifies our ad hoc artificial boundary conditions.

Problem 2 (Computational)

Continue with the initial value problem (IVP-1) in Problem 1.

We implement the high-resolution method below to solve (IVP-1).

$$F_{i+1/2}^{(\mathrm{HR})} = \underbrace{F_{i+1/2}^{(Up)}}_{\text{Upwind}} + \phi_{i+1/2} \underbrace{\left[F_{i+1/2}^{(\mathrm{LW})} - F_{i+1/2}^{(Up)} \right]}_{\text{Correction}}$$

where

$$F_{i+1/2}^{(\mathrm{LW})} - F_{i+1/2}^{(Up)} = \frac{1}{2}\left(\psi_{i+1/2} - r\alpha(u_i^n, u_{i+1}^n)^2 \right)(u_{i+1}^n - u_i^n), \qquad r = \frac{\Delta t}{\Delta x}$$

$$\phi_{i+1/2} = \phi\left(\frac{\Delta u_{i-1/2}^n}{\Delta u_{i+1/2}^n}, \frac{\Delta u_{i+3/2}^n}{\Delta u_{i+1/2}^n} \right), \qquad \Delta u_{i+1/2}^n = u_{i+1}^n - u_i^n$$

$$\phi(c_L, c_R) = \max\left(0,\ \min(1, qc_L, qc_R)\right)$$

We set $q = 1.5$ in the high-resolution method.

In each time step, we calculate $\left\{ u_i^{n+1}, 1 \le i \le N-1 \right\}$. We need u^n at x_{-1}, x_0, x_N and x_{N+1}. We use artificial boundary conditions: $u_{-1}^n = u_0^n = u_1^n$, $u_{N+1}^n = u_N^n = u_{N-1}^n$.

In simulations, we need a vector holding $\left\{ u_i^{n+1}, -1 \le i \le N+1 \right\}$.

Part 1:

Use $N = 300$ and $r = \Delta t / \Delta x = 0.5$.

Plot the exact solution and numerical solutions of the upwind and the high-resolution methods at $t = 1$ in one figure.

Part 2:

We explore the effect of $r = \Delta t / \Delta x$ on the numerical stability.

Use $N = 300$ and $r = \Delta t / \Delta x = 10/8$.

Plot numerical solutions at t = 0.45 of, respectively, the upwind and the high-resolution methods, in separate figures.

You will see oscillations of huge magnitude.

Part 3:

Use N = 300 and $r = \Delta t/\Delta x = 9/8$.

Plot the exact solution and numerical solutions of the upwind and the high-resolution methods at t = 1 in one figure.

You will see that in a non-linear problem, "instability" may not lead to numerical solution blowing up; it may manifest as other defects in numerical solution.

Problem 3 (Computational)

Consider the IVP

$$\begin{cases} u_t + u_x = 0, & t > 0 \\ u(x,0) = u_0(x) \end{cases} \qquad \text{(IVP-2)}$$

where $u_0(x)$ is a periodic function with period = 1.

$$u_0(x) = \begin{cases} 1, & 0.25 < x \le 0.75 \\ 0, & \text{Otherwise in } (0,1] \end{cases} \quad \text{for } x \in (0,1]$$

The exact solution is $u(x,t) = u_0(x - t)$.

We implement the upwind method to solve (IVP-2).

We select [0, 1] as the computational domain and use the numerical grid

$$\Delta x = \frac{1}{N}, \qquad x_i = i\Delta x, \qquad x_0 = 0, \qquad x_N = 1$$

In each time step, we calculate $\left\{u_i^{n+1}, 1 \le i \le N\right\}$. Note that the index range includes the right end of the domain. We need u^n at x_0. We use the periodic boundary condition

$$u_0^n = u_N^n$$

Solve (IVP-2) to T = 5 using N = 100 and $r = \Delta t/\Delta x = 0.5$.

Plot the exact solution and the numerical solution at T = 5 in one figure.

Problem 4 (Computational)

Continue with the initial value problem (IVP-2) in Problem 3.

The Fourier series of $u_0(x)$ is

$$u_0(x)=\frac{1}{2}+\sum_{k=0}^{\infty}\frac{2(-1)^{k+1}}{\pi(2k+1)}\cos\left(2\pi(2k+1)x\right)$$

For $u_t + u_x = 0$, the modified PDE of the upwind method is

$$w_t+w_x=\sigma w_{xx}\,,\qquad \sigma=\frac{\Delta x}{2}(1-r) \tag{ME-1}$$

The exact solution of (ME-1) with initial value $w(x, 0) = u_0(x)$ is

$$w(x,t)=\frac{1}{2}+\sum_{k=0}^{\infty}\frac{2(-1)^{k+1}}{\pi(2k+1)}\exp\left(-\sigma 4\pi^2(2k+1)^2 t\right)\cos\left(2\pi(2k+1)(x-t)\right)$$

where σ is from the modified PDE given above.

Use the first 200 terms of the Fourier series to approximate $w(x, T)$.

Use the numerical grid from Problem 3 to represent $w(x, T)$.

In one figure, plot 3 functions:

- exact solution of (IVP-2) at $T = 5$,
- numerical solution of (IVP-2) from Problem 3, and
- $w(x, T)$ at $T = 5$ (which is the exact solution of modified PDE).

Observe that the behavior of numerical solution is much more accurately captured by the modified PDE than by the original PDE.

Problem 5 (Computational)

Continue with the initial value problem (IVP-2) in Problem 3.

We implement the high-resolution method to solve (IVP-2). We use the same computational domain and the same numerical grid as in Problem 3.

In each time step, we calculate $\left\{u_i^{n+1},\, 1\le i\le N\right\}$. We need u^n at x_0, x_{-1}, x_{N+1} and x_{N+2} We use the periodic boundary condition

$$u_0^n=u_N^n\,,\quad u_{-1}^n=u_{N-1}^n$$
$$u_{N+1}^n=u_1^n\,,\quad u_{N+2}^n=u_2^n$$

In simulations, we need a vector holding $\left\{u_i^{n+1},\, -1\le i\le N+2\right\}$.

Solve (IVP-2) to $T = 5$ using $N = 100$, $r = \Delta t/\Delta x = 0.5$ and $q = 1.5$.

Plot the exact solution and numerical solutions of the upwind and the high-resolution methods at $T = 5$ in one figure.