AM213B Assignment #1

Problem 1 (Theoretical)

Suppose E_n satisfies the recursive inequality

$$E_{n+1} \le (1+Ch)E_n + h \qquad \text{for } n \ge 0$$
$$E_0 = 0$$

where $C > 0$ is a constant independent of h and n.

Derive that $\quad E_N \le \dfrac{e^{CT}-1}{C} \quad$ for $Nh \le T$

Problem 2

Use the composite trapezoidal rule and the composite Simpson's rule, respectively, to approximate the integral

$$\int_0^3 \sqrt{1+\cos^2(x)}\,\exp\left(\cos(x)\right)dx$$

For each method, carry out numerical simulations for $N = 2^2, 2^3, 2^4, \ldots, 2^{10}$.

For each method, use numerical results to do numerical error estimations.

For each method, plot the estimated error (absolute value) as a function of h. Use log-log plot to accommodate the wide ranges of h and error.

Plot the two curves in ONE figure to compare the performance of the two methods.

Problem 3

Implement the Euler method and the backward Euler method to solve the IVP below.

$$\begin{cases} u' = -\lambda \sinh\left(u-\cos(t)\right), & \lambda = 10^6 \\ u(0) = 0 \end{cases}$$

where sinh() is the hyperbolic sine function: $\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right)$.

Part 1: For the Euler method, solve the IVP to $T = 2^{-10}$. Try $h = 2^{-18}, 2^{-19}, 2^{-20}, \ldots$

At what time step size, the numerical solution is "well-behaved" (i.e. numerical solution is bounded)?

Plot one representative figure showing the behavior of numerical solution when the time step is not small enough.

Plot another representative figure for a time step small enough.

Part 2: For the backward Euler method, solve the IVP to $T = 10$. Use Newton's method to solve the non-linear equation in each step. Use $h = 0.1$ in your simulations.

Plot the numerical solution vs t for the backward Euler method.

Hint: Look at the sample code on how to implement Newton's method.

Problem 4:

Implement the trapezoidal method to solve the IVP in Problem 3.

Use Newton's method to solve the non-linear equation in each step.

Solve the IVP to $T = 10$. Use $h = 0.1$ in your simulations.

Part1: Plot the numerical solution vs t.

Is the numerical solution bounded?

Do you observe any oscillation in the numerical solution with $h = 0.1$?

Part2: Try $h = 2^{-7}, 2^{-8}, 2^{-9}, \ldots$.

What happens to the oscillation as the time step is refined?

Problem 5:

Use the Euler method and the 2-step midpoint method, respectively, to solve the IVP

$$\begin{cases} u' = -u \\ u(0) = 1 \end{cases}$$

The exact solution is $u_{\text{exact}}(t) = \exp(-t)$. In the midpoint method, use exact solution $u_1 = \exp(-t_1)$ to get started. Use $h = 0.2$ in simulations.

Part 1: First solve the IVP to $T = 2$. Compare the numerical results of the two methods and the exact solution in ONE figure.

Is the midpoint method more accurate than the Euler in this time period?

Part 2: Then solve the IVP to $T = 20$. Compare the numerical results of the two methods and the exact solution in ONE figure.

Is the result of midpoint method well behaved over this longer period?

Try decreasing the time step. Does that reduce the growing error mode?