Midterm Exam

AM213A

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Problem 1 (definitions)

a.

Definition 1 (Full Rank Matrix)**.** Let the matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. The matrix $\mathbf{A}$ is said to be full-rank if the rank of $\mathbf{A}$ is equal to the minimum of m and n.

$$\mathbf{A} := \text{full-rank} \iff \text{rank}(\mathbf{A}) = \min(m, n) \quad \forall \mathbf{A} \in \mathbb{R}^{m \times n} \tag{1}$$

Definition 2 (Rank Deficient Matrix)**.** Let the matrix $\mathbf{B} \in \mathbb{R}^{m \times n}$. The matrix $\mathbf{B}$ is said to be a rank-deficient (or not full-rank) matrix if the rank of $\mathbf{B}$ is strictly less than but not equal to the minimum between m and n.

$$\mathbf{B} := \text{rank-deficient} \iff \text{rank}(\mathbf{B}) < \min(m, n) \quad \forall \mathbf{B} \in \mathbb{R}^{m \times n} \tag{2}$$

b.

Definition 3 (Orthogonal Matrix)**.** Let $\mathbf{Q} \in \mathbb{R}^{m \times m}$. The matrix $\mathbf{Q}$ is said to be orthogonal if and only if $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix.

$$\mathbf{Q} := \text{orthogonal} \iff \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad \forall \mathbf{Q} \in \mathbb{R}^{m \times m} \tag{3}$$

It immediately follows then that $\mathbf{Q}^T = \mathbf{Q}^{-1}$.

Additionally, since $\mathbf{Q}$ is square, it is also true that $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T$

Definition 4 (Unitary Matrix)**.** Let $\mathbf{U} \in \mathbb{C}^{m \times m}$. The matrix $\mathbf{U}$ is said to be unitary if and only if $\mathbf{Q}^*\mathbf{Q} = \mathbf{I} = \mathbf{Q}\mathbf{Q}^*$

$$\mathbf{Q} := \text{unitary} \iff \mathbf{Q}^*\mathbf{Q} = \mathbf{Q}\mathbf{Q}^* = \mathbf{I} \quad \forall \mathbf{Q} \in \mathbb{C}^{m \times m} \tag{4}$$

c.

Definition 5 (Singular Value Decomposition)**.** Let $\mathbf{A} \in \mathbb{C}^{m \times n}$ with $\text{rank}(\mathbf{A}) = k \leq \min(m, n)$. A Singular value decomposition (SVD) of $\mathbf{A}$ is a factorization

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^* \tag{5}$$

where

$$\mathbf{U} \in \mathbb{C}^{m \times m} \quad \text{is unitary} \tag{6}$$
$$\mathbf{V} \in \mathbb{C}^{n \times n} \quad \text{is unitary} \tag{7}$$
$$\Sigma \in \mathbb{R}^{m \times n} \quad \text{is diagonal with positive real entries.} \tag{8}$$

Additionally, the diagonal entries σ_j of Σ are called the singular values of $\mathbf{A}$ and are in nonincreasing order, such that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0 \tag{9}$$

where $p = \min(m, n)$.

d.

Definition 6 (Algebraic Multiplicity of the Eigenvalue)**.** Let $\mathbf{A} \in \mathbb{C}^{m\times m}$. Let

$$p_A(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) \tag{10}$$

$$= \prod_{i=1}^{m} (\lambda - \lambda_i) \tag{11}$$

be the characteristic polynomial associated with the matrix $\mathbf{A}$, and the roots of this characteristic polynomial are the eigenvalues of the matrix $\mathbf{A}$. The algebraic multiplicity of a particular eigenvalue $\lambda_k \in \sigma(\mathbf{A})$ where $\sigma(\mathbf{A})$ is the spectrum of $\mathbf{A}$, the set of all eigenvalues of $\mathbf{A}$, is the multiplicity of the corresponding root of $p_A(\lambda)$, i.e., how many times the factor $(\lambda - \lambda_k)$ appears in the characteristic polynomial $p_A(\lambda)$.

Definition 7 (Geometric Multiplicity of the Eigenvalue)**.** Let $\mathbf{A} \in \mathbb{R}^{m\times m}$. The geometric multiplicity, $\gamma_A(\lambda)$ of a particular eigenvalue $\lambda_k \in \sigma(\mathbf{A})$ is the dimension of the eigenspace $E_\lambda(A)$ associated with λ_k. It is the maximal number of linearly independent eigenvectors corresponding to λ_k, i.e.,

$$\gamma_A(\lambda) = m - \operatorname{rank}(\mathbf{A} - \lambda \mathbf{I}) \tag{12}$$

e.

Definition 8 (Defective Matrix)**.** Let $\mathbf{A} \in \mathbb{R}^{m\times m}$. An eigenvalue λ_k is said to be defective, if the algebraic multiplicity is strictly greater than the geometric multiplicity of that particular eigenvalue. Additionally, the matrix $\mathbf{A}$ is said to be defective if it has fewer than m linearly independent eigenvectors. Thus, a non-defective matrix must have no defective eigenvalues.

f.

Definition 9 (Relative Condition Number of a vector valued function)**.** Let $\mathbf{A} \in \mathbb{C}^{m\times n}$. Consider a function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ when $\mathbf{x} = \mathbf{x_0}$. The condition number of the function f is given by

$$\kappa = \kappa(\mathbf{x_0}) = \|\mathbf{A}\| \frac{\|\mathbf{x_0}\|}{\|\mathbf{A}\mathbf{x_0}\|} \tag{13}$$

Additionally, when $\mathbf{A}$ is non-singular, the upper bound of the relative condition number κ is such that

$$\kappa(\mathbf{x_0}) \leq \|\mathbf{A}\|\|\mathbf{A}^{-1}\| \tag{14}$$

g.

Definition 10 (Condition Number of a Matrix)**.** Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be full-rank, such that $\operatorname{rank}(\mathbf{A}) = m$. The condition number κ of the matrix $\mathbf{A}$ is given by,

$$\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|. \tag{15}$$

If $\|\cdot\| = \|\cdot\|_2$, then $\|\mathbf{A}\| = \sigma_1$ and $\|\mathbf{A}^{-1}\| = \frac{1}{\sigma_m}$. Thus in terms of the singular values of $\mathbf{A}$ the condition number is,

$$\kappa(A) = \frac{\sigma_1}{\sigma_m} \tag{16}$$

h.

Definition 11 (Diagonalizable Matrix)**.** Let $\mathbf{A} \in \mathbb{R}^{m\times m}$. The matrix $\mathbf{A}$ is said to be diagonalizable if it is non-defective such that all the eigenvectors of $\mathbf{A}$ are linearly independent. If $\mathbf{A}$ is diagonalizable then the eigenvectors form a basis for all space.

Definition 12 (Similarity Transformation)**.** Let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}^{m\times m}$. $\mathbf{A}$ is said to be similar to $\mathbf{B}$ if there is a non-singular matrix $\mathbf{P}$ for which

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}. \tag{17}$$

The operation $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is called a similarity transformation of $\mathbf{A}$. If $\mathbf{B}$ is similar to $\mathbf{A}$ then the reciprocal is also true ($\mathbf{A}$ is similar to $\mathbf{B}$).

i.

Definition 13 (Machine accuracy)**.** Machine accuracy is represented by the smallest number we can store for different lengths of binary numbers. For example a 64-bit Floating point number, which is often refered to as a double precision number can represent only down to

$$\epsilon_{\text{mach}} = 1 \times 10^{-16}. \tag{18}$$

The smallest number a 32-bit Floating point number can represent is

$$\epsilon_{\text{mach}} = 1 \times 10^{-7} \tag{19}$$

j.

Definition 14 (Backwards stability)**.** An algorithm $\tilde{f}$ for a problem f is backwards stable if, for each possible input X,

$$\tilde{f}(\tilde{X}) \quad \text{for some} \quad \tilde{X} \quad \text{with} \quad \frac{\|\tilde{X} - X\|}{\|X\|} = \mathcal{O}(\epsilon_{\text{mach}}). \tag{20}$$

Accuracy is guaranteed for any backwards stable algorithm as long as the condition number $\kappa(X)$ is globally bounded for all X as well. This implies that for the function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ discussed earlier in Definition 9, the accuracy of a backward stable algorithm that computes the solution of this function can be estimated with

$$\frac{\|\tilde{f}(\tilde{X}) - f(X)\|}{\|f(X)\|} \leq \mathcal{O}(\kappa(\mathbf{A})\epsilon_{\text{mach}}) \tag{21}$$

Problem 2

Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ is a square and symmetric matrix with only real entries.

The matrix $\mathbf{A}$ is then diagonalizable by an orthogonal matrix.

Proof. The Schur decomposition theorem states that if $\mathbf{A} \in \mathbb{C}^{m \times m}$, then there exists a unitary matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{T}$ such that

$$\mathbf{A} = \mathbf{Q}\mathbf{T}\mathbf{Q}^{-1}. \tag{22}$$

Given that $\mathbf{A}$ is both a real square matrix and symmetric, recalling that real square matrices are simply a subset of complex square matrices and symmetric matrices have the property that $\mathbf{A} = \mathbf{A}^T$, let us prove that the matrix $\mathbf{A}$ is diagonalizable by an orthogonal matrix by starting from (22) where $\mathbf{Q}$ is an orthogonal matrix and has the properties defined in definition 3.

$$\mathbf{A} = \mathbf{Q}\mathbf{T}\mathbf{Q}^{-1} \tag{23}$$
$$\mathbf{A}^T = \left(\mathbf{Q}\mathbf{T}\mathbf{Q}^{-1}\right)^T \tag{24}$$
$$\mathbf{A} = \left(\mathbf{Q}^{-1}\right)^T \mathbf{T}^T \mathbf{Q}^T \tag{25}$$
$$\mathbf{A} = \mathbf{Q}\mathbf{T}^T\mathbf{Q}^T \tag{26}$$
$$\mathbf{Q}\mathbf{T}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{T}^T\mathbf{Q}^T \tag{27}$$
$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{Q}\mathbf{T}^T\mathbf{Q}^T \tag{28}$$
$$\implies \mathbf{T} = \mathbf{T}^T \tag{29}$$

If $\mathbf{T}$ is an upper triangular matrix and $\mathbf{T} = \mathbf{T}^T$ then it follows that the entries in the lower diagonal are equal to the entries in the upper diagonal which must be equal to zero. Hence, $\mathbf{T} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix.

Thus we have that $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}$. Furthermore we can show that the entries along the diagonal of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$. Let each diagonal entry along Λ be the eigenvalues of the matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{V} = \Lambda\mathbf{V}$ where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{A}$.

$$\mathbf{AV} = \Lambda \mathbf{V} \tag{30}$$
$$\mathbf{QDQ}^{-1}\mathbf{V} = \Lambda \mathbf{V} \tag{31}$$
$$\mathbf{Q}^T\mathbf{QDQ}^{-1}\mathbf{V} = \mathbf{Q}^T\Lambda \mathbf{V} \tag{32}$$
$$\mathbf{DQ}^{-1} = \Lambda \mathbf{Q}^T\mathbf{V} \tag{33}$$
$$\mathbf{DU} = \Lambda \mathbf{U} \tag{34}$$

where $\mathbf{U} = \mathbf{Q}^T\mathbf{V}$ is a change of basis. Finally, $\mathbf{D} = \Lambda$ and by the definition 11 $\mathbf{A}$ is therefore diagonalizable. □

Problem 3

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $m > n$ and let $\mathbf{A}$ be full-rank, i.e., $\text{rank}(\mathbf{A}) = n$ ($\mathbf{A}$ is non-defective by definition 8).

Let the matrix

$$\mathbf{C} = \mathbf{A}^T\mathbf{A} \tag{35}$$

a.

The matrix $\mathbf{C}$ is symmetric.

Proof. Suppose for the sake of contradiction that $\mathbf{C} \neq \mathbf{C}^T$. Starting from equation (35) we have

$$\mathbf{C} = \mathbf{A}^T\mathbf{A} \tag{36}$$
$$\mathbf{C}^T = \left(\mathbf{A}^T\mathbf{A}\right)^T \tag{37}$$
$$= \mathbf{A}^T\left(\mathbf{A}^T\right)^T \tag{38}$$
$$= \mathbf{A}^T\mathbf{A} \tag{39}$$
$$= \mathbf{C} \tag{40}$$

We have arrived at a contradiction, therefore $\mathbf{C}$ is symmetric. □

b.

$\mathbf{C}$ is positive-definite.

Proof. Recall that a matrix

$$\mathbf{M} := \text{positive definite} \iff \mathbf{x}^T\mathbf{M}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq \mathbf{0} \tag{41}$$

Let $\mathbf{x}$ be any positive vector such that $\mathbf{x} \neq \mathbf{0}$. This means that $\|\mathbf{x}\| > 0$. Starting from the definition (35) we have

$$\mathbf{C} = \mathbf{A}^T\mathbf{A} \tag{42}$$
$$\mathbf{x}^T\mathbf{C}\mathbf{x} = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} \tag{43}$$
$$= \left(\mathbf{A}\mathbf{x}\right)^T\mathbf{A}\mathbf{x} \tag{44}$$
$$= \|\mathbf{A}\mathbf{x}\|^2 > 0 \tag{45}$$

Additionally $\mathbf{x}^T\mathbf{C}\mathbf{x} \neq 0$ because for any $\mathbf{x} \in \mathbb{R} \quad \forall \mathbf{x} \neq \mathbf{0}$ if $\mathbf{A}\mathbf{x} = 0$ then $\mathbf{x}$ would be in the null space of $\mathbf{A}$, but this cannot be true because $\mathbf{A}$ is full-rank and the columns of $\mathbf{A}$ are linearly independent.

Therefore we have that $\mathbf{x}^T\mathbf{C}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}$ which satisfies the definition for $\mathbf{C}$ being a positive definite matrix. □

c.

Solving $\mathbf{Cx} = \mathbf{b}$ could be very inaccurate when $\mathbf{A}$ is ill-conditioned. To understand why, let us examine the condition number of the matrix $\mathbf{C}$ given by definition 10.

Since $\mathbf{A} \in \mathbb{R}^{m\times n}$ and is full-rank (since $\mathbf{A}$ is also full-rank), the condition number of the matrix $\mathbf{A}$ is given by,

$$\begin{aligned}\kappa(\mathbf{A}) &= \|\mathbf{A}\|\|\mathbf{A}^{-1}\| && (46)\\ &= \frac{\sigma_1}{\sigma_n} && (47)\end{aligned}$$

where σ_1 is the largest singular value and σ_n is the smallest singular value of the matrix $\mathbf{A}$.

Well, since the singular values of the matrix $\mathbf{C}$ are the eigenvalues of the matrix $\mathbf{C}^T\mathbf{C}$ then it is easy to show that the singular values of $\mathbf{C}$ are simply the singular values of the matrix $\mathbf{A}$ squared.

$$\begin{aligned}\mathbf{C}^T\mathbf{C} &= \left(\mathbf{A}^T\mathbf{A}\right)^T\mathbf{A}^T\mathbf{A} && (48)\\ &= \mathbf{A}^T\left(\mathbf{A}^T\right)^T\mathbf{A}^T\mathbf{A} && (49)\\ &= \left(\mathbf{A}^T\mathbf{A}\right)\left(\mathbf{A}^T\mathbf{A}\right) && (50)\\ &= \left(\mathbf{A}^T\mathbf{A}\right)^2 && (51)\end{aligned}$$

Thus the condition number $\kappa(\mathbf{C})$ can be expressed in terms of the singular values of the matrix $\mathbf{A}$ which are just the singular values of the matrix $\mathbf{A}$ squared.

$$\begin{aligned}\kappa(\mathbf{C}) &= \|\mathbf{C}\|\|\mathbf{C}^{-1}\| && (52)\\ &= \|\mathbf{A^TA}\|\|\mathbf{A^TA}^{-1}\| && (53)\\ &= \frac{\sigma_1^2}{\sigma_n^2} && (54)\\ &= \kappa(\mathbf{A})^2 && (55)\end{aligned}$$

Problem 4

Let $\mathbf{A} \in \mathbb{R}^{m\times m}$.

a.

If $\mathbf{A}$ has non-negative entries such that $\sum_{j=1}^{m} a_{ij} = 1$ for $1 \leq i \leq m$, then no eigenvalue of $\mathbf{A}$ has an absolute value greater than 1.

Proof. By the property of the spectral radius of the matrix $\mathbf{A}$,

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\| \quad \text{for any p-norm} \tag{56}$$

In order to show that $\rho(\mathbf{A}) \leq 1$ or that the largest eigenvalue of the matrix $\mathbf{A}$ is less than or equal to 1, we choose to use the infinity-matrix norm or the max-matrix norm, defined by

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \tag{57}$$

From equations (56) and (57) we can already see that

$$\rho(\mathbf{A}) \leq \max_i \sum_{j=1}^{m} |a_{ij}| = 1 \tag{58}$$

□

b.

Suppose that $\sum_{j=1}^{m} |a_{ij}| < 1$ for each i. Prove that $\mathbf{B} = \mathbf{I} - \mathbf{A}$ is invertible.

Proof. In order to prove that the matrix $\mathbf{B}$ is invertible. We will show that 0 is not an eigenvalue of $\mathbf{B}$.

For the sake of contradiction, suppose that 0 is an eigenvalue of $\mathbf{B}$.

Then

$$\mathbf{B}\mathbf{x} = 0 \tag{59}$$

Substituting the definition of $\mathbf{B}$ we have:

$$(\mathbf{I} - \mathbf{A})\,\mathbf{x} = 0 \tag{60}$$
$$\mathbf{I}\mathbf{x} - \mathbf{A}\mathbf{x} = 0 \tag{61}$$
$$\mathbf{A}\mathbf{x} = \mathbf{x} \tag{62}$$

By the definition of the eigenvalue, $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue 1.

Thus we have arrived at a contradiction, because by the propety of the spectral radius outlined in part a of this problem if the sum of the absolute value of each entry in every column of the matrix $\mathbf{A}$ is strictly less than 1, then the spectral radius of $\mathbf{A}$ or its largest eigenvalue must also be less than 1.

So $\mathbf{A}\mathbf{x} \neq \mathbf{x}$. Therefore 0 is not an eigenvalue of $\mathbf{B}$ and by the Equivlance Theorem for non-singular matrices, $\mathbf{B}$ is invertible. □

Problem 5

Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be symmetric positive definite (SPD).

a.

Definition 15 (Symmetric Positive Definite)**.** A real matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is symmetric if it has the same entries below the diagonal as above. Mainly,

$$a_{ij} = a_{ji} \quad \forall i, j. \tag{63}$$

Hence, $\mathbf{A} = \mathbf{A}^T$.

The matrix is positive definite if for any vector $\mathbf{x} \neq 0$ and $\mathbf{x} \in \mathbb{R}^m$,

$$\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \tag{64}$$

If the matrix satisfies both properties then it is SPD.

b.

If **A** is SPD, then every diagonal entry of **A** are strictly positive.

Proof. Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a real symmetric positive definite matrix. Let $\mathbf{e}_i$ be a column-vector such that only the i-th element in the column vector has the value 1 and all other entries are 0, where $1 \leq i \leq m$.

Since $\mathbf{e}_i$ is clearly not equal to the zero vector $\mathbf{0}$, and $\mathbf{A}$ satisfies the property that $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \quad \forall x \neq 0$, we have that

$$\mathbf{e}_i^T\mathbf{A}\mathbf{e}_i > 0 \tag{65}$$

Each basis vector contains only zero entries for all entries but the i-th, thus during matrix multiplication all values in the matrix **A** will contribute nothing to the total sum during matrix multiplication except the a_{ii} term. If we look at the component wise form for matrix vector multiplication,

$$y_i = \sum_{j=1}^{m} a_{ij}\delta_{ij} \quad \text{for all} \quad i = 1, 2, \cdots, m \tag{66}$$

where $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

thus the diagonal entries of the matrix **A**, a_{ii} must be strictly positive if **A** is SPD. □

c.

The eigenvalues of a symmetric positive definite matrix **A** are positive.

Proof. Let **A** be SPD. Let v, λ be an eigenvector, eigenvalue pair of the matrix **A**.

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{67}$$

$$(\mathbf{A}\mathbf{v})^T = (\lambda\mathbf{v})^T \tag{68}$$

$$\mathbf{v}^T\mathbf{A}^T = \lambda\mathbf{v}^T \tag{69}$$

Since **A** is symmetric, and satisfies the property that $\mathbf{A}^T = \mathbf{A}$,

$$\mathbf{v}^T\mathbf{A} = \lambda\mathbf{v}^T \tag{70}$$

$$\mathbf{v}^T\mathbf{A}\mathbf{v} = \lambda\mathbf{v}^T\mathbf{v} \tag{71}$$

$$= \lambda\|\mathbf{v}\|^2 \tag{72}$$

Since $\mathbf{v} \neq 0$, $\|\mathbf{v}\|^2 > 0$, and since $\mathbf{v}^T\mathbf{A}\mathbf{v} > 0$ (because of the matrix **A** being positive definite) we have,

$$\lambda\|\mathbf{v}\|^2 > 0 \tag{73}$$

Hence, $\lambda > 0$. Therefore every eigenvalue of the SPD matrix **A** is positive. □

Problem 6

a.

The algorithm is for basic Gaussian elimination, it converts the original matrix $\mathbf{A}$ into an upper triangular matrix $\mathbf{U}$.

b.

The number of floating point operations for gaussian elimination of a matrix $\mathbf{A}$ of size $m \times m$ is approximately $\frac{2}{3}m^3$ flops. **chapter 20, page 152, equation (20.8), numerical linear algebra textbook**

c.

The first problem is that the algorithm stops if any terms along the diagonal entries are 0. This can happen if the matrix is non-singular. The second problem is that even for a non-singular, well-conditioned matrix, this algorithm is not stable. Furthermore, it calculates entries that we already know to be zero.

example (from numerical linear algebra text book page 153, chapter 20, 20)

Let the matrix

$$\mathbf{A} = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix}. \tag{74}$$

Applying Gaussian elimination to $\mathbf{A}$ yields

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & 1 - 10^{20} \end{bmatrix} \tag{75}$$

if we use double precision as defined in 13, $\epsilon_{mach} = 10^{-16}$ then the value $1 - 10^{20}$ will only be represented by -10^{20}, which would yield the matrices

$$\tilde{\mathbf{L}} = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix} \quad \tilde{\mathbf{U}} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix} \tag{76}$$

If we try and recompute the matrix product $\tilde{\mathbf{L}}\tilde{\mathbf{U}} = \mathbf{A}$ we have,

$$\tilde{\mathbf{L}}\tilde{\mathbf{U}} = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 0 \end{bmatrix} \tag{77}$$

This matrix $\mathbf{A}$ is not the same as the original. The Gaussian elimination algorithm without pivoting is neither backward stable or stable as a general algorithm for $\mathbf{LU}$ factorization.

d.

Gaussian Elimination with Partial Pivoting (from textbook page 160, chapter 21, numerical linear algebra)

Algorithm 1 Gaussian Elimination with Partial Pivoting

Input: Matrix $\mathbf{A}$ of size $m \times m$
Output: Matrices $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{LU}$
Initialize $\mathbf{U} = \mathbf{A}$, $\mathbf{L} = \mathbf{I}$, $\mathbf{P} = \mathbf{I}$
for $k = 1$ to $m - 1$ **do**
  Select $i \geq k$ to maximize $|u_{ik}|$
  Interchange rows $u_{k,k:m} \leftrightarrow u_{i,k:m}$
  Interchange rows $l_{k,1:k-1} \leftrightarrow l_{i,1:k-1}$
  Interchange rows $p_{k,:} \leftrightarrow p_{i,:}$
  for $j = k + 1$ to m **do**
    $l_{jk} = \frac{u_{jk}}{u_{kk}}$
    $u_{j,k:m} = u_{j,k:m} - l_{jk} u_{k,k:m}$
  end for
end for

Problem 7.

Suppose $\mathbf{A} \in \mathbb{R}^{m \times m}$ is SPD.

a.

The matrix $\mathbf{A}_{i+1}$ is also symmetric and positive definite

Proof. In order to prove that $\mathbf{A}_{i+1}$ is SPD, we first need to show that it is symmetric, i.e., that it satisfies $\mathbf{A}_{i+1} = \mathbf{A}_{i+1}^T$.

$$
\begin{aligned}
(\mathbf{A}_{i+1})^T &= (\mathbf{L}_i^T \mathbf{L}_i)^T && (78)\\
&= \mathbf{L}_i^T \left(\mathbf{L}_i^T\right)^T && (79)\\
&= \mathbf{L}_i^T \mathbf{L}_i && (80)\\
&= \mathbf{A}_{i+1} && (81)
\end{aligned}
$$

We also need to establish that it is positive definite, mainly that for any $\mathbf{x} \neq 0$, $\mathbf{x}^T \mathbf{A_{i+1}} \mathbf{x} > 0$.

starting with the equality,

$$
\begin{aligned}
\mathbf{A}_{i+1} &= \mathbf{L}_i^T \mathbf{L}_i && (82)\\
\mathbf{x}^T \mathbf{A}_{i+1} \mathbf{x} &= \mathbf{x}^T \mathbf{L}_i^T \mathbf{L}_i \mathbf{x} && (83)\\
&= (\mathbf{L}_i \mathbf{x})^T (\mathbf{L}_i \mathbf{x}) && (84)\\
&= \|\mathbf{L}_i \mathbf{x}\|^2 && (85)
\end{aligned}
$$

Since $\mathbf{L}_i$ is a lower triangular matrix with non-zero diagonal elements,

$$
\mathbf{A}_{i+1} = \|\mathbf{L}_i \mathbf{x}\|^2 > 0 \tag{86}
$$

Therefore $\mathbf{A}_{i+1}$ is also SPD. □

b.

$\mathbf{A}_{i+1}$ is similar to $\mathbf{A}_0$, i.e., $\mathbf{A}_{i+1} = \mathbf{B}^{-1}\mathbf{A}_0\mathbf{B}$ for some non-singular matrix $\mathbf{B}$.

c.

Let $\mathbf{A}_0 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

The matrix $\mathbf{A_1} = \begin{bmatrix} \frac{10}{3} & \frac{\sqrt{5}}{3} \\ \frac{\sqrt{5}}{3} & \frac{5}{3} \end{bmatrix}$