AM213B Assignment #9

Problem 1 (Computational)

We use the method of characteristics to solve the 2D IVP

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} + \nabla\cdot\left(\vec{a}(x,y)u(x,y,t)\right) = 0 \\ u(x,y,0) = u_0(x,y) \equiv \sin^2(x+y) \end{cases} \qquad \text{(IVP-2D)}$$

where

$$\vec{a}(x,y) = \begin{pmatrix} a_1(x,y) \\ a_2(x,y) \end{pmatrix} \equiv \begin{pmatrix} \sin(x)\sin(y) \\ 1-\exp\left(\sin(x+y)\right) \end{pmatrix}$$

The whole problem is periodic in both x and y directions with period = 2π.

We first write out the divergence and write the PDE as

$$\frac{\partial u}{\partial t} + a_1(x,y)\frac{\partial u}{\partial x} + a_2(x,y)\frac{\partial u}{\partial y} = b(x,y)u$$

where

$$b(x,y) = -\frac{\partial a_1}{\partial x} - \frac{\partial a_2}{\partial y} = -\cos(x)\sin(y) + \exp\left(\sin(x+y)\right)\cos(x+y)$$

Our goal is to calculate the solution of the 2D IVP at any given point (ξ, η, T).

The method of characteristics consists of the two steps below.

- Tracing back the C-line from (ξ, η, T) to time 0

$$\begin{aligned} &\frac{dX}{dt} = a_1(X,Y) \\ &\frac{dY}{dt} = a_2(X,Y) \\ &X(T) = \xi, \quad Y(T) = \eta \end{aligned} \qquad \text{(FVP-1)}$$

  We use an ODE solve to solve this FVP from $t = T$ to $t = 0$.

  After solution, we set $x_0 = X(0)$ and $y_0 = Y(0)$ as the starting point for next step.

- Advancing from $(x_0, y_0, 0)$ to (ξ, η, T).

$$\frac{dx}{dt}=a_1(x,y)$$
$$\frac{dy}{dt}=a_2(x,y) \qquad \text{(IVP-1)}$$
$$\frac{dv}{dt}=b(x,y)v$$
$$x(0)=x_0\,,\quad y(0)=y_0\,,\quad v(0)=u_0(x_0,y_0)$$

We use an ODE solve to solve this IVP from t = 0 to $t = T$.

The solution of the 2D IVP at (ξ, η, T) is $u(\xi, \eta, T) = v(T)$.

Write a code to calculate $u(x, y, T)$ at any given point (x, y, T).

In your implementation, use RK4 with time step $h = 0.01$ ($h = -0.01$ in tracing back).

Test your code at $(x, y, T) = (3.9, 2.3, 1.2)$. You should get $u(3.9, 2.3, 1.2) \approx 5.340824$

Part 1:

Set $x_1 = 3.9$. Calculate and plot $u(x_1, y, T)$ as a function of y for $T = 0.75$, 1.0, and 1.25 in one figure. Use about 300 points for y in $[0, 2\pi]$.

Part 2:

Set $x_1 = 2.5$. Calculate and plot $u(x_1, y, T)$ as a function of y for $T = 0.75$, 1.0, and 1.25 in one figure. Use about 300 points for y in $[0, 2\pi]$.

Problem 2 (Computational)

Continue with the IVP in problem 1.

Calculate $u(x_1, y_1, t)$ as a function of t for $t \in [0, 1.25]$, for 3 sets of (x_1, y_1) below

$(x_1, y_1) = (3.9, 2.3), (2.7, 4.0), (2.0, 3.0)$

Plot $u(x_1, y_1, t)$ vs t for the 3 sets of (x_1, y_1) in one figure.

Problem 3 (Computational)

Continue with the IVP in problem 1. Consider the numerical grid on (x, y):

$$\Delta x=\Delta y=\frac{2\pi}{N}\,,\qquad x_i=i\,\Delta x\,,\quad 0\le i\le N\,,\qquad y_j=j\,\Delta x\,,\quad 0\le j\le N$$

Use $N = 80$ and calculate $u(x, y)$ on the grid for $T = 0.0$, 0.25, 0.5, 0.75, 1.0, and 1.25.

Plot $u(x, y)$ using contourf with colorbar (see sample code).

Plot 6 panels, one panel for each time level specified above.

Problem 4 (Computational)

Consider the general linear 1D conservation law

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial}{\partial x}\Big(a(x)u(x,t)\Big)=0$$

The Lax-Wendroff method based on the Taylor expansion is

$$u_i^{n+1}=u_i^n-r\Big(F_{i+1/2}-F_{i-1/2}\Big)$$

$$F_{i+1/2}=\frac{1}{2}\Big(a_i u_i^n+a_{i+1}u_{i+1}^n\Big)-\frac{r}{2}\left(\frac{a_i+a_{i+1}}{2}\right)\Big(a_{i+1}u_{i+1}^n-a_i u_i^n\Big)$$

Implement the Lax-Wendroff method (see sample code) to solve

$$\begin{cases}\dfrac{\partial u(x,y,t)}{\partial t}+\dfrac{\partial}{\partial x}\Big(\sin(x)\sin(y)u(x,y,t)\Big)=0\\ u(x,y,0)=\sin^2(x+y)\end{cases}\qquad\text{(IVPX-1D)}$$

where y is a parameter. The whole problem is periodic with period = 2π.

Select [0, 2π] as the computational domain. Set $\Delta x = 2\pi/N$ as the grid size.

Solve (IVPX-1D) for y = 1.5, 4, and 5. Use N = 300 and r = 0.5 in simulations.

Plot u vs x at T = 0.32π for y = 1.5, 4, and 5 in one figure.

Problem 5 (Computational)

Continue with (IVPX-1D) in Problem 4. Consider the numerical grid on (x, y):

$$\Delta x=\Delta y=\frac{2\pi}{N},\qquad x_i=i\,\Delta x,\quad 0\le i\le N,\qquad y_j=j\,\Delta x,\quad 0\le j\le N$$

Solve (IVPX-1D) for all y_j's to calculate $\{u(x_i, y_j)\}$ at T = 0.32π. Use N = 300 and r = 0.5.

Plot $u(x, y)$ at T = 0.32π, using contourf with colorbar.

Remark:

This is a component in the split-operator method.