AM213B Assignment #7

Problem 1 (Theoretical)

Consider the Lax-Friedrichs method for solving $u_t + a\, u_x = 0$

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

We write $(u_{i+1}^n + u_i^n)/2$ on the RHS as

$$\frac{u_{i+1}^n + u_{i-1}^n}{2} = u_i^n + \underbrace{\frac{1}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}_{\text{Added viscosity}}$$

The second part is the added viscosity. We know that the Lax-Friedrichs method tends to have too much added viscosity. So we consider a modified version of Lax-Friedrichs in which the added viscosity is multiplied by a factor of $0 < q \le 1$.

$$u_i^{n+1} = u_i^n + \frac{q}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}, \qquad 0 < q \le 1 \qquad \text{(LF-2)}$$

Part 1:

Use Taylor expansion to find the modified PDE of (LF-2).

Part 2:

Carry out von Neumann stability analysis to show that (LF-2) is stable if and only if

$$|ar| \le \sqrt{q}\,.$$

Can you make a connection between the modified PDE and the stability?

Hint:

Consider function $f(z) \equiv 1 - |\rho|^2$ where $z \equiv \sin^2 \frac{\xi \Delta x}{2}$.

The method is stable if and only if $f(z) \ge 0$ for $z \in [0, 1]$.

Problem 2 (Computational)

Consider the IBVP

$$\begin{cases} u_t + u_x = 0, & x \in (-0.5, 2.5), \quad t > 0 \\ u(x,0) = f(x), & x \in (-0.5, 2.5) \\ u(-0.5, t) = g(t), & t > 0 \end{cases}$$

where

$$f(x) = -\cos(\pi x), \qquad g(t) = \sin(\pi t)$$

The exact solution is expressed in terms of $f(x)$ and $g(t)$ as

$$u_{\text{ext}}(x,t) = \begin{cases} f(x-t), & x-t > -0.5 \\ g(t-x-0.5), & x-t \le -0.5 \end{cases}$$

Implement the 3 methods below to solve this IBVP.

- Upwind method
- Lax-Friedrichs method
- Lax-Wendroff method

Use the numerical grid

$$\Delta x = \frac{3}{N}, \quad x_j = -0.5 + j\Delta x, \quad x_0 = -0.5, \quad x_N = 2.5$$

The IBVP specifies a boundary condition only at $x_0 = -0.5$.

In the Lax-Friedrichs and Lax-Wendroff methods, we need an *artificial numerical boundary condition* at $x_N = 2.5$. In this problem, we use the exact solution.

$$u_N^n = u_{\text{ext}}(x_N, t_n)$$

In simulations, use $N = 300$ and $\Delta t = r\,\Delta x$ with, respectively, $r = 0.4$ and $r = 0.8$.

Use the exact solution to calculate the error of each method.

$$E_i^n = \left| u_{\text{ext}}(x_i, t_n) - u_i^n \right|$$

For each r value, plot errors vs x of the 3 methods at $t = 1.6$ in one figure. Plot two figures, one for $r = 0.4$ and the other for $r = 0.8$. Use log scale for the errors.

Which method has the smallest error? Which r value yields a smaller error?

Problem 3 (Computational)

Continue with the numerical solution of IBVP in Problem 2.

Change the initial value to

$$f(x) = \cos(\pi x), \quad x \in (-0.5, 2.5)$$

Change the artificial numerical boundary condition at $x_N = 2.5$ to

Condition 1: $\quad u_N^n = 2u_{N-1}^n - u_{N-2}^n \quad$ (extrapolation)

In simulations, use $r = 0.4$.

Part 1:

Plot numerical solutions of the 3 methods and the exact solution at $t = 1.6$ in one figure. Use linear scale for solutions.

Part 2:

Plot errors vs x of the 3 methods at $t = 1.6$ in one figure. Use log scale for the errors.

Part 3:

Observe the connection between the location of maximum error and the location of the cusp in the exact solution.

Observe whether or not the effect of artificial numerical boundary condition encroaches into the interior of computational region.

Problem 4 (Computational)

Continue with the numerical solution of IBVP in Problem 2.

Change the initial value to

$$f(x)=\begin{cases} \cos(\pi x), & x>0 \\ -\cos(\pi x), & x\le 0 \end{cases}$$

Note that the initial condition is discontinuous at $x = 0$.

In simulations, use $r = 0.4$ and use the artificial numerical boundary condition

$$u_N^n = 2u_{N-1}^n - u_{N-2}^n \quad \text{at } x_N = 2.5.$$

Part 1:

Plot numerical solutions of the 3 methods and the exact solution at $t = 1.6$ in one figure. Use linear scale for solutions.

Observe how the discontinuity is represented in the 3 numerical solutions.

Part 2:

Plot errors vs x of the 3 methods at $t = 1.6$ in one figure. Use log scale for the errors.

Observe the connection between the location of maximum error and the location of the discontinuity in the exact solution.

Problem 5 (Computational)

Consider the IVP

$$\begin{cases} u_t + \big(\sin(x)+\cos(x)\big)u_x = -\cos(x)u, & x\in(-\infty,+\infty), \quad t>0 \\ u(x,0)=\cos^2(x), & x\in(-\infty,+\infty) \end{cases}$$

Implement the method of characteristics to solve this IVP.

Write a code for calculating $u(x, t)$ at one point ($x = \xi$, $t = T$). In your implementation, use RK4 ODE solver with time step $h = 0.005$ ($h = -0.005$ when tracing characteristics from $t = T$ back to $t = 0$). Test your code at ($x = 1$ and $t = 0.5$). You should get

$$u(1, 0.5) \approx 0.6129394$$

Use your code to calculate $u(x, t)$ at three time levels: $t = 0.2$, $t = 0.5$, $t = 1$. At each time level, use about 200 points to represent u for $x \in [0, 2\pi]$.

Plot $u(x, t)$ vs x at these 3 time levels in one figure.