AM213B Assignment #5

Problem 1 (Theoretical)

Consider the BTCS and Crank-Nicolson methods for solving $u_t = u_{xx}$.

Part 1:

Carry out von Neumann stability analysis to show that

- The BTCS method is unconditionally stable
- The Crank-Nicolson method is unconditionally stable.

Part 2:

Carry out Taylor expansions to show that the local truncation error of the Crank-Nicolson method is

$$e_i^n(\Delta x, \Delta t) = \Delta t\, O\left((\Delta t)^2 + (\Delta x)^2\right)$$

In the final expression, convert r back to $\Delta t/(\Delta x)^2$.

Problem 2 (Theoretical)

Consider matrix

$$A = \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}_{N\times N}, \qquad \Delta x = \frac{1}{N+1}$$

Part 1:

Verify that the set below are eigenvalues and eigenvectors of matrix A.

$$\left.\begin{array}{l} \lambda^{(k)} = \dfrac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x) - 1\right) \\ w^{(k)} = \left\{\sin(k\pi i\Delta x),\quad i = 1,2,\ldots,N\right\} \end{array}\right\}, \quad k = 1,2,\ldots,N$$

Part 2:

Verify that

$$\frac{1}{(\Delta x)^2}\left(\cos(k\pi(i-1)\Delta x) - 2\cos(k\pi i\Delta x) + \cos(k\pi(i+1)\Delta x)\right)$$
$$= \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x) - 1\right)\cdot\cos(k\pi i\Delta x)$$

Explain why $u^{(k)} = \left\{ \cos(k\pi i \Delta x),\quad i = 1, 2, \ldots, N \right\}$ is NOT an eigenvector of A.

Problem 3 (Computational)

Consider the IBVP of the heat equation:

$$\begin{cases} u_t = u_{xx}, & x \in (0,\ 2),\quad t > 0 \\ u(x,0) = f(x), & x \in (0,\ 2) \\ u(0,t) = g_L(t), & u(2,t) = g_R(t) \end{cases}$$

where $f(x) = 0.5x$, $g_L(t) = \cos(2t)$, $g_R(t) = \sin(2t)$

Implement the FTCS method to solve the IBVP.

Solve the IBVP to T = 3. Use Δx = 0.01 and Δt specified below:

- Try $\Delta t = \frac{(\Delta x)^2}{2} \cdot \frac{1}{0.99}$, which is slightly above the stability threshold;
- Try $\Delta t = \frac{(\Delta x)^2}{2} \cdot \frac{1}{1.01}$, which is slightly below the stability threshold.

For each Δt, plot $u(x, t)$ vs x at t = 0.02, 0.5, 1, and 3 in one figure.

Hint:

Find N (# of internal points) according to the prescribed Δx.

Problem 4 (continue with the IBVP in problem 3)

Discretize the IBVP in the framework of **method of lines (MOL)**.

Implement the 2-statge DIRK method to solve the resulting ODE system.

Part 1:

Solve the IBVP to T = 3 with Δx = 0.01 and Δt = 0.01

Plot $u(x, t)$ vs x at t = 0.02, 0.5, 1, and 3 in one figure.

Part 2:

Repeat the calculation with Δx = 0.01 and Δt = 0.01/2. Use the two numerical solutions to estimate the error, which is the error associated with the time discretization.

Plot the estimated error vs x at t = 0.5, 1, and 3 in one figure. Use linear scales for both the error and x. Plot the estimated error vs x at t = 0.02 in a separate figure.

Hint:

The MOL ODE system has the matrix-vector form

$$\frac{d\vec{u}(t)}{dt} = A\vec{u}(t) + \vec{b}(t)$$

$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N-1}(t) \\ u_N(t) \end{pmatrix}, \qquad \vec{b}(t) = \frac{1}{(\Delta x)^2}\begin{pmatrix} g_L(t) \\ 0 \\ \vdots \\ 0 \\ g_R(t) \end{pmatrix}$$

$$A = \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}_{N\times N}, \qquad \Delta x = \frac{L}{N+1}$$

Find N according to the prescribed Δx. Implement the 2-statge DIRK method.

Problem 5

Consider the IBVP with an insulated boundary condition:

$$\begin{cases} u_t = u_{xx}, & x \in (0,\ 2),\quad t > 0 \\ u(x,0) = p(x), & x \in (0,\ 2) \\ u_x(0,t) = 0, & u(2,t) = q(t) \end{cases}$$

where $p(x) = (1-x)^2$, $q(t) = \cos(2t)$

<u>Numerical grid:</u>

$$\Delta x = \frac{L}{N+0.5}, \quad x_i = (i-0.5)\Delta x, \quad x_0 = -0.5\Delta x, \quad x_1 = 0.5\Delta x, \quad x_{N+1} = L$$

<u>The FTCS method:</u>

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{(\Delta x)^2}\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right), \qquad i = 1, 2, \ldots, N$$

$$u_0^n = u_1^n, \qquad u_{N+1}^n = q(n\Delta t)$$

where N = # of internal points.

Use the FTCS method to solve the IBVP to $T = 3$ with $N = 199$ and $\Delta t = 4\times 10^{-5}$.

<u>Plot $u(x, t)$ vs x at t = 0.02, 0.5, 1, and 3 in one figure.</u>