AM213B Numerical Methods for the Solution of Differential Equations

Lecture 18

List of topics in this lecture

- Jacobi iterative method for solving the discretization, residual, convergence of Jacobi iteration, slowest decaying modes in the error, weighted Jacobi iteration for reducing high wave number modes, mapping the system to a coarse grid to reduce low wave number modes
- Multigrid method: mapping between two grids using the residual-increment formulation, discretization at multiple levels, solution operator, the two-grid V-cycle

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Review:

The split-operator method for solving 2D problems

Goal: Solve the 2D problem accurately using a good 1D solver

$$u_t = (L_X + L_Y)[u], \qquad L_X[u] \equiv \frac{\partial F_1(u)}{\partial x}, \quad L_Y \equiv \frac{\partial F_2(u)}{\partial y}$$

Strategy: We split the 2D solution operator as follows:

$$\exp\left((L_X + L_Y)\Delta t\right) = \underbrace{\exp(L_X \frac{\Delta t}{2})}_{\substack{\text{One } \Delta t/2 \text{ step} \\ \text{of } L_X \text{ only}}} \underbrace{\exp(L_Y \Delta t)}_{\substack{\text{One } \Delta t \text{ step} \\ \text{of } L_Y \text{ only}}} \underbrace{\exp(L_X \frac{\Delta t}{2})}_{\substack{\text{One } \Delta t/2 \text{ step} \\ \text{of } L_X \text{ only}}} + O\left((\Delta t)^3\right)$$

Numerical solution of the Poisson equation

1-D BVP

$$\begin{cases} -u''(x) = s(x), & 0 < x < L \\ u(0) = c, \quad u(L) = d \end{cases}$$

Numerical discretization

$$h = \frac{L}{N}, \quad x_i = ih, \quad u_i \approx u(x_i), \quad u = \left(u_1, u_2, \ldots, u_{N-1}\right) \quad \text{(internal points)}$$

$$T_1 u = s + \beta$$

where vector s corresponds to $s(x)$ and vector β contains the effect of BCs.

Operator T_1 is defined in the difference form

$$T_1 : u \rightarrow T_1 u$$

$$(T_1 u)_i = -\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \right), \qquad 1 \le i \le (N-1)$$

with zero BCs $u_0 = u_N = 0$

Truncation error

$e(h) \equiv T_1 v - (s + \beta)$ where v is the exact solution

Error in numerical solution

$E(h) \equiv v - u$

Error analysis

$$\left\| E(h) \right\|_2 \le \frac{L^2}{4} \left\| e(h) \right\|_2$$

End of review

Our next task is to develop/analyze iterative methods for solving $T_1 u = s + \beta$.

Recall that operator T_1 is defined in the difference form. For any w, the calculation of $T_1 w$ is done conveniently using the difference form, without using the matrix form. To make a method practical for solving 2-D and 3-D problems, we want to avoid using the matrix form.

Framework of iterative methods

Start with an initial approximation u^0.

Map it to an improved approximation: $u^0 \rightarrow u^1$

Repeat this process: $u^k \rightarrow u^{k+1}$ until convergence

We study iterative methods that use only evaluations of $T_1 w$.

Jacobi iterative method for solving $T_1 u = s + \beta$

We start with the component-wise form of $T_1 u = s + \beta$.

$$-\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \right) = s_i + \beta_i , \qquad 1 \le i \le N-1 , \quad u_0 = u_N = 0$$

Keeping u_i on the LHS and moving others to the RHS, we obtain

$$u_i = \frac{u_{i-1} + u_{i+1}}{2} + \frac{h^2}{2}(s_i + \beta_i)$$

We view the RHS as a mapping and we use it to map $u^k \rightarrow u^{k+1}$.

The Jacobi iterative method in the component-wise form:

$$u_i^{k+1} = \frac{u_{i-1}^k + u_{i+1}^k}{2} + \frac{h^2}{2}\left(s_i + \beta_i\right)$$

We write the Jacobi iterative method in the operator-vector form.

We first write T_1 as

$$T_1 = L + D + U$$

where L is the lower triangular part of T_1, U is the upper triangular part of T_1, and D is the diagonal of T_1. Recall the definition of operator T_1.

$$(T_1 u)_i = -\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}\right), \qquad 1 \le i \le (N-1)$$

Operator D is very simple. Dw and $D^{-1}w$ are very easy to calculate.

$$D = \frac{2}{h^2} I\,, \qquad Dw = \frac{2}{h^2} w\,, \qquad D^{-1}w = \frac{h^2}{2} w \qquad \text{for any vector } w.$$

We write linear system $T_1 u = s + \beta$ as

$$(L + D + U)u = s + \beta$$

$$\Longrightarrow \quad Du = -\underbrace{\left(L + U\right)}_{T_1 - D} u + s + \beta \qquad \text{(We want to work with } T_1\text{)}$$

$$\Longrightarrow \quad u = D^{-1}\left[-\left(T_1 - D\right)u + s + \beta\right]$$

$$\Longrightarrow \quad u = u + D^{-1}\left[-T_1 u + s + \beta\right]$$

The Jacobi iterative method in the operator-vector form:

$$u^{k+1} = u^k + D^{-1}\left(-T_1 u^k + s + \beta\right)$$

Remark:

The Jacobi iterative method is in the form of $u^{k+1} = u^k$ + increment.

The increment can be used to determine when to stop the iteration.

The increment is calculated based on how well u^k satisfies the linear system $(-T_1 u^k + s + \beta)$, which leads to the definition of residual.

Definition (Residual):

Residual of u^k is defined as $r^k \equiv -T_1 u^k + s + \beta$.

The Jacobi iterative method in the residual-increment form

$$u^{k+1} = u^k + D^{-1} r^k$$

where $r^k = -T_1 u^k + s + \beta$ is the residual of u^k.

Remark:

This residual-increment form is very useful numerically and theoretically.

To analyze the convergence of Jacobi iterative method, it is more convenient to do it in the framework of a general iterative method.

Convergence of a general iterative method $u^{k+1} = Au^k + f$

Let u be the solution of the linear system $u = Au + f$ and let $\varepsilon^k = u - u^k$.

$u = Au + f$ linear system

$u^{k+1} = Au^k + f$ iterative method for solving the linear system

==> $\varepsilon^{k+1} = A\varepsilon^k$

==> $\varepsilon^k = A^k \varepsilon^0$

Theorem (convergence of iteration)

If $\|A\|_2 < 1$, then the iteration $u^{k+1} = A\,u^k + f$ converges for any u^0.

Proof:

$$\|\varepsilon^k\|_2 = \|A^k \varepsilon^0\|_2 \le \|A^k\|_2 \|\varepsilon^0\|_2 \le \left(\|A\|_2\right)^k \|\varepsilon^0\|_2 \to 0 \qquad \text{as } k \to +\infty$$

We use this theorem to analyze the Jacobi iteration.

Convergence of Jacobi iterative method

We write the Jacobi iteration in the form of $u^{k+1} = A\,u^k + f$,

$$u^{k+1} = u^k + D^{-1} r^k, \qquad r^k = -T_1 u^k + s + \beta$$

==> $$u^{k+1} = \underbrace{(I - D^{-1}T_1)}_{A} u^k + \underbrace{D^{-1}(s+\beta)}_{f} \equiv Au^k + f$$

Using $D = (2/h^2)I$, we write the iteration matrix A as

$$A = I - D^{-1}T_1 = I - \frac{h^2}{2}T_1$$

Recall that the eigenvalues of T_1 are

$$\lambda^{(n)} = \frac{4}{h^2}\sin^2\left(\frac{n\pi}{2N}\right), \qquad 1 \le n \le (N-1)$$

The eigenvalues of A are

$$\mu^{(n)} = 1 - \frac{h^2}{2}\lambda^{(n)} = 1 - \frac{h^2}{2}\frac{4}{h^2}\sin^2\left(\frac{n\pi}{2N}\right) = \cos\left(\frac{n\pi}{N}\right), \qquad 1 \le n \le (N-1)$$

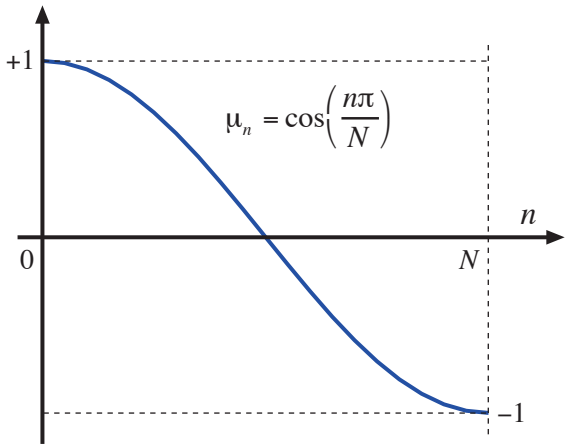
All eigenvalues of A satisfy $|\mu^{(n)}| < 1$, which leads to the theorem below.

Theorem:

The Jacobi iterative method converges for any u^0.

Remark:

We will see that the Jacobi iterative method converges very slowly. This is evident from the plot of $\mu^{(n)}$ vs n shown below. Although all eigenvalues are strictly below 1 in magnitude, two of them are very close to 1 in magnitude.



Let $\varepsilon^k = (u - u^k)$. We examine the speed of convergence $\varepsilon^k \to 0$ in more details.

Slowest decaying modes in $\varepsilon^k = (u - u^k)$

Let $\varepsilon^0 = \sum_{n=1}^{N-1} c_n w^{(n)}$.

This is called the eigenvector expansion of ε^0. Matrix $A = I - D^{-1}T_1$ is real and symmetric. Consequently, it has a complete and orthonormal set of eigenvectors. Thus, the eigenvector expansion is justified.

After k iterations, $\varepsilon^k = u - u^k$ has the expression

$$\varepsilon^k = (A)^k \varepsilon^0 = \sum_{n=1}^{N-1} (\mu^{(n)})^k c_n w^{(n)}$$

Each mode in $\varepsilon^k = (u - u^k)$ decays with a different rate.

The slowest decaying modes are $w^{(1)}$ and $w^{(N-1)}$.

$$\mu^{(1)} = \cos\left(\frac{\pi}{N}\right) \approx 1 - \frac{\pi^2}{2N^2} \approx \exp\left(\frac{-\pi^2}{2N^2}\right)$$

$$\Rightarrow \quad (\mu^{(1)})^k \approx \exp\left(\frac{-k\pi^2}{2N^2}\right)$$

$$\mu^{(N-1)} = \cos\left(\frac{(N-1)\pi}{N}\right) = -\cos\left(\frac{\pi}{N}\right) \approx -\exp\left(\frac{-\pi^2}{2N^2}\right)$$

$$\Rightarrow \quad (\mu^{(N-1)})^k \approx (-1)^k \exp\left(\frac{-k\pi^2}{2N^2}\right)$$

We look at a few numbers.

- To make $\exp\left(\frac{-k\pi^2}{2N^2}\right) \le \frac{1}{2}$, we need $k \ge N^2 \frac{2}{\pi^2}\log(2) \approx 0.1405N^2$.
- To make $\exp\left(\frac{-k\pi^2}{2N^2}\right) \le \frac{1}{10}$, we need $k \ge N^2 \frac{2}{\pi^2}\log(10) \approx 0.4666N^2$.
- To make $\exp\left(\frac{-k\pi^2}{2N^2}\right) \le \frac{1}{10^5}$, we need $k \ge N^2 \frac{2}{\pi^2}\log(10^5) \approx 2.333N^2$.

It takes O(N^2) iterations to reduce the slowest decaying modes by a given factor.

In summary, the Jacobi iterative method has two properties:

1) It converges for any u^0.
2) It takes O(N^2) iterations to converge. The slowest decaying modes in $\varepsilon^k = (u - u^k)$ are the lowest wave number $w^{(1)}$ and the highest wave number $w^{(N-1)}$.

Property 1) is good. Property 2) is not good.

We need to speed up the decay of the lowest and the highest wave numbers.

Below, we first modify the iteration method to make all high wave number modes decay rapidly. Then we introduce the multigrid technique to speed up the decay of low wave number modes in $\varepsilon^k = (u - u^k)$.

Weighted Jacobi iteration (damped Jacobi iteration)

$$u^{k+1} = u^k + \underbrace{\omega D^{-1} r^k}_{\text{increment}}, \qquad 0 < \omega \le 1$$

where $r^k = -T_1 u^k + s + \beta$ is the residual of u^k.

We write it in the general form of $u^{k+1} = Au^k + f$. The iteration matrix A is

$$A = I - \omega D^{-1} T_1$$

Eigenvalues of A are

$$\mu^{(n)}(\omega) = 1 - 2\omega \sin^2\left(\frac{n\pi}{2N}\right) = (1-\omega) + \omega\cos\left(\frac{n\pi}{N}\right), \quad 1 \le n \le N-1$$

We select $\omega = 2/3$. A plot of $\mu^{(n)}$ vs n for $\omega = 2/3$ is shown below.

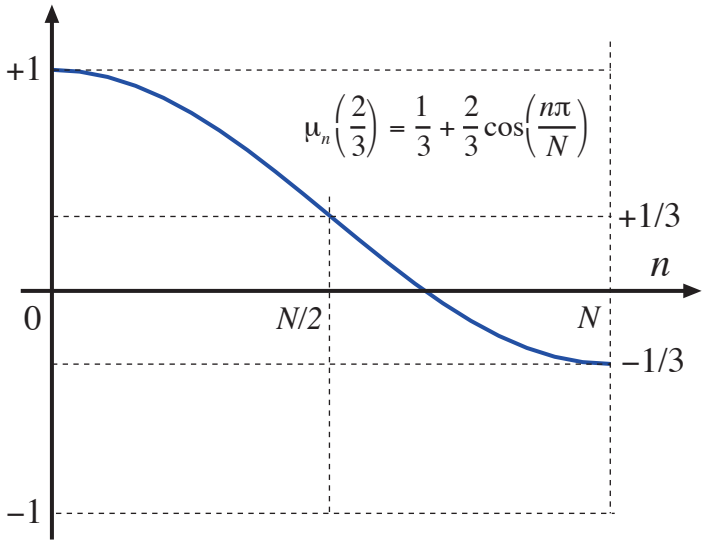
Behavior of high wave numbers under weighted Jacobi

All wave numbers between $N/2$ and N decay as $(1/3)^k$ or faster (see plot below).

That is a very fast decay. The decay rate is independent of N.

To make $(1/3)^k \le 10^{-5}$, we need $k \ge \log_3(10^5) \approx 10.48$, independent of N.

It takes O(1) iterations to reduce high wave number modes by a given factor.



Behavior of low wave numbers under weighted Jacobi

We focus on the lowest wave number $w^{(1)}$. Its decay rate is

$$\mu^{(1)}(\omega) = (1-\omega) + \omega\cos\left(\frac{\pi}{N}\right) \approx (1-\omega) + \omega\left(1 - \frac{\pi^2}{2N^2}\right) \approx 1 - \omega\frac{\pi^2}{2N^2}$$

The decay of $w^{(1)}$ is slightly slower than that for $\omega = 1$ (regular Jacobi).

We need to speed up the decay of low wave number modes in $\varepsilon^k = (u - u^k)$.

Basic idea for speeding up the decay of low wave number modes

Key observation on the decay of $w^{(1)}$

$(\mu^{(1)}(\omega))^k$ decays faster when N is smaller (on a coarser grid)!

Strategy:

We map the system to a coarser grid.

Caution:

Mapping u^k to a coarser grid will introduce an error that does not decrease with k.

Remedy:

We map residual and increment down and up between two grids.

Multigrid method

We expand the general strategy stated above.

Mapping down and up using the residual-increment formulation

Let $u^{(ap)}$ be the current approximate solution of linear system $T_1 u = b$.

Task:

To reduce low wave number modes in the error of $u^{(ap)}$

Approach:

- We write the true solution as current approximation + increment

  $$u = u^{(ap)} + \underline{u}$$

  The increment $\underline{u}$ satisfies the residual-increment system

  $$T_1 \underline{u} = r \quad \text{where} \quad r \equiv \underbrace{-T_1 u^{(ap)} + b}_{\text{residual}}$$

  We work with the residual-increment system $T_1 \underline{u} = r$.

  The current approximate solution of $T_1 \underline{u} = r$ is $\underline{u}^{(ap)} = 0$.

  Our task is to reduce low wave number modes in the error of $\underline{u}^{(ap)} = 0$.
- We map residual r from the fine grid down to the coarse grid.
- On the coarse grid, we do one step of weight Jacobi on $\underline{u}^{(ap)} = 0$ to reduce some low wave number modes (low relative to the fine grid).

  We obtain $\underline{u}^{(ap)}$(new) on the coarse grid.
- We map $\underline{u}^{(ap)}$(new) from the coarse grid up to the fine grid.
- On the fine grid, update the approximate solution of $T_1 u = b$ using $u = u^{(ap)} + \underline{u}$.

  $$u^{(ap)}(\text{new}) = u^{(ap)} + \underline{u}^{(ap)}(\text{new})$$

Remark:

This mapping down and up is for reducing low wave number modes in the error. For reducing high wave number modes in the error, we need to do one step of weighted Jacobi on the fine grid.

Advantage of the residual-increment formulation

When the starting approximation $u^{(ap)}$ is already the true solution of $T_1 u = b$, we have

- residual is zero: $r \equiv -T_1 u^{(ap)} + b = 0$;
- the approximate solution of residual-increment system $T_1 \underline{u} = r$ is

  $\underline{u}^{(ap)}(\text{new}) = 0$;
- there is no error introduced in mapping down and up!

In general, when we directly map the linear system $T_1 u = s + \beta$ between the coarse and fine grids, the error associated with the mapping is roughly proportional to the magnitude of $s + \beta$, which does not decrease as $k \to \infty$. This mapping error will ruin the convergence of u^k.

When we map the residual-increment system $T_1 \underline{u} = r$ between the coarse and fine grids, the error associated with the mapping is roughly proportional to the magnitude of r, which decreases to zero as $k \to \infty$ (as u^k converges to u).

We expand the general strategy more and describe the procedure in details.

Discretization at multiple grid levels

Numerical grid at level i:

G_i: $\quad N_i = 2^i, \quad h_i = \dfrac{L}{N_i}, \quad x_j^{(i)} = j h_i$

$$u^{(i)} = \left(u_1^{(i)}, u_2^{(i)}, \ldots, u_{N_i-1}^{(i)}\right)^T$$

Note:

Here index i refers to the grid level.

At each new grid level, N is doubled.

Numerical discretization on grid G_i:

$$T_1^{(i)} u^{(i)} = b^{(i)}$$

where operator $T_1^{(i)}$ is defined in the difference form on grid G_i.

$$T_1^{(i)} : u^{(i)} \to T_1^{(i)} u^{(i)}$$

$$\left(T_1^{(i)} u^{(i)}\right)_j = -\frac{u_{j-1}^{(i)} - 2u_j^{(i)} + u_{j+1}^{(i)}}{h_i^2}, \qquad j = 1, 2, \ldots, N_i - 1$$

with zero BCs $u_0^{(i)} = u_{N_i}^{(i)} = 0$

Here the RHS $b^{(i)}$ is the residual of the current approximate solution.

Notations

True solution and approximate solutions

$u^{(i)}$: true solution of $T_1^{(i)} u^{(i)} = b^{(i)}$

$v^{(i)}, v^{(i)*}, \underline{v}^{(i)}\ V^{(i)}, V^{(i)*}$: approximate solutions

Question: Why do we need so many symbols for approximate solutions?

We need distinct symbols for

*) current approximation to true solution $u^{(i)}$,

*) new approximation to $u^{(i)}$,

*) approximation to increment ($u^{(i)} - v^{(i)}$), and

*) these approximations before and after mapping down and up …

We write one step of weighted Jacobi as a solution operator.

The solution operator

$$S : \left(b^{(i)}, v^{(i)}\right) \longrightarrow v^{(i)*}$$

where

$b^{(i)}$: RHS of the linear system (from residual)

$v^{(i)}$: current approximation before the iteration

$v^{(i)*}$: new approximation after the iteration

The mapping is just one step of weighted Jacobi method

$$v^{(i)*} = v^{(i)} + \omega D^{-1}\left(-T_1^{(i)} v^{(i)} + b^{(i)}\right)$$

The key building block (the two-grid V-cycle)

Start with $v^{(i)}$, the current approximate solution of $T_1^{(i)} u^{(i)} = b^{(i)}$.

Step 1:

Do one step of weight Jacobi (WJ) on G_i

(This is for reducing high wave number modes in the error).

$$v^{(i)*} = S\left(b^{(i)}, v^{(i)}\right)$$

Calculate residual of $v^{(i)*}$

$$r^{(i)} = -T_1^{(i)} v^{(i)*} + b^{(i)}$$

Switch to the residual-increment system

$$T_1^{(i)} \underline{u}^{(i)} = r^{(i)}$$

The true solutions of the two systems are related by

$$u^{(i)} = v^{(i)*} + \underline{u}^{(i)}$$

Step 2:

Map the residual-increment system from G_i down to G_{i-1}

$$r^{(i)} \longrightarrow r^{(i-1)}$$

(this mapping will be described later)

Step 3:

Do one step of WJ on G_{i-1} with current approximation = 0.

(This is for reducing some low wave number modes, low relative to G_i).

$$\underline{v}^{(i-1)} = S\left(r^{(i-1)}, 0\right)$$

Step 4:

Map the new approximation from G_{i-1} up to G_i

$$\underline{v}^{(i-1)} \longrightarrow \underline{v}^{(i)}$$

(this mapping will be described later)

On G_i, update the approximate solution of $T_1^{(i)} u^{(i)} = b^{(i)}$ using $u^{(i)} = v^{(i)*} + \underline{u}^{(i)}$

$$V^{(i)} = v^{(i)*} + \underline{v}^{(i)}$$

Step 5:

Do one step of WJ on G_i.

(This is for reducing high wave number modes in the error).

$$V^{(i)*} = S\left(b^{(i)}, V^{(i)}\right)$$

Ending with $V^{(i)*}$, the new approximate solution of $T_1^{(i)} u^{(i)} = b^{(i)}$.

Remarks:

- The two grid V-cycle can be viewed/used as an iteration mapping.

  The input $v^{(i)}$ is an approximate solution of $T_1^{(i)} u^{(i)} = b^{(i)}$.

  The output $V^{(i)*}$ is a new approximate solution of $T_1^{(i)} u^{(i)} = b^{(i)}$.

The process can be repeated.

- In the two grid V-cycle, on G_{i-1}, we do one step of WJ. We can replace that one step of WJ with a two grid V-cycle between grids G_{i-1} and G_{i-2}.
- The two grid V-cycle can be applied recursively.