**AM213B Numerical Methods for the Solution of Differential Equations**

**Lecture 13**



**List of topics in this lecture**

- Lax-Wendroff method, Leap-frog method
- The CFL condition, domain of dependence, necessary condition for convergence
- Method of characteristics
- Modified PDE of a numerical method

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**Review of numerical solution of hyperbolic PDEs**

$$u_t + a u_x = 0$$

The upwind method

$$u_i^{n+1} = u_i^n - ar\left(u_i^n - u_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}, \quad a > 0$$

1st order in time and in space; stable if and only if $0 \le (a\,r) \le 1$.

The downwind method

$$u_i^{n+1} = u_i^n - ar\left(u_{i+1}^n - u_i^n\right), \quad r = \frac{\Delta t}{\Delta x}, \quad a > 0$$

unstable for all $r$.

The FTCS method

$$u_i^{n+1} = u_i^n - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}, \quad \text{regardless of the sign of } a$$

1st order in time and 2nd order in space; unstable for all $r$.

Lax-Friedrichs method

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}, \quad \text{regardless of the sign of } a$$

1st order in time and in space; stable if and only if $|\,a\,r\,| \le 1$.

We want a method having i) a unified form and ii) 2nd order in time and space.

**Lax-Wendroff method** for solving $u_t + au_x = 0$

We derive the Lax-Wendroff method using Taylor expansion.

$$u(x_i,t_{n+1})-u(x_i,t_n)=u_t\big|_{(x_i,t_n)}\Delta t+u_{tt}\big|_{(x_i,t_n)}\frac{(\Delta t)^2}{2}+O\left((\Delta t)^3\right)$$

We use PDE $u_t + au_x = 0$ to re-write $u_t$ and $u_{tt}$

$$u_t=-au_x$$
$$u_{tt}=(-au_x)_t=-a(u_t)_x=-a(-au_x)_x=a^2u_{xx}$$

Substituting into Taylor expansion, we get

$$u(x_i,t_{n+1})-u(x_i,t_n)=-a\Delta t\,u_x\big|_{(x_i,t_n)}+a^2\frac{(\Delta t)^2}{2}u_{xx}\big|_{(x_i,t_n)}+O\left((\Delta t)^3\right)$$

We replace the derivatives using central differences.

$$u_x\big|_{(x_i,t_n)}=\frac{u(x_{i+1},t_n)-u(x_{i-1},t_n)}{2\Delta x}+O\left((\Delta x)^2\right)$$
$$u_{xx}\big|_{(x_i,t_n)}=\frac{u(x_{i+1},t_n)-2u(x_i,t_n)+u(x_{i-1},t_n)}{(\Delta x)^2}+O\left((\Delta x)^2\right)$$

The Taylor expansion becomes

$$u(x_i,t_{n+1})-u(x_i,t_n)=-\frac{a}{2}\cdot\frac{\Delta t}{\Delta x}\left(u(x_{i+1},t_n)-u(x_{i-1},t_n)\right)$$
$$+\frac{a^2}{2}\cdot\left(\frac{\Delta t}{\Delta x}\right)^2\left(u(x_{i+1},t_n)-2u(x_i,t_n)+u(x_{i-1},t_n)\right)+\underbrace{\Delta t\,O\left((\Delta t)^2+(\Delta x)^2\right)}_{\text{LTE}}$$

<u>Lax-Wendroff method:</u>

$$u_i^{n+1}=u_i^n-\frac{ar}{2}\left(u_{i+1}^n-u_{i-1}^n\right)+\frac{(ar)^2}{2}\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right),\qquad r=\frac{\Delta t}{\Delta x}$$

<u>Local truncation error</u> (LTE) of Lax-Wendroff method:

$$e_i^n(\Delta x,\Delta t)=\Delta t\,O\left((\Delta t)^2+(\Delta x)^2\right)$$

It is 2nd order in time and 2nd order in space.

<u>von Neumann stability analysis</u> on Lax-Wendroff method:

Substituting $u_i^n=\rho^n e^{\sqrt{-1}\xi i\Delta x}$ into the numerical method yields

$$\rho = 1 - \frac{ar}{2}\underbrace{\left(e^{\sqrt{-1}\xi\Delta x} - e^{-\sqrt{-1}\xi\Delta x}\right)}_{=2\sqrt{-1}\sin(\xi\Delta x)} + \frac{(ar)^2}{2}\underbrace{\left(e^{\sqrt{-1}\xi\Delta x} - 2 + e^{-\sqrt{-1}\xi\Delta x}\right)}_{=-4\sin^2\frac{\xi\Delta x}{2}}$$

==> $\rho = 1 - 2(ar)^2\sin^2\frac{\xi\Delta x}{2} - \sqrt{-1}(ar)\sin(\xi\Delta x)$

==> $|\rho|^2 = \left(1 - 2(ar)^2\sin^2\frac{\xi\Delta x}{2}\right)^2 + (ar)^2\sin^2(\xi\Delta x)$

Using $\sin^2\alpha = 4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} = 4\left(\sin^2\frac{\alpha}{2} - \sin^4\frac{\alpha}{2}\right)$, we write $|\rho|^2$ as

$$|\rho|^2 = 1 - 4(ar)^2\sin^2\frac{\xi\Delta x}{2} + 4(ar)^4\sin^4\frac{\xi\Delta x}{2} + 4(ar)^2\left(\sin^2\frac{\xi\Delta x}{2} - \sin^4\frac{\xi\Delta x}{2}\right)$$

$$= 1 - 4(ar)^2\left(1 - (ar)^2\right)\sin^4\frac{\xi\Delta x}{2}$$

The case of $|ar| \le 1$:

$|\rho|^2 \le 1$

The case of $|ar| > 1$:

At $\xi\Delta x = \pi$, we have $\sin(\xi\Delta x/2) = 1$ and

$|\rho|^2 = 1 + 4(ar)^2\left((ar)^2 - 1\right) > 1$

In summary, Lax-Wendroff method is stable if and only if $|a\,r| \le 1$.

Next, we examine a multi-step method.

**Leap-frog method** for solving $u_t + au_x = 0$

We use central difference in both the $x$-direction and the $t$-direction.

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

==> $u_i^{n+1} = u_i^{n-1} - ar\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$

Local truncation error of leap-frog method:

$$e_i^n(\Delta x, \Delta t) = \Delta t\, O\left((\Delta t)^2 + (\Delta x)^2\right)$$

It is 2nd order in time and 2nd order in space.

<u>von Neumann stability analysis on the leap-frog method</u> (skip in lecture)

Substituting $u_i^n = \rho^n e^{\sqrt{-1}\xi i \Delta x}$ into the numerical method yields

$$\rho^2 = 1 - \rho(ar)\left(e^{\sqrt{-1}\xi\Delta x} - e^{-\sqrt{-1}\xi\Delta x}\right)$$

==> $\quad \rho^2 + \rho\sqrt{-1}\cdot 2(ar)\sin(\xi\Delta x) - 1 = 0$

This is a quadratic equation of ρ. We use the quadratic formula

$$\rho_{1,2} = -\sqrt{-1}\,(ar)\sin(\xi\Delta x) \pm \sqrt{1-(ar)^2\sin^2(\xi\Delta x)}$$

We discuss three cases.

<u>The case of $|a\,r| < 1$</u>:

$$1-(ar)^2\sin^2(\xi\Delta x) \ge 1-(ar)^2 > 0$$

Two distinct roots satisfying $|\rho_{1,2}| = 1$.

The numerical method is stable for $|a\,r| < 1$.

<u>The case of $|a\,r| = 1$</u>:

At $\xi\Delta x = \pi/2$, we have $\sin(\xi\Delta x) = 1$ and $1-(ar)^2\sin^2(\xi\Delta x) = 0$.

$\rho_{\text{double}} = -\sqrt{-1}\,(ar)$ is a <u>double root.</u>

The numerical method has a solution

$$u_i^n = n(\rho_{\text{double}})^n e^{\sqrt{-1}\xi i\Delta x} = n(-\sqrt{-1}(ar))^n e^{\sqrt{-1}\, i\pi/2} = n(-\sqrt{-1}(ar))^n(\sqrt{-1})^i .$$

$|u_i^n| = n$ which grows unbounded.

The numerical method is unstable for $|a\,r| = 1$.

<u>The case of $|a\,r| > 1$</u>:

At $\xi\Delta x = \pi/2$, we have $\sin(\xi\Delta x) = 1$ and $1-(ar)^2\sin^2(\xi\Delta x) = 1-(ar)^2 < 0$.

Two distinct imaginary roots: $\quad \rho_{1,2} = -\sqrt{-1}\left((ar) \pm \sqrt{(ar)^2-1}\right)$

==> $\quad \max|\rho_{1,2}| = |ar| + \sqrt{(ar)^2-1} > 1$

The numerical method is unstable for $|a\,r| > 1$.

<u>In summary,</u> the leap-frog method is stable if and only if $|a\,r| < 1$.

Note that the stability condition of the leap-frog method is slightly different from those of upwind, Lax-Friedrichs, and Law-Wendroff:

$|a\,r| < 1 \qquad$ vs $\qquad |a\,r| \le 1$

**The CFL condition (Courant -Friedrichs-Lewy condition)**

To describe the CFL condition, we introduce the concept of domain of dependence.

Domain of dependence

The exact solution of $u_t + au_x = 0$ ($a > 0$) is

$$\underbrace{u(x,t)}_{\substack{\text{Solution} \\ \text{after } t}} = \underbrace{u(x-at,0)}_{\text{Initial condition}}$$

==> $u(x, t_1+\Delta t) = u(x-a\Delta t, t_1)$

Solution at position $x$ at time $(t_1+\Delta t)$ is determined by the solution at position $(x-a\Delta t)$ at time $t_1$. We say that the domain of dependence of position $x$ tracing back time $\Delta t$ is position $(x-a\Delta t)$.

Notation:

$D_{\text{PDE}}(x, \Delta t)$ = domain of dependence of position $x$ tracing back time $\Delta t$ for the PDE

$D_{\text{num}}(x, \Delta t)$ = domain of dependence of position $x$ tracing back time $\Delta t$ for the numerical method

Example:

For PDE $u_t + au_x = 0$ ($a > 0$), we have

$D_{\text{PDE}}(x_i, \Delta t) = \{ x_i - a\Delta t \}$

(**Draw characteristics to show $D_{\text{PDE}}$ and solution propagation**)

Consider the upwind method,

$$u_i^{n+1} = u_i^n - ar(u_i^n - u_{i-1}^n), \qquad r = \frac{\Delta t}{\Delta x}, \qquad a > 0$$

$D_{\text{num}}(x_i, \Delta t) = [x_{i-1}, x_i] = [x_i - \Delta x, x_i]$

(**Draw the stencil of upwind method to show $D_{\text{num}}$**)

Recall that the upwind method is stable if and only if $a\,r \le 1$

if and only if $a\Delta t \le \Delta x$

if and only if $(x_i - \Delta x) \le (x_i - a\Delta t)$.

We compare $D_{\text{PDE}}(x_i, \Delta t) = \{ x_i - a\Delta t \}$ and $D_{\text{num}}(x_i, \Delta t) = [x_i - \Delta x, x_i]$.

The upwind method is stable if and only if $D_{\text{PDE}} \subseteq D_{\text{num}}$.

Example:

For PDE $u_t + au_x = 0$, we consider the downwind method,

$$u_i^{n+1} = u_i^n - ar(u_{i+1}^n - u_i^n), \qquad r = \frac{\Delta t}{\Delta x}, \qquad a > 0$$

$D_{\text{num}}(x_i, \Delta t) = [x_i, x_{i+1}] = [x_i, x_i + \Delta x]$

**(Draw the <u>stencil</u> of downwind method to show $D_{\text{num}}$)**

Recall that the downwind method is unstable for all $r$.

We compare $D_{\text{PDE}}(x_i, \Delta t) = \{ x_i - a\Delta t \}$ and $D_{\text{num}}(x_i, \Delta t) = [x_i, x_i + \Delta x]$.

For the downwind method, $D_{\text{PDE}} \subseteq D_{\text{num}}$ is always false for $a > 0$.

Example

For PDE $u_t + au_x = 0$, we consider the Lax-Friedrichs method,

$$u_i^{n+1} = \frac{u_{i-1}^n + u_{i+1}^n}{2} - \frac{ar}{2}(u_{i+1}^n - u_{i-1}^n), \qquad r = \frac{\Delta t}{\Delta x}$$

$D_{\text{num}}(x_i, \Delta t) = [x_{i-1}, x_{i+1}] = [x_i - \Delta x, x_i + \Delta x]$

**(Draw the <u>stencil</u> of Lax-Friedrichs method to show $D_{\text{num}}$)**

Recall that the Lax-Friedrichs method is stable if and only if $|a\, r| \le 1$

if and only if $|a\Delta t| \le \Delta x$

if and only if $(x_i - \Delta x) \le (x_i - a\Delta t) \le (x_i + \Delta x)$.

We compare $D_{\text{PDE}}(x_i, \Delta t) = \{ x_i - a\Delta t \}$ and $D_{\text{num}}(x_i, \Delta t) = [x_i - \Delta x, x_i + \Delta x]$.

The Lax-Friedrichs method is stable if and only if $D_{\text{PDE}} \subseteq D_{\text{num}}$.

Example

For PDE $u_t + au_x = 0$, we consider the Lax-Wendroff method,

$$u_i^{n+1} = u_i^n - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{(ar)^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

$D_{\text{num}}(x_i, \Delta t) = [x_{i-1}, x_{i+1}] = [x_i - \Delta x, x_i + \Delta x]$

**(Draw the <u>stencil</u> of Lax-Wendroff method to show $D_{\text{num}}$)**

We just showed that the Lax-Wendroff method is stable if and only if $|a\, r| \le 1$

if and only if $|a\Delta t| \le \Delta x$

if and only if $(x_i - \Delta x) \le (x_i - a\Delta t) \le (x_i + \Delta x)$.

We compare $D_{\text{PDE}}(x_i, \Delta t) = \{ x_i - a\Delta t \}$ and $D_{\text{num}}(x_i, \Delta t) = [x_i - \Delta x, x_i + \Delta x]$.

The Lax-Wendroff method is stable if and only if $D_{\text{PDE}} \subseteq D_{\text{num}}$.

The 4 examples above seem to indicate that $D_{\text{PDE}} \subseteq D_{\text{num}}$ is a necessary and sufficient condition for stability. The example below shows it is only a necessary condition.

Example:

For PDE $u_t + au_x = 0$, we consider the FTCS method,

$$u_i^{n+1} = u_i^n - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

$D_{\text{num}}(x_i, \Delta t) = [x_{i-1}, x_{i+1}] = [x_i - \Delta x, x_i + \Delta x]$

(**Draw the <u>stencil</u> of FTCS method to show $\boldsymbol{D}_{\text{num}}$**)

Recall that the FTCS method is unstable for all $r$.

We compare $D_{\text{PDE}}(x_i, \Delta t) = \{\, x_i - a\Delta t \,\}$ and $D_{\text{num}}(x_i, \Delta t) = [x_i - \Delta x, x_i + \Delta x]$.

For the FTCS method, $D_{\text{PDE}} \subseteq D_{\text{num}}$ is true when $|a\, r| \le 1$.

For the FTCS method, $D_{\text{PDE}} \subseteq D_{\text{num}}$ does not imply stability.

<u>Theorem</u> (the CFL condition):

For a numerical method, a <u>necessary</u> condition for convergence is

$D_{\text{PDE}} \subseteq D_{\text{num}}$

<u>Remarks:</u>

- The condition $D_{\text{PDE}} \subseteq D_{\text{num}}$ is called the CFL condition, named after Richard **Courant**, Kurt **Friedrichs**, and Hans **Lewy**.
- When a numerical method is consistent with the PDE, stability is equivalent to convergence (Lax equivalence theorem).
- If we disregard the consistency requirement, we can always have a trivial numerical method that is stable but is not consistent with the PDE.

  $$u_i^{n+1} = u_i^n$$

- For consistent methods, the CFL condition is a <u>necessary condition</u> for stability.

We look at another example where the CFL condition describes the stability.

<u>Example</u>:

For PDE $u_t + au_x = 0$, we consider the Beam-Warming method (for $a > 0$)

$$u_i^{n+1} = u_i^n - \frac{ar}{2}\left(3u_i^n - 4u_{i-1}^n + u_{i-2}^n\right) + \frac{(ar)^2}{2}\left(u_i^n - 2u_{i-1}^n + u_{i-2}^n\right), \quad r = \frac{\Delta t}{\Delta x}, \quad a > 0$$

$D_{\text{num}}(x_i, \Delta t) = [x_{i-2}, x_i] = [x_i - 2\Delta x, x_i]$

(**Draw the <u>stencil</u> of Beam-Warming method to show $D_{\text{num}}$**)

The Beam-Warming method is stable if and only if $a\,r \le 2$ (derivation not included)

if and only if $a\Delta t \le 2\Delta x$

if and only if $(x_i - 2\Delta x) \le (x_i - a\Delta t)$.

We compare $D_{\text{PDE}}(x_i, \Delta t) = \{ x_i - a\Delta t \}$ and $D_{\text{num}}(x_i, \Delta t) = [x_i - 2\Delta x, x_i]$.

The Beam-Warming method is stable if and only if $D_{\text{PDE}} \subseteq D_{\text{num}}$.

**Method of characteristics**

Consider the IVP of linear hyperbolic PDE

$$\begin{cases} u_t + a(x)u_x = b(x)u + c(x) \\ u(x,0) = u_0(x) \end{cases}$$

<u>Note:</u> $u_t + \left(a(x)u\right)_x = 0$ can be written into this form.

<u>Basic idea</u>

The CFL condition says we need to include the location where the information comes from. Then why not go directly to that exact location?

<u>Formulation of the exact solution</u>

<u>Characteristics</u> (going forward)

$$\begin{aligned} &\frac{dx}{dt} = a(x(x_0,t)) \\ &\frac{dv}{dt} = b(x(x_0,t))v(x_0,t) + c(x(x_0,t)) \qquad \text{(E01)} \\ &x(x_0,0) = x_0, \qquad v(x_0,0) = u_0(x_0) \end{aligned}$$

Here $x(x_0, t)$ is the C-line that has position $x_0$ at time 0, and $v(x_0, t)$ is the solution of PDE along the C-line: $v(x_0, t) = u(x(x_0, t), t)$.

<u>Key observation:</u>

The solution of (E01) at time $T$ gives the solution of PDE at $(x(x_0, T), T)$. To find the solution of PDE at a specified $(\xi, T)$, we need to find $x_0$ such that $x(x_0, T) = \xi$. For that purpose, we trace the C-line from $(\xi, T)$ back to time $t = 0$.

Formulation of the method of characteristics

Characteristics (tracing backward)

$$\begin{aligned}&\frac{dX}{dt}=a(X)\\&X(T)=\xi\end{aligned}\qquad\text{(FVP-1)}$$

(FVP-1) is a final value problem (FVP). We can solve a FVP by using a negative time step $h$ in an RK solver.

When we trace the C-line from $(\xi, T)$ back to time 0, we calculate only $X$, not $v$.

Here $X(\xi, t)$ denotes the C-line that has position $\xi$ at time $T$. We use $X$ to distinguish it from $x(x_0, t)$, which is the C-line that has position $x_0$ at time 0. $X(\xi, t)$ and $x(x_0, t)$ are two different C-lines.

**(Draw these two C-lines to show the difference.)**

Evolution of $v$ along C-line starting at $X(\xi, 0)$

Once $x_0 \equiv X(\xi, 0)$ is known, we calculate $u(\xi, T)$, by solving the forward evolution of the C-line starting at $x_0$ at time 0 and the evolution of $v$ along the C-line.

Set $x_0 \equiv X(\xi, 0)$. We solve the IVP below

$$\begin{aligned}&\frac{dx}{dt}=a(x)\\&\frac{dv}{dt}=b(x)v+c(x)\\&x(0)=x_0,\quad v(0)=u_0(x_0),\quad x_0\equiv X(\xi,0)\end{aligned}\qquad\text{(IVP-1)}$$

Solution of (IVP-1) at time $T$ gives the solution of PDE at $(\xi, T)$.

$u(\xi, T) = v(T)$

Procedure of the method of characteristics

Given time $T$ and position $\xi$, we follow the steps below to calculate $u(\xi, T)$.

- Solve (FVP-1) numerically using an RK solver to calculate $X(\xi, 0)$.

  The sole purpose of this step is to trace the C-line from $(\xi, T)$ back to time 0.
- Set $x_0 \equiv X(\xi, 0)$. Solve (IVP-1) to time $T$ to obtain $u(\xi, T) = v(T)$.
- Repeat the process (or carry them out in parallel) at all $(\xi, T)$ where we need to calculate the solution of PDE.

Remarks:

- Method of characteristics is (relatively) easy to implement and is very accurate (due to the RK solver) for linear hyperbolic PDEs.

- Method of characteristics is NOT for solving nonlinear hyperbolic PDEs.

**Modified PDEs**

Background: our general view of a numerical approximation.

Example 1:

Recall the approximation-error framework

$$\underbrace{T(h)}_{\substack{\text{Numerical}\\ \text{approximation}}} = \underbrace{I}_{\text{Exact}} + \underbrace{E(h)}_{\text{Error}}, \qquad E(h) = O(h^p)$$

Instead of viewing $T(h)$ simply as an approximation, we view it as the exact value of a perturbed quantity $(I + E(h))$. The perturbation term $E(h)$ is not exactly known. But we know some aspects of $E(h)$.

Example 2:

Recall the mathematical form of round-off error

$$\underbrace{\mathrm{fl}(x)}_{\substack{\text{Floating point}\\ \text{representation}}} = \underbrace{x}_{\text{Exact}}(1+\underbrace{\varepsilon}_{\substack{\text{Round-off}\\ \text{error}}}), \qquad \varepsilon \sim 10^{-16}$$

Instead of viewing $\mathrm{fl}(x)$ simply as an approximation, we view is as the exact value of a perturbed quantity $x(1+\varepsilon)$. The perturbation term $\varepsilon$ is not exactly known. But we know some aspects of $\varepsilon$.

We adopt this approach in our study of numerical solution of PDEs.

Case study: Solution of the upwind method for PDE $u_t + a\, u_x = 0$, $a > 0$.

$$u_i^{n+1} = u_i^n - ar\left(u_i^n - u_{i-1}^n\right)$$

Key concept:

The upwind method is an approximation of the PDE.

We view the upwind method as the exact of a perturbed PDE.

Suppose $u_i^n$ is the exact value of an underlying smooth function $w$ at $(x_i, t_n)$. Specifically, we write

$$u_i^n = w(x_i, t_n; \Delta x, \Delta t)$$

Short notation: $u_i^n = w(x_i, t_n)$

We expect that $w(x, t)$ satisfies a perturbed PDE. We won't know the exact perturbation. But we will find some aspects of the perturbation.

<u>Governing equation</u> for $w(x, t)$

We use Taylor expansion to find the governing equation.

We substitute $u_i^n = w(x_i, t_n)$ into the upwind method.

$$w(x_i, t_{n+1}) - w(x_i, t_n) = -a\frac{\Delta t}{\Delta x}\Big(w(x_i, t_n) - w(x_{i-1}, t_n)\Big)$$

We expand every term around ($x_i$, $t_n$)

$$\text{LHS} = w_t\big|_{(x_i,t_n)}\Delta t + w_{tt}\big|_{(x_i,t_n)}\frac{(\Delta t)^2}{2} + O\Big((\Delta t)^3\Big)$$

$$\text{RHS} = -a\frac{\Delta t}{\Delta x}\left[w_x\big|_{(x_i,t_n)}\Delta x - w_{xx}\big|_{(x_i,t_n)}\frac{(\Delta x)^2}{2} + O\Big((\Delta x)^3\Big)\right]$$

Dividing by $\Delta t$ and moving $w_{tt}$ term to the right side, we get

$$w_t = -a w_x - \frac{\Delta t}{2}w_{tt} + \frac{a\Delta x}{2}w_{xx} + O\Big((\Delta t)^2 + (\Delta x)^2\Big) \qquad \text{(T01)}$$

We convert $w_{tt}$ on the RHS to a spatial derivative. We use an iterative approach.

The leading-term version of (T01) is

$$w_t = -a w_x + O(\Delta t + \Delta x) \qquad \text{(T01B)}$$

Differentiating (T01B) with respect to $t$ gives us

$$w_{tt} = (-a w_x)_t + O(\Delta t + \Delta x) = -a(w_t)_x + O(\Delta t + \Delta x)$$

Using (T01B) again for ($w_t$) yields

$$w_{tt} = -a(-a w_x)_x + O(\Delta t + \Delta x) = a^2 w_{xx} + O(\Delta t + \Delta x) \qquad \text{(T01C)}$$

Substituting (T01C) back into (T01), we arrive at

$$w_t = -a w_x + \left(\frac{a\Delta x}{2} - \frac{a^2\Delta t}{2}\right)w_{xx} + O\Big((\Delta t)^2 + (\Delta x)^2\Big)$$

Neglecting the $O((\Delta t)^2+(\Delta x)^2)$ term, we write the equation for $w(x, t)$ as

$$\boxed{w_t = -a w_x + \sigma w_{xx}} \qquad \text{(M01)}$$

where the coefficient $\sigma$ is given by

$$\sigma = \frac{\Delta x}{2}a(1 - a r), \qquad r = \frac{\Delta t}{\Delta x}$$

<u>(M01) is called the modified PDE of the upwind method.</u>

Remarks:

- The modified PDE is a perturbation of the original PDE
- The numerical method is more consistent with the modified PDE than with the original PDE.
- Consequently, the numerical solution is better described by the behavior of the modified PDE than by that of the original PDE.

Behavior of solution of modified PDE

- wave dissipation
- wave dispersion
- phase velocity
- group velocity

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