CS280–FALL 2013 — Solutions to Homework 1

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September 18, 2013

1a.

A plane of points in 3 D can be represented as $X \cdot N = d$, where N is a unit vector in the same direction as the normal of the plane. Now for any point on the plane we know we can transform it to the image plane with the formula $p = fX/Z$. Now we can dot both sides with the normal and we get $p \cdot N = fX \cdot N/Z$. Now as we take the limit to infinity we should get the set of points on the image plane that are mapped when $Z \to \infty$. The right hand side goes to 0. Now p can be generally represented as $p = (x, yf)$, where f is the focal distance. Now we take the dot product and get $xN_x + yN_y + fN_z = 0$. Since the last term is really a constant, the set of points are a line!

1b.

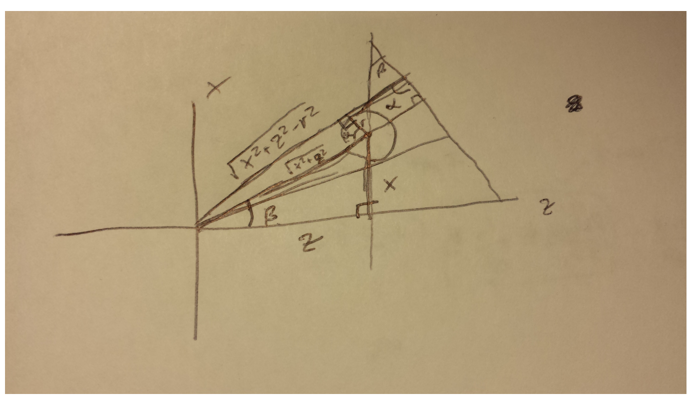

Figure 1: figure shows a cut of the cone with the plane y = 0

To imagine this problem, draw a line from the center of the sphere to 0. Draw the tip of the cone coming out of the origin and intersecting the sphere tangentially. Now the sphere will be mapped to the image plane with a plane perpendicular to the Z axis.

From wikipedia there is a formula that gives the eccentricity of an ellipse as $e = \frac{sin\beta}{sin\alpha}$, where α and β are the angle between the slant of the cone and the plane of intersection and the slant angle of the cone. From the picture we can see that $\sin\alpha = \frac{\sqrt{X^2+Z^2-r^2}}{\sqrt{X^2+Z^2}}$ and that $sin\beta = \frac{X}{\sqrt{X^2+Z^2}}$. Thus we get an eccentricity of $e = \frac{X}{\sqrt{X^2+Z^2-r^2}}$. In order to get a parabola the sphere has to lie a distance r in the Z direction. Then the $e = 1$ and we get a parabola. This might not be possible since then the sphere is tangential to the aperature and it is impossible to get a good focus that close. A hyperbola would result if the sphere is partially behind the aperature which is definitely not possible.

1c.

We can set up a triangle where the height of the observer h is looking down on the ground plane at a point Z away. In this representation we have $\tan(\theta) = \frac{Z}{h}$. Then moving the tangent over to the other side and then taking the derivative with respect to Z we get:

$$\frac{d\theta}{dZ} = (1/h) * (\frac{1}{1+(Z/h)^2})$$

. Then solving for dZ we get:

$$dZ = d\theta h(1 + \frac{Z}{h}^2)$$

Now let's plug in some numbers. Let's say the person is 5'8" or 1.72 m and looking 10m away we get a spatial resolution of 1.7 cm.

3.1.

The matrix given for a reflection across a line that has an angle of a:

$$\begin{pmatrix} \cos 2a & \sin 2a \\ \sin 2a & -\cos 2a \end{pmatrix}$$

Therefore the equation of a line flipped over a line with an angle of a and b is:

$$\begin{pmatrix} \cos 2b & \sin 2b \\ \sin 2b & -\cos 2b \end{pmatrix} * \begin{pmatrix} \cos 2a & \sin 2a \\ \sin 2a & -\cos 2a \end{pmatrix}$$

This gives us:

$$\begin{pmatrix} \cos 2b * \cos 2a + \sin 2b * \sin 2a & \cos 2b * \sin 2a - \sin 2b * \cos 2a \\ \cos 2a * \sin 2b - \sin 2a * \cos 2b & \cos 2b * \cos 2a + \sin 2b * \sin 2a \end{pmatrix}$$

This simplifies using trig identities to:

$$\begin{pmatrix} \cos 2(b-a) & -\sin 2(b-a) \\ \sin 2(b-a) & \cos 2(b-a) \end{pmatrix}$$

This looks exactly like the rotation equation using an angle of rotation of $2(b-a)$.

3.2.

We start with the equation $e^{\tilde{w}\theta}$. Using a taylor expansion series we get

$$1 + \tilde{w} * \theta + \frac{\tilde{w}^2 * \theta^2}{2!} + \frac{\tilde{w}^3 * \theta^3}{3!} + \frac{\tilde{w}^4 * \theta^4}{4!} ...$$

In order to simplify this we must consider the exponentiation of $\tilde{w}$. *tildew* represents the cross product of a matrix, with a vector w being the vector that the axis of rotation.

$$\tilde{w} = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix}$$

$$\tilde{w}^2 = \begin{pmatrix} -(w_z^2 + w_y^2) & w_y w_x & w_z w_x \\ w_x w_y & -(w_z^2 + w_x^2) & w_y w_z \\ w_x w_z & w_z w_y & -(w_x^2 + w_y^2) \end{pmatrix}$$

$$\tilde{w}^3 = \begin{pmatrix} 0 & w_z & -w_y \\ -w_z & 0 & w_x \\ w_y & -w_x & 0 \end{pmatrix} = -\tilde{w}$$

$$\tilde{w}^4 = -\tilde{w}^2 and \tilde{w}^5 = \tilde{w}$$

Therefore our original taylor series simplifies to:

$$1 + \tilde{w} * \theta + \frac{\tilde{w}^2 * \theta^2}{2!} - \frac{\tilde{w} * \theta^3}{3!} - \frac{\tilde{w}^2 * \theta^4}{4!} + ...$$

Now if we look at the even and the odd terms they look very similar to the taylor series expansion of sine and cosine. Therefore we combine the even and odd terms and factor out $\tilde{w}$ and $\tilde{w}^2$. We get:

$$1 + \tilde{w}(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} ...) + \tilde{w}^2(1 - (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} ...))$$

$$1 + \tilde{w} * \sin(\theta) + \tilde{w}^2 * (1 - \cos(\theta))$$

This is Rodriguez formula.

3.3.

Let us construct matices with all the points in u and call it U and all the points in v and call it V. Then the error is $Error = (E * U - V)^T (E * U - V)$. This is the least square error. To find the solution we take a derivative with respect to E. The solution to this is always $E = VU'(UU')^{-1}$. After plugging in the points we get the transformation matrix E as:

$$\begin{pmatrix} 0 & 1 & 0 \\ -0.75 & 0 & 0.75 \\ 0 & 0 & 1 \end{pmatrix}$$

3.4.

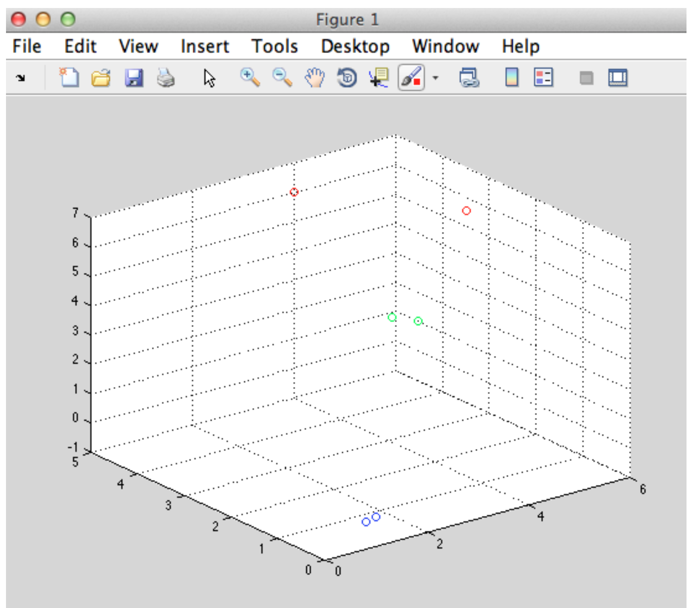

Figure 2: same color dots been rotated to the right

One of the Eigenvectors is the same direction as the rotation matrix. It has an eigenvalue of 1. The matrix R is nothing but:

$$\begin{pmatrix} \cos\theta + w_x^2(1-\cos\theta) & w_y w_x(1-\cos\theta) - w_z\sin\theta & w_z w_x(1-\cos\theta) + w_y\sin\theta \\ w_y w_x(1-\cos\theta) + w_z\sin\theta & \cos\theta + w_y^2(1-\cos\theta) & w_y w_z(1-\cos\theta) - w_x\sin\theta \\ w_z w_x(1-\cos\theta) - w_y\sin\theta & w_y w_z(1-\cos\theta) + w_x\sin\theta & \cos\theta + w_z^2(1-\cos\theta) \end{pmatrix}$$

The trace of the matrix is the sum of the diagonals which gives us $2\cos(\theta) + 1$ and therefore it is simple to see that $\cos(\theta) = 1/2(trace(R) - 1)$. I attached computeH.m that has the function. The plot shows points in the same color that have been rotated by the vector $[4, 3, 1]$ around 30 degrees.

3.5

. In order to justify the function, I showed that $R - R^T = 2\sin\theta\tilde{w}$. Now we have R stated in the prev problem. If we subtract them we get:

$$2\sin\theta \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix} = 2\sin\theta\tilde{w}$$

In order to find theta we just use the fact that the vector w squared sums to 1. Then we can get theta and divide out the constant factor and we get both w and theta. I attached reverseR.m which has the function.