High-Gain Observers in Nonlinear Feedback Control

(notes based on seminar by Prof. Hassan Khalil)

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Introduction

Control of non-linear systems in real world is an interesting problem due to two major challenges - uncertainity in the model and noise in measurements. If this is solved, we can skip the linearization and gain scheduling approaches that are generally followed in control of non-linear systems. In this note, we learn about high gain observers in output feedback control. In general, we start with developing a state feedback controller that is stabilizing the system. Then, we build an observer and use it to develop a output feedback controller that drives the error dynamics to zero while stabilizing the system.

Motivating example

Consider the following state dynamics with states x, control input u and disturbances w:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \phi\,(x, u, w)$$

Measurement/ Output equation:

$$y = x_1$$

Simple state feedback controller:

$$u = -\phi\,(x, w) - x_1 - x_2$$

Control objective: To find a stabilizing output feedback controller $u = \gamma\,(\hat{x})$ such that the we drive the system to $\dot{x} = 0$ in the closed-loop.

High Gain Observer

We start with building an observer such that

$$\dot{\hat{x}}_1 = \hat{x}_2 + h_1\left(y - \hat{x}_1\right)$$

$\dot{\hat{x}}_2 = \phi_n(\hat{x}, u) + h_2\left(y - \hat{x}_1\right)$ where ϕ_n represents the nominal model of the system.

Now, the measurement error is given by $\tilde{x} = x - \hat{x}$. Looking at the error dynamics, we have

$$\dot{\tilde{x}}_1 = -h_1 \tilde{x}_1 + \tilde{x}_2$$

$\dot{\tilde{x}}_2 = -h_2 \tilde{x}_1 + \delta\left(x, \tilde{x}, w\right)$ where $\delta\left(x, \tilde{x}, w\right) = \phi\left(x, \gamma\left(\hat{x}\right), w\right) - \phi_n(\hat{x}, \gamma\left(\hat{x}\right))$ is approximation error/ uncertainity in the system model.

The main crux of the problem is to find the gains h_1 and h_2 that will drive the error dynamics to zero (while also driving the system to $\dot{x} = 0$).

In high gain observer design, we choose:

$h_1 = \dfrac{\alpha_1}{\epsilon}$ and $h_2 = \dfrac{\alpha_2}{\epsilon}$

where α_1 and α_2 are used to drive the system to zero and choosen be in the left half of the plane. ϵ is a small number less than 1 which drives the gains to be high number.

Taking $\eta_1 = \dfrac{\tilde{x}_1}{\epsilon}$ and $\eta_2 = \tilde{x}_2$, we can rewrite the above equation as

$$\epsilon\, \dot{\eta}_1 = -\alpha_1 \eta_1 + \eta_2$$

$$\epsilon\, \dot{\eta}_2 = -\alpha_2 \eta_2 + \epsilon\, \delta$$

Here comes the crucial observation in above dynamics: the small value of ϵ supresses the uncertainity in system dynamics and also, by multiplying ϵ on the left side of the equation, we have a faster dynamics for η. Hence, we are able to simultaneously drive the error to zero and stabilize the system. Hence, reducing the value of ϵ would solve our problem (achieving very high gain). Yay, success! No, wait. There is a caveat.

Peaking phenomenon

Since $\eta_1 = \dfrac{x - \hat{x}}{\epsilon}$ when initially $x(0) \neq \hat{x}(0)$, we have $\eta_1 \propto O\left(\dfrac{1}{\epsilon}\right)$ and for small values of ϵ, the system may get destabilized due to impulsive input ie. the system could diverge in finite time which is undesirable. This is called as Peaking phenomenon. A easy workaround for this problem is to **saturate** the control input to a safe limit (or $\hat{x}$ outside a compact set of interest).

Another way to understand the compact set of interest would be through comparison of system dynamics with state feedback controller. Under state feedback controller, we have a **region of attraction** within which all state trajectories will lead to asymptotically stable equilibrium (here it is origin). The saturated controller ensures that the system is within this region of attraction. Hence, choosing the appropriate saturation value will relate to the conservative usage of of the region of attraction. We can reduce the value of ϵ to match to the boundary of region of attraction.

Thus, the output feedback controller described above recovers the following properties of the (saturated) state feedback controller as ϵ tends to zero:

1. Asymptotic stability of the origin (also called as **Nonlinear separation principle**)
2. Recover the Region of attraction
3. Achieve State trajectories similar to state feedback controller

Extended High Gain Observer

Motivating example: Consider the following dynamic system with states x, control input u, disturbance w, measurements y. The functions $a(x, w)$, $b(x, w)$, w are not known.

$$\dot{x_1} = x_2$$
$$\dot{x_2} = a(x, w) + b(x, w)u$$
$$y = x_1$$

Let us rewrite the system dynamics using nominal models of $a(x, w)$ and $b(x, w)$.

$$\dot{x_2} = a_n(x) + b_n(x)u + \sigma \text{ where } \sigma = a(x, w) - a_n(x) + [b(x, w) - b_n(x)]u$$

Now, we extend the original system dynamics of order n to $n + 1$.

$$\dot{x_1} = x_2$$

$$\dot{x_2} = a_n(x) + b_n(x)u + \sigma$$

$$\dot{\sigma} = \phi\left(x, w, \dot{w}, u, \dot{u}\right)$$

$$y = x_1$$

High Gain Observer:

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{\alpha_1}{\epsilon}\left(y - \hat{x}_1\right)$$

$$\dot{\hat{x}}_2 = a_n(\hat{x}) + b_n(\hat{x})u + \hat{\sigma} + \frac{\alpha_2}{\epsilon^2}\left(y - \hat{x}_1\right)$$

$$\dot{\hat{\sigma}} = \frac{\alpha_3}{\epsilon^3}\left(y - \hat{x}_1\right)$$

here, the high gain observer has additional order and hence the name extended high gain observer. This additional order comes from modelling the changes in input and disturbances.

Challenges

Like there is no free lunch, high gain observers have following issues:

Numerical challenge

With the high gain observer of dimension n, we have the observer gain in the order of $O\left(\frac{1}{\epsilon^n}\right)$ and the peaking signal in the order of $O\left(\frac{1}{\epsilon^{n-1}}\right)$. This can cause numerical instability when implementing in an embedded device with limited precision capabilities. In order to overcome this challenge, the idea of cascade realization of high-gain observer with saturation is applied. Thus, it prevents from blowing up issues.

Measurement Noise

By applying high-gain on our measurement, it not only decreases the error due to model (desired) but also increases the error due to noise (undesired). High-gain yields faster response but increases the steady-state error due to noise. Thus, there needs to be a trade off between transient and steady-state performance. One can resort to higher gain during transient phase and lower gain at steady state (like switching gains in piecewise linear function) or gain can be adopted. Also, one can use a low pass filter to reduce the effect of high frequency noise. Using nonlinear gain function, we can achieve error comparable to linear gain function but at lower value of ϵ.

Conclusion

The high-gain observer is promising for handling uncertainities in non-linear system control. One of the competing methods for nonlinear observers is sliding mode observers. Although both methods have similar performance, high-gain observers are simple to implement. In theory, the sliding mode observers can achieve zero error in finite time in the absence of measurement error while high-gain observers achieve error in order of $O\left(\frac{1}{\epsilon}\right)$ in finite time. But in the presence of noise, both the methods give comparable performance. A detail study on sliding mode observer will be of immediate interest.