Compactness in Metric Spaces

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Here, we wish to prove the equivalence of a number of seemingly unrelated notions of compactness in metric spaces. We first recall the definition of (topological) compactness, which deals with open covers:

Definition. A space X is *compact* if every open cover $\{U_\alpha\}_{\alpha \in J}$ of X contains a finite subcover of X.

There is another notion known as limit point compactness:

Definition. A space X is *limit point compact* if every infinite set $A \subseteq X$ has a limit point in X.

Recall that a limit point x of a set A is such that every neighborhood of x intersects A at some point other than x itself. An equivalent definition is x is a limit point of A if $x \in \mathrm{Cl}(A - \{x\})$.

Yet another form of compactness deals with sequences:

Definition. A space X is *sequentially compact* if every sequence $(x_n)_{n \in \mathbb{Z}^+}$ has a convergent subsequence.

Finally, there is a form of compactness that is in general weaker than compactness, but which will prove to be very useful in deriving relationships among the other types of compactness:

Definition. A *countably compact* space X is one for which every countable (open) cover of X has a finite subcover.

Proposition. If a space X is T_1, then countable compactness and limit point compactness are equivalent.

Proof. The converse requires X to be T_1. The forward direction does not.

($\implies$) Suppose X is not limit point compact. Then there exists some infinite set $A \subseteq X$ that does not have a limit point. Consider any countably infinite subset $B = \{b_n\}_{n \in \mathbb{Z}^+} \subseteq A$. It also cannot have a limit point. Using this set, we find a countable cover of X with no finite subcover. Indeed, for each $b_n \in B$, because b_n is not a limit point of B, there is some neighborhood U_n of b_n such that $B \cap U_n = \{b_n\}$; that is to say, singletons are open in B. Then, simply consider the open cover of B formed by the singletons $\{b_n\}$; this cover has no finite subcover.

($\impliedby$) Suppose X is not countably compact. Suppose that the countable open cover $\{U_n\}_{n \in \mathbb{Z}^+}$ of X has no finite subcover. We use this cover to formulate an infinite set A with no limit point. Select arbitrarily, for each $n \in \mathbb{Z}^+$, an element $a_n \in X - \bigcup_{i \le n} U_i$. Such a choice is always possible since $\{U_n\}$ contains no finite subcover.

First, it is easy to check that the set $A = \{a_n\}_{n \in \mathbb{Z}^+}$ is infinite, for if not, then some element $x = a_n$ for infinitely many n. Since $\{U_n\}$ is a cover of X, then $x \in U_{n_x}$ for *some* $n_x \in \mathbb{Z}^+$. Then $a_i \notin U_{n_x}$ for each $i \ge n_x$, which is a contradiction.

Then, we show that A has no limit point. For any $x \in X$, let $n_x \in \mathbb{Z}^+$ be such that $x \in U_{n_x}$ (this is possible because $\{U_n\}$ is a cover of X). Note that for any $i \ge n_x$, $a_i \notin U_n$. However, it could be the case that $x_i \in U_n$ for $i < n_x$. But because X is T_1, the finite collection $\{a_1, \dots, a_{n_x - 1}\} - \{x\}$ is closed, so its complement $V_x = (X - \{a_1, \dots, a_{n_x-1}\}) \cup \{x\}$ is a neighborhood of x. Then, consider that the neighborhood $U_{n_x} \cap V$ of x excludes all elements of B (except possibly x itself), which shows that x is not a limit point of B.

Definition. An ω-*accumulation point* of a set $A \subseteq X$ is a point $p \in X$ such that each neighborhood of p intersects A in infinitely many points.

Proposition. The following are equivalent:

1. X is countably compact.
2. Every infinite set $A \subseteq X$ has an ω-accumulation point in X.
3. Every sequence in X has an accumulation point in X.

Proof. We show that (1) implies (2), (2) implies (3), and (3) implies (1).

($1 \implies 2$) Let $A \subseteq X$ be infinite. Let $B \subseteq A$ be a countably infinite subset of A. Suppose for the sake of contradiction that B has no ω-accumulation points. Then for any $x \in X$, there is a neighborhood U_x of x that intersects B in only finitely many points. For each finite subset K of B (there are countably many of them), define V_K to be the union of all U_x such that $K = B \cap U_x$; note that each V_K is open. Also, the collection $\{V_K\}$ covers X because for each $x \in X$, the chosen $U_x \subseteq V_{K_x}$, where $K_x = B \cap U_x$. Since X is countably compact, there exists a finite subcover $\{V_{K_1}, \ldots, V_{K_N}\} \subseteq \{V_K\}$ that covers X. Note, however, that

$$B \cap \bigcup_{i=1}^{N} V_{K_i} = \bigcup_{i=1}^{N} (B \cap V_{K_i})$$

being a finite union of finite sets, is finite. This is a contradiction because B is infinite. Therefore, B must have an ω-accumulation point, which is obviously also an ω-accumulation point of A.

($2 \implies 3$) Let $(a_n)_{n \in \mathbb{Z}^+}$ be a sequence in X. Consider the set $A = \{a_n\}$. If A is finite, then there is some $x \in X$ such that $x = a_n$ for infinitely many $n \in \mathbb{Z}^+$. Such a point is clearly an accumulation point of (a_n). If A is infinite, then by hypothesis, it has an ω-accumulation point $p \in X$. This point is clearly an accumulation point of (a_n).

($3 \implies 1$) Suppose that X is not countably compact. Let $\{U_n\}_{n \in \mathbb{Z}^+}$ be a countable cover of X that does not contain a finite cover. Choose, for each $i \in \mathbb{Z}^+$, $a_i \in X - \bigcup_{n=1}^{i} U_n$. This is always possible because $\{U_n\}$ has no finite subcover. We claim that (a_i) has no accumulation point. Suppose for the sake of contradiction that a is an accumulation point of (a_i). Since $\{U_n\}$ cover X, there is some $n_a \in \mathbb{Z}^+$ such that $a \in U_{n_a}$. But by construction, U_{n_a} misses all elements a_i with $i > n_a$. Therefore, the neighborhood U_{n_a} intersects at most a finite number of elements of (a_i), which contradicts the hypothesis that a is an accumulation point of (a_i).

Proposition. If a space X is first countable, then countable compactness and sequential compactness are equivalent.

Proof. The forward direction requires X to be first countable. The converse does not.

($\implies$) Suppose that X is countably compact. Let $(a_n)_{n \in \mathbb{Z}^+}$ be a sequence, and consider the set $A = \{a_n\}$. If A is finite, then there is some $x \in X$ such that $x = a_n$ for infinitely many $n \in \mathbb{Z}^+$. Then a subsequence formed by taking just these elements converges, trivially, to x. The more interesting case is when A is infinite. Then, we can apply the previous proposition ($1 \implies 2$) to show that A has an ω-accumulation point $a \in X$. We claim that there is a subsequence $(a_{n_j})_{j \in \mathbb{Z}^+}$ that converges to a.

Let $\{B_i\}_{i\in\mathbb{Z}^+}$ be a countable neighborhood basis of a (guaranteed because X is first countable). From this basis, we can create a nested neighborhood basis $\{C_i\}_{i\in\mathbb{Z}^+}$ by taking, for each $i \in \mathbb{Z}^+$, $C_i = \bigcap_{j=1}^{i} B_i$. Then, for each $j \in \mathbb{Z}^+$, since C_j intersects A in infinitely many points, select $a_{n_j} \in A \cap C_j$ such that n_j is the smallest element of $\mathbb{Z}^+$ where $n_j > n_i$ for each $i < j$. The subsequence (a_{n_j}) converges to a, since any neighborhood U of a contains C_{j_U} for some $j_U \in \mathbb{Z}^+$, and for any $j > j_U$, we have that $a_{n_j} \in C_j \subseteq C_{j_U} \subseteq U$.

($\Longleftarrow$) Suppose that X is sequentially compact. Let $(a_n)_{n\in\mathbb{Z}^+}$ be an arbitrary sequence in X. Then there exists a subsequence $(a_{n_j})_{j\in\mathbb{Z}^+}$ that converges to some point $p \in X$. This point is clearly an accumulation point of the sequence (a_n), since for any neighborhood U of p, there exists some $j_U \in \mathbb{Z}^+$ such that $a_{n_j} \in U$ for each $j > j_U$. By the previous proposition ($3 \implies 1$), X is countably compact.

Now, we arrive at what we wish to prove, which is that all of the forms of compactness defined above are equivalent for metric spaces.

Proposition. For a metric space X, the following are equivalent:

1. X is compact.
2. X is sequentially compact.
3. X is limit point compact.

Proof. We show that (1) implies (3), (3) implies (2), and finally that (2) implies (1).

($1 \implies 3$) This is true for topological spaces in general and follows immediately from previous propositions.

($3 \implies 2$) We note that all metric spaces are T_1, since given any $x \neq y \in X$, the neighborhood $B_d(x, d(x, y))$ of x misses y and the neighborhood $B_d(y, d(x, y))$ of y misses x. This shows that X is countably compact. Also, all metric spaces are first countable, since for any element $x \in X$, the collection

$$\left\{ B_d\left(x, \frac{1}{n}\right) | n \in \mathbb{Z}^+ \right\}$$

is a countable neighborhood basis of x. By a previous proposition, X is sequentially compact.

($2 \implies 1$) First, we show the following claim.

Proposition. Let X be a sequentially compact metric space. Let $\mathcal{A}$ be an open cover of X. Then there exists a number $\delta > 0$ such that every set $C \subseteq X$ with diameter smaller than δ is contained in some $A \in \mathcal{A}$.

Proof. Assume the hypotheses of the claim. Suppose for the sake of contradiction that for every $\varepsilon > 0$, there exists a subset $C \subseteq X$ with diameter smaller than ε that is not contained in any $A \in \mathcal{A}$. We can use this fact to construct a sequence as follows. For each $n \in \mathbb{Z}^+$, choose an arbitrary subset $C_n \subseteq X$ with diameter smaller than $\frac{1}{n} > 0$ that is not contained in any element of $\mathcal{A}$, and choose an arbitrary $x_n \in C_n$. Since X is sequentially compact, the sequence (x_n) has a subsequence, say, $(x_{n_j})_{j\in\mathbb{Z}^+}$, that converges to $x_0 \in X$. Suppose that the open set $A_0 \in \mathcal{A}$ contains x_0. Then, there is some $\varepsilon > 0$ such that $B_d(x_0, \varepsilon) \subseteq A_0$. Since (x_{n_j}) converges to x_0, there must exist some $J_1 \in \mathbb{Z}^+$ such that $x_{n_j} \in B_d\left(x_0, \frac{1}{2}\varepsilon\right)$ for each $j > J_1$. Also, take $J_2 \in \mathbb{Z}^+$ to be such that $n_{J_2} > \frac{2}{\varepsilon} \implies \frac{1}{n_{J_2}} < \frac{1}{2}\varepsilon$; for each $j > J_2$, it follows that the set C_{n_j}, containing the element x_{n_j} and with diameter less than $\frac{1}{2}\varepsilon$, is contained in $B_d\left(x_{n_j}, \frac{1}{2}\varepsilon\right)$. For $j > \max\{J_1, J_2\}$, it follows that the set $C_{n_j} \subseteq B_d(x_0, \varepsilon) \implies C_{n_j} \subseteq A_0$, which contradicts the hypothesis that no C_n is contained in any $A \in \mathcal{A}$.

Next, we prove that a sequentially compact metric space X is totally bounded.

Definition. A metric space X is *totally bounded* if for every $\varepsilon > 0$, there exists a finite collection $\{B_d(x_n, \varepsilon)\}_{n\in\mathbb{Z}^+}$ that covers X.

Proposition. A sequentially compact metric space X is totally bounded.

Proof. Suppose that X is not totally bounded; i.e., there exists some $\varepsilon > 0$ for which no finite collection $\{B_d(x_1, \varepsilon), \ldots, B_d(x_N, \varepsilon)\}$ covers X. We can construct a sequence as follows. Select $x_1 \in X$ arbitrarily. Now, for each positive integer $n > 1$, select $x_n \in X - \bigcup_{i=1}^{n-1} B_d(x_i, \varepsilon)$ (such a choice is always possible because of the hypothesis that no finite collection $\{B_d(x_1, \varepsilon), \ldots, B_d(x_N, \varepsilon)\}$ covers X). Note that by construction, for any pair $k \neq m \in \mathbb{Z}^+$, $d(x_k, x_m) \geq \varepsilon$. We claim that the sequence $(x_n)_{n \in \mathbb{Z}^+}$ does not have a convergent subsequence. Suppose for the sake of contradiction that (x_n) has a convergent subsequence, say, $(x_{n_j})_{j \in \mathbb{Z}^+}$, that converges to $x_0 \in X$. Then there must exist some $J \in \mathbb{Z}^+$ such that for each $j > J$, $x_{n_j} \in B_d\left(x_0, \frac{1}{2}\varepsilon\right)$. Since $x_{n_{J+1}}, x_{n_{J+2}} \in B_d\left(x_0, \frac{1}{2}\varepsilon\right)$, then

$$d(x_{n_{J+1}}, x_{n_{J+2}}) \leq d(x_0, x_{n_{J+1}}) + d(x_0, x_{n_{J+2}}) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

which contradicts the previous implication $n_{J+1} \neq n_{J+2} \implies d(x_{n_{J+1}}, x_{n_{J+2}}) \geq \varepsilon$.

Now we return to the original statement that was to be proved.

Proof. Assume that X is a sequentially compact metric space. Let $\mathcal{A}$ be an open cover of X. The first proposition guarantees some $\delta > 0$ such that every subset of X with diameter smaller than δ is contained in some element of $\mathcal{A}$. Use the second proposition to guarantee a finite collection $\{B_d(x_1, \frac{1}{3}\delta), \ldots, B_d(x_N, \frac{1}{3}\delta)\}$ that covers X. Each element $B_d(x_i, \frac{1}{3}\delta)$ of this collection has a diameter that is no greater than $\frac{2}{3}\delta < \delta$, so each of them is contained in some element $A_i \in \mathcal{A}$. Then the finite subcollection $\{A_1, \ldots, A_N\} \subseteq \mathcal{A}$ covers X.