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EFFICIENT AND ACCURATE FINITE DIFFERENCE METHOD FOR THE FOUR UNDERLYING ASSET ELS

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Abstract. In this study, we consider an efficient and accurate finite difference method for the four underlying asset equity-linked securities (ELS). The numerical method is based on the operator splitting method with non-uniform grids for the underlying assets. Even though the numerical scheme is implicit, we solve the system of discrete equations in explicit manner using the Thomas algorithm for the tri-diagonal matrix resulting from the system of discrete equations. Therefore, we can use a relatively large time step and the computation of the ELS option pricing is fast. We perform characteristic computational test. The numerical test confirm the usefulness of the proposed method for pricing the four underlying asset equity-linked securities.

1. Introduction

The most common types of derivatives that can be invested in Korea are equity-linked securities (ELS), derivative linked securities (DLS), equity linked warrant (ELW), exchange traded note (ETN). These various structured products are traded in large quantities with the advantage of allowing customers to choose the product they want. Financial markets agree that the era of near zero interest rates will last for a long time as US Federal Reserve announced to keep rates near zero through 2023 in order to control the risks in the market caused by the aftermath of COVID-19. Under such circumstances, ELS is attracting more investors for higher gains than the interest on deposit even at heightened risks.

The more number of underlying assets with lower correlations and higher volatility, the higher yields ELS can generally provide. Therefore, issuers can increase

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the number of underlying assets t in order to structure ELS medium-risk with high stable coupons, even if it could increase the risk of knock-in. In the past, most of ELSs were used to be issued based on two underlying assets, but currently, ELSs issued based on three underlying assets are dominating. Especially the number of ELSs issued based on four underlying assets has also increased.

Finite difference method (FDM) is one of the most important evaluation tools in quantitative finance, and unlike Monte-Carlo Simulation (MCS), the result is stable. Therefore, FDM is one of the preferred methods for calculating the Greeks required for hedging ELS, which has become a popular instrument in the Korean capital market. In addition, there are various researches of numerical studies of ELS using FDM [7, 8, 10, 14, 21, 22, 23]. Fazlollah Soleymani [19] suggested that three high order semi-discretization techniques can be used to deal with European and American style options to address the computational performance of multi asset option pricing problems. Wen Chen and Song Wang [1] have developed the Crank–Nicolson alternating direction implicit (ADI) method to solve 2D fractional BS equation. Fazlollah Soleymani and Ali Akgul [20] have found the calculation method with the application of the Krylov method to resolve multi-asset option pricing problem which led the reduction the elapsed time and effort. In addition, there are various option pricing studies using FDM [2, 15, 13, 6, 4].

We consider an accurate and efficient FDM for the four underlying asset ELS. The numerical method is based on the operator splitting method with non-uniform grids for the underlying assets. Even though the numerical scheme is implicit, we solve the system of discrete equations in explicit manner using the Thomas algorithm for the tri-diagonal matrix resulting from the system of discrete equations. Therefore, we can use a relatively large time step and the computation of the ELS option pricing is fast. We perform characteristic computational xtest. The numerical test confirm the usefulness of the proposed scheme for pricing the four underlying asset ELS.

The outline of this article is as follows. In Section 2, the four underlying asset equity-linked securities is described. In Section 3, the four-dimensional Black–Scholes equation is given. We provide a numerical solution algorithm in Section 4. Characteristic numerical experiments are performed in Section 5. Finally, conclusion is given in Section 6.

2. Four asset ELS

Let $K_1,\ K_2,\ K_3,\ K_4,\ K_5,\ K_6$ be strike prices, where $K_i \geq K_{i+1}$ for $i = 1, \ldots, 5$. Let $c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6$ be coupon rates at times $T_1,\ T_2,\ T_3,\ T_4,\ T_5,\ T_6$, respectively, where $c_i < c_{i+1}$ and $T_i < T_{i+1}$ for $i = 1, \ldots, 5$. Let us define the scaled underlying assets: $x(t) = 100S_1(t)/S_1(0)$, $y(t) = 100S_2(t)/S_2(0)$, $z(t) = 100S_3(t)/S_3(0)$, $w(t) = 100S_4(t)/S_4(0)$, where $S_k(t)$ is the k-th underlying asset value at time t for $k = 1, 2, 3,$ and 4. Let us define the worst performer $(WP(t))$ among four asset paths:

$$WP(t) = \min(x(t), y(t), z(t), w(t)). \tag{2.1}$$

While $k = 1, 2, 3, 4, 5$, if $WP(T_k) \geq K_k$ at $t = T_k$, then $(1 + c_k)F$ is paid, where F is the face value. Otherwise, the contract will be continued. At $t = T_6$, if $WP(T_6) \geq K_6$, then $(1 + c_6)F$ is paid. Otherwise, if $\min_{0 \leq t \leq T_6} WP(t) \leq D$, then $WP(T_6)F/100$ is paid. Otherwise, the payment is $(1 + d)F$, where d is a dummy rate. Figure 1 illustrates the four underlying asset step-down ELS option payoff.

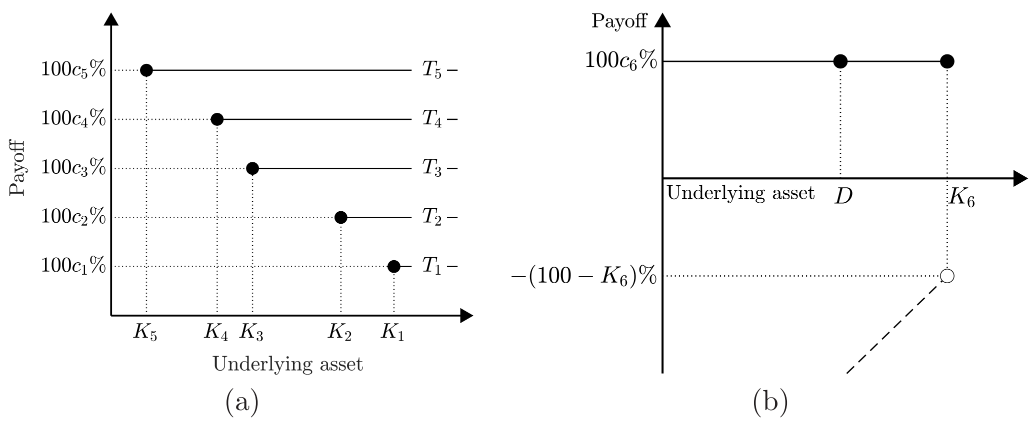

Figure 1. Schematic illustration of the payoff structures of the four underlying asset step-down ELS at times (a) $t = T_1,\ T_2,\ T_3,\ T_4,\ T_5$ and (b) $t = T_6$.

3. Four-dimensional Black–Scholes equation

The evaluation of options on multi-underlying assets is important in the financial industry [16, 17]. The option price $u(x, y, z, w, t)$ follows the multi-dimensional

Black–Scholes equation:

$$
\begin{aligned}
u_t(x,y,z,w,t) \;=\;& -\frac{1}{2}\sigma_x^2 x^2 u_{xx}(x,y,z,w,t) - \frac{1}{2}\sigma_y^2 y^2 u_{yy}(x,y,z,w,t) \\
& -\frac{1}{2}\sigma_z^2 z^2 u_{zz}(x,y,z,w,t) - \frac{1}{2}\sigma_w^2 w^2 u_{ww}(x,y,z,w,t) \\
& -\rho_{xy}\sigma_x\sigma_y xy u_{xy}(x,y,z,w,t) - \rho_{xz}\sigma_x\sigma_z xz u_{xz}(x,y,z,w,t) \\
& -\rho_{xw}\sigma_x\sigma_w xw u_{xw}(x,y,z,w,t) - \rho_{yz}\sigma_y\sigma_z yz u_{yz}(x,y,z,w,t) \\
& -\rho_{yw}\sigma_y\sigma_w yw u_{yw}(x,y,z,w,t) - \rho_{zw}\sigma_z\sigma_w zw u_{wz}(x,y,z,w,t) \\
& -rxu_x(x,y,z,w,t) - ryu_y(x,y,z,w,t) - rzu_z(x,y,z,w,t) \\
& -rwu_w(x,y,z,w,t) + ru(x,y,z,w,t)
\end{aligned} \tag{3.1}
$$

with $u(x,y,z,w,T) = \Phi(x,y,z,w)$, where Φ is the payoff function [18]. σ_x, σ_y, σ_z and σ_w are volatilities. ρ_{xy}, ρ_{xz}, ρ_{xw}, ρ_{yz}, ρ_{yw}, and ρ_{zw} are correlation values. r is interest rate. Let $\tau = T - t$, then we have the following equations:

$$
\begin{aligned}
u_\tau(x,y,z,w,\tau) \;=\;& \frac{1}{2}\sigma_x^2 x^2 u_{xx}(x,y,z,w,\tau) + \frac{1}{2}\sigma_y^2 y^2 u_{yy}(x,y,z,w,\tau) \\
& +\frac{1}{2}\sigma_z^2 z^2 u_{zz}(x,y,z,w,\tau) + \frac{1}{2}\sigma_w^2 w^2 u_{ww}(x,y,z,w,\tau) \\
& +\rho_{xy}\sigma_x\sigma_y xy u_{xy}(x,y,z,w,\tau) + \rho_{xz}\sigma_x\sigma_z xz u_{xz}(x,y,z,w,\tau) \\
& +\rho_{xw}\sigma_x\sigma_w xw u_{xw}(x,y,z,w,\tau) + \rho_{yz}\sigma_y\sigma_z yz u_{yz}(x,y,z,w,\tau) \\
& +\rho_{yw}\sigma_y\sigma_w yw u_{yw}(x,y,z,w,\tau) + \rho_{zw}\sigma_z\sigma_w zw u_{wz}(x,y,z,w,\tau) \\
& rxu_x(x,y,z,w,\tau) + ryu_y(x,y,z,w,\tau) + rzu_z(x,y,z,w,\tau) \\
& +rwu_w(x,y,z,w,\tau) - ru(x,y,z,w,\tau), \qquad (3.2) \\
u(x,y,z,w,0) \;=\;& \Phi(x,y,z,w).
\end{aligned}
$$

4. COMPUTATIONAL METHOD

Let $\Omega = [0,L]\times[0,M]\times[0,N]\times[0,O]$ be the numerical domain with space steps $h^x_{i-1} = x_i - x_{i-1}$, $h^y_{j-1} = y_j - y_{j-1}$, $h^z_{k-1} = z_k - z_{k-1}$, and $h^w_{p-1} = w_p - w_{p-1}$. Here, $x_0 = y_0 = z_0 = w_0 = 0$, $x_{N_x} = L$, $y_{N_y} = M$, $z_{N_z} = N$, and $w_{N_w} = O$. Let $\Delta\tau = T/N_\tau$ be the temporal step size. The numbers of grid points in the x-, y-, z-, w- and τ-directions are denoted by N_x, N_y, N_z, N_w, and N_τ, respectively. Figure 2 shows the non-uniform mesh on x-axis, y-axis, z-axis, and w-axis, from top to bottom row, respectively. In addition, we define the extra points x_{N_x+1}, y_{N_y+1},

z_{N_z+1}, and w_{N_w+1} as $x_{N_x}+h^x_{N_x-1}$, $y_{N_y}+h^y_{N_y-1}$, $z_{N_z}+h^z_{N_z-1}$, and $w_{N_w}+h^w_{N_w-1}$, respectively.

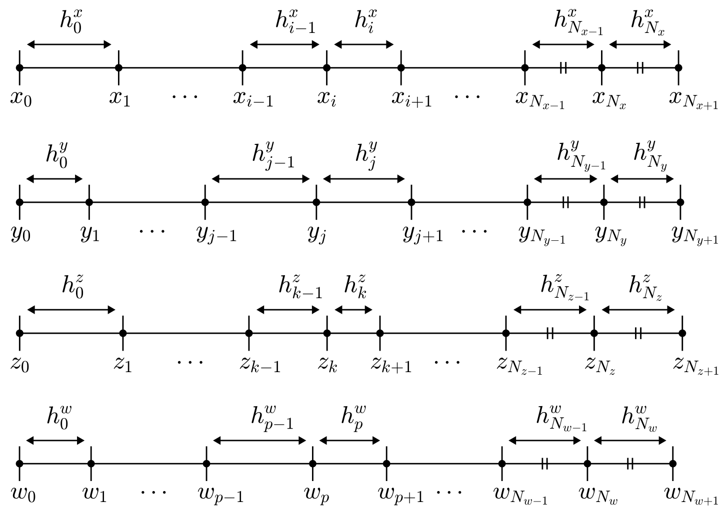

Figure 2. Schematic representation of the non-equidistant grid on x-axis, y-axis, z-axis, and w-axis, from top to bottom row, respectively.

Let $u^n_{ijkp} \equiv u(x_i, y_j, z_k, w_p, n\Delta\tau)$, where $i = 0,\dots,N_x$, $j = 0,\dots,N_y$, $k = 0,\dots,N_z$, $p = 0,\dots,N_w$, and $n = 0,\dots,N_\tau$. We use the homogeneous Dirichlet boundary conditions at left end points and the linear boundary conditions at right end points. Now, we apply operator splitting method (OSM) [3, 9] to numerically solve Eq. (3.2). We extend the three dimensional scheme [12] to four dimensional scheme and consider

$$(4.1)\quad \frac{u^{n+1}_{ijkp}-u^n_{ijkp}}{\Delta\tau} = (\mathcal{L}^x_{BS}u)^{n+\frac{1}{4}}_{ijkp} + \left(\mathcal{L}^y_{BS}u\right)^{n+\frac{2}{4}}_{ijkp} + (\mathcal{L}^z_{BS}u)^{n+\frac{3}{4}}_{ijkp} + (\mathcal{L}^w_{BS}u)^{n+1}_{ijkp},$$

where $\mathcal{L}^x_{BS}$, $\mathcal{L}^y_{BS}$, $\mathcal{L}^z_{BS}$,and $\mathcal{L}^w_{BS}$ are defined by

$$\begin{aligned}
(\mathcal{L}^x_{BS}u)^{n+\frac{1}{4}}_{ijkp} &= \frac{(\sigma_x x_i)^2}{2}D_{xx}u^{n+\frac{1}{4}}_{ijkp} + rx_iD_xu^{n+\frac{1}{4}}_{ijkp} + \sigma_x\sigma_y\rho_{xy}x_iy_jD_{xy}u^n_{ijkp}\\
&\quad +\sigma_x\sigma_z\rho_{xz}x_iz_kD_{xz}u^n_{ijkp} + \sigma_x\sigma_w\rho_{xw}x_iw_pD_{xw}u^n_{ijkp}\\
&\quad +\sigma_y\sigma_z\rho_{yz}y_jz_kD_{yz}u^n_{ijkp} + \sigma_y\sigma_w\rho_{yw}y_iw_pD_{yw}u^n_{ijkp}\\
&\quad +\sigma_z\sigma_w\rho_{zw}z_kw_pD_{zw}u^n_{ijkp} - ru^{n+\frac{1}{4}}_{ijkp},\\
\left(\mathcal{L}^y_{BS}u\right)^{n+\frac{2}{4}}_{ijk} &= \frac{(\sigma_y y_j)^2}{2}D_{yy}u^{n+\frac{2}{4}}_{ijkp} + ry_jD_yu^{n+\frac{2}{4}}_{ijkp},
\end{aligned}$$

$$
\begin{aligned}
(\mathcal{L}^z_{BS}u)^{n+\frac{3}{4}}_{ijkp} &= \frac{(\sigma_z z_k)^2}{2} D_{zz} u^{n+\frac{3}{4}}_{ijkp} + r z_k D_z u^{n+\frac{3}{4}}_{ijkp},\\
(\mathcal{L}^w_{BS}u)^{n+1}_{ijkp} &= \frac{(\sigma_w w_p)^2}{2} D_{ww} u^{n+1}_{ijkp} + r w_p D_w u^{n+1}_{ijkp}.
\end{aligned}
$$

Here, we use

$$
\begin{aligned}
D_x u_{ijkp} &= -\frac{h^x_i}{h^x_{i-1}(h^x_{i-1}+h^x_i)} u_{i-1,jkp} + \frac{h^x_i - h^x_{i-1}}{h^x_{i-1}h^x_i} u_{ijkp} + \frac{h^x_{i-1}}{h^x_i(h^x_{i-1}+h^x_i)} u_{i+1,jkp},\\
D_{xx} u_{ijkp} &= \frac{2}{h^x_{i-1}(h^x_{i-1}+h^x_i)} u_{i-1,jkp} - \frac{2}{h^x_{i-1}h^x_i} u_{ijkp} + \frac{2}{h^x_i(h^x_{i-1}+h^x_i)} u_{i+1,jkp},\\
D_{xy} u_{ijkp} &= \frac{u_{i+1,j+1,kp} - u_{i-1,j+1,kp} - u_{i+1,j-1,kp} + u_{i-1,j-1,kp}}{h^x_i h^y_j + h^x_{i-1} h^y_j + h^x_i h^y_{j-1} + h^x_{i-1} h^y_{j-1}}.
\end{aligned}
$$

Then, OSM consists of the following four discrete equations

$$
\frac{u^{n+\frac{1}{4}}_{ijkp} - u^n_{ijkp}}{\Delta\tau} = (\mathcal{L}^x_{BS}u)^{n+\frac{1}{4}}_{ijkp}, \tag{4.2}
$$

$$
\frac{u^{n+\frac{2}{4}}_{ijkp} - u^{n+\frac{1}{4}}_{ijkp}}{\Delta\tau} = \left(\mathcal{L}^y_{BS}u\right)^{n+\frac{2}{4}}_{ijkp}, \tag{4.3}
$$

$$
\frac{u^{n+\frac{3}{4}}_{ijkp} - u^{n+\frac{2}{4}}_{ijkp}}{\Delta\tau} = (\mathcal{L}^z_{BS}u)^{n+\frac{3}{4}}_{ijkp}, \tag{4.4}
$$

$$
\frac{u^{n+1}_{ijkp} - u^{n+\frac{3}{4}}_{ijkp}}{\Delta\tau} = (\mathcal{L}^w_{BS}u)^{n+1}_{ijkp}. \tag{4.5}
$$

Next, given u^n_{ijkp}, Eq. (4.2) is rewritten as follows:

$$
\alpha_i u^{n+\frac{1}{4}}_{i-1,jkp} + \beta_i u^{n+\frac{1}{4}}_{ijkp} + \gamma_i u^{n+\frac{1}{4}}_{i+1,jkp} = f_{ijkp} \text{ for } i = 1, \cdots, N_x, \tag{4.6}
$$

where

$$
\begin{aligned}
\alpha_i &= -\frac{(\sigma_x x_i)^2}{h^x_{i-1}(h^x_{i-1}+h^x_i)} + r x_i \frac{h^x_i}{h^x_{i-1}(h^x_{i-1}+h^x_i)},\\
\beta_i &= \frac{(\sigma_x x_i)^2}{h^x_{i-1}h^x_i} - r x_i \frac{h^x_i - h^x_{i-1}}{h^x_{i-1}h^x_i} + \frac{1}{\Delta\tau} + r,\\
\gamma_i &= -\frac{(\sigma_x x_i)^2}{h^x_i(h^x_{i-1}+h^x_i)} - r x_i \frac{h^x_{i-1}}{h^x_i(h^x_{i-1}+h^x_i)},\\
f_{ijkp} &= \sigma_x\sigma_y\rho_{xy}x_iy_jD_{xy}u^n_{ijkp} + \sigma_x\sigma_z\rho_{xz}x_iz_kD_{xz}u^n_{ijkp}\\
&\quad +\sigma_x\sigma_w\rho_{xw}x_iw_pD_{xw}u^n_{ijkp} + \sigma_y\sigma_z\rho_{yz}y_jz_kD_{yz}u^n_{ijkp}\\
&\quad +\sigma_y\sigma_w\rho_{yw}y_jw_pD_{yw}u^n_{ijkp} + \sigma_z\sigma_w\rho_{zw}z_kw_pD_{zw}u^n_{ijkp} + \frac{1}{\Delta\tau}u^n_{ijkp}.
\end{aligned}
$$

For fixed index j, k and p, the solution vector $u^{n+\frac{1}{4}}_{1:N_x,jkp} = [u^{n+\frac{1}{4}}_{1jkp}\ u^{n+\frac{1}{4}}_{2jkp}\ \cdots\ u^{n+\frac{1}{4}}_{N_xjkp}]^T$ can be obtained by solving the tridiagonal system

$$A_x u^{n+\frac{1}{4}}_{1:N_x,jkp} = f_{1:N_x,jkp},$$

where A_x is a tridiagonal matrix constructed from Eq. (4.6) with the zero Dirichlet ($u^{n+\frac{1}{4}}_{0jkp} = 0$ at $x = 0$) and linear boundary ($u^{n+\frac{1}{4}}_{N_x+1,jkp} = 2u^{n+\frac{1}{4}}_{N_xjkp} - u^{n+\frac{1}{4}}_{N_x-1,jkp}$ at $x = L$) conditions, i.e.,

$$A_x = \begin{pmatrix} \beta_1 & \gamma_1 & 0 & \cdots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \cdots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{N_x-1} & \gamma_{N_x-1} \\ 0 & 0 & 0 & \cdots & \alpha_{N_x} - \gamma_{N_x} & \beta_{N_x} + 2\gamma_{N_x} \end{pmatrix}.$$

Similarly, Eqs. (4.3), (4.4), and (4.5) are solved. For more details about the solution algorithm, see references [10, 9, 12].

5. Numerical experiments

To confirm the performance of the proposed scheme, we consider a convergence test with Monte Carlo simulation for pricing four underlying asset step-down ELS. We perform all simulations on MATLAB version R2020a on a Intel(R) Core(TM) i5-7400 CPU @ 3.00GHz 3.00 GHz PC with 12.0 GB RAM. Figure 3(a) and (b) show payoff functions at (a) maturity and (b) early redemption for four underlying asset step-down ELS.

Let D be the knock-in barrier level and d be a dummy. Let $u(x, y, z, w, \tau)$ and $v(x, y, z, w, \tau)$ be the numerical approximations with and without knock-in event, respectively. Let $\min_{xyzw} = \min(x, y, z, w)$. The initial conditions are defined as

$$u(x, y, z, w, \tau = 0) = \begin{cases} F \min_{xyzw}/100 & \text{if } \min_{xyzw} < K_6, \\ (1 + c_6)F & \text{otherwise.} \end{cases} \tag{5.1}$$

$$v(x, y, z, w, \tau = 0) = \begin{cases} (1 + c_6)F & \text{if } \min_{xyzw} \geq K_6, \\ (1 + d)F & \text{if } D < \min_{xyzw} < K_6, \\ F \min_{xyzw}/100 & \text{otherwise.} \end{cases} \tag{5.2}$$

First, we update u and v by solving Eqs. (4.2)–(4.5) with Eqs. (5.1) and (5.2). We redefine v by u as $v^1_{ijkp} = u^1_{ijkp}$ for $(x_i, y_j, z_k, w_p) \in \Omega_{ki} = \{(x_i, y_j, z_k, w_p) |\ x_i < D, y_j < D, z_k < D, w_p < D\}$. In addition, $v^n_{ijkp} = u^n_{ijkp}$ for $(x_i, y_j, z_k, w_p) \in \Omega_{ki}$

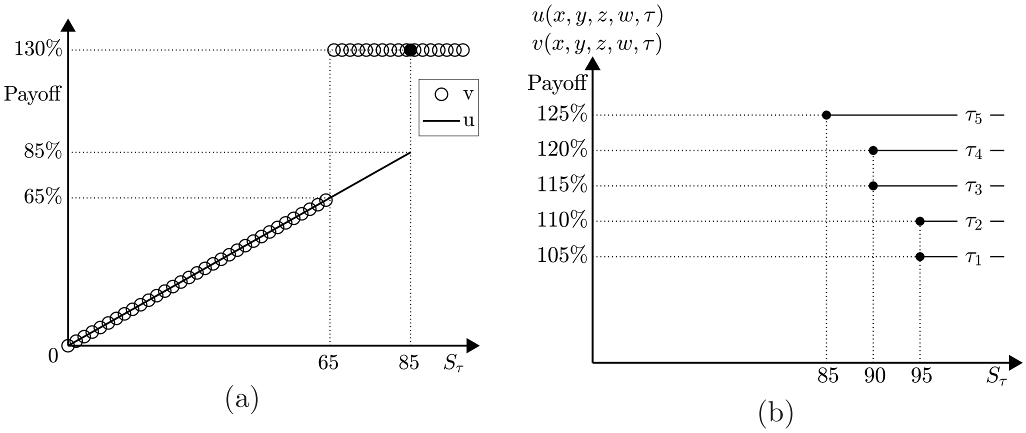

Figure 3. Pay-off functions at (a) maturity and (b) early redemption for four underlying asset step-down ELS.

and $n = 1, \ldots, N_t$. Let $\Omega_m = \{(x, y, z, w)|x \geq K_m, y \geq K_m, z \geq K_m, w \geq K_m\}$. At $\tau_1 = T/6$, we reset values of u and v as $u_{ijkp}^{n_1} = v_{ijkp}^{n_1} = (1+c_5)F$ for $(x_i, y_j, z_k, w_p) \in \Omega_5$. Similarly, we define $u_{ijkp}^{n_q} = v_{ijkp}^{n_q} = (1 + c_{6-q})F$ for $(x_i, y_j, z_k, w_p) \in \Omega_{6-q}$ for $q = 2,\ 3,\ 4$ and 5. The parameters are listed in Table 1.

Observation date (years)	Exercise price	Return rate
$\tau_1 = 1T/6$	$K_1 = 90$	$c_1 = 0.1$
$\tau_2 = 2T/6$	$K_2 = 90$	$c_2 = 0.2$
$\tau_3 = 3T/6$	$K_3 = 85$	$c_3 = 0.3$
$\tau_4 = 4T/6$	$K_4 = 85$	$c_4 = 0.4$
$\tau_5 = 5T/6$	$K_5 = 80$	$c_5 = 0.5$
$\tau_6 = 6T/6$	$K_6 = 80$	$c_6 = 0.6$

Table 1. Parameters of four underlying asset step-down ELS.

We use the following parameters: strike prices $K_1 = 90$, $K_2 = 90$, $K_3 = 85$, $K_4 = 85$, $K_5 = 80$, $K_6 = 80$, knock-in barrier $D = 60$, the interest rate $r = 0.01$, coupon rates $c_1 = 0.1$, $c_2 = 0.2$, $c_3 = 0.3$, $c_4 = 0.4$, $c_5 = 0.5$, $c_6 = 0.6$, the volatilities of the underlying assets $\sigma_x = 0.2$, $\sigma_y = 0.3$, $\sigma_z = 0.25$, $\sigma_w = 0.24$ and the correlation $\rho_{xy} = 0.7$, $\rho_{xz} = 0.48$, $\rho_{xw} = 0.27$, $\rho_{yz} = 0.45$, $\rho_{yw} = 0.3$, $\rho_{zw} = 0.5$. ELS option prices using MCS are computed with a temporal step size $\Delta\tau = 1/360$ and 16×10^2, 8×10^3, 4×10^4, 2×10^5, and 10^6 samples. We give MCS algorithm in Appendix. In FDM, a non-uniform mesh [0 30 59:1:61 70 79:1:81 84:1:86 89:1:91 99:1:102 110 120 130 140 160 180] is used with each $\Delta\tau = 1/10,\ 1/20,\ 1/50,\ 1/70$ until $\tau = 3$.

In Fig. 4, the open circles are the results of $u(100, 100, 100, 100, 3)$ from MCS with varying number of samples. For each number of samples, we plot the results obtained from 100 trials. In the legend in Fig. 4, $\Delta\tau = 1/10,\ 1/20,\ 1/50,\ 1/70$ from FDM on a non-uniform grid [0 30 59:1:61 70 79:1:81 84:1:86 89:1:91 99:1:102 110 120 130 140 160 180]. The CPU time of FDM ($\Delta\tau = 1/70$) and MCS (10^6 sample) are about $70s$ and $100s$, respectively. This simulation result indicates that FDM converges to the analytic solution faster than MCS does with the same computational cost.

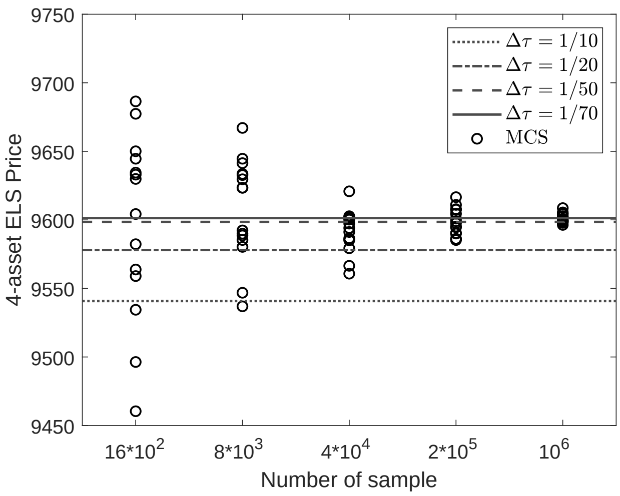

Figure 4. Results from MCS (open circles), and FDM (line) at $(x, y, z, w) = (100, 100, 100, 100)$ at $\tau = 3$.

6. Conclusions

In this article, we proposed an efficient and accurate FDM for the four underlying asset ELS. The numerical scheme is based on the operator splitting method with non-uniform grids for the underlying assets. Even though the numerical scheme is implicit, we solve the system of discrete equations in explicit manner using the Thomas algorithm for the tri-diagonal matrix resulting from the system of discrete equations. Therefore, we can use a relatively large time step and obtain the fast computational result of the ELS option pricing. We performed characteristic computational test and the numerical results confirmed the usefulness of the proposed method for pricing the four underlying asset ELS.

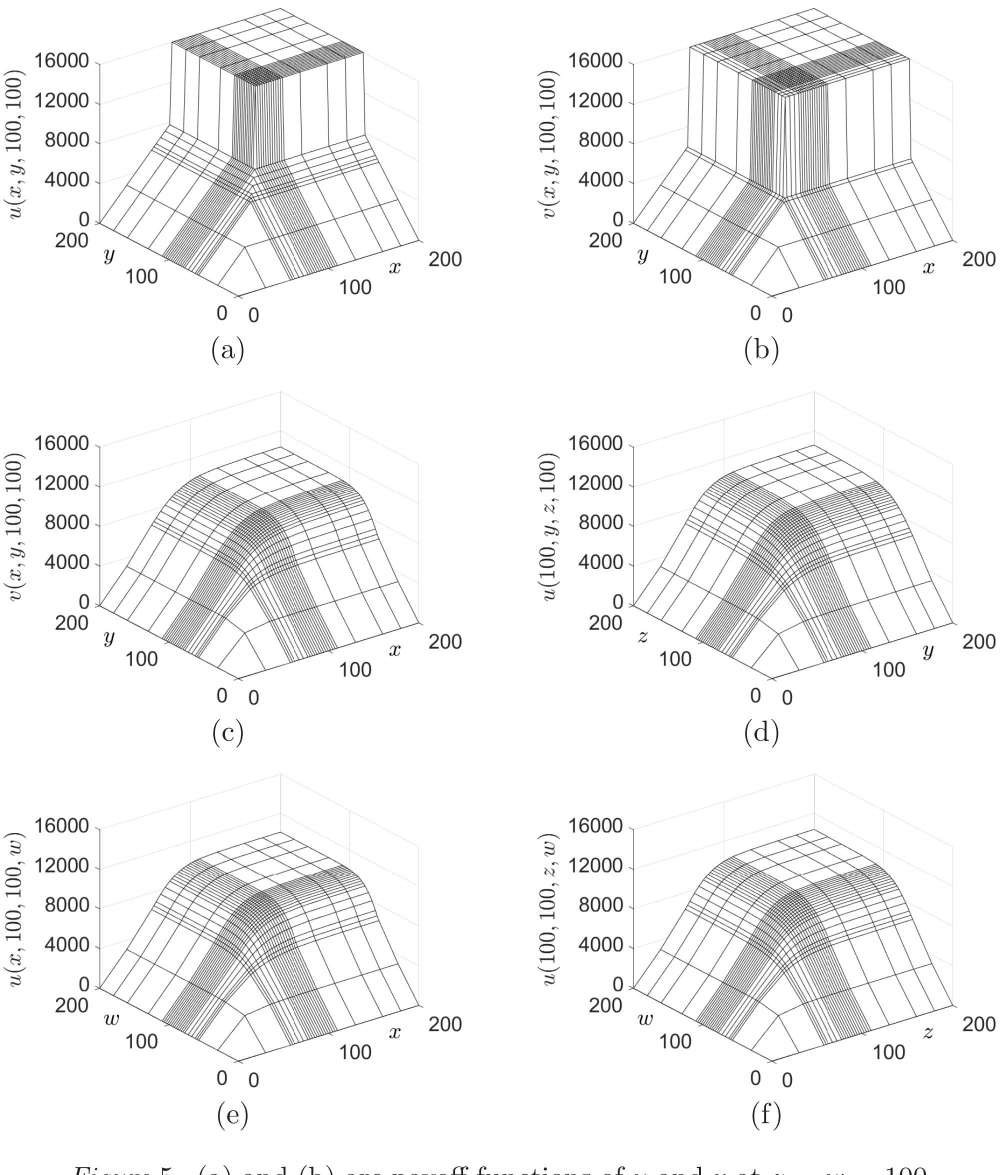

Figure 5. (a) and (b) are payoff functions of u and v at $z = w = 100$, respectively. (c), (d), (e), and (f) are the final solutions of v at $z = w = 100$, $x = w = 100$, $y = z = 100$, and $x = y = 100$, respectively. Here, $\tau = 3$.

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Appendix

For completeness of exposition, let us consider the computational algorithm of MCS for the four underlying asset ELS. Let

$$A = \begin{pmatrix} 1 & \rho_{xy} & \rho_{xz} & \rho_{xw} \\ \rho_{xy} & 1 & \rho_{yz} & \rho_{yw} \\ \rho_{xz} & \rho_{yz} & 1 & \rho_{zw} \\ \rho_{xw} & \rho_{yw} & \rho_{zw} & 1 \end{pmatrix} \tag{6.1}$$

be the correlation coefficient matrix between i and j underlying assets. We can decompose the matrix A using the Cholesky factorization [5] as follows:

$$A = LL^T, \tag{6.2}$$

where L is a lower triangular matrix. We generate correlated random numbers Z_1^*, Z_2^*, Z_3^*, and Z_4^* from a standard multivariate normal distribution Z_1, Z_2, Z_3, and Z_4 using

$$(Z_1^* \; Z_2^* \; Z_3^* \; Z_4^*)^T = L(Z_1 \; Z_2 \; Z_3 \; Z_4)^T. \tag{6.3}$$

We make the following four correlated asset paths:

$$\begin{aligned} X_1(t_{i+1}) &= X_1(t_i)e^{(r-0.5\sigma_1^2)\Delta t+\sigma_1\sqrt{\Delta t}Z_{1i}^*}, \\ X_2(t_{i+1}) &= X_2(t_i)e^{(r-0.5\sigma_2^2)\Delta t+\sigma_2\sqrt{\Delta t}Z_{2i}^*}, \\ X_3(t_{i+1}) &= X_3(t_i)e^{(r-0.5\sigma_3^2)\Delta t+\sigma_3\sqrt{\Delta t}Z_{3i}^*}, \\ X_4(t_{i+1}) &= X_4(t_i)e^{(r-0.5\sigma_4^2)\Delta t+\sigma_4\sqrt{\Delta t}Z_{4i}^*}. \end{aligned}$$

Let $WP(t_i)$ be the worst performer among four asset paths:

$$WP(t_i) = \min(X_1(t_i), X_2(t_i), X_3(t_i), X_4(t_i)). \tag{6.4}$$

We make stock prices at $T_1,\ T_2,\ T_3,\ T_4,\ T_5,\ T_6$. That is,

$$WP(T_i),\ i = 0, \ldots, 6, \tag{6.5}$$

where $WP(T_0) = 100$ and $T_0 = 0$. If the early redemptions and the maturity conditions are not satisfied and $\min\{WP(T_1),\ WP(T_2),\ WP(T_3),\ WP(T_4),\ WP(T_5),\ WP(T_6)\} \le D$, then the payoff is $WP(T_6)F/100$. If $\min\{WP(T_1), WP(T_2), WP(T_3),$

$WP(T_4)$, $WP(T_5)$, $WP(T_6)\} > D$, then if $\min_{1 \leq i \leq T_6/\Delta t} WP(t_i) \leq D$, then the return is $WP(T_6)F/100$. Otherwise, it is $(1+d)F$, where d is a dummy rate. For more detail about MCS for multi-asset ELS, please see [7].

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