

Research Article

Benchmark Problems for the Numerical Discretization of the Cahn–Hilliard Equation with a Source Term

Sungha Yoon,¹ Hyun Geun Lee,² Yibao Li,³ Chaeyoung Lee,¹ Jintae Park,¹ Sangkwon Kim,¹ Hyundong Kim,¹ and Junseok Kim¹

¹Department of Mathematics, Korea University, Seoul 02841, Republic of Korea

²Department of Mathematics, Kwangjuon University, Seoul 01897, Republic of Korea

³School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China

Correspondence should be addressed to Junseok Kim; cfdkim@korea.ac.kr

DWVhV \$&EVWV TVd\$" \$ #- 3UWVW \$" @ahV TVd\$" \$ #- BgTTeZW (6VWV TVd\$" \$ #

Academic Editor: Nikos I. Karachalios

Copyright © 2021 Sungha Yoon et al. is is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we present benchmark problems for the numerical discretization of the Cahn–Hilliard equation with a source term. If the source term includes an isotropic growth term, then initially circular and spherical shapes should grow with their original shapes. However, there is numerical anisotropic error and this error results in anisotropic evolutions. Therefore, it is essential to use isotropic space discretization in the simulation of growth phenomenon such as tumor growth. To test numerical discretization, we present two benchmark problems: one is the growth of a disk or a sphere and the other is the growth of a rotated ellipse or a rotated ellipsoid. The computational results show that the standard discrete Laplace operator has severe grid orientation dependence. However, the isotropic discrete Laplace operator generates good results.

1. Introduction

We consider the Cahn–Hilliard (CH) equation with the reaction term:

$$\frac{z \dots \mathbf{x}; t}{z t} \sim \mu \dots \mathbf{x}; t \dagger c \dots \mathbf{x}; t \dagger; \mathbf{x} \quad ; t > 0; \quad \dots 1 \dagger$$

$$\mu \dots \mathbf{x}; t \dagger \wedge F \dots \mathbf{x}; t \dagger \quad ^2 \quad \dots \mathbf{x}; t \dagger;$$

where $\dots \mathbf{x}; t$ is the order parameter in the domain \mathbb{R}^d , $d \in \{2, 3\}$, $F \dots \dagger \in [0, 25 \dots ^2 \quad 1 \dagger^2]$, and \dots is a constant. The reaction term $c \dots \dagger$ is specified in Section 3. In this paper, we present two benchmark problems for the numerical discretization of the CH equation with a source term in 2D and 3D.

The CH equation without a source term was derived for minimizing the interface in binary alloys [1, 2]. This topic has been studied in various fields such as multiphase flows [3–5], image inpainting [6], phase-field model [7–10], and microstructures with elastic inhomogeneity [11]. Since the common behavior of the CH equation without source terms is minimizing the interface, the anisotropic space

discretization error is not noticeable. However, in the case of a growth model, anisotropic discretization may have critical problems. In particular, tumor growth simulation with the CH equation is one of the cases. Several research studies have been conducted numerically by using the Galerkin finite element method [12–15]. Fakhri [16, 17] analyzed the asymptotic behavior of the CH equation with source in both Dirichlet and Neumann boundary conditions by using the P1 element method. Khain and Sander [18] studied the logistic growth model of the CH equation. In recent years, many studies for tumor growth simulation using the finite difference method have been actively conducted [19–23].

However, the conventional space discretization scheme has a high probability of having grid-related artifact because it has anisotropic properties. Therefore, we need to use the isotropic discretization. For instance, Kumar introduced the isotropic finite difference method [24]. We use the method proposed by Kumar in this paper. The isotropic finite difference scheme was derived for symmetric dendritic solidification and reduces an observable computational bias of the numerical anisotropy. In [25], the authors introduced

