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A Practical Monte Carlo Method for Pricing Equity-Linked Securities with Time-Dependent Volatility and Interest Rate

Sangkwon Kim[1] · Jisang Lyu[1] · Wonjin Lee[2] · Eunchae Park[1] · Hanbyeol Jang[2] · Chaeyoung Lee[1] · Junseok Kim[1]

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Abstract
We develop a fast Monte Carlo simulation (MCS) for pricing equity-linked securities (ELS) with time-dependent volatility and interest rate. In this paper, we extend a recently developed fast MCS for pricing ELS. In the previous model, both the volatility and interest rate were constant. However, in the real finance market, volatility and interest rate are time-dependent parameters. In this work, we approximate the time-dependent parameters by piecewise constant functions and apply Brownian bridge technique. We present some numerical results of the proposed method. The computational results demonstrate the fastness of the proposed algorithm with equivalent accuracy with standard MCS. It is important for traders and hedgers considering derivatives to evaluate prices and risks quickly and accurately. Therefore, our algorithm will be very useful to practitioners in the ELS market.

Keywords Fast Monte Carlo method · Time-dependent volatility · Time-dependent interest rate · Brownian bridge · Black–Scholes equation

1 Introduction

In finance, barrier options are being studied in many ways (Kontosakos et al., 2021; Jia & Chen, 2020; Liu & Yang, 2021; Kim et al., 2021b) because they are looking for a large proportion of derivatives. Equity-linked security (ELS) is one of the most common barrier and auto-callable options for financial derivatives and issuance amount is growing rapidly in South Korea.

As shown in Fig. 1a, ELS issuance is on a growing trend, and it has grown about 23.6 times in 2019 compared to 2003. Since 2004, an early repayment step-down structure with more than two underlying assets has been issued, and as shown in Fig. 1b, products with three underlying assets account for more than 80%. In 2020,

✉ Junseok Kim
cfdkim@korea.ac.kr

Extended author information available on the last page of the article

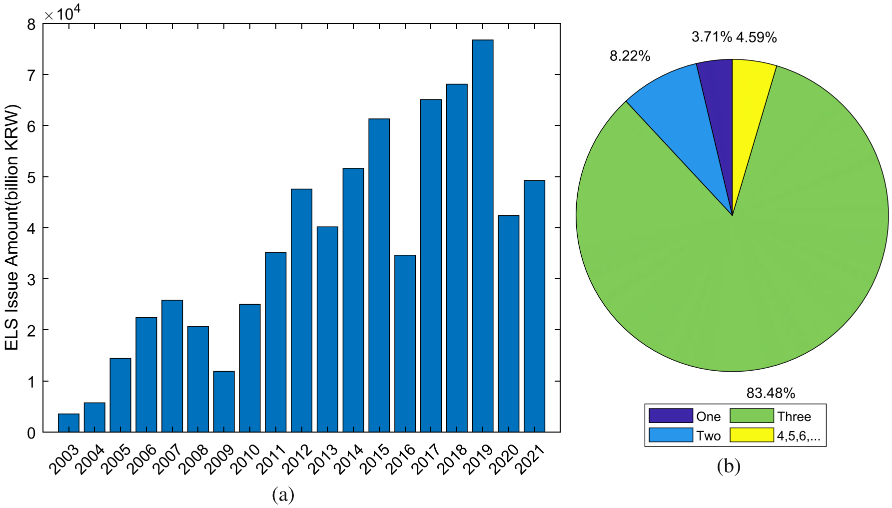


Fig. 1 **a** Amount of ELS issuance from 2003 to 2021 (billion KRW). **b** Share of issuance according to the number of underlying assets in 2019

ELS products with 9 underlying assets were issued in South Korea. The ELS market in South Korea is popular, and the proportion of ELS products with more than three underlying assets is increasing. Therefore, pricing the ELS efficiently and accurately is very important. To price the ELS, it is common to use either the Black–Scholes (BS) partial differential equation (PDE) using finite difference method (FDM) (Abdi-Mazraeh et al., 2020; Koffi & Tambue, 2020; Li & Tourin, 2022) or the Monte Carlo simulation (MCS) (Jeong et al., 2019; Jang et al., 2020; Lux, 2022). These methods have high computational costs for pricing the ELS. To overcome this disadvantage, derived methods are being studied (de Lima & Samanez, 2016; Jerbi, 2016; Nugroho, 2016). Recently, methods of solving PDE using deep learning techniques have been developed. Wang et al. (2022) solved the BS model using Physics-Informed Neural Networks (PINNs) to option pricing. Jang et al. (2020) presented the fast MCS. It has significantly low computational cost, i.e., it is approximately more than 20 times faster than a standard MCS. The fast MCS for pricing the ELS consists of two steps. The first step is generating a stock path at the redemption dates only. If early redemption does not occur and the stock price at predetermined redemption dates does not touch the knock-in barrier, then we go on to the next step. The next step is regenerating a daily stock path using Brownian bridge. In the previous fast MCS, the volatility and interest rate were constant. However, they are generally time-dependent parameters. For that reason, we need to develop a method which can deal with time-dependent variables.

Naz and Johnpillai (2018) adopted a method to convert the BS model with time dependent parameters to a classical heat equation. Chen et al. (2019)

examined time-dependent spot volatility model to price futures contract options. They verified the use of time-dependent spot volatility model through the Schwartz two-factor model. Farnoosh et al. (2016) considered the time-dependent parameters to price the discrete barrier option under the BS PDE. They presented a numerical method which values an option by converting the BS PDE with time-dependent variables into the BS PDE with constant parameters in each monitoring time interval of the discrete barrier option. Georgiev and Vulkov (2020) studied the recovery of time-dependent volatility using an average in time linearization of diffusion terms. Kim et al. (2021a) proposed the local volatility surface which is reconstructed by using a nonlinear fitting function. Lee and Hong (2021) studied the semi-closed pricing method by applying the closed non-exit probability of the Brownian bridge to the step-down ELS pricing for a single underlying asset, an automatic early repayment product. To expand the dimension, Lee et al. (2022) explored the non-exit probability of a two-dimensional brown bridge for the pricing of barrier options and auto-callable products with two underlying assets.

The main purpose of this paper is to propose a fast MCS for pricing the ELS with given time-dependent interest rate and volatility functions. ELS products are used as an important investment tool in the capital market. As risk management and hedge strategies become important, algorithms that can accurately and quickly calculate the price and profit or loss of a product are considered to be key elements of risk management and investment. However, although volatility and interest rates are generally time-dependent variables, volatility and interest rates were used at constant values in the previous fast MCS. Therefore, we develop a method that can handle time-dependent variables to verify the accuracy and speed of the method through numerical tests. The key of the proposed algorithm for pricing ELS is to apply the time-dependent variables to the previous fast MCS by organizing them into functions with constant values between exercise times, that is, the piecewise constant function. Here, using the parameters, we determine the piecewise constant approximation, interest rate $r(t)$ and volatility $\sigma(t)$, in the least-squares sense.

The contents of this paper are as follows: In Sect. 2, we present the proposed method. In Sect. 3, computational experiments are presented. Conclusions are given in Sect. 4.

2 Numerical Algorithm

In this study, we assume that time-dependent volatility $\sigma(t)$ and interest rate $r(t)$ are given. Then, the BS partial differential equation with time-dependent interest rate and volatility is as follows (Lo et al., 2003):

$$\frac{\partial u(S,t)}{\partial t} = -\frac{1}{2}[\sigma(t)S]^2\frac{\partial^2 u(S,t)}{\partial S^2} - r(t)S\frac{\partial u(S,t)}{\partial S} + r(t)u(S,t), \tag{1}$$

for $(S, t) \in \mathbb{R}^+ \times [0, T)$, where $u(S,\ t)$ is the option value of the underlying price S and time t, $\sigma(t)$ is the volatility of underlying asset at time t, $r(t)$ is the riskless interest rate at time t. The final condition is the payoff function $u(S, T) = \Phi(S)$ at expiry T.

Let $\tau = T - t$ be the time to expiry, then Eq. (1) can be given as the following initial-value problem:

$$\frac{\partial u(S,\tau)}{\partial \tau} = \frac{1}{2}[\sigma(\tau)S]^2\frac{\partial^2 u(S,\tau)}{\partial S^2} + r(\tau)S\frac{\partial u(S,\tau)}{\partial S} - r(\tau)u(S,\tau), \tag{2}$$

for $(S, \tau) \in \Omega \times (0, T]$ with an initial condition $u(S, 0) = \Phi(S)$ for $S \in \Omega = (0, L)$, where the infinite domain is truncated to a finite computational domain (Tavella & Randall, 2000). Now, to solve Eq. (2) numerically, we apply to a fully implicit FDM. Let us denote the numerical approximation of the solution of Eq. (2) by $u_i^n \equiv u(S_i, \tau_n) = u(ih,\ n\Delta\tau)$, for $i = 1,\ 2, \ldots,\ N_S$ and $n = 0,\ 1, \ldots,\ N_\tau$. Here, $h = L/(N_S - 1)$ and $\Delta\tau = T/N_\tau$. The variable volatility σ^n and interest rate r^n are defined as $\sigma^n \equiv \sigma(\tau_n)$ and $r^n \equiv r(\tau_n)$, respectively.

By applying FDM to Eq. (2), we obtain that

$$\begin{aligned}\frac{u_i^{n+1} - u_i^n}{\Delta\tau} =& \frac{(\sigma^{n+1}S_i)^2}{2}\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{h^2} + r^{n+1}S_i\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2h} \\ &- r^{n+1}u_i^{n+1}.\end{aligned} \tag{3}$$

We can rewrite the above Eq. (3) as

$$\alpha_i u_{i-1}^{n+1} + \beta_i u_i^{n+1} + \gamma_i u_{i+1}^{n+1} = b_i, \quad \text{for} \quad i = 2, \cdots, N_S - 1, \tag{4}$$

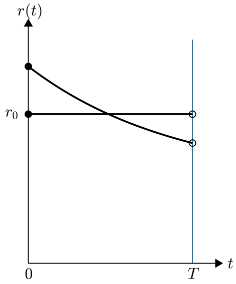
where $\alpha_i = \frac{r^{n+1}S_i}{2h} - \frac{(\sigma^{n+1}S_i)^2}{2h^2}$, $\beta_i = \frac{1}{\Delta\tau} + \frac{(\sigma^{n+1}S_i)^2}{h^2} + r^{n+1}$, $\gamma_i = -\frac{r^{n+1}S_i}{2h} - \frac{(\sigma^{n+1}S_i)^2}{2h^2}$, and $b_i = \frac{u_i^n}{\Delta\tau}$. We use the zero Dirichlet boundary condition at S_1, that is, $u_1^n = 0$ and the linear boundary condition at S_{N_S}, that is, $u_{N_S}^n = 2u_{N_S-1}^n - u_{N_S-2}^n$ for all n (Windcliff et al., 2004). To solve the discrete equations (4), we use the Thomas algorithm (Thomas, 1949).

2.1 Piecewise Constant Interest Rate

If volatility $\sigma(t)$ is zero, then a stochastic process $S(t)$ can be expressed as the following equation:

$$\frac{dS(t)}{S(t)} = r(t)dt. \tag{5}$$

Fig. 2 Time-dependent interest rate $r(t)$ and constant interest rate r_0



By integrating Eq. (5) from $t = 0$ to $t = T$, we have

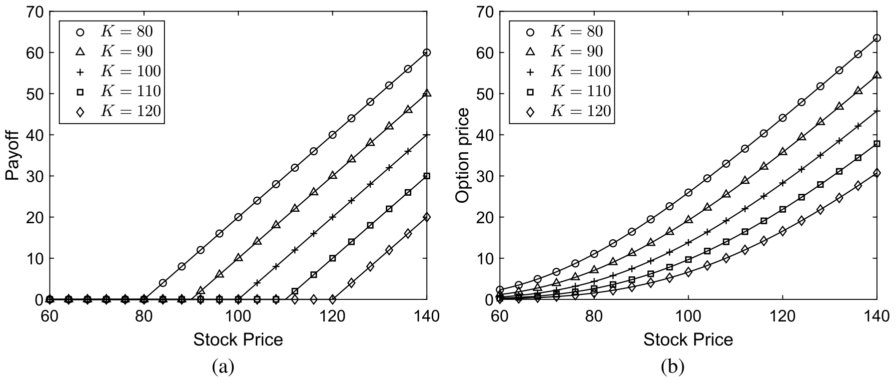
$$S(T) = S(0)e^{\int_0^T r(t)dt}. \tag{6}$$

If we assume constant interest rate, i.e., $r(t) = r_0$, then we have

$$S(T) = S(0)e^{r_0 T}. \tag{7}$$

Equating the two Eqs. (6) and (7) gives $r_0 = \frac{1}{T}\int_0^T r(t)dt$, which is schematically illustrated in Fig. 2.

To verify the equivalency between time-dependent and constant interest rates, we test with a European call option. We consider a time-dependent



(a) (b)

Fig. 3 **a** Payoff functions, $u(S, 0) = \max(S - K,\ 0)$, where $K = 80, 90,\ \ 100,\ \ 110,$ and 120. **b** Option prices with constant interest rate r_0 with different strike prices. Solid lines are the corresponding option prices with $r(t)$

Table 1 European call option prices at underlying asset $S = 100$ with time-dependent and constant interest rates

K	80	90	100	110	120
$r(t)$	25.9611	19.2292	13.8214	9.6812	6.6371
r_0	25.9633	19.2313	13.8232	9.6826	6.6382

interest rate $r(t) = 0.01 + 0.05e^{-t}$ (Lo et al., 2003). The payoff functions are given as $\Phi_\beta(S) = \max(S - K_\beta, 0)$ where strike prices $K_\beta = 70 + 10\beta$ for $\beta = 1, \dots, 5$. The other parameters used are underlying asset price $S \in [0, 400]$, underlying asset price step size $h = 1$, the volatility $\sigma = 0.3$, time to maturity $T_\alpha = 0.5\alpha$ for $\alpha = 1, \dots, 6$, and the time step size $\Delta t = 1/360$. Figure 3a, b show the payoff functions and option prices with constant interest rate $r_0 = \frac{1}{T}\int_0^T r(t)dt = 0.06 - 0.05e^{-1}$ with different strike prices using Eq. (3), respectively. Here, solid lines and makers are option prices that correspond with $r(t)$ and r_0, respectively.

Table 1 lists European call option prices at underlying asset $S = 100$ with time-dependent and constant interest rates, which confirms that r_0 is a good approximation for $r(t)$.

We will price ELS by using a fast MCS. The method first generates a path for early redemption dates only using constant interest rate and in the case that there is no early redemption and is never below the barrier, then it recreates a daily path using Brownian bridge with variable interest rate. Hence, to construct a piecewise constant function, we compute the average interest rate between early redemption dates. Figure 2 shows that a constructed piecewise constant interest function.

2.2 Piecewise Constant Volatility

We need a piecewise constant volatility function to apply the fast MCS for pricing the ELS. We construct it by using the steepest descent method to approximate time-dependent volatility function (Fig. 4). We describe the proposed algorithm for piecewise constant approximation for time-dependent volatility function, $\sigma(t)$. Suppose that we have a set of option values $\{\omega_\beta^\alpha\}$ with the time-dependent volatility function, $\sigma(t)$, where ω_β^α is the option price with the exercise time T_α for $\alpha = 1, \dots, M_t$ and the strike price K_β for $\beta = 1, \dots, M_k$. Using the parameters, we determine the piecewise

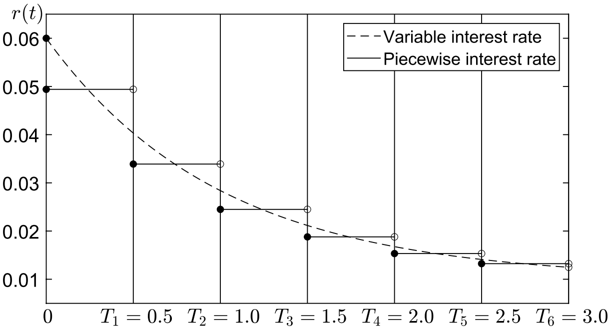


Fig. 4 Time-dependent interest rate function $r(t)$ and constructed piecewise constant interest function $r_0(t)$

constant approximation, $\sigma(t)$, in the least-squares sense. For $\alpha = 1, \dots, M_t$, we define the piecewise constant approximation as

$$\sigma_\alpha(t) = \sigma_i \quad \text{if } t \in [T_{i-1},\ T_i), \ \text{ for } \ i = 1, \dots, \alpha, \tag{8}$$

where $T_0 = 0$ and constant values σ_i for $i = 1, \dots, \alpha$. We assume $\sigma_p(t) = \sigma_q(t)$ for $0 \le t < T_q$ if $p > q$. Let $\Gamma_0(\sigma_0(t)) = 0$. Then, for $\alpha = 1, \dots, M_t$, we minimize the following mean-square error:

$$\Gamma_\alpha(\sigma_\alpha(t)) = \Gamma_{\alpha-1}(\sigma_{\alpha-1}(t)) + \frac{1}{M_k}\sum_{\beta=1}^{M_k}[u_{K_\beta}(\sigma_\alpha(t);S_0, T_\alpha) - \omega_\beta^\alpha]^2, \tag{9}$$

where $u_{K_\beta}(\sigma_\alpha(t);S_0, T_\alpha)$ is the numerical solution at $S = S_0$ of Eq. (3) with the strike price K_β at time T_α and the piecewise constant approximation $\sigma_\alpha(t)$. We apply the steepest descent method (Burden & Faires, 2001) to find $\sigma_{M_t}(t)$ that minimizes $\Gamma_{M_t}(\sigma_{M_t}(t))$. The process is as follows:

Step 1 Find the constant volatility function on $[0, T_1)$: By assuming $\sigma_1(t) = \sigma_1$ on $[0, T_1)$, we find that σ_1 minimizes the following cost function:

$$\Gamma_1(\sigma_1(t)) = \frac{1}{M_k}\sum_{\beta=1}^{M_k}\left[u_{K_\beta}(\sigma_1(t);S_0, T_1) - \omega_\beta^1\right]^2. \tag{10}$$

Figure 5a illustrates the constant volatility function at this step.

Step 2 Find the piecewise constant volatility function on $[0, T_2)$: With σ_1 obtained in *Step 1*, we define the piecewise constant volatility function $\sigma_2(t)$ on $[0, T_2)$ as

$$\sigma_2(t) = \begin{cases} \sigma_1(t) & \text{if} \quad 0 \le t < T_1, \\ \sigma_2 & \text{if} \quad T_1 \le t < T_2. \end{cases} \tag{11}$$

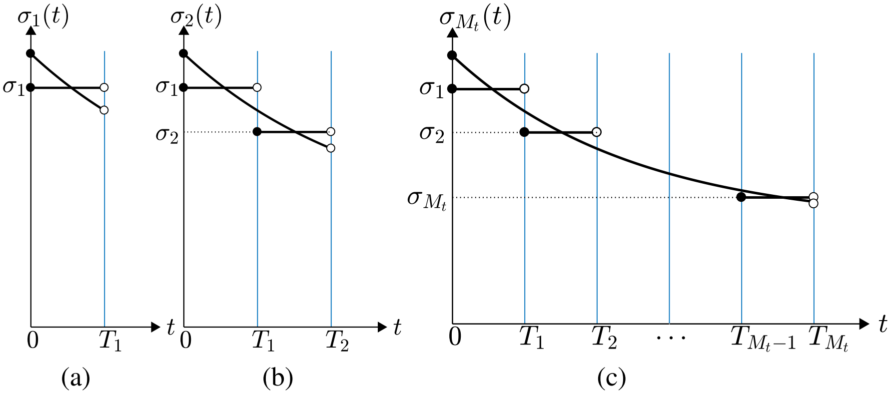


Fig. 5 Schematic illustration for the construction of the piecewise constant volatility function

Here, Eq. (11) represents a piecewise constant function having the optimal value σ_1 on $[0, T_1)$ and having the optimal value σ_2 on $[T_1, T_2)$ that minimizes the following cost function.

$$\Gamma_2(\sigma_2(t)) = \Gamma_1(\sigma_1(t)) + \frac{1}{M_k}\sum_{\beta=1}^{M_k}\left[u_{K_\beta}(\sigma_2(t);S_0,T_2) - \omega_\beta^2\right]^2. \tag{12}$$

Note that in this *Step*, we only estimate a single parameter σ_2 using the given value σ_1. Figure 5b illustrates the piecewise constant volatility function $\sigma_2(t)$ on $[0,\ T_2)$.

Step 3 Find the piecewise constant volatility function on $[0, T_{M_t})$: The following process is repeated from $\alpha = 3$ to $\alpha = M_t$. By using $\sigma_{M_t-1}(t)$ obtained in previous step, we set the piecewise constant volatility function $\sigma_{M_t}(t)$ on $[0, T_{M_t})$ as

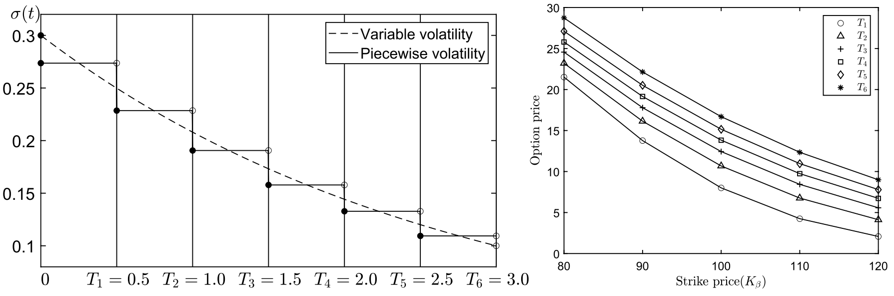
$$\sigma_{M_t}(t) = \begin{cases} \sigma_{M_t-1}(t) & \text{if} \quad 0 \le t < T_{M_t-1} \\ \sigma_{M_t} & \text{if} \quad T_{M_t-1} \le t < T_{M_t}, \end{cases} \tag{13}$$

which minimizes the cost function

$$\Gamma_{M_t}(\sigma_{M_t}(t)) = \Gamma_{M_t-1}(\sigma_{M_t-1}(t)) + \frac{1}{M_k}\sum_{\beta=1}^{M_k}\left[u_{K_\beta}(\sigma_{M_t}(t);S_0,T_{M_t}) - \omega_\beta^\alpha\right]^2. \tag{14}$$

Figure 5c illustrates the piecewise constant volatility function $\sigma_{M_t}(t)$ on $[0,\ T_{M_t})$. Note that it is very robust to compute the piecewise volatility function in the proposed method because we only need to find a single parameter value using the steepest descent algorithm.

For example, let us consider a time-dependent volatility function, $\sigma(t) = 0.3/3^{t/3}$ (Jin et al., 2018). To find the piecewise constant volatility function for $\sigma(t)$, we solve Eq. (4) to obtain the price of a European call option using the fully implicit FDM with the payoff functions $\Phi_\beta(S) = \max(S - K_\beta,\ 0)$ for $\beta = 1, \ldots, 5$. The other parameters used are strike prices $K_\beta = 70 + 10\beta$ for $\beta = 1, \ldots, 5$, underlying asset



(a) (b)

Fig. 6 **a** Time-dependent volatility function $\sigma(t)$ and constructed piecewise constant volatility function. **b** European call option prices with time-dependent (solid lines) and constant volatility (markers)

Table 2 European call option prices at underlying asset $S = 100$ with time-dependent and constant volatility

K		80	90	110	110	120
T_1	$\sigma(t)$	21.5158	13.7744	8.0091	4.2526	2.0846
	$\sigma_1(t)$	21.5146	13.7699	8.0024	4.2457	2.0791
T_2	$\sigma(t)$	23.2039	16.1400	10.6820	6.7651	4.1270
	$\sigma_2(t)$	23.2051	16.1401	10.6812	6.7638	4.1256
T_3	$\sigma(t)$	24.5636	17.7863	12.4360	8.4364	5.5806
	$\sigma_3(t)$	24.5653	17.7871	12.4359	8.4357	5.5796
T_4	$\sigma(t)$	25.7970	19.1545	13.8272	9.7446	6.7324
	$\sigma_4(t)$	25.7956	19.1505	13.8213	9.7377	6.7254
T_5	$\sigma(t)$	27.1091	20.5184	15.1557	10.9652	7.7991
	$\sigma_5(t)$	27.1061	20.5116	15.1460	10.9539	7.7873
T_6	$\sigma(t)$	28.7359	22.1426	16.6910	12.3482	8.9948
	$\sigma_6(t)$	28.7404	22.1448	16.6910	12.3464	8.9919

Table 3 Parameter values for the step-down ELS

Redemption date	$T_1 = 0.5$	$T_2 = 1$	$T_3 = 1.5$	$T_4 = 2$	$T_5 = 2.5$	$T_6 = 3$
Strike percentage	$K_1 = 95$	$K_2 = 95$	$K_3 = 95$	$K_4 = 90$	$K_5 = 90$	$K_6 = 90$
Coupon rate	$c_1 = 0.025$	$c_2 = 0.05$	$c_3 = 0.075$	$c_4 = 0.1$	$c_5 = 0.125$	$c_6 = 0.15$

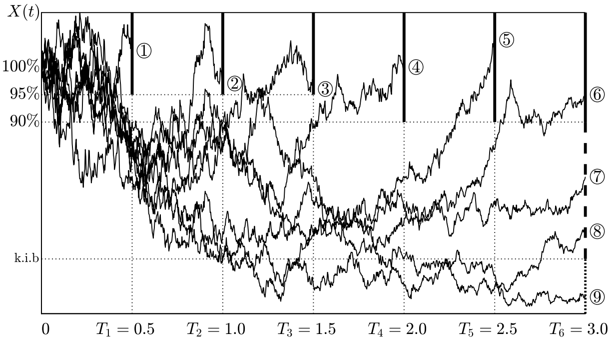


Fig. 7 Nine paths in the step-down ELS

price $S \in [0, 400]$, underlying asset price step size $h = 1$, the time-dependent interest rate function $r(t) = 0.01 + 0.05e^{-t}$, time to maturity $T_\alpha = 0.5\alpha$ for $\alpha = 1, \dots, 6$, and the time step size $\Delta t = 1/360$.

Figure 6a, b show a piecewise constant volatility function by using the steepest descent method and the value ω_β^α and $u_{K_\beta}(\sigma_\alpha(t))$, respectively. It can be seen that the computational results from continuous and piecewise constant parameters are similar to each other as shown in Fig. 6b and Table 2. Therefore, we can verify that the piecewise constant volatility function is a good approximation for a given time-dependent continuous volatility function.

2.3 Algorithm for Pricing ELS

For completeness of presenting the proposed algorithm, we include parts of the previous algorithm (Jang et al., 2020) of the fast MCS for ELS with constant parameters. This section describes the ELS products of the step-down type with a maturity of three years. Table 3 lists strike percentages (K) and coupon rates (c) for each early repayment dates (T). We consider nine repayment cases for the step-down ELS. The face value $F = 100$, knock-in barrier $D = 65$, dummy rate $d = 0.15$, the time-dependent interest rate function $r(t) = 0.01 + 0.05e^{-t}$, time-dependent volatility function $\sigma(t) = 0.3/3^{t/3}$, and time-step size $\Delta t = 1/360$ are used.

Figure 7 shows the nine paths in the step-down ELS.

2.4 Fast MCS for Pricing ELS

The Brownian bridge construction was used to compute the price of the step-down ELS in (Jang et al., 2020). However, in the previous research, the interest rate and the volatility were assumed to be constants, which is not an adequate assumption because they are time-dependent parameters in the real-world financial market. Therefore, in this study, we use piecewise time-dependent interest rate and volatility to evaluate reasonable ELS prices. To determine the price of the ELS product described in the previous section, we make the sample paths at times $t = t_i = i\Delta t$ using

$$S(t_{i+1}) = S(t_i)e^{(r(t_i)-0.5\sigma(t_i)^2)\Delta t+\sigma(t_i)\sqrt{\Delta t}W_{t_i}}, \tag{15}$$

where $S(t_i)$ is the underlying asset price, $r(t_i)$ is time-dependent interest rate, $\sigma(t_i)$ is time-dependent volatility, W_{t_i} is a standard normal distribution at time $t = t_i$ (Higham, 2004) and Δt is the time-step size.

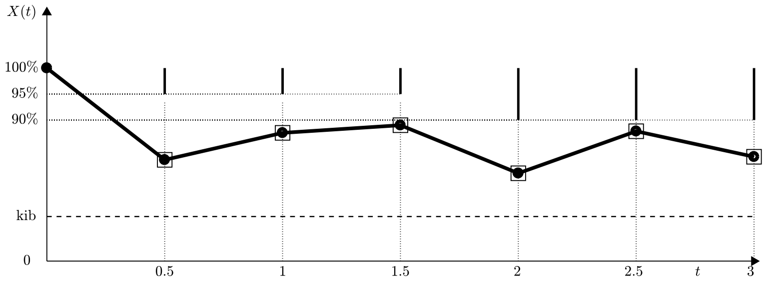


Fig. 8 Semiannual stock path for three years

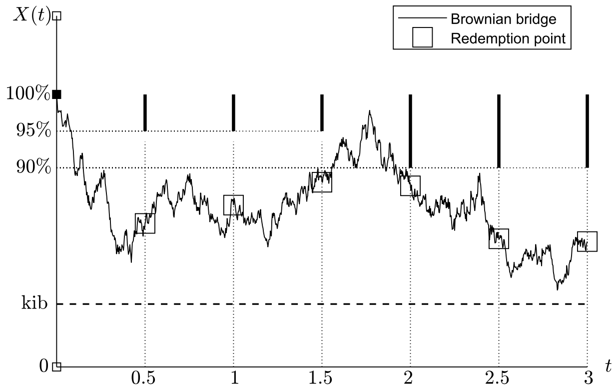


Fig. 9 Stock paths with Brownian motion ('+') and Brownian bridge (solid line)

Now, we consider the Brownian bridge construction. We only make stock prices at redemption and maturity dates, see Fig. 8.

At this time, we use piecewise constant interstate rate and volatility instead of time-dependent interstate rate. That is,

$$S(T_{i+1}) = S(T_i)e^{(r_i-0.5\sigma_i^2)(T_{i+1}-T_i)+\sigma_i\sqrt{T_{i+1}-T_i}Z_i}, i = 0, \ldots, 5, \tag{16}$$

where $S(T_0) = 100$, $T_0 = 0$ and r_i and σ_i are constant interest rates and volatilities in $[T_i,\ T_{i+1})$ for $i = 0, \ldots, 5$. We check redemptions and the maturity condition. If all conditions are not satisfied and $\min\{S(T_1), S(T_2), \ldots, S(T_6)\} \le D$, then the return is $S(T_6)F/100$. If $\min\{S(T_1), S(T_2), \ldots, S(T_6)\} > D$, then we make a full path using the Brownian bridge method (Shreve, 2004; Deshpande et al., 2019; Oh & Lee, 2019) with time-dependent interest rate and volatility, see Fig. 9.

If $\min_{1\le i\le T_6/\Delta t} S(t_i) \le D$, then the payoff is $S(T_6)F/100$; else, it is $(1+d)F$. Let $S(T_i)$ and $S(T_{i+1})$ be given, then we make a path from $\hat{S}(T_i) = S(T_i)$.

$$\hat{S}(t_{j+1}) = \hat{S}(t_j)e^{w_j},\ j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1, \tag{17}$$

where $w_j = (r(t_j) - 0.5\sigma(t_j)^2)\Delta t + \sigma(t_j)\sqrt{\Delta t}W_{t_j}$ and $t_j = T_i + j\Delta t$. Let $W_{t_j} = \sum_{i=0}^{j} w_{t_i}$, then $\hat{S}(t_{j+1}) = \hat{S}(T_i)e^{W_{t_j}}$, $j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1$. In general, $\hat{S}(T_{i+1}) \ne S(T_{i+1})$.

To make a path from $S(T_i)$ to $S(T_{i+1})$, we apply the Brownian bridge technique to W_{t_j}. Let

$$B_j = W_{t_j} + \frac{t_j - T_i}{T_{i+1} - T_i} \log \frac{S(T_{i+1})}{\hat{S}(T_{i+1})},\ j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1. \tag{18}$$

Then, we have

$$S(t_{j+1}) = S(T_i)e^{B_{t_j}},\ j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1. \tag{19}$$

A pseudo code for the fast Brownian bridge MCS is described in Algorithm 1.

Algorithm 1 Fast MCS algorithm

```
Require: $S_0$, $T$, the number of checking days $N_c$, the number of sample paths $N_m$, $N_T$, $\Delta t = T/N_T$,
  face value $F$, $\sigma(t)$, constant volatility $\sigma_i$, $r(t)$, constant interest rate $r_i$, $T_i$, coupon rates $c_i$, $K_i$, dummy
  $d$, and knock-in barrier $D$. Set $M_i = 0$ and $X(t) = 100S(t)/S_0$. Here, $1 \leq i \leq N_c$ and $T_0 = 0$.
  for $k = 1$ to $N_m$ do
      ▷ Generate stock path for $T_i$ only as
      for $i = 0$ to $N_c - 1$ do
          $X(T_{i+1}) = X(T_i)\exp((r_i - 0.5\sigma_i^2)(T_{i+1} - T_i) + \sigma_i\sqrt{T_{i+1} - T_i}W_{T_i}),\ \ W_{T_i} \sim N(0,1)$
      end for
      ▷ Check the conditions
      if $X(T_1) \geq K_1$ then $M_1 = M_1 + (1 + c_1)F$
      else if $X(T_2) \geq K_2$ then $M_2 = M_2 + (1 + c_2)F$
                              ⋮
      else if $X(T_{N_c}) \geq K_{N_c}$ then $M_{N_c} = M_{N_c} + (1 + c_{N_c})F$
      else if $\min_{1 \leq i \leq N_c}\{X(T_i)\} \leq D$ then $M_{N_c} = M_{N_c} + FX(T_{N_c+1})/100$
      else
          ▷ Generate a daily path
          for $i = 0$ to $N_c - 1$ do
              Set $Y(T_i) = X(T_i)$
              for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ do
                  $Y(t_{j+1}) = Y(t_j)\exp(w_j)$,
                  where $w_j = (r(t_j) - 0.5\sigma(t_j)^2)\Delta t + \sigma(t_j)\sqrt{\Delta t}W_{t_j}$, $Z_j \sim N(0,1)$, and $t_j = j\Delta t$
              end for
              ▷ Use the Brownian bridge
              for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ do
                  $Y(t_{j+1}) = Y(T_i)\exp(W_j),\ \ W_j = \sum_{p=T_i/\Delta t}^{j} w(t_p)$
              end for
              for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ do
                  $X(t_j) = X(T_i)\exp(B_j),\ \ B_j = W_j + \frac{t_j - T_i}{T_{i+1} - T_i}\log\frac{X(T_{i+1})}{Y(T_{i+1})}$
              end for
          end for
          if $\min_{1 \leq j \leq N_c/\Delta t}\{X(t_j)\} \leq D$ then $M_{N_c} = M_{N_c} + FX(T_{N_c+1})/100$
          else
              $M_{N_c} = M_{N_c} + (1 + d)F$
          end if
      end if
  end for
  ▷ Find option value.
  $V^0 = \sum_{i=1}^{N_c} e^{-r_i T_i} M_i/N_m$
```

3 Numerical Experiments

We demonstrate the performance of the proposed piecewise constant parameters by numerical experiments with given interest rate function and volatility function. We present numerical convergence tests between the standard MCS and fast MCS. and we calculate the CPU time for both the standard MCS and fast MCS. All computations were performed on a 2.7 GHz Intel PC with 16 GB of RAM loaded with MATLAB 2019b.

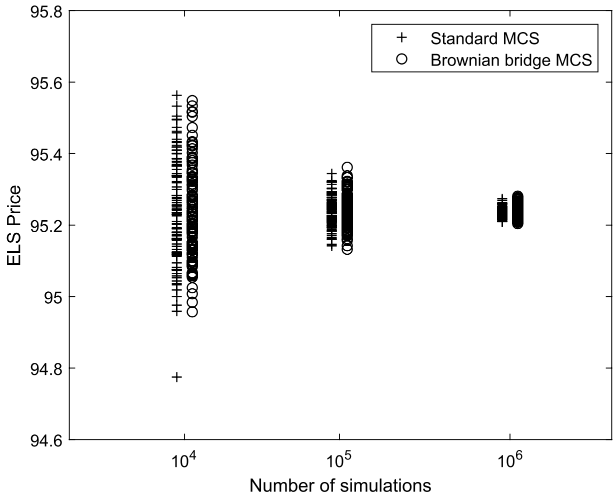


Fig. 10 ELS prices with different number of samples

Table 4 Average and variance of the ELS prices with the two methods

Case	Mean	Variance
Standard MCS	95.2339	0.0021
Fast MCS	95.2433	0.0023

3.1 Convergence Test

We compute the ELS price using the standard MCS with given time-dependent interest rate function ($r(t) = 0.01 + 0.05e^{-t}$) and volatility function ($\sigma(t) = 0.3/3^{t/3}$). We calculate the ELS price using the fast MCS with given two functions and constructed piecewise constant functions. For each number of samples, we show 100 simulation results. As the number of samples increases, one and the other converge to the same value.

Figure 10 shows the distribution of ELS price for different number of samples and it can be seen that ELS price converge as the number of samples increases. Here, Plus and open symbols denote the ELS prices obtained using the standard MCS and proposed method, respectively.

Table 4 lists the mean and variance of ELS prices using two methods: a 10^5 sample and a 500 simulation. The absolute error of mean and variance for the two methods is 0.0094 and 0.0002, respectively, so the equivalence of the two methods can be confirmed.

3.2 Complexity of Calculating ELS Price Using Two Methods

It is difficult to analyze the computational cost of numerical experiments using the MCS. Therefore, we estimate the probabilities of the nine cases as shown in Fig. 7. We present the computational cost of fast MCS by using estimated probabilities and compare the complexities of standard MCS with fast MCS. The probabilities of the nine cases, $W_i = n_i/N_m$ for $i = 1, 2, \cdots, 9$, are estimated by counting the number of paths n_i in each case where N_m is the number of sample paths. To measure computational cost, we define the work unit WU, which is the number of operations used to calculate Eq. (15) once. The fast MCS, first, only finds stock prices at redemption and maturity dates. Therefore, the cost of cases ①–⑥requires $i \times WU$ for $i = 1, 2, \cdots, 6$, respectively. Case ⑨ does not need to make a full path because the stock price at maturity dates is less than the knock-in barrier. Hence, the cost is $6 \times WU$. In cases ⑦ and ⑧, the full path shall be made because the stock prices at redemption and maturity dates do not satisfy early redemptions throughout the contract period. The cost of checking early repayment added to the costs of making a full path is $(N_T + 6) \times WU$, where $N_T = T/\Delta t$. Finally, the computational cost of fast MCS is as follows:

$$Cost_f = N_d \times N_m \times \left(\sum_{i=1}^{6} iW_i + (N_T + 6)W_7 + (N_T + 6)W_8 + 6W_9 \right) \times WU,$$

where $\sum_{i=1}^{9} W_i = 1$ and N_d and N_m are the numbers of underlying asset and sample paths, respectively. Herein, the cost of using the Brownian bridge technique, which is a small amount compared to the cost of the entire cycle, is neglected. The standard MCS makes always the full paths, therefore, the computational cost of standard MCS is as follows:

$$Cost_s = N_d \times N_m \times N_T \times WU.$$

Finally, the ratio of the cost of fast MCS to the cost of standard MCS is

$$Ratio = \frac{Cost_f}{Cost_s} = \frac{\sum_{i=1}^{6} iW_i + (N_T + 6)W_7 + (N_T + 6)W_8 + 6W_9}{N_T}.$$

We compute the cost with different numbers of samples: 10^4, , 10^5, and 10^6. Table 5 lists the computational cost for standard MCS, fast MCS, and the ratio of the

Table 5 Computational cost of the two different methods for the three assets ELS

Case	10^4	10^5	10^6
Standard MCS	3240.0000×10^4	3240.0000×10^5	3240.0000×10^6
Fast MCS	267.4338×10^4	267.9098×10^5	265.1260×10^6
Ratio	12.1151	12.0936	12.0839

Table 6 CPU time (in seconds) for the two different methods

Case	10^4	10^5	10^6
Standard MCS	0.5302	5.2961	52.9718
Fast MCS	0.0335	0.3302	3.3039
Ratio	15.8489	16.0403	16.0333

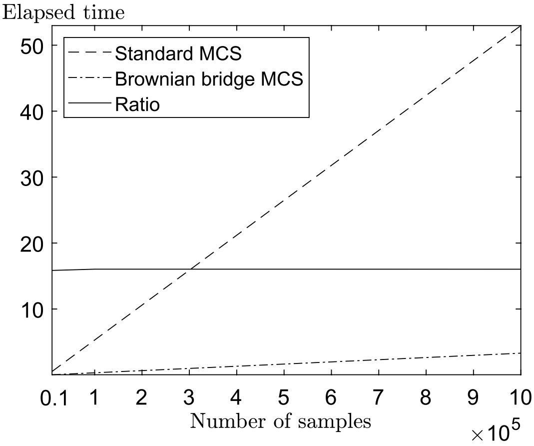


Fig. 11 Elapsed CPU time for the two different methods

computational cost with two methods. The result of Table 5 is the average of the results obtained from hundred times.

3.3 CPU Time of Calculating ELS Price Using Two Methods

We compare the CPU times for computing ELS prices with the standard MCS and fast MCS with computed piecewise constant parameters. We compute the CPU times with different number of samples: 10^4, 10^5, and 10^6. Table 6 lists the elapsed CPU time for the standard MCS and fast MCS. Calculating the ratio of elapsed CPU time to compare the results of the two methods confirmed that the fast MCS is relatively 16 times faster than the standard MCS. Furthermore, Fig. 11 shows the results in Table 6.

4 Conclusions

The standard BS equation is the most widely known model of option pricing methods. This model has the disadvantage of assuming interest rate and volatility as constant. In particular, constant volatility does not reflect the feature, skew or smile of market volatility. Therefore, interest rate and volatility should be assumed as time-dependent variable. However, the previous fast MCS by Jang et al. (2020) used constant interest rate and volatility. Therefore, in this article, we presented the fast MCS with given time-dependent interest rate function and volatility function for pricing ELS. First, we construct the piecewise constant parameter functions for given interest rate and volatility functions to apply the Brownian bridge technique. In the fast MCS, we compute the price of ELS with given time-dependent two parameter functions and piecewise constant parameter functions. The results showed that the proposed MCS was relatively approximately 16 times faster than the standard MCS for pricing ELS. The results demonstrated the fastness of the proposed MCS. Also, we present the pseudo algorithm that is more suitable for the financial market using the constructed piecewise constant parameter functions and the Brownian bridge technique.

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Declarations

Conflicts of interest The author declares that there is no conflict of interest.

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Authors and Affiliations

Sangkwon Kim[1] · Jisang Lyu[1] · Wonjin Lee[2] · Eunchae Park[1] · Hanbyeol Jang[2] · Chaeyoung Lee[1] · Junseok Kim[1]

Sangkwon Kim
ksk8863@korea.ac.kr

Jisang Lyu
lyujis96@korea.ac.kr

Wonjin Lee
wonjin381@korea.ac.kr

Eunchae Park
012dmsco@korea.ac.kr

Hanbyeol Jang
styliststar@korea.ac.kr

Chaeyoung Lee
chae1228@korea.ac.kr

[1] Department of Mathematics, Korea University, Seoul 02841, Republic of Korea

[2] Department of Financial Engineering, Korea University, Seoul 02841, Republic of Korea