

## Research Article

# Nonuniform Finite Difference Scheme for the Three-Dimensional Time-Fractional Black–Scholes Equation

Sangkwon Kim,<sup>1</sup> Chaeyoung Lee,<sup>1</sup> Wonjin Lee,<sup>2</sup> Soobin Kwak,<sup>1</sup> Darae Jeong,<sup>3</sup> and Junseok Kim <sup>1</sup>

<sup>1</sup>Department of Mathematics, Korea University, Seoul 02841, Republic of Korea

<sup>2</sup>Department of Financial Engineering, Korea University, Seoul 02841, Republic of Korea

<sup>3</sup>Department of Mathematics, Kangwon National University, Gangwon-do 24341, Republic of Korea

Correspondence should be addressed to Junseok Kim; cfdkim@korea.ac.kr

DVWfHW \$&EVWfW TVd\$ " \$ #- 3UWbfW \$+ @ahW TVd\$ " \$ #- BgTfeZW \$&6WfW TVd\$ " \$ #

Academic Editor: Youssri Hassan Youssri

Copyright © 2021 Sangkwon Kim et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this study, we present an accurate and efficient nonuniform finite difference method for the three-dimensional (3D) time-fractional Black–Scholes (BS) equation. The operator splitting scheme is used to efficiently solve the 3D time-fractional BS equation. We use a nonuniform grid for pricing 3D options. We compute the three-asset cash-or-nothing European call option and investigate the effects of the fractional-order in the time-fractional BS model. Numerical experiments demonstrate the efficiency and fastness of the proposed scheme.

## 1. Introduction

We consider the following 3D version of the time-fractional Black–Scholes (BS) model [1]:

$$\begin{aligned} \frac{\partial u}{\partial t} \delta x, y, z, t + L_{BS} u \delta x, y, z, t \\ = 0 \quad \text{for } \delta x, y, z, t \in [0, T], \end{aligned} \quad (1)$$

$$u \delta x, y, z, T = u_T \delta x, y, z, \quad (2)$$

where  $u \delta x, y, z, t$  is the option value at time  $t$  and  $u_T \delta x, y, z$  is the payoff function at time  $t = T$ ,

$$\frac{\partial u}{\partial t} \delta x, y, z, t = \frac{1}{\delta t} \frac{d}{dt} \bigg|_{t=t+\delta t} u \delta x, y, z, t - u \delta x, y, z, t, \quad (3)$$

where  $0 < \alpha < 1$  and

$$\begin{aligned} L_{BS} = & \frac{1}{2} \sigma_x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \sigma_y^2 \frac{\partial^2 u}{\partial y^2} + \frac{1}{2} \sigma_z^2 \frac{\partial^2 u}{\partial z^2} \\ & + \rho_{xy} \sigma_x \sigma_y \frac{\partial^2 u}{\partial x \partial y} + \rho_{yz} \sigma_y \sigma_z \frac{\partial^2 u}{\partial y \partial z} \\ & + \rho_{zx} \sigma_z \sigma_x \frac{\partial^2 u}{\partial z \partial x} + r x \frac{\partial u}{\partial x} + r y \frac{\partial u}{\partial y} + r z \frac{\partial u}{\partial z} - r u. \end{aligned} \quad (4)$$

Here,  $x$ ,  $y$ , and  $z$ , and  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$  are the prices and volatilities of the underlying assets  $x$ ,  $y$ , and  $z$ , respectively. Additionally,  $\rho_{xy}$ ,  $\rho_{yz}$ , and  $\rho_{zx}$  are the correlation values between two subscript asset variables, and  $r$  is the interest rate. Black and Scholes published in 1973 their paper which described the BS model and option pricing formula [2]. This has become an important fundamental topic for studying financial engineering and financial theory. However, the option pricing formula is based on the assumption that the returns of asset prices follow a Gaussian distribution. This means that the volatility of the underlying asset price is





















