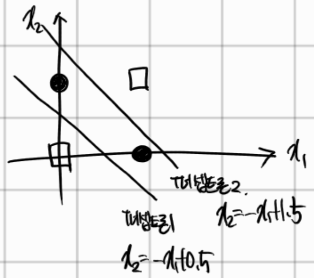
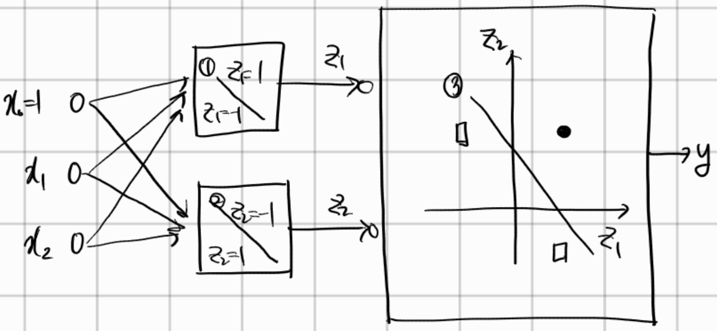
[ML_07] NN & MLP

XOR 문제의 해결



x_1	x_2	$y = x_1$ XOR x_2
0	0	-1 (□)
0	1	1 (●)
1	0	1 (●)
1	1	-1 (□)



→ $y=1$이 되기 위한 조건

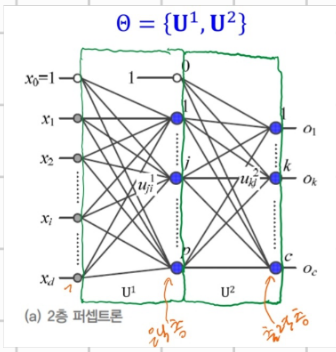
① : $x_2 > -x_1 + 0.5 \Rightarrow z_{s1} = -0.5 + x_1 + x_2 > 0$

② : $x_2 < -x_1 + 1.5 \Rightarrow z_{s2} = 1.5 - x_1 - x_2 > 0$

③ : $z_1 > 0,\ z_2 > 0 \Rightarrow z_2 > -z_1 + 1$

$\Rightarrow y_s = -1 + z_1 + z_2 > 0$

Model Parameters



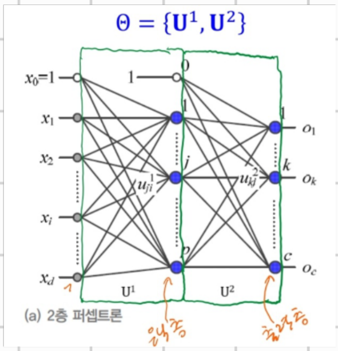
$$U^1 = \begin{pmatrix} u^1_{10} & u^1_{11} & \cdots & u^1_{1d} \\ u^1_{20} & u^1_{21} & \cdots & u^1_{2d} \\ \vdots & & & \\ u^1_{p0} & u^1_{p1} & \cdots & \underbrace{u^1_{pd}} \end{pmatrix} = \begin{pmatrix} u^1_1 \\ u^1_2 \\ \vdots \\ u^1_p \end{pmatrix}$$

u^1_{pd}: 0층의 d번째 노드 → 1층의 p번째 노드

$$U^2 = \begin{pmatrix} u^2_{10} & u^2_{11} & \cdots & u^2_{1p} \\ u^2_{20} & u^2_{21} & \cdots & u^2_{2p} \\ \vdots & & & \vdots \\ u^2_{c0} & \cdots & & \underbrace{u^2_{cp}} \end{pmatrix} = \begin{pmatrix} u^2_1 \\ u^2_2 \\ \vdots \\ u^2_c \end{pmatrix}$$

u^2_{cp}: (2-1)층의 p번째 노드 → 2층의 c번째 노드

Forward computation



Layer 1 (hidden):

$\underline{z}_{sum} = \underline{U}^1 \underline{x}$

$\underline{z} = \tau(\underline{z}_{sum})$

Layer 2 (output):

$\underline{o}_{sum} = \underline{U}^2 \underline{z}$

$\underline{o} = \tau(\underline{o}_{sum})$

Training set & Cost Function

$$\underline{X} = \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{pmatrix}, \quad \underline{Y} = \begin{pmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{pmatrix}$$

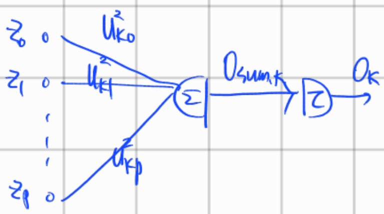
MSE cost : $J(\Theta) = \begin{cases} \text{온라인모드}: \frac{1}{2}\|\underline{y} - \underline{o}\|_2^2 \\ \text{배치모드}: \frac{1}{2n}\sum_{i=1}^{n} \|\underline{y}^i - \underline{o}^i\|_2^2 \end{cases}$

Gradient (Output layer)

$$\frac{\partial J}{\partial \underline{U}^2} : \left[\frac{\partial J}{\partial u^2_{kj}}\right]_{\substack{1\le k\le c \\ 0\le j\le p}}, \quad J(\Theta) = \frac{1}{2}\|\underline{y} - \underline{o}\|_2^2 = \sum_{q=1}^{c} \frac{1}{2}(o_q - y_q)^2$$

$$\frac{\partial J}{\partial u^2_{kj}} = \frac{\partial}{\partial u^2_{kj}}\left(\frac{1}{2}(o_k - y_k)^2\right)$$

$$= (o_k - y_k)\frac{\partial o_k}{\partial u^2_{kj}}$$



$$= (o_k - y_k)\frac{\partial(\tau(o_{sum,k}))}{\partial o_{sum,k}} \times \frac{\partial o_{sum,k}}{\partial u^2_{kj}}$$

$$= \underbrace{(o_k - y_k)\tau'(o_{sum,k})}_{\triangleq \delta_k} \cdot z_j = \delta_k z_j$$

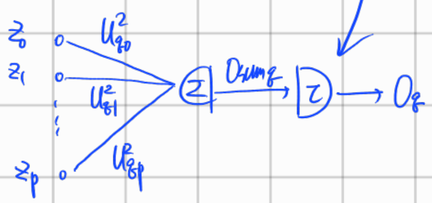
Gradient (hidden layer)

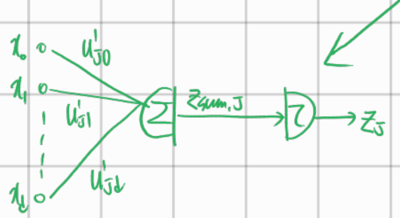
$$\frac{\partial J}{\partial U^1} : \left[\frac{\partial J}{\partial U^1_{ji}}\right] \quad \begin{matrix} 1 \le j \le p \\ 0 \le i \le d \end{matrix}, \quad J(\theta) = \frac{1}{2}\|y - o\|_2^2 = \sum_{q=1}^{c} \frac{1}{2}(o_q - y_q)^2$$

$$\frac{\partial J}{\partial U^1_{ji}} = \sum_{q=1}^{c} (o_q - y_q)\frac{\partial o_q}{\partial U^1_{ji}}$$

$$= \sum_{q=1}^{c} (o_q - y_q)\frac{\partial o_q}{\partial z_j} \cdot \frac{\partial z_j}{\partial U^1_{ji}}$$

$$= \sum_{q=1}^{c} (o_q - y_q)\, \tau'(o_{sum,q}) \frac{\partial o_{sum,q}}{\partial z_j} \cdot \tau'(z_{sum\,j}) \frac{\partial z_{sum\,j}}{\partial U_{ji}}$$





$$= \underbrace{\left(\sum_{q=1}^{c} \underbrace{(o_q - y_q)\,\tau'(o_{sum\,q})}_{=\delta_q} U^2_{qj}\right) \tau'(z_{sum\,j})}_{\triangleq \eta_j} x_i$$

$$= \eta_j x_i$$

요약

- 출력층

$\frac{\partial J}{\partial U^2}$: $\delta_k = (o_k - y_k)\,\tau'(o_{sum\,k})$, $1 \le k \le c$

$\frac{\partial J}{\partial U^2_{kj}} = \Delta U^2_{kj} = \delta_k z_j$ $\quad 0 \le j \le p$, $1 \le k \le c$

- 은닉층

$\frac{\partial J}{\partial U^1}$: $\eta_j = \tau'(z_{sum\,j}) \sum_{q=1}^{c} \delta_q U^2_{qj}$ $\quad 1 \le j \le p$

$\frac{\partial J}{\partial U^1_{ji}} = \Delta U^1_{ji} = \eta_j x_i$ $\quad 0 \le i \le d$, $1 \le j \le p$