

# Search of Involution in General Graphs

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## 1 Introduction

In this report, we describe two algorithm for searching valid involutions in a general graph. The purpose of these algorithms is to identify symmetry properties of a given graph that lead to fast graph Fourier transform (GFT) implementations discussed in the companion paper [1]. Essentially, this problem is a combinatorial problem, where a brute force search requires an exponential complexity. However, given the knowledge of the graph structure and associated properties (e.g., when the graph is a tree), we can apply useful conditions to prune the brute force algorithm, or even design an algorithm in polynomial time.

In the following, we will present two methods for searching a valid involution in a graph that leads to a graph symmetry. The first method takes advantage of the degree list, and prune the exhaustive search based on the degrees of the nodes. The intuition behind it is that two nodes paired by the involution  $\phi$  in a  $\phi$ -symmetric graph must have the same degree. This provides a necessary criterion for pruning involutions that are not valid during the search. The second approach is designed particularly for trees, which are special cases where  $\mathcal{O}(n)$  complexity can be achieved. In this method, we consider a property that

### 1.1 Notations and Definitions

We denote a graph as  $\mathcal{G}$  with vertex set  $\mathcal{V}$  of  $n$  nodes, edge set  $\mathcal{E}$ , and weight matrix  $\mathbf{W}$ . We allow self-loops in the graph, and define the self-loop matrix  $\mathbf{S}$  as a diagonal matrix whose diagonal elements are the self-loop weights  $w_{1,1}, w_{2,2}, \dots, w_{n,n}$ . The degree of node  $i$  is defined as  $d_i = \sum_{(i,j) \in \mathcal{E}} w_{i,j}$ . The degree matrix is a diagonal matrix whose diagonal elements are  $d_1, d_2, \dots, d_n$ . The Laplacian matrix is defined as  $\mathbf{L} = \mathbf{D} - \mathbf{W} + \mathbf{S}$ .

**Definition 1** ([2]). A permutation on a finite set  $\mathcal{V}$  is called an involution if it is its own inverse, i.e.,  $\phi(\phi(i)) = i$  for all  $i \in \mathcal{V}$ .

By definition, an involution uniquely characterizes a pairing function. That is, if  $i \in \mathcal{V}$  is mapped to  $j \in \mathcal{V}$  ( $\phi(i) = j$ ), then  $j$  is mapped to  $i$  ( $\phi(j) = \phi(\phi(i)) = i$ ). Unpaired elements ( $k \in \mathcal{V}$  with  $\phi(k) = k$ ) are allowed in this definition.

**Definition 2.** Let  $\phi$  be an involution defined on the vertex set  $\mathcal{V}$  of a graph  $\mathcal{G}$ . We say  $\mathcal{G}$  is  $\phi$ -symmetric if  $w_{i,j} = w_{\phi(i),\phi(j)}$  for all  $i \in \mathcal{V}$ ,  $j \in \mathcal{V}$ .

In what follows, we say  $\phi$  is a valid involution for graph  $\mathcal{G}$  if  $\mathcal{G}$  is  $\phi$ -symmetric.

## 1.2 Background

The number of all involutions, also known as the telephone number, on a set of  $n$  elements can be expressed as a sum

$$T(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k (n-2k)! k!} \sim \left(\frac{n}{e}\right)^{n/2} \frac{e^{\sqrt{n}}}{(4e)^{1/4}},$$

which asymptotically grows faster than polynomials in  $n$ . Thus, an exhaustive search of valid involutions among  $T(n)$  of possible candidates is a combinatorial problem with an exponential (or higher) complexity. This exhaustive search can be described as in Algorithm 1. The telephone numbers with  $n$  from 1 to 12 are shown in Table 1.2.

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**Algorithm 1** Exhaustive search of graph symmetry

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1: procedure EXHAUSTIVE( $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ )
2:    $\Pi \leftarrow \emptyset$ 
3:    $\mathcal{T} \leftarrow$  set of all involutions on  $\mathcal{V}$ 
4:   while  $\phi \in \mathcal{T}$  do ▷ Loop over all involutions
5:     if  $w_{i,j} = w_{\phi(i),\phi(j)}$  for all  $i, j \in \mathcal{V}$  then
6:        $\Pi \leftarrow \Pi \cup \{\phi\}$ 
7:   return  $\Pi$  ▷ Set of all valid involutions

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Here, we would like to design efficient algorithms for searching the valid involutions and discuss conditions of the graph for having such an algorithm in polynomial time. In fact, from the graph topology we can obtain useful information, such as local characteristics of nodes or graph sparsity, for pruning invalid involutions. For example, we may consider the degree or the

Table 1: Telephone numbers  $T(n)$  with  $n$  from 1 to 12

$n$	1	2	3	4	5	6	7	8	9	10	11	12
$T(n)$	1	2	4	10	26	76	232	764	2620	9496	35696	140152

number of neighbors of each graph node. Also, when the graph is sparse, it may be easier to obtain the involutions by partitioning the graph into two or three connected subgraphs because the number of partitions may be small. Motivated by these intuitions, we propose two methods as in Sections 2 and 3.

## 2 Pruning based on Degree Lists

From Definition 2, we obtain a necessary condition for nodes  $i$  and  $\phi(i)$  being paired: nodes  $i$  and  $\phi(i)$  have the same *weighted degree* (sum of weights on edges adjacent to a node). This is given by

$$d_i = \sum_{(i,j) \in \mathcal{E}} w_{i,j} = \sum_{(i,j) \in \mathcal{E}} w_{\phi(i),\phi(j)} = \sum_{(\phi(i),\phi(j)) \in \mathcal{E}} w_{\phi(i),\phi(j)} = d_{\phi(i)}. \quad (1)$$

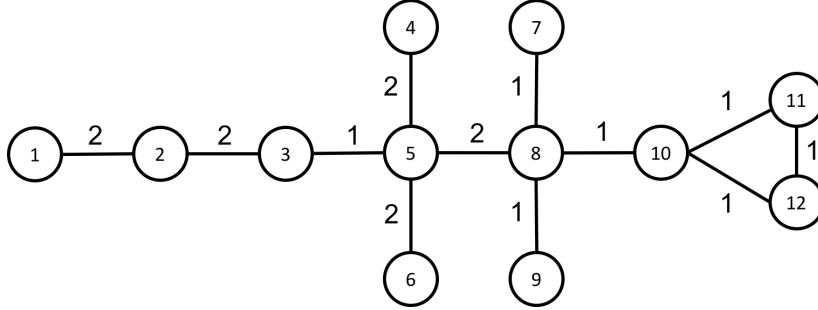
In addition, we can also consider the *unweighted degree* (number of neighbors of a node). We denote this number of neighbors for node  $i$  as  $f_i$ . Similar to (1), we have

$$f_i = \sum_{(i,j) \in \mathcal{E}} \mathbb{1}(w_{i,j}) = \sum_{(i,j) \in \mathcal{E}} \mathbb{1}(w_{\phi(i),\phi(j)}) = \sum_{(\phi(i),\phi(j)) \in \mathcal{E}} \mathbb{1}(w_{\phi(i),\phi(j)}) = f_{\phi(i)}, \quad (2)$$

where  $\mathbb{1}$  is the indicator function with  $\mathbb{1}(w) = 1$  if  $w \neq 0$  and  $\mathbb{1}(w) = 0$  otherwise.

Based on these two conditions (1), (2), we obtain a useful pruning rule for the exhaustive search: *only search those involutions in which all pairs of nodes have the same degrees and numbers of neighbors*. To apply this criterion, we first partition the vertex set  $\mathcal{V}$  into subsets  $\{\mathcal{V}_{d,f}\}$ , each of which contains nodes that share the common weighted and unweighted degrees ( $d$  and  $f$ ). Then, for each vertex subset  $\mathcal{V}_{d,f}$ , we list all  $T(|\mathcal{V}_{d,f}|)$  possible involutions on it. Finally, we search all combinations of the listed involutions across all vertex subsets. Note that, the number of such involutions is  $\Pi_{d,f} T(|\mathcal{V}_{d,f}|)$ , which is typically significantly smaller than  $T(n)$  for graphs that are not regular (node degrees are not all equal).

An example is shown in Figure 1. We note that the graph nodes have weighted degrees  $d \in \{1, 2, 3, 4, 5, 7\}$  and unweighted degrees  $f \in \{1, 2, 3, 4\}$ . As in Figure 1(c), there are 7 combinations of  $(d, f)$ , so the vertex set is



(a) Graph

$d_1$	$d_2$	$d_3$	$d_4$	$d_5$	$d_6$	$d_7$	$d_8$	$d_9$	$d_{10}$	$d_{11}$	$d_{12}$
2	4	3	2	7	2	1	5	1	3	2	2

$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$
1	2	2	1	4	1	1	4	1	3	2	2

(b) Degree lists

Involutions on  $\mathcal{V}_{1,1} = \{7, 9\}$   
( $d = 1, f = 1$ )

$\phi(7)$	$\phi(9)$
7	9
9	7

Involutions on  $\mathcal{V}_{2,2} = \{11, 12\}$   
( $d = 2, f = 2$ )

$\phi(11)$	$\phi(12)$
11	12
12	11

Involutions on  $\mathcal{V}_{4,2} = \{2\}$   
( $d = 4, f = 2$ )

$\phi(2)$
2

Involutions on  $\mathcal{V}_{2,1} = \{1, 4, 6\}$   
( $d = 2, f = 1$ )

$\phi(1)$	$\phi(4)$	$\phi(6)$
1	4	6
1	6	4
4	1	6
6	4	1

Involutions on  $\mathcal{V}_{3,2} = \{3\}$   
( $d = 3, f = 2$ )

$\phi(3)$
3

Involutions on  $\mathcal{V}_{5,4} = \{8\}$   
( $d = 5, f = 4$ )

$\phi(8)$
8

Involutions on  $\mathcal{V}_{3,3} = \{10\}$   
( $d = 3, f = 3$ )

$\phi(10)$
10

Involutions on  $\mathcal{V}_{7,4} = \{5\}$   
( $d = 7, f = 4$ )

$\phi(5)$
5

(c) Tables of involutions

Figure 1: An example of pruning based on degree lists. (a) An example graph, (b) its weighted and unweighted degree lists, and (d) its vertex partitions and involutions on them.

partitioned into 7 subsets with sizes 2, 3, 2, 1, 1, 1, and 1. For the node in each class with size 1, all valid involutions must map it to itself, which reduces the dimension of the search. On the other hand, those classes  $\mathcal{V}_{1,1}$ ,  $\mathcal{V}_{2,1}$ , and  $\mathcal{V}_{2,2}$  have 2, 3, and 2 elements, respectively, so there are  $T(2) = 2$ ,  $T(3) = 4$ , and  $T(2) = 2$  candidates of involutions to be searched. As a result, the number of total involutions to be searched is reduced from  $T(12) = 140152$  to  $T(2)^2 T(3) = 2 \times 2 \times 4 = 16$ .

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**Algorithm 2** Truncated search of graph symmetry based on degree lists

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1: procedure TRUNCATED( $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$ )
2:   Compute the weighted degree list  $\mathbf{d} = (d_1, \dots, d_n)^\top$  with  $d_i = \sum_{j=1}^n w_{i,j}$ 
3:   Compute the unweighted degree list  $\mathbf{f} = (f_1, \dots, f_n)^\top$  with  $f_i = \sum_{j=1}^n \mathbb{1}(w_{i,j})$ 
4:    $\mathcal{S} \leftarrow \text{GetTruncatedList}(\mathbf{d}, \mathbf{f})$ 
5:    $\Pi \leftarrow \emptyset$ 
6:   while  $\phi \in \mathcal{S}$  do  $\triangleright$  Loop over all involutions that are not pruned
7:     if  $w_{i,j} = w_{\phi(i), \phi(j)}$  for all  $i, j \in \mathcal{V}$  then
8:        $\Pi \leftarrow \Pi \cup \{\phi\}$ 
9:   return  $\Pi$   $\triangleright$  Set of all valid involutions
10: procedure GETTRUNCATEDLIST( $\mathbf{d}, \mathbf{f}$ )
11:    $\mathcal{V}_{d,f} \leftarrow \emptyset$  for all  $d$  and  $f$ 
12:   for  $i = 1, \dots, n$  do
13:      $\mathcal{V}_{d_i, f_i} \leftarrow \mathcal{V}_{d_i, f_i} \cup \{i\}$ 
14:   for  $\mathcal{V}_{d,f} \neq \emptyset$  do  $\triangleright$  Loop over all  $\mathcal{V}_{d,f}$ 
15:      $\mathcal{S}_{d,f} \leftarrow$  set of all involutions on  $\mathcal{V}_{d,f}$ 
16:    $\mathcal{S} \leftarrow \bigotimes_{d,f} \mathcal{S}_{d,f}$   $\triangleright$  Direct product of all involution sets
17:   return  $\mathcal{S}$   $\triangleright$  Set of all valid involutions after pruning

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We show the procedures based on this pruning criterion in Algorithm 2. The code for MATLAB implementation can be found in our github repository<sup>1</sup>.

## 2.1 Complexity Analysis

Let the number of nodes be  $n$  and the number of edges be  $m$ . We note that, for a given involution  $\phi$ , checking whether it is a valid involution (lines 5 and 6 in Algorithm 1) requires  $\mathcal{O}(m)$  time. This means that Algorithm 1 takes an overall complexity of  $\mathcal{O}(mT(n))$ .

To analyze the complexity of Algorithm 2, we break it into several parts:

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<sup>1</sup><http://github.com/kslu/fastgft/>

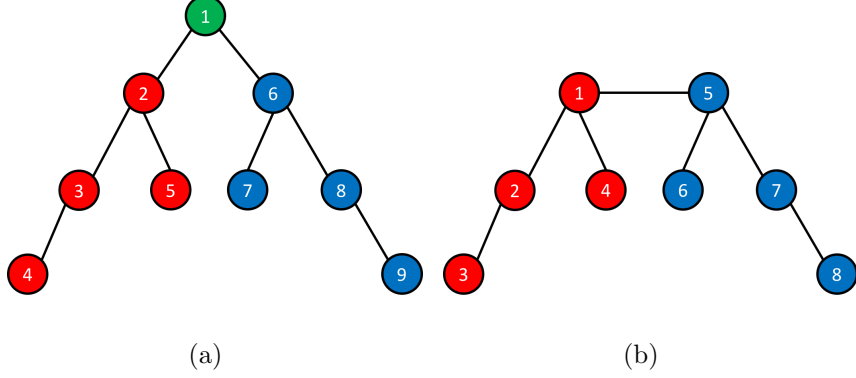


Figure 2: Symmetry of tree characterized by identical branches. (a) Two branches with a common root. (b) Two branches with adjacent roots. Their associated involutions are  $\phi_a = (1, 6, 8, 9, 7, 2, 5, 3, 4)$  and  $\phi_b = (5, 7, 8, 6, 1, 4, 2, 3)$ .

- Retrieval of the degree lists (lines 2 and 3). This can be done by accumulating all edge weights, so the complexity is  $\mathcal{O}(m)$  of each of  $\mathbf{d}$  and  $\mathbf{f}$ .
- Obtaining the truncated list of involutions (lines 10-17). The algorithm requires scanning through all nodes (lines 12-13), visiting all possible involutions for each  $\mathcal{V}_{d,f}$  (lines 14-15), and apply the direct product of all sets of involutions (line 16). Let the number of partitions be  $q$  and the sizes of partitions be  $k_1, k_2, \dots, k_q$ . Then, these three procedures take  $\mathcal{O}(n)$ ,  $\mathcal{O}(\sum_{i=1}^q T(k_i))$ , and  $\mathcal{O}(\prod_{i=1}^q T(k_i))$ .
- Searching over the truncated set of involutions (lines 6-8). This takes  $\mathcal{O}(m \prod_{i=1}^q T(k_i))$ .

Combining all the complexities above, we obtain an overall complexity of  $\mathcal{O}(m \prod_{i=1}^q T(k_i))$ . This reduces the complexity of Algorithm 1 by a factor of  $T(n) / \prod_{i=1}^q T(k_i)$ , where  $n = \sum_{i=1}^q k_i$ .

### 3 Searching of Identitcal Tree Branches

Here, we provide another algorithm for the search of involution on trees in polynomial time ( $\mathcal{O}(n \log n)$ , or  $\mathcal{O}(n)$  with optimization). This algorithm arises from the fact that a symmetry in a tree can be uniquely characterized by identical branches of the tree whose roots are common (as in Figure 2(a))

or adjacent (as in Figure 2(b)). In what follows, we refer to such branches as *adjacent identical branches* (AIB). The main algorithm is based on the fact that adjacent branches can be identified efficiently using a bottom-up traversal in a tree.

This approach is motivated by related work in [3, 4], where the so-called *subtree repeats* can be found in linear time. The subtree repeat problem and our problem are similar, but different in two aspects. First, we do not allow the subtrees to overlap. Second, from the condition of graph symmetry, we require that the roots of the target subtrees to be adjacent. Thus, the algorithm we design is a slightly simplified version of the forward/non-overlapping stage in [3], and has the same time and space complexity.

First, we show that if a tree  $\mathcal{G}$  is  $\phi$ -symmetry, then this symmetry can be characterized by a pair, or several pairs of adjacent identical branches in  $\mathcal{G}$ :

**Lemma 1.** *Let  $\phi$  be a non-trivial involution ( $\phi(\phi(i)) = i$  for some  $i$ ) and let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$  be a  $\phi$ -symmetric. Then, the tree nodes can be partitioned into  $\mathcal{V} = \mathcal{V}_X \cup \mathcal{V}_Y \cup \mathcal{V}_Z$  such that  $\phi(i) \in \mathcal{V}_Y$  if  $i \in \mathcal{V}_X$  and  $\phi(j) = j$  if  $j \in \mathcal{V}_Z$ , and there is at most one edge  $(k, l) \in \mathcal{E}$  with  $k \in \mathcal{V}_X$ ,  $l \in \mathcal{V}_Y$ .*

*Proof.* We define  $\mathcal{V}_Z = \{i \in \mathcal{V} | \phi(i) = i\} \neq \mathcal{V}$ , and discuss two cases:  $\mathcal{V}_Z \neq \emptyset$  and  $\mathcal{V}_Z = \emptyset$ .

If  $\mathcal{V}_Z$  is non-empty, there must be a (possibly non-unique) node  $k_1 \in \mathcal{V}_Z$  connected to two nodes  $i_1$  and  $j_1 = \phi(i_1)$ . First, we start with  $\mathcal{V}_X = \{i_1\}$  and  $\mathcal{V}_Y = \{j_1\}$ . Then, we apply a depth-first search (DFS) to traverse the branch rooted by  $i_1$  toward its other neighbor than  $k_1$ , and include those nodes being visited into  $\mathcal{V}_X$ . Note that, from the  $\phi$ -symmetry, every step of the DFS corresponds to a traversal step at  $j_1$  side. In this way, we can obtain an identical branch rooted by  $j_1$ , and include all nodes in this branch into  $\mathcal{V}_Y$ . If there exists another node  $k_2 \in \mathcal{V}_Z$  other than  $k_1$  connected to two nodes  $i_2$  and  $j_2 = \phi(i_2)$ , we apply the same traversal procedure again and append  $\mathcal{V}_X$  and  $\mathcal{V}_Y$  in the same way. This procedure can be repeated until all nodes are classified into  $\mathcal{V}_X$ ,  $\mathcal{V}_Y$ , and  $\mathcal{V}_Z$ . The resulting  $\mathcal{V}_X$  and  $\mathcal{V}_Y$  will end up containing one or several identical branches of the original tree.

If  $\mathcal{V}_Z$  is empty, then by connectivity of the tree graph, there must be an edge connecting two nodes paired by  $\phi$ ,  $i$  and  $\phi(i)$ . Then, we apply a DFS to traverse the branch rooted by  $i$  toward its other neighbor than  $\phi(i)$ , and include all visited nodes into  $\mathcal{V}_X$ . Similarly, the  $\phi$ -symmetry also gives an identical branch rooted by  $\phi(i)$ . We include nodes in this branch into  $\mathcal{V}_Y$ . The connectivity of the graph implies that all nodes are included in these two identical branches. □ Note that

the converse of this theorem is straightforward: when there are two identical

branches whose roots are common or adjacent, we can identify an involution  $\phi$  that pairs the nodes of the branches such that the graph is  $\phi$ -symmetric.

### 3.1 Algorithm

Since tree symmetry can be characterized by adjacent identical branches, we can build branch descriptors for all branches in the tree, and compare branches that are joined or adjacent. Note that, to apply this method, the tree needs to be rooted, i.e., has a root so that there is a hierarchy with different levels of nodes.

The reasonable choice of root would be a node in the *center* of the tree. The center of a graph is the set of vertices  $v$  where the greatest distance  $d(u, v)$  to other vertices  $v$  is minimal. In fact, if  $(p_1, \dots, p_L)$  is a longest path of the graph (whose length is called the *diameter* of the graph), then the center node(s) of this path must belong to the center of the graph. An important consequence of this property is given as follows.

**Lemma 2.** *A node in the center of a tree cannot be an internal node of a branch that has an adjacent identical branch.*

(Otherwise, joining these two branches leads to a path whose length is greater than the diameter, yielding a contradiction.). To find the center of a tree, we can apply breadth-first search (BFS) twice to obtain the longest path in the tree, and identify the nodes in the center of the longest path. An algorithm for obtaining the center is shown in Algorithm 3.

With Lemma 2, we design a bottom-up branch matching algorithm on a tree rooted with its center, which is summarized in Algorithm 4.

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#### Algorithm 3 Finding the center of a tree

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1:	<b>procedure</b> TREECENTER( $\mathcal{G}$ )	$\triangleright \mathcal{G}$ is a tree
2:	Pick any node $v_1 \in \mathcal{V}$	
3:	Run BFS on $\mathcal{G}$ from $v_1$ , and obtain the furthest node $v_2$	
4:	Run BFS on $\mathcal{G}$ from $v_2$ , and obtain the furthest node $v_3$	
5:	Retrieve the $v_2 - v_3$ path ( $p_1 = v_2, p_2, \dots, p_L = v_3$ ) from the previous BFS solution	$\triangleright$ This path is the diameter of the tree
6:	<b>if</b> $L = 2n + 1$ is odd <b>then</b>	
7:	<b>return</b> $\{p_n\}$	$\triangleright$ The tree has one center
8:	<b>if</b> $L = 2n$ is even <b>then</b>	
9:	<b>return</b> $\{p_n, p_{n+1}\}$	$\triangleright$ The tree has two centers

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Here we provide an example with graph in Figure 3 to demonstrate the algorithm. In Table 2 we show the iterations in the order of reverse topological



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**Algorithm 4** Finding valid involutions in a tree

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1: procedure FINDINVOLUTIONINTREE( $\mathcal{G}$ )
2:   Take a node  $v \in \mathcal{C} = \text{TreeCenter}(\mathcal{G})$ 
3:   Run BFS from  $v$ , and obtain the node list with topological order
   ( $u_1, \dots, u_n$ )
4:    $\mathcal{I} \leftarrow \emptyset$  ▷ List of valid involutions
5:   for  $i = n, \dots, 1$  do ▷ Build branch descriptor bottom-up
6:     if  $u_i$  has no childs then
7:        $\mathcal{S}_i \leftarrow ()$  ▷ Empty descriptor
8:     if  $u_i$  has childs then
9:       ( $t_{i,1}, \dots, t_{i,c_i}$ )  $\leftarrow$  list of childs, sorted w.r.t. their descriptors.
10:      If any two branch descriptors are equal, append  $\mathcal{I}$  with the
      involution characterized by the two identical branches
11:       $\mathcal{S}_i \leftarrow ((w_{u_i, t_{i,1}}, \mathcal{S}_{t_{i,1}}), (w_{u_i, t_{i,2}}, \mathcal{S}_{t_{i,2}}), \dots, (w_{u_i, t_{i,c_i}}, \mathcal{S}_{t_{i,c_i}}))$  ▷ Build
      the descriptor of  $u_i$  by concatenating the descriptors of its childs
12:      if  $|\mathcal{C}| = 2$  then ▷ The center has two nodes
13:        if Two branches rooted by the two center nodes are identical then
14:          Append  $\mathcal{I}$  with the involution characterized by these two
          branches
15:      return  $\mathcal{I}$ 

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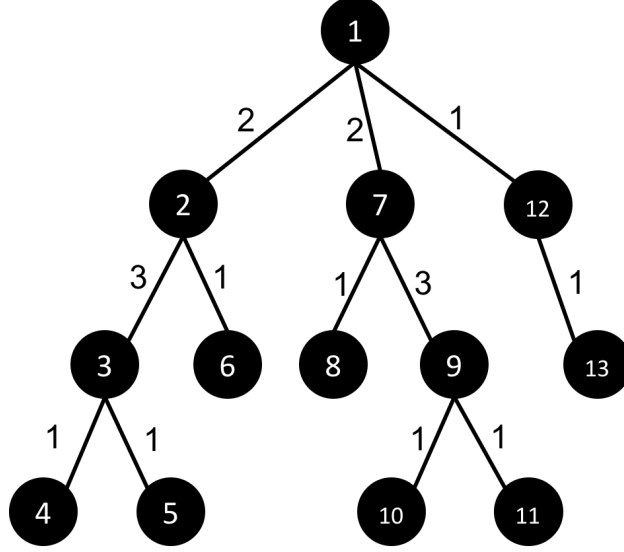


Figure 3: An example graph for demonstration of Algorithm 4. The center of this graph is  $\{1\}$ . Note that a topological order given by BFS from node 1 would be  $(1, 2, 7, 12, 3, 6, 8, 9, 13, 4, 5, 10, 11)$ .

order (from bottom to top of the tree rooted at node 1). Note that in iteration 5, an involution associated to the two AIBs is  $\phi_1 = (1, 2, 3, 5, 4, 6, 7, \dots)$  (pairing of nodes 4 and 5), while the involution found in iteration 8 is  $\phi_2 = (1, \dots, 9, 11, 10, 12, 13)$  (pairing of nodes 10 and 11). In the last iteration, the involution corresponding to the largest AIBs is

$$\phi_3 = (1, 7, 9, 10, 11, 8, 2, 6, 3, 4, 5, 12, 13),$$

that pairs nodes 2, 3, 4, 5, and 6 with nodes 7, 9, 10, 11, and 12, respectively. Note that, when each descriptor is built, the sorting of branch descriptors from the lower level will fix the order, so two AIBs must have identical branch descriptors.

### 3.2 Complexity Analysis

The overall complexity of the algorithm include the following:

- Finding the center (line 2). This requires two BFS's and a traversal of a path. For a tree, the number of edges is  $m = n - 1$ , so the overall complexity of these operations on a tree is  $\mathcal{O}(m + n) = \mathcal{O}(n)$ .
- Finding a topological order using a BFS (line 3). The complexity of this step is  $\mathcal{O}(n)$ .

Table 2: Demonstration of Algorithm 4.

#Iteration	Node	Branch descriptor	Comment
1	4	$\mathcal{S}_4 = ()$	
2	5	$\mathcal{S}_5 = ()$	
3	10	$\mathcal{S}_{10} = ()$	
4	11	$\mathcal{S}_{11} = ()$	
5	3	$\mathcal{S}_3 = ((w_{3,4}, \mathcal{S}_4), (w_{3,5}, \mathcal{S}_5))$ $= ((1, ()), (1, ()))$	AIB found
6	6	$\mathcal{S}_6 = ()$	
7	8	$\mathcal{S}_8 = ()$	
8	9	$\mathcal{S}_9 = ((w_{9,10}, \mathcal{S}_{10}), (w_{9,11}, \mathcal{S}_{11}))$ $= ((1, ()), (1, ()))$	AIB found
9	13	$\mathcal{S}_{13} = ()$	
10	2	$\mathcal{S}_2 = ((w_{2,3}, \mathcal{S}_3), (w_{2,6}, \mathcal{S}_6))$ $= ((3, ((1, ()), (1, ()))), (1, ()))$	
11	7	$\mathcal{S}_7 = ((w_{7,9}, \mathcal{S}_9), (w_{7,8}, \mathcal{S}_8))$ $= ((3, ((1, ()), (1, ()))), (1, ()))$	
12	12	$\mathcal{S}_{12} = ((1, ()))$	
13	1	$\mathcal{S}_1 = ((w_{1,2}, \mathcal{S}_2), (w_{1,7}, \mathcal{S}_7), (w_{1,12}, \mathcal{S}_{12}))$ $= (\text{omitted})$	AIB found

- Building descriptors for all nodes (line 5-11). Here, it take  $n$  iterations to loop over all tree nodes. Let  $p_i$  be the number of childs of node  $u_i$ . In iteration  $i$ , sorting the descriptors takes  $\mathcal{O}(p_i \log(p_i))$ , and concatenating them takes  $\mathcal{O}(p_i)$ . The overall complexity of this loop is

$$\begin{aligned}
\sum_{i=1}^n p_i \log(p_i) &= (n-1) \left[ \sum_{i=1}^n \frac{p_i}{n-1} \log(p_i) \right] \\
&= (n-1) \left[ \sum_{i=1}^n \frac{p_i}{n-1} \log\left(\frac{p_i}{n-1}\right) + \sum_{i=1}^n \frac{p_i}{n-1} \log(n-1) \right] \\
&= (n-1) \left[ -H_p + \frac{\sum_{i=1}^n p_i}{n-1} \log(n-1) \right] \\
&\leq (n-1) \log(n-1)
\end{aligned} \tag{3}$$

where the equation  $\sum_{i=1}^n p_i = n-1$  has been applied, and  $H_p \geq 0$  because it is an entropy quantity. From (3), we obtain a complexity of  $\mathcal{O}(n \log n)$ .

- Check complete symmetry (line 13-14). This requires a comparison of two descriptors, and takes  $\mathcal{O}(n)$  time.

In fact, as mentioned in [3], sorting of descriptors can be implemented by radix sort, which requires only  $\mathcal{O}(p_i)$ , due to the finite range of strings to be sorted. As a result, the overall complexity is  $\mathcal{O}(n \log n)$  with an  $\mathcal{O}(n \log n)$  sort, or  $\mathcal{O}(n)$  with a  $\mathcal{O}(n)$  sort.

## 4 Conclusion

In this report, we discuss two methods for searching a valid involution  $\phi$  in a graph for graph symmetry. The first method is based on the fact that two symmetric nodes must have the same weighted and unweighted degrees. The second approach takes advantage of the sparsity of tree graphs, in which case the graph symmetry can be characterized by adjacent identical branches. We show the algorithms of both methods as well as examples for illustration. The complexities of both methods are analyzed. The MATLAB implementations can be found in the companion github repository.

## References

- [1] K.-S. Lu and A. Ortega, “Fast graph Fourier transforms based on symmetry and bipartition,” *submitted, IEEE Trans. Signal Processing*, 2019.
- [2] Donald E. Knuth, *The Art of Computer Programming, Volume 3: (2Nd Ed.) Sorting and Searching*, Addison Wesley Longman Publishing Co., Inc., Redwood City, CA, USA, 1998.
- [3] T. Flouri, K. Kobert, S. P. Pissis, and A. Stamatakis, “An optimal algorithm for computing all subtree repeats in trees,” in *Combinatorial Algorithms*, T. Lecroq and L. Mouchard, Eds., Berlin, Heidelberg, 2013, pp. 269–282, Springer Berlin Heidelberg.
- [4] M. Christou, M. Crochemore, T. Flouri, C. S. Iliopoulos, J. Janouek, B. Melichar, and S. P. Pissis, “Computing all subtree repeats in ordered trees,” *Information Processing Letters*, vol. 112, no. 24, pp. 958 – 962, 2012.