MAXIMUM FLEXIBILITY INTERVAL SCHEDULES

1. Naive Flexibility (NF), Decoupling Problem (DP), Concurrent Flexibility (CF)

STN $S = (T, C \cup C^x)$ with distance matrix $D = [d_{ij}]$, where

$$T = \cup_a T_a = \{t_1, \dots, t_n\}$$

includes a cluster of variables per agent $a \in A$,

$$C = \cup_a C_a$$

includes an intra-agent constraint set per agent, and

$$C^x = \cup_{a,b} C^x_{ab}$$

contains an inter-agent constraint set for each pair of agents $(a, b) \in A \times A$.

Lemma 1. *The following LP has a single optimal solution (l, u) with $l_i = est_i$ and $u_i = lst_i$.*

$$\text{(NF)} \qquad \begin{aligned} \max_{l,u\geq 0} \quad & \sum_i (u_i - l_i) & \\ s.t. \quad & u_i \geq l_i & \forall i \in T \\ & u_j - u_i \leq d_{ij} & \forall (i,j) \in T \times T \\ & l_j - l_i \leq d_{ij} & \forall (i,j) \in T \times T \end{aligned}$$

Proof. Consider optimal (l, u) such that $l_i \neq \text{est}_i$ and/or $u_i \neq \text{lst}_i$. Letting $l_i = \text{est}_i$ and $u_i = \text{lst}_i$ is also feasible and with a better objective. □

We say an arc $(i, j) \in T \times T$ is a *crossing arc* when $i \in T_a, j \in T_b$ with $a, b \in A$ and $a \neq b$. Let $P \subseteq T \times T$ be the set of all crossing arcs. We are interested in the *decoupling problem*:

$$\text{(DP)} \qquad \begin{aligned} \max_{l,u\geq 0} \quad & \sum_i (u_i - l_i) & \\ \text{s.t.} \quad & u_i \geq l_i & \forall i \in T \\ & u_j - u_i \leq d_{ij} & \forall (i,j) \in T \times T \\ & l_j - l_i \leq d_{ij} & \forall (i,j) \in T \times T \\ & u_j - l_i \leq d_{ij} & \forall (i,j) \in P \end{aligned}$$

Problem (NF) is effectively a special-case of problem (DP) with only a single agent, that is, with $|A| = 1$, in which case $P = \emptyset$. Another special-case of (DP) is that with $|A| = n$, in which case $P = T \times T$, stated as follows:

$$\text{(CF)} \qquad \begin{aligned} \max_{l,u\geq 0} \quad & \sum_i (u_i - l_i) & \\ \text{s.t.} \quad & u_i \geq l_i & \forall i \in T \\ & u_j - l_i \leq d_{ij} & \forall (i,j) \in T \times T \end{aligned}$$

Note that $u_j - l_i \leq d_{ij}$ and $u_j \geq l_i$ implies $u_j - u_i \leq d_{ij}$ and $l_j - l_i \leq d_{ij}$.

Summarizing, (NV) is a special-case of (DP) with no crossing-edge constraints, while (CF) is a special-case of (DP) with a crossing-edge constraint for every pair of time variables.

2. Rewritting (DP)

Note that, in (DP), assignments u and l must correspond to feasible STN schedules. Together with requiring that $u_i \geq l_i$ for all i, this implies $u_j - l_i \leq \text{lst}_j - \text{est}_i$ for all $(i, j) \in T \times T$. As such, defining $\bar{P} := T \times T - P$ and $\Delta = [\delta_{ij}]$ with

$$\delta_{ij} = \begin{cases} d_{ij} & \text{for } (i,j) \in P \\ w_{ij} = \text{lst}_j - \text{est}_i & \text{for } (i,j) \in \bar{P} := T \times T - P \end{cases}$$

we may rewrite the decoupling problem as follows:

$$\text{(DP)} \qquad \max_{l,u\geq 0} \sum_i (u_i - l_i)$$

$$\begin{aligned}
&(1) && \text{s.t.} \quad u_i \geq l_i && \forall i \in T \\
&(2) && u_j - u_i \leq d_{ij} && \forall (i,j) \in \bar{P} \\
&(3) && l_j - l_i \leq d_{ij} && \forall (i,j) \in \bar{P} \\
&(4) && u_j - l_i \leq \delta_{ij} && \forall (i,j) \in T \times T
\end{aligned}$$

since $u_j - u_i \leq d_{ij}$ and $l_j - l_i \leq d_{ij}$ is implied by $u_j - l_i \leq d_{ij}$ for all $(i,j) \in P$.

This formulation of (DP) is effectively problem (CF) for an STN with distances matrix Δ, but with additional constraint-sets (2) and (3).

3. Necessary optimality conditions for (DP)

Lemma 2. *If (l,u) is optimal, then $l_i = \max_{k\in T} \gamma_{ik}$ for all $i \in T$ where*

$$\gamma_{ik} := \begin{cases} u_k - d_{ik} & (i,k) \in P \\ \max(l_k - d_{ik}, u_k - w_{ik}) & (i,k) \in \bar{P} \end{cases}$$

Proof. Suppose (l,u) is optimal and $l_i = \max_k \gamma_{ik} + c$ for some particular $i \in T$. It can be shown that (l,u) is not feasible unless $c \geq 0$.

Suppose that $c > 0$ for this particular i and consider (l',u) that differs from (l,u) only in that $l'_i = l_i - c = \max_k \gamma_{ik}$. Clearly, (l',u) yields a better objective. However, is it feasible?
Constraint (1) and (2): Trivially, (l',u) satisfies these constraints.
Constraint (3): To be feasible, (l',u) must satisfy $l_j - l'_i \leq d_{ij}$ for all j; that is

$$\max_k \gamma_{ik} \geq l_j - d_{ij}$$

for all j. By definition, $\max_k \gamma_{ik} \geq u_j - d_{ij} \geq l_j - d_{ij}$ for all $j : (i,j) \in P$. Also, $\max_k \gamma_{ik} \geq \max(l_j - d_{ij}, u_j - w_{ij}) \geq l_j - d_{ij}$ for all $j : (i,j) \in \bar{P}$. Thus, (l',u) satisfies (3).
Constraint (4): To be feasible, (l',u) must satisfy $u_j - l'_i \leq \delta_{ij}$ for all j; that is,

$$\max_k \gamma_{ik} \geq u_j - \delta_{ij}$$

for all j. By definition, $\max_k \gamma_{ik} \geq u_j - w_{ij} = u_j - \delta_{ij}$ for all $j : (i,j) \in \bar{P}$. Also, $\max_k \gamma_{ik} \geq u_j - d_{ij} = u_j - \delta_{ij}$ for all $j : (i,j) \in P$. Thus, (l',u) satisfies (4).

In conclusion, (l',u) is also feasible but with a better objective, contradicting our assumption that an optimal (l,u) with $l_i > \max_k \gamma_{ik}$ for some i exists. Since no feasible solution exists with $l_i < \max_k \gamma_{ik}$, we conclude that every optimal solution has $l_i = \max_k \gamma_{ik}$ for all $i \in T$. □

Lemma 3. *If (l,u) is optimal, then $u_j = \min_{k\in T} \zeta_{kj}$ for all $j \in T$ where*

$$\zeta_{kj} := \begin{cases} l_k + d_{kj} & (k,j) \in P \\ \min(u_k + d_{kj}, l_k + w_{kj}) & (k,j) \in \bar{P} \end{cases}$$

Proof. Similar line of reasoning with the proof of the previous Lemma. □

4. Reduction of (DP) to (DP')

Consider the following relaxation of (DP)

$$\text{(DP')} \qquad \max_{l,u\geq 0} \sum_i (u_i - l_i)$$

$$\begin{aligned}
&(5) && \text{s.t.} \quad u_i \geq l_i && \forall i \in T \\
&(6) && u_j - \min_k \zeta_{ki} \leq d_{ij} && \forall (i,j) \in \bar{P} \\
&(7) && \max_k \gamma_{jk} - l_i \leq d_{ij} && \forall (i,j) \in \bar{P} \\
&(8) && u_j - l_i \leq \delta_{ij} && \forall (i,j) \in T \times T
\end{aligned}$$

Lemma 4. *(DP') encompasses the solution-space of (DP).*

Proof. If (l, u) is feasible for (DP), then $u_i \leq \min_k \zeta_{ki}$ for all i. As such, if $u_j - u_i \leq d_{ij}$ (constraint (2) of (DP)), then $u_j - \min_k \zeta_{ki} \leq d_{ij}$ and (6) is satisfied.

If (l, u) is feasible for (DP), then $l_j \geq \max_k \gamma_{jk}$ for all j. As such, if $l_j - l_i \leq d_{ij}$ (constraint (3) of (DP)), then $\max_k \gamma_{jk} - l_i \leq d_{ij}$ and (7) is satisfied.

Finally, constraints (5) and (8) are also part of (DP), so if (l, u) is feasible for (DP), it is also feasible for (DP'). □

Lemma 5. *Every optimal (l, u) for (DP') satisfies the necessary optimality conditions of (DP).*

Proof. Not shown yet but not disproven by experiments. □

5. Reduction of (DP') to (CF)

5.1. Rewriting constraint (6):

$$u_j - \min_k \zeta_{ki} \leq d_{ij} \quad (i,j) \in \bar{P}$$

$$\text{(6')} \qquad \Leftrightarrow \begin{cases} u_j \leq l_k + d_{ki} + d_{ij} & (i,j,k) \in T^3 : (i,j) \in \bar{P}, (k,i) \in P \\ \begin{cases} u_j \leq u_k + d_{ki} + d_{ij} \\ u_j \leq l_k + w_{ki} + d_{ij} \end{cases} & (i,j,k) \in T^3 : (i,j) \in \bar{P}, (k,i) \in \bar{P} \end{cases}$$

Recall that (8) implies $u_j - u_k \leq d_{kj}$ for all $(k,j) \in \bar{P}$. Thus, the $u_j \leq u_k + d_{ki} + d_{ij}$ part of (6') above is redundant for all $\{(i,j,k) \in T^3 : (i,j) \in \bar{P}, (k,i) \in \bar{P}, (k,j) \in \bar{P}\}$, since D being a APSP matrix, $d_{kj} \leq d_{ki} + d_{ij}$.

Furthermore, note that $(i,j) \in \bar{P}$ (i.e. t_i and t_j belong to the same agent) together with $(k,i) \in \bar{P}$ imply that also $(k,j) \in \bar{P}$, if we assume that *each time-variable belongs to exactly one agent.* Under the same assumption, $(i,j) \in \bar{P}$ and $(k,i) \in P$ implies $(k,j) \in P$. As such, (6') can be rewritten as:

$$\text{(6')} \qquad \begin{cases} u_j - l_k \leq d_{ki} + d_{ij} & (i,j,k) \in T^3 : (i,j) \in \bar{P}, (k,i) \in P, (k,j) \in P \\ u_j - l_k \leq w_{ki} + d_{ij} & (i,j,k) \in T^3 : (i,j) \in \bar{P}, (k,i) \in \bar{P}, (k,j) \in \bar{P} \end{cases}$$

by ommitting what is already implied by (8).

5.2. Rewriting constraint (7):

$$\text{(7')} \qquad \max_k \gamma_{jk} - l_i \leq d_{ij} \quad (i,j) \in \bar{P}$$

$$\Leftrightarrow \begin{cases} u_k - l_i \leq d_{ij} + d_{jk} & (i,j,k) \in T^3 : (i,j) \in \bar{P}, (j,k) \in P \\ \begin{cases} l_k - l_i \leq d_{ij} + d_{jk} \\ u_k - l_i \leq d_{ij} + w_{jk} \end{cases} & (i,j,k) \in T^3 : (i,j) \in \bar{P}, (j,k) \in \bar{P} \end{cases}$$

Again, we note that $(i,j) \in \bar{P}$ and $(j,k) \in \bar{P}$ imply $(i,k) \in \bar{P}$. Therefore, $l_k - l_i \leq d_{ij} + d_{jk}$ is implied again by (8) and we can rewrite (7) as

$$\text{(7')} \qquad \begin{cases} u_k - l_i \leq d_{ij} + d_{jk} & (i,j,k) \in T^3 : (i,j) \in \bar{P}, (j,k) \in P, (i,k) \in P \\ u_k - l_i \leq d_{ij} + w_{jk} & (i,j,k) \in T^3 : (i,j) \in \bar{P}, (j,k) \in \bar{P}, (i,k) \in \bar{P} \end{cases}$$

5.3. Rewriting (DP') as (CF).

We managed to rewrite (6) and (7) in the form of (8). After trivial manipulations, we can rewrite (DP') as an instance of (CF), according to the following construction.

For each t_i and t_j that belong to different agents, let K_{ij} index all t_k that belong to the same agent with either t_i or t_j. That is, associate each $(i,j) \in P$ with:

$$\begin{aligned} K_{ij} &:= \{k \in T : (k,j) \in \bar{P} \text{ or } (i,k) \in \bar{P}\} \\ &= \{k \in T : (k,j) \in \bar{P}\} \cup \{k \in T : (i,k) \in \bar{P}\} \end{aligned}$$

Furthermore, associate each triplet (t_i, t_k, t_j) with t_i and t_j belonging to different agents and t_k belonging to the same agent with either t_i or t_j, with constant:

$$c_{ikj} := d_{ik} + d_{kj}$$

For each t_i and t_j that belong to the same agent, let $\bar{K}_{ij}$ index all t_k that also belong to the same agent. That is, associate each $(i,j) \in \bar{P}$ with: $\bar{K}_{ij} := \{k \in T : (k,i) \in \bar{P}\}$. Furthermore, associate each triplet (t_i, t_k, t_j) belonging to the same agent with constant:

$$\bar{c}_{ikj} = \min(w_{ik} + d_{kj}, d_{ik} + w_{kj})$$

Problem (DP') for STN with distances matrix D can be cast as problem (CF), shown below: for STN with distances matrix $X = [x_{ij}]$ defined as follows:

$$x_{ij} = \begin{cases} \min(d_{ij}, \min_{k \in K_{ij}} c_{ikj}) & (i,j) \in P \\ \min(w_{ij}, \min_{k \in \bar{K}_{ij}} \bar{c}_{ikj}) & (i,j) \in \bar{P} \end{cases} \tag{9}$$

However, assuming D is a APSP matrix, $x_{ij} = \min(d_{ij}, \min_{k \in K_{ij}} c_{ikj}) = d_{ij}$ for all $(i,j) \in P$. Thus, we may refine X as follows:

$$x_{ij} = \begin{cases} d_{ij} & (i,j) \in P \\ \min(w_{ij}, \min_{k \in \bar{K}_{ij}} \bar{c}_{ikj}) & (i,j) \in \bar{P} \end{cases} \tag{10}$$

References