Fourier analysis

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1 Fourier

Main formula

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n}{T} t) + \sum_{n=1}^{\infty} a_n \sin(\frac{2\pi n}{T} t)$$

Orthogonality

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f_1(t) f_2(t) dt = \begin{cases} \alpha \text{ if } f_1 = f_2 \\ 0 \text{ if } f_1 \neq f_2 \end{cases}$$

The oscillation is named Gibbs phenomenon. The equality does not always maintain when operating in both sides.

2 Sin and cosine

Using $\frac{2}{T}$ to normalize:
cos · cos

$$\frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\frac{2n\pi}{T} t) \cos(\frac{2m\pi}{T} t) dt = \begin{cases} 0 \text{ if } n \neq 0 \\ 1 \text{ if } n = 0 \end{cases}$$

sin · cos

$$\frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(\frac{2n\pi}{T} t) \cos(\frac{2m\pi}{T} t) dt = 0$$

sin · sin

$$\frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(\frac{2n\pi}{T} t) \sin(\frac{2m\pi}{T} t) dt = \begin{cases} 0 \text{ if } n \neq 0 \\ 1 \text{ if } n = 0 \end{cases}$$

Applied in Fourier

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(\frac{2n\pi}{T} t) dt = \sum_{n=1}^{\infty} a_n \cdot \frac{T}{2}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(\frac{2n\pi}{T} t) dt = \sum_{n=1}^{\infty} b_n \cdot \frac{T}{2}$$

3 Geometric Fourier series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n}{T} t) + \sum_{n=1}^{\infty} a_n \sin(\frac{2\pi n}{T} t)$$

Where:

$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(\frac{2\pi n}{T} t) dt$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(\frac{2\pi n}{T} t) dt$$

4 Parity functions

Even:

$$f(t) = f(-t)$$

For integrals in origin you can calculate half and multiply by 2
Odd:

$$f(t) = -f(-t)$$

Integrals in origin are equal to 0.

Properties:

$$f_p(t) \cdot f_p(t) = f_p(t)$$

$$f_i(t) \cdot f_i(t) = f_p(t)$$

$$f_i(t) \cdot f_p(t) = f_i(t)$$

In Fourier
If $f(t)$ is even:

$$a_0 = \frac{2}{T} \int_0^{\frac{T}{2}} f_p(t) dt$$

$$a_n = \frac{4}{T} \int_0^{\frac{T}{2}} f_p(t) \cos(\frac{2n\pi}{T} t) dt$$

$$b_n = 0$$

If $f(t)$ is odd:

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{T} \int_0^{\frac{T}{2}} f_p(t) \sin(\frac{2n\pi}{T} t) dt$$

5 Complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{i2n\pi}{T}t}$$

Where:

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\frac{i2n\pi}{T}t}dt$$

$$a_n = 2\mathbb{R}\mathrm{e}(C_n)$$

$$b_n = -2\mathbb{I}\mathrm{m}(C_n)$$
