Fourier analysis

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1 Fourier

Main formula

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T}t\right)$$

Orthogonality

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f_1(t) f_2(t) dt = \begin{cases} \alpha \text{ if } f_1 = f_2 \\ 0 \text{ if } f_1 \neq f_2 \end{cases}$$

Orthonormality

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} (f_1(t))^2 dt = 1$$

The oscillation is named Gibbs phenomenon. The equality does not always maintain when operating in both sides.

2 Sin and cosine

Using $\frac{2}{T}$ to normalize:

cos · cos

$$\frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2n\pi}{T}t\right) \cos\left(\frac{2m\pi}{T}t\right) dt = \begin{cases} 0 \text{ if } n \neq 0 \\ 1 \text{ if } n = 0 \end{cases}$$

sin · cos

$$\frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin\left(\frac{2n\pi}{T}t\right) \cos\left(\frac{2m\pi}{T}t\right) dt = 0$$

sin · sin

$$\frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin\left(\frac{2n\pi}{T}t\right) \sin\left(\frac{2m\pi}{T}t\right) dt = \begin{cases} 0 \text{ if } n \neq 0 \\ 1 \text{ if } n = 0 \end{cases}$$

Applied in Fourier

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos\left(\frac{2n\pi}{T}t\right) dt = \sum_{n=1}^{\infty} a_n \cdot \frac{T}{2}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin\left(\frac{2n\pi}{T}t\right) dt = \sum_{n=1}^{\infty} b_n \cdot \frac{T}{2}$$

3 Geometric Fourier series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T}t\right)$$

Where:

$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt$$

a_0 is the average value.

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos\left(\frac{2\pi n}{T}t\right) dt$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin\left(\frac{2\pi n}{T}t\right) dt$$

4 Parity functions

Even:

$$f(t) = f(-t)$$

For integrals in origin you can calculate half and multiply by 2.

Odd:

$$f(t) = -f(-t)$$

Integrals in origin are equal to 0.

Properties:

$$f_p(t) \cdot f_p(t) = f_p(t)$$
$$f_i(t) \cdot f_i(t) = f_p(t)$$
$$f_i(t) \cdot f_p(t) = f_i(t)$$

In Fourier

If $f(t)$ is even:

$$a_0 = \frac{2}{T} \int_0^{\frac{T}{2}} f_p(t) dt$$

$$a_n = \frac{4}{T} \int_0^{\frac{T}{2}} f_p(t) \cos\left(\frac{2n\pi}{T}t\right) dt$$

$$b_n = 0$$

If $f(t)$ is odd:

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{T} \int_0^{\frac{T}{2}} f_p(t) \sin\left(\frac{2n\pi}{T}t\right) dt$$

5 Complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{i2n\pi}{T}t}$$

Where:

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-\frac{i2n\pi}{T}t} dt$$

$$a_n = 2\mathbb{R}e(C_n)$$

$$b_n = -2\mathbb{I}m(C_n)$$

Amplitude spectrum: The value of $|C_n|$ as y and $(2\pi/T)n$ as x.

Phase spectrum: The value of $arg(Cn)$ as y and $(2\pi/T)n$ as x.

6 Fourier transform

$$\mathcal{F}(f(t)) = F(W) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$

$$\mathcal{F}^{-1}(F(W)) = f(t) = \int_{-\infty}^{\infty} F(W) e^{iwt} dw$$

7 Trigonometric integral

$$F(W) = \int_{-\infty}^{\infty} f(t) \cos(wt) dt - i \int_{-\infty}^{\infty} f(t) \sin(wt) dt$$

$$F_c(W) = \int_{-\infty}^{\infty} f(t) \cos(wt) dt$$

$$F_s(W) = \int_{-\infty}^{\infty} f(t) \sin(wt) dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [F_c(W) - iF_s(W)][\cos(wt) + i\sin(wt)] dt$$

8 Steps to solve Fourier transform with tables

1. Identify the properties that can be applied.

2. Algebraic work to make the problem look as a propertie.
3. The properties should reduce the complexity of the problem to solve.

9 Convolution

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(x) f_2(t-x) dx$$

$$f_1(t) * H(t) = \int_{-\infty}^{\infty} f(t-x) dx$$

$$f_1(t) * \delta(t) = f(t)$$

$$f_1(t) * f_2(t) = \mathcal{F}_1(W) \mathcal{F}_2(W)$$

$$f_1(t) f_2(t) = \frac{1}{2\pi} [\mathcal{F}_1(W) * \mathcal{F}_2(W)]$$

10 Miscellaneous

$$\text{sinc} = \frac{\sin(x)}{x}$$

$$H(t) = u(t)$$

If you reduce time the function goes to the right, if you add time the function goes to the left.

$$t^n \cdot f(t) = \frac{1}{(-i)^n} \cdot \frac{\delta^n}{dw^n} F(W)$$