Signals and Systems 1.0

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1 Signals and Systems

1.1 Periodic Signals

A signal $x(t)$ is periodic if there exists a positive number T such that $x(t) = x(t+T)$ for all t.
A signal $x[n]$ is periodic if there exists a positive number N such that $x[n] = x[n+N]$ for all n.

1.2 Basic Signals in Continuous Time

- Unit Step: $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$
- Unit Impulse: $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$
- Ramp: $r(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$
- Exponential: e^{at}
- Sinusoidal: $x(t) = A\cos(\omega_0 t + \phi)$, where $\omega_0 = 2\pi f_0$, it is periodic with fundamental period $T_0 = \frac{1}{f_0}$
- Complex Exponential: $e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$ where $\omega_0 = 2\pi f_0$ is the radian frequency and f_0 is the frequency in Hz. It is periodic with fundamental period $T_0 = \frac{1}{f_0}$

1.3 Basic Signals in Discrete Time

- Unit Step: $u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$
- Unit Impulse: $\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$
- Ramp: $r[n] = \begin{cases} 0 & n < 0 \\ n & n \geq 0 \end{cases}$
- Exponential: a^n
- Sinusoidal: $x[n] = A\cos(\Omega_0 n + \phi)$, where $\Omega_0 = \frac{2\pi}{N}$, it is periodic with fundamental period N when $\frac{m}{n}$ is rational.

1.4 Signal Operations and Transformations

This operations are defined for both continuous and discrete time signals.

- Sum: $(f+g)(t) = f(t) + g(t)$
- Scalar Multiplication: $(\alpha f)(t) = \alpha f(t)$
- Linear Combination: $(\alpha f + \beta g)(t) = \alpha f(t) + \beta g(t)$
- Product: $(f \cdot g)(t) = f(t) \cdot g(t)$
- Time Shift: $(f(t-t_0))(t) = f(t-t_0)$ if $t_0 > 0$ then the signal is moved to the right, if $t_0 < 0$ then the signal is moved to the left.
- Inversion: $(f(-t))(t) = f(-t)$ the signal is reflected around the y-axis. If it contains a time shift, then the signal is reflected around the line $t = t_0$.

Just for the discrete signals we can add the following operations:

- Sum: $\sum_{n=a}^{b} x[k] = x[a] + x[a+1] + ... + x[b]$
- Util sum: $\sum_{n=0}^{N} b^n = \frac{1-b^{N+1}}{1-b}$
- Backward sum: $\Delta x[n] = \frac{\Delta x[n]}{\Delta n} = x[n] - x[n-1]$

1.5 Convolution

The convolution integral of two signals $f(t)$ and $g(t)$ is defined as:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

The convolution sum of two signals $f[n]$ and $g[n]$ is defined as:

$$(f * g)[n] = \sum_{k=-\infty}^{\infty} f[k]g[n-k]$$

1.6 Properties of Convolution

- Commutative: $f * g = g * f$
- Associative: $f * (g * h) = (f * g) * h$
- Distributive: $f * (g + h) = f * g + f * h$
- Asociative with scalar: $\alpha(f) * \beta(g) = (\alpha\beta)(f * g)$
- Shift: $x(t) * \delta(t-t_0) = x(t-t_0)$ and $x[n] * \delta[n-n_0] = x[n-n_0]$

1.7 Sampling and Reconstruction

Sampling is the process of converting a continuous time signal into a discrete time signal. The sampling frequency is the number of samples per second.

$$x[n] = x_c(nT)$$

Where T is the sampling period and $F_s = \frac{1}{T}$ is the sampling frequency.

1.8 System Properties

- Causality: A system is causal if the output at any time t_0 depends only on the input at times $t \leq t_0$.
- Linearity: A system is linear if it satisfies the principle of superposition. $T\alpha x_1 + \beta x_2 = \alpha T x_1 + \beta T x_2$
- Time Invariance: A system is time invariant if a time shift in the input signal causes a corresponding time shift in the output signal. $Tx(t) = y(t) \Rightarrow Tx(t-t_0) = y(t-t_0)$
- Stability: A system is stable if the output is bounded for any bounded input.

2 Linear Time-Invariant Systems

A system is linear if it satisfies the principle of superposition. A system is time-invariant if a time shift in the input signal causes a corresponding time shift in the output signal. A system is linear time-invariant (LTI) if it is both linear and time-invariant.

2.1 Impulse Response

The impulse response of an LTI system is the output of the system when the input is an impulse function.

$$y(t) = x(t) * h(t)$$

2.2 Response

We can find:

$$y = y_p + y_h$$

Where y_p is the permanent response and y_h is the transient response. The first one persists while the input persists, the second one decays to zero as time goes to infinity.

2.3 Causality

A system is causal if the output at any time t_0 depends only on the input at times $t \leq t_0$.

2.4 LTI represented as EDLs

An LTI system can be represented as a differential equation in continuous time or as a difference equation in discrete time.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

2.5 Discrete Time FIR

A discrete time FIR system is a system whose output is the sum of a finite number of weighted samples of the input signal.

$$y[n] = x[n] * h[n] = \sum_{k-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{L_h-1} h[k]x[n-k]$$

2.6 Discrete Time IIR

A discrete time IIR system is a system whose output is the sum of a finite number of weighted samples of the input signal and a finite number of weighted samples of the output signal. It is recursive.

$$y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right)$$

3 LTI System Analysis with Laplace and Z-Transforms

3.1 Laplace Transform

The Laplace transform of a signal $x(t)$ is defined as:

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

The inverse Laplace transform is defined as:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

3.2 Properties of Laplace Transform

- Linearity: $a_1x_1(t) + a_2x_2(t) \Leftrightarrow a_1X_1(s) + a_2X_2(s)$
- Convolution: $x_1(t) * x_2(t) \Leftrightarrow X_1(s)X_2(s)$
- Differentiation: $\frac{d^n x(t)}{dt^n} \Leftrightarrow s^n X(s) - \sum_{k=0}^{n-1} s^{n-1-k}\frac{d^k x(0)}{dt^k}$
- Delay in S: $e^{-at}x(t) \Leftrightarrow X(s+a)$
- Differentiation in S: $-tx(t) \Leftrightarrow \frac{dX(s)}{ds}$

3.3 Laplace elemental transforms

$x(t)$	$X(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$\frac{dx^a(t)}{dt^a}$	$s^a X(s) - x(0) - ... - x^a(0)$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$
$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

A transformation for partial fractions is:

$$\mathcal{L}^{-1}\left\{\frac{Bs+C}{s^2+\beta s+\lambda}\right\} = e^{-\alpha t}\left(A_1\cos(\omega_0 t) + A_2\sin(\omega_0 t)\right)u(t)$$

Where:

- $\alpha = 0.5\beta$
- $\omega_0 = 0.5\sqrt{4\lambda - \beta^2}$
- $A_1 = B$
- $A_2 = \frac{2C-\beta B}{2\omega_0}$

3.4 Transfer Function

The transfer function of an LTI system is the Laplace transform of the impulse response.

$$H(s) = \frac{Y(s)}{X(s)}$$

3.5 Stability

A system is stable if all the poles of the transfer function have negative real parts.

3.6 Z-Transform

The Z-transform of a signal $x[n]$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

For the inverse Z-transform we have:

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

3.7 Properties of Z-Transform

- Linearity: $\alpha x_1[n] + \beta x_2[n] \Leftrightarrow \alpha X_1(z) + \beta X_2(z)$
- Convolution: $x_1[n] * x_2[n] \Leftrightarrow X_1(z)X_2(z)$
- Time shift: $x[n-a] \Leftrightarrow z^{-a}X(z)$
- Scaling in Z: $a^n x[n] \Leftrightarrow X(a^{-1}z)$
- Differentiation in Z: $nx[n] \Leftrightarrow -z\frac{dX(z)}{dz}$

3.8 Z-Transform elemental transforms

$x[n]$	$X(z)$
$\delta[n]$	1
$u[n]$	$\frac{z}{z-1}$
$x[n-a]$	$z^{-a}X(z)$
$a^n u[n]$	$\frac{z}{z-a}$
$na^n u[n]$	$\frac{az}{(z-a)^2}$
$\rho^n \cos{(\Omega_0 n)} u[n]$	$\frac{z^2 - z(\rho\cos{(Omega_0)} + \rho\sin{(\Omega_0)})}{z^2 - 2\rho\cos{(\Omega_0)}z + \rho^2}$
$\rho^n \sin{(\Omega_0 n)} u[n]$	$\frac{\rho\sin{(\Omega_0)}z}{z^2 - 2\rho\cos{(\Omega_0)}z + \rho^2}$

A transformation for partial fractions is:

$$\mathcal{Z}^{-1}\left\{\frac{Bz^2 + Cz}{z^2 + \beta z + \lambda}\right\} = \rho^n \left(A_1 \cos{(\Omega_0 n)} + A_2 \sin{(\Omega_0 n)}\right) u[n]$$

Where:

- $\rho = \sqrt{\lambda}$
- $\Omega_0 = \cos^{-1}\left(\frac{-\beta}{2\sqrt{\lambda}}\right)$
- $A_1 = B$
- $A_2 = \frac{2C - \beta B}{2\rho\sin{(\Omega_0)}}$

3.9 Transfer Function in Z

The transfer function of an LTI system is the Z-transform of the impulse response.

$$H(z) = \frac{Y(z)}{X(z)}$$

3.10 Stability in Z

A system is stable if all the poles of the transfer function have magnitude less than 1.

3.11 Discretization using backwards difference

The backwards difference is defined as:

$$Y'(z) = \frac{z-1}{Tz} Y(z)$$

3.12 Discretization of the derivative

Using Barrow difference we have:

$$Y'_c = \frac{2}{T}\left(\frac{z-1}{z+1}\right) Y(z)$$

3.13 Discretization of Systems through bilinear transformation

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

4 Miscellaneous

4.1 Complex magnitude and fase

The magnitude of a complex number $z = a + jb$ is defined as:

$$|z| = \sqrt{a^2 + b^2}$$

The phase of a complex number $z = a + jb$ is defined as:

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

To keep it in the range $-\pi < \phi \leq \pi$ we use:

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$

4.2 Solve a LTI IIR system

To solve a LTI IIR system we can use the following steps:

1. Get $y[n] = ...$
2. We start with $n = 0$ and we get $y[0] = ...$ substituting the values given for $x[n]$.
3. The value $y[n-1]$ is the output of the system at time $n-1$
4. We substitute the value of $y[n-a]$ until we get the desired $y[n]$

4.3 Solve a convolution sum

To solve a convolution sum we can use the following steps:

$$x[n] * y[n]$$

1. Get $y[n] = ...$ and $x[n] = ...$
2. Invert the signal $x[n]$ to get $x[-n]$
3. Shift the signal $x[-n]$ to get $x[n-a]$
4. Multiply the signals $x[n-a]y[n]$
5. Sum the results to get the result, you will have $|x[n]| + |y[n]| - 1$ results.