No homogeneous second grade differential equations

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1 Antecedents

Final solution

$$y = y_h + y_p$$

Homogeneous
Variable change

$$Ay'' + By' + Cy = 0$$

$$A\lambda'' + B\lambda' + C = 0$$

The solution to the differential equation is the values of λ
Real roots solution

$$y_h = C_1 e^{\lambda_1} + C_2 e^{\lambda_2}$$

$$C.F. = \{\lambda_1, \lambda_2\}$$

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$$C.F. = \{e_1^{\lambda}, e_2^{\lambda}\}$$

Imaginary roots solution

$$\lambda_1 = a + bi, \lambda_2 = a - bi$$

$$y = C_1 e^{ax} cos(bx) + C_1 e^{ax} sen(bx)$$

$$C.F. = \{e^{ax} cos(bx), e^{ax} sen(bx)\}$$

2 Indeterminate coefficients

Solving steps:

1. Solve homogeneous equation
2. Obtain a possible particular solution

   Polynomial

$$x = (Ax + B)$$

$$x^2 = (Ax + B)(Cx + D)$$

$$x^3 = (Ax + B)(Cx + D)(Ex + F)$$

   Exponential

$$e^{nx} = Ae^{nx}$$

$$mxe^{nx} = (Ax + B)e^{nx}$$

$$mx^2 e^{nx} = (Ax + B)(Cx + D)e^{nx}$$

3. Combine both solutions to obtain y.

$$y = y_h + y_p$$

Superposition theory

$$a_n y^n + a_{n-1} y^{n-1} + ...a_1 y' + a_0 y = g(x) \mid a_n \neq 0$$

$$y_p = y_{p1} + y_{p2} + ...y_{pk}$$

3 Annihilator operator

Operator	Functions Annihilated (including linear combinations of the functions)
D^n	$1, x, x^2, \ldots, x^{n-1}$
$(D - \alpha)^n$	$e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}, \ldots, x^{n-1} e^{\alpha x}$
$[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$	$e^{\alpha x} \cos \beta x, xe^{\alpha x} \cos \beta x, \ldots, x^{n-1} e^{\alpha x} \cos \beta x$ $e^{\alpha x} \sin \beta x, xe^{\alpha x} \sin \beta x, \ldots, x^{n-1} e^{\alpha x} \sin \beta x$

$$a_n y^n + a_{n-1} y^{n-1} + ...a_1 y' + a_0 y = g(x)$$

Solving solutions:

1. Solve homogeneous equation.
2. Represent equation in terms of operators.
3. Find annihilator operator for $g(x)$.
4. Apply annihilator operator to both sides of the original operator to obtain all the solutions.
5. The λ that were not solved by homogeneous are solved with undetermined coefficients because this is the particular solution.

4 Parameter variation

1. Normalize equation

$$\frac{\alpha_1}{\alpha_1}y^{''} + \frac{\alpha_2}{\alpha_1}y^{'} + \frac{\alpha_3}{\alpha_1}y = \frac{g(x)}{\alpha_1}$$

2. Solve homogeneous equation

$$y_h = C_1 y_1 + C_2 y_2$$

3. Particular solution

$$y_p = U_1(x)y_1 + U_2(x)y_2$$

4. You need Wronskian (W)

$$\begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix}$$

5. We find

$$U_1^{'}(x) = \frac{-y_2 f(x)}{W}$$

$$U_2^{'}(x) = \frac{y_1 f(x)}{W}$$

6. We find U1 y U2

$$U_1(x) = \int U_1^{'}(x) \cdot dx$$

$$U_2(x) = \int U_2^{'}(x) \cdot dx$$

7. Remember that, final solution (Note: remember to simplify constants)

$$y = y_h + y_p$$