Complex Analysis

Enrique Calderon

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1 Basic operators

Complex Number:

$$z = x + iy$$

Addition and subtraction:

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

Product:

$$z_1 \cdot z_2 = [(x_1 \cdot x_2) - (y_1 \cdot y_2)] + i[(x_1 \cdot y_2) + (y_1 \cdot x_2)]$$

Conjugate:

$$z \cdot \overline{z} = (x + iy)(x - iy) = x^2 + y^2$$

Division:

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2}$$

Power:

$$z^w = e^{w \cdot \log z} \quad | \quad \forall \quad z, w \in \mathbb{C}$$

Logarithm:

$$\log z = \ln |z| + i \cdot \tan^{-1} \frac{y}{x}$$

$$\ln |z| = \ln(r) + i \ln(\theta + 2kr)$$

Exponential product:

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}$$

Exponential division:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)}$$

Exponential root:

$$\sqrt[n]{z} = (z)^{\frac{1}{n}} = r^{\frac{1}{n}} \cdot e^{i[\frac{\theta + 2kr}{n}]}$$

Where there are n roots, $k = 0, 1, 2, ..., n - 1$

2 Set

The module if z is $|z|$, where $z \in \mathbb{R}$
Circumference with radius k:

$$|z + z_0| = k$$

To the set z that exists near the point we know it as **Neighborhood** with radius ρ

$$z - z_0 = \rho$$

Points

- *Frontier*: Point which neighborhood is both inside and outside z
- *Interior*: Point which neighborhood is only inside z

Sets

- *Open*: Exclusively of interior points
- *Closed*: Inner and frontier points, is limited by $\mathbb{R}$ and $\mathbb{C}$
- *Neither open nor closed*

3 Complex functions

Linear mapping

$$f(z) = az + b$$

For a, r modifies magnitude and θ the angle:

$$f(z) = az = (r_a e^{i\theta_a}) \cdot (re^{i\theta}) = (r_a r) \cdot (e^{i(\theta_a + \theta)})$$

For b, $\mathbb{R}(b)$ move in $\mathbb{R}$ and $\mathbb{I}(b)$ move in $\mathbb{I}$:

$$f(z) = z + b$$

Potential mapping

$$f(z) = z^n$$

$f(z) = z^2$: $x = k$ and $y = k$ generate parabolas.

$f(z) = z^3$: generate curves.

Inverse mapping

$$f(z) = \frac{1}{z} = \frac{1}{r} \cdot e^{-i\theta}$$

4 Trigonometry

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\cos(iz) = \cosh(z)$$

$$\cosh(z) = \frac{e^{-z} + e^{z}}{2}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2}$$

$$-\sin(iz) = \sinh(z)$$

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

$$e^z = e^{x+iy} = e^x \cdot (\cos(y) + isin(y))$$

$$e^{iz} = e^{ix-y} = e^{-y} \cdot (\cos(x) + isin(x))$$

5 Limits

Infinite possible directions, if not all the directions converge in the same point, the limit does not exist.

$$\lim_{z \to z_0} = f(z) = f(z_0)$$

6 Derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

Horizontally, $\Delta z = \Delta x$ and $\Delta y = 0$:

$$f'(z) = \frac{\delta u(x,y)}{\delta x} + i\frac{\delta v(x,y)}{\delta x}$$

Vertically, $\Delta z = \Delta y$ and $\Delta x = 0$:

$$f'(z) = -i\frac{\delta u(x,y)}{\delta y} + \frac{\delta v(x,y)}{\delta y}$$

Exponential:

$$f'(z) = \frac{1}{r}e^{-i\theta}\left[\frac{\delta v}{\delta \theta} - i\frac{\delta u}{\delta \theta}\right]$$

Multiple:

$$f''(z) = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 v}{\delta x^2} \text{ or } \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 v}{\delta y^2} \text{ or } \frac{\delta^2 u}{\delta x \delta y} + \frac{\delta^2 v}{\delta x \delta y}$$

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7 Cauchy - Riemann

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \text{ and } \frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$$

If and only if $f(z)$ is continuous in the domain D and C-R is true, them $f(z)$ is analytic in D.

If it is analytic, then it is derivable in the same region. Once a function is harmonic, it is conjugate if C-R is true.

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8 Laplace

If $f(z)$ is analytic in a region R and Laplace is true in the same region, the function is harmonic.

Binomial:

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \text{ and } \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0$$

Exponential:

$$\frac{\delta^2 u}{\delta r^2} + \frac{1}{r^2}\frac{\delta^2 u}{\delta \theta^2} + \frac{1}{r}\frac{\delta u}{\delta \theta}$$

$$\frac{\delta^2 v}{\delta r^2} + \frac{1}{r^2}\frac{\delta^2 v}{\delta \theta^2} + \frac{1}{r}\frac{\delta v}{\delta \theta}$$

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9 Integral

In analytic functions the calculus fundamental theorem is true but in non analytic there is a infinity of results.

$$\int_c f(z)dz = \int_{t1}^{t2} f(z(t)) \cdot z'(t)dt$$

If C contains z_0:

$$\oint_c \frac{1}{(z-z_0)^n}dz = \begin{cases} 2\pi i & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Cauchy theorem

$$\oint_c f(z)dz = \oint_{c_1} f(z)dz + \oint_{c_2} f(z)dz + ... + \oint_{c_n - 1} f(z)dz$$

If there are no discontinuities in C, the integral is 0. If there are discontinuities:

$$\oint_c^1 \frac{1}{z-z_0}dz = 2\pi i$$

If $f(z)$ is analytic except in $z = z_0$

$$f(z) = \frac{g(z)}{z-z_0} \to \oint_{c1} f(z)dz = 2\pi i \cdot g(z)|_{z=z_0}$$

For indeterminate, we get the singularity order, n, which is the times we need to derive in order to get rid off the indeterminate.

$$\oint f(z)dz = \oint \frac{g(z)}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \cdot \frac{\delta^{n-1}}{\delta z^{n-1}} g(z)|_{z=z_0}$$

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10 Series and successions

Succession

Function with integers as domain.

Convergence : $\lim_{n\to\infty} z_n = a + ib$

Cauchy criteria : $|z_n - z_m| < \epsilon$

Series

Sum of a succession

$$\sum_{n=1}^{\infty} z_n$$

Geometric function :

$$\sum_{n=1}^{\infty} az^{n-1} = a + az + az^2 + ...$$

Series convergence :

$$S_k = \frac{a(1-z^k)}{(1-z)}$$

$$\lim_{k\to\infty} z^k \begin{cases} |z| > 1 \to z^k \to \infty \text{ so it diverges} \\ |z| < 1 \to z^k \to 0 \text{ so it converges} \end{cases}$$

So if $|z| < 1$, it is convergent in $S_k = \frac{a}{1-z}$

Series:

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + ...$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + ...$$

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + ...$$

$$sen(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...$$

$$cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + ...$$

Convergence criteria:
The proportion criteria

$$\lim_{n\to\infty} \frac{|Z_{n+1}|}{|Z_n|}$$

The root criteria

$$\lim_{n\to\infty} \sqrt[n]{|Z_n|}$$

Where:

$$\begin{cases} L < 1 \rightarrow \text{ converges} \\ L > 1 \rightarrow \text{ diverges} \\ L = 1 \rightarrow \text{ unknown} \end{cases}$$

Power series:

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ...$$

We can use root or proportion criteria with a_n to determine the convergence

Convergence radio, in power series a_n is the coefficient and z_0 the center of the series. It is important because there the equality is true, to calculate it:

$$R = \frac{1}{L}$$

$$a_n = \frac{1}{n!} \cdot \frac{\delta^n}{\delta z^n} f(z)|_{z=z_0}$$

Taylor: in a point.
Maclaurin: in origin.
Laurent:

$$f(z) = ... + \frac{a_{-k}}{(z-z_0)^k} + ... + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + ...$$

Where the principal part is:

$$... + \frac{a_{-k}}{(z-z_0)^k} + ... + \frac{a_{-1}}{z-z_0}$$

And the analytic part is:

$$a_0 + a_1(z-z_0) + ...$$

To obtain a_k:

$$a_k = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^{k+1}} dz$$

11 Residue theory

By theory:

$$Res(f, z_0) = \lim_{z\to z_0} \frac{1}{(k-1)!} \frac{\delta^{k-1}}{\delta z^{k-1}}[(z-z_0)^k f(z)])$$

The residue of $f(z)$ in $z = z_0$ is:

$$a_{-1} = \frac{1}{2\pi i} \oint f(z)dz$$

In a curve with N discontinuities:

$$\oint_c f(z)dz = 2\pi i \cdot \sum_{k=1}^{N} \text{Res}\{f(z), z = z_0)\}$$

Función	Criterio	Tipo de singularidad	Residuo en z_0
f(z)	$\text{lím}_{z\to z_0}(z-z_0)f(z) = 0$	removible	0
$\frac{g(z)}{h(z)}$	g y h tienen ceros mismo orden	removible	0
f(z)	$\text{lím}_{z\to z_0}(z-z_0)f(z)$ existe y es$\neq 0$	polo simple	$\text{lím}_{z\to z_0}(z-z_0)f(z)$
$\frac{g(z)}{h(z)}$	$g(z_0) \neq 0,\ h(z_0) = 0,\ h'(z_0) \neq 0$	polo simple	$\frac{g(z_0)}{h'(z_0)}$
$\frac{g(z)}{h(z)}$	g tiene cero orden k h tiene cero orden $k+1$	polo simple	$(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$
$\frac{g(z)}{h(z)}$	$g(z_0) \neq 0$ $h(z_0)=h'(z_0)=0$ $h''(z_0) \neq 0$	polo orden 2	$2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{(h''(z_0))^2}$
$\frac{g(z)}{(z-z_0)^2}$	$g(z_0) \neq 0$	polo orden 2	$g'(z_o)$
$\frac{g(z)}{h(z)}$	$g(z_0) = 0,\ g'(z_0) \neq 0$ $h(z_0)=h'(z_0)=h''(z_0)=0$ $h'''(z_0) \neq 0$	polo orden 2	$3\frac{g''(z_0)}{h'''(z_0)} - \frac{3}{2}\frac{g'(z_0)h^{(IV)}(z_0)}{(h'''(z_0))^2}$
$f(z)$	k es el entero más pequeño tal que $\text{lím}_{z\to z_0}\Theta(z)$ existe, donde $\Theta(z) = (z-z_0)^k f(z)$	polo orden k	$\text{lím}_{z\to z_0}\frac{\Theta^{(k-1)}(z)}{(k-1)!}$
$\frac{g(z)}{h(z)}$	g tiene un cero de orden n f tiene un cero de orden $n+m$	polo orden m	$\text{lím}_{z\to z_0}\frac{\Theta^{(k-1)}(z)}{(k-1)!}$ donde $\Theta(z) = \frac{(z-z_0)^k g(z)}{h(z)}$