Ph 21 Project 2: Introduction to Fourier Transforms

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Part I

(1)

Prove to yourself that the definition of the Fourier series is consistent.

$$h(x) = \sum_{k=-\infty}^{\infty} \tilde{h}_k e^{-2\pi i \frac{1}{L} x}$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{L} \int_0^L h(y) e^{2\pi i \frac{k}{L} y} d_y \right) e^{-2\pi i \frac{1}{L} x}$$

$$= \frac{1}{L} \sum_{n=-\infty}^{\infty} \delta \left(\frac{-ix}{L} - n \right) \left(\int_0^L h(y) e^{2\pi i \frac{k}{L} y} d_y \right)$$

$\delta\left(\frac{-ix}{L} - n\right) = 1$ when $x = iLn$ and 0 for all other values of x

$$h(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \left(\int_0^L h(y) e^{2\pi i \frac{k}{L} y} C \right)$$

$$= \frac{1}{L} \sum_{n=-\infty}^{\infty} \left(\int_0^L h(y) \delta \left(\frac{iy}{L} - n \right) d_y \right)$$

$\delta\left(\frac{iy}{L} - n\right) = 1$ when $y = -iLn$ and 0 for all other values of y, therefore if $y = x$ then the sum is only non-zero when $n = 0$

$$h(x) = \frac{1}{L} \int_0^L h(x) d_y$$

$$h(x) = h(x) \frac{1}{L} \int_0^L d_y$$

$$h(x) = h(x) \frac{L - 0}{L}$$

$$h(x) = h(x)$$

(2)

Show that a linear combination of $e^{\frac{-2i\pi x}{L}}$ and $e^{\frac{2i\pi x}{L}}$ can represent any equation of the form $Asin(\frac{2i\pi x}{L}+\phi)$

$$Asin(\frac{2i\pi x}{L}+\phi)$$

$$=\frac{iA}{2}e^{\frac{-2i\pi x}{L}-i\phi}-\frac{iA}{2}e^{\frac{2i\pi x}{L}+i\phi}$$

$$=\frac{iA}{2}(e^{\frac{-2i\pi x}{L}}e^{-i\phi}-e^{\frac{2i\pi x}{L}}e^{i\phi})$$

$$=Be^{\frac{-2i\pi x}{L}}+Ce^{\frac{2i\pi x}{L}}$$

Where $B=\frac{iA}{2}e^{-i\phi}$ and $C=\frac{-iA}{2}e^{i\phi}$

(3)

Show that for real $h(x)$, $\tilde{h}_k$ must satisfy $\tilde{h}_{-k}=\tilde{h}_k^*$

$$\tilde{h}_k=\frac{1}{L}\int_0^L h(x)e^{2\pi i\frac{k}{L}x}d_x$$

$$\tilde{h}_{-k}=\frac{1}{L}\int_0^L h(x)e^{2\pi i\frac{-k}{L}x}d_x$$

$$\tilde{h}_k^*=\frac{1}{L}\int_0^L h(x)e^{2\pi-i\frac{k}{L}x}d_x$$

So $\tilde{h}_{-k}=\tilde{h}_k^*$

(4)

Convince yourself of the convolution theorem

$$H(x)=h^{(1)}(x)h^{(2)}(x)$$

$$=\left(\sum_{k_1=-\infty}^{\infty}\tilde{h}_{k_1}^{(1)}e^{-2\pi i\frac{k_1}{L}x}\right)\left(\sum_{k_2=-\infty}^{\infty}\tilde{h}_{k_2}^{(2)}e^{-2\pi i\frac{k_2}{L}x}\right)$$

$$=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\tilde{h}_{k_1}^{(1)}\tilde{h}_{k_2}^{(2)}e^{-2\pi i\frac{k_1+k_2}{L}x}$$

$$=\sum_{(k-k')=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\tilde{h}_{(k-k')}^{(1)}\tilde{h}_{k'}^{(2)}e^{-2\pi i\frac{k}{L}x}$$

where $k-k'=k_1$ and $k'=k_2$

$$\sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{L} - n\right) = \sum_{k=-\infty}^{\infty} e^{2\pi i \frac{k}{L} x}$$

therefore

$$H(x) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{-x}{L} - n\right) \sum_{k'=-\infty}^{\infty} \tilde{h}^{(1)}_{(k-k')} \tilde{h}^{(2)}_{k'} e^{-2\pi i \frac{k'}{L} x}$$

$$x = -Ln$$

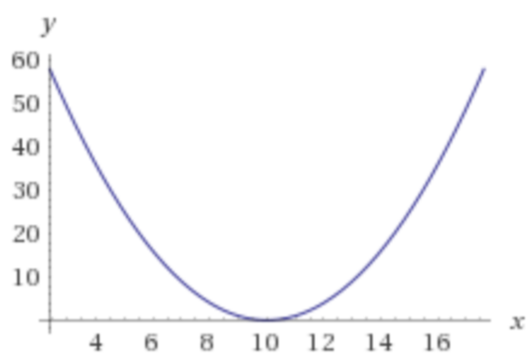
$$H(x) = \sum_{n=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \tilde{h}^{(1)}_{(k-k')} \tilde{h}^{(2)}_{k'} e^{-2\pi i k' n}$$

$$= \sum_{n=-\infty}^{\infty} \delta\left(\frac{n}{L} - n\right) \sum_{k'=-\infty}^{\infty} \tilde{h}^{(1)}_{(k-k')} \tilde{h}^{(2)}_{k'}$$

$\delta\left(\frac{n}{L} - n\right) = 1$ when $n = 0$ therefore

$$H(x) = \sum_{k'=-\infty}^{\infty} \tilde{h}^{(1)}_{(k-k')} \tilde{h}^{(2)}_{k'}$$

For a smooth $\tilde{h}^{(1)}_k$ centered at $k = 0$ and $\tilde{h}^{(2)}_k = \delta(k - 10)$ the product is zero except when $k' = 10$ so the graph is just $(x - 10)^2$



(5)

Test the numpy fft on the functions $C + Acos(ft + \phi)$ and $Ae^{-B\left(t-\frac{L}{2}\right)^2}$ For a point of comparison I evaluated $\tilde{h}_k\left(t\right) = \frac{1}{L}\int_0^L h(t)e^{2\pi i\frac{k}{L}t}d_t$ analytically for both $h^1\left(t\right)$ and $h^2\left(t\right)$

$$\tilde{h}_k^1\left(t\right) = \frac{1}{L}\int_0^L \left(C + Acos\left(\frac{k}{L}t + \phi\right)\right)e^{2\pi i\frac{k}{L}t}d_t$$

$$= \frac{-ie^{2i\pi\frac{k}{L}t}\left(-2i\pi Asin\left(\frac{k}{L}t+\phi\right) + 4\pi^2 Acos\left(\frac{k}{L}t+\phi\right) + C\left(4\pi^2-1\right)\right)}{\left(8\pi^3 - 2\pi\right)}\Bigg|_0^L$$

$$\tilde{h}_k^2\left(t\right) = \frac{1}{L}\int_0^L \left(Ae^{-B\left(t-\frac{L}{2}\right)^2}\right)e^{2\pi i\frac{k}{L}t}d_t$$

$$= \frac{\sqrt{\pi}Ae^{\frac{\pi i}{B}\frac{k}{L}\left(BL+\pi i\frac{k}{L}\right)}erf\left(\frac{-BL+2Bt-2\pi i\frac{k}{L}}{2\sqrt{B}}\right)}{2L\sqrt{B}}\Bigg|_0^L$$

For plotting purposes I chose variables such that:

$$h^1\left(t\right) = 1 + 2cos(f\frac{3}{4}t + \pi)$$

$$h^2\left(t\right) = 2e^{-.005(t-2)^2}$$

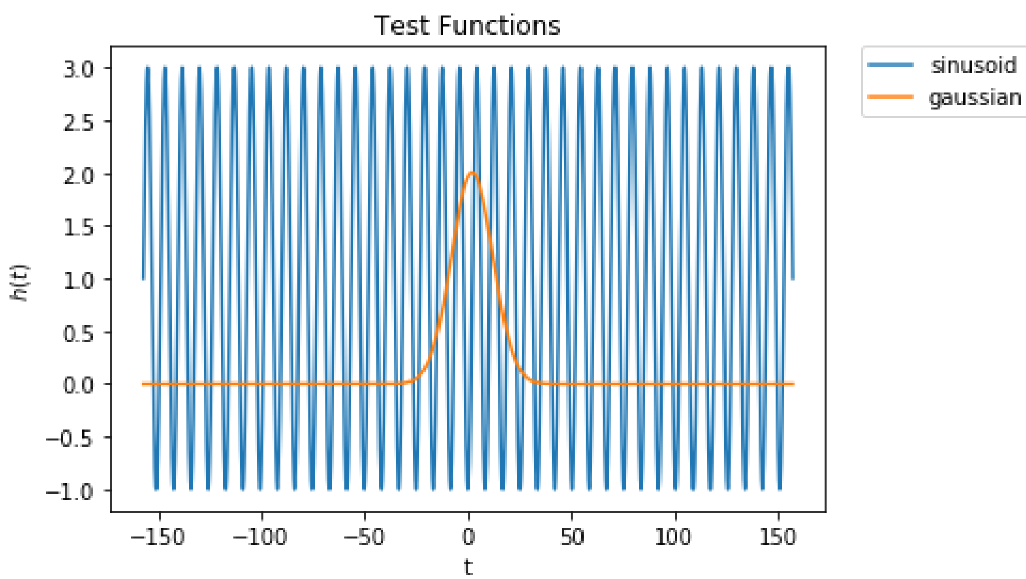


Figure 1: Plot of $h^1\left(t\right)$ a sinusoid and $h^2\left(t\right)$ a Gaussian

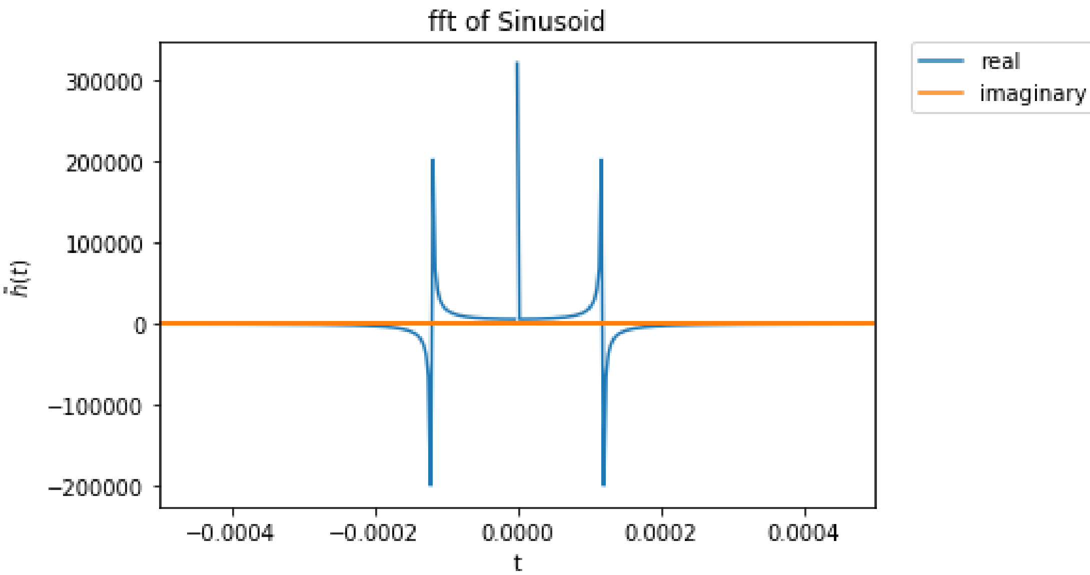


Figure 2: Plot of $\tilde{h}^1(t)$, the numpy fft function of $h^1(t)$

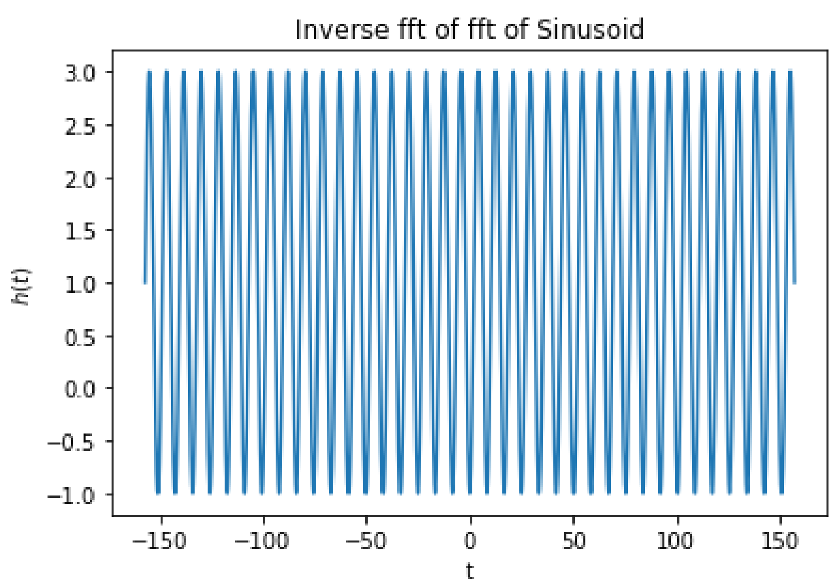


Figure 3: Plugging the results of fft into ifft returns the original function $h^1(t)$

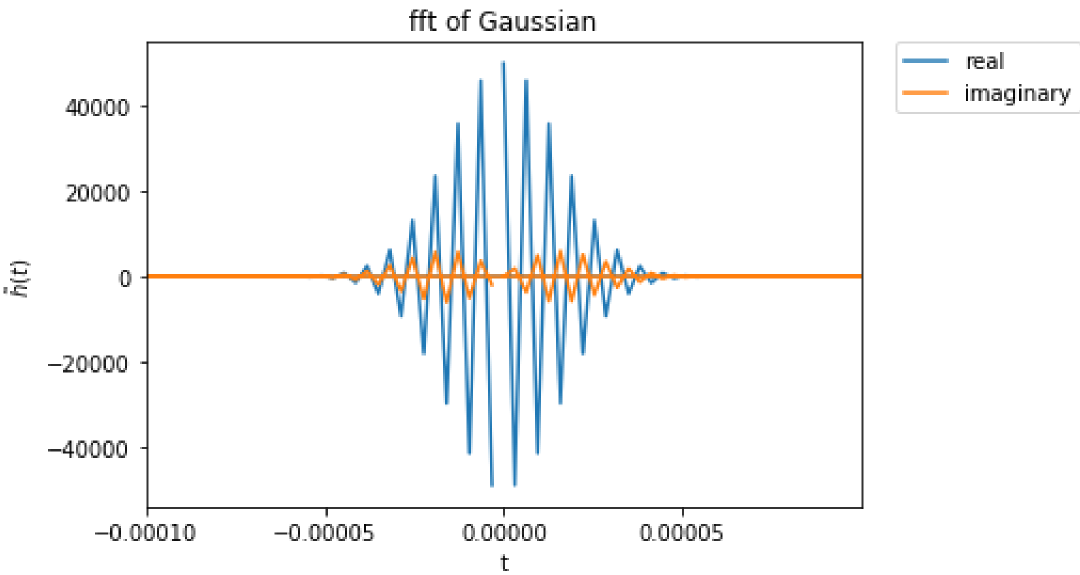


Figure 4: Plot of $\tilde{h}^2(t)$, the numpy fft function of $h^2(t)$

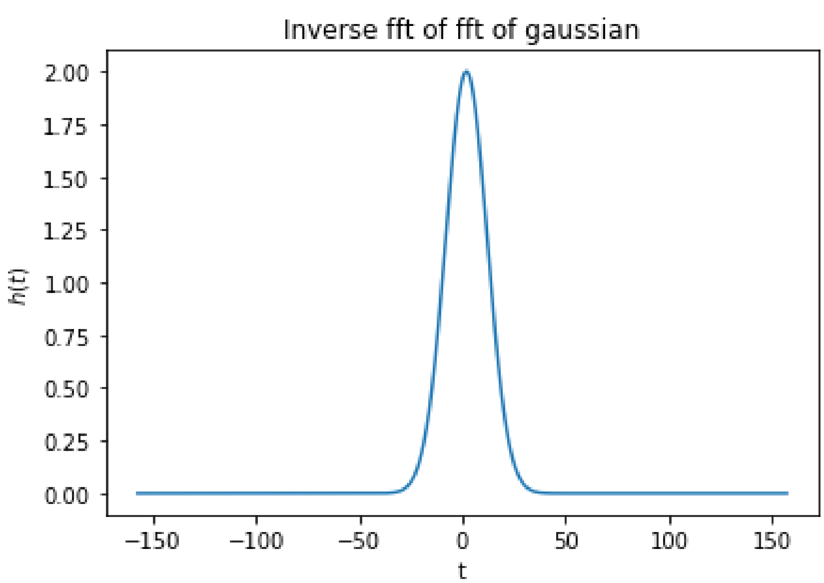


Figure 5: Plugging the results of fft into ifft returns the original function $h^2(t)$

The numpy fft and ifft functions behave as expected

Part II

Lets examine the data collected from the Aricebo radiotelescope.

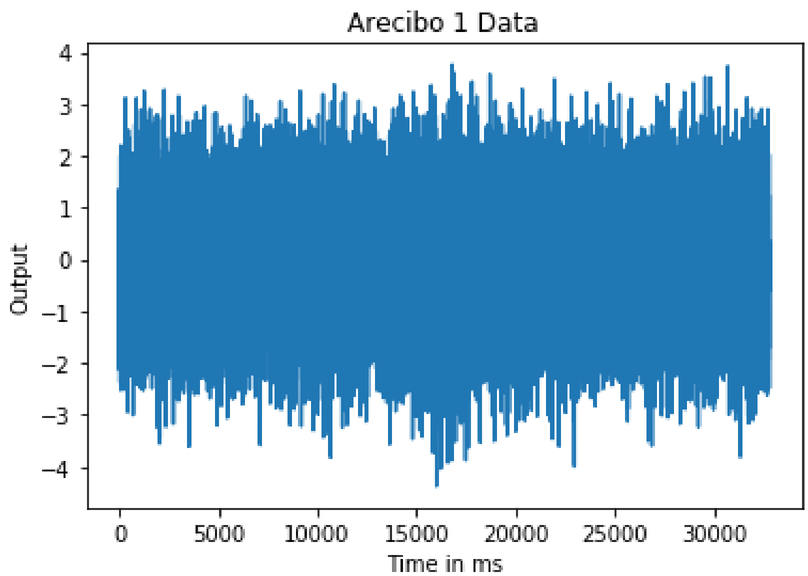


Figure 6: As we can see there is no clear pattern when we plot the signal with respect to time.

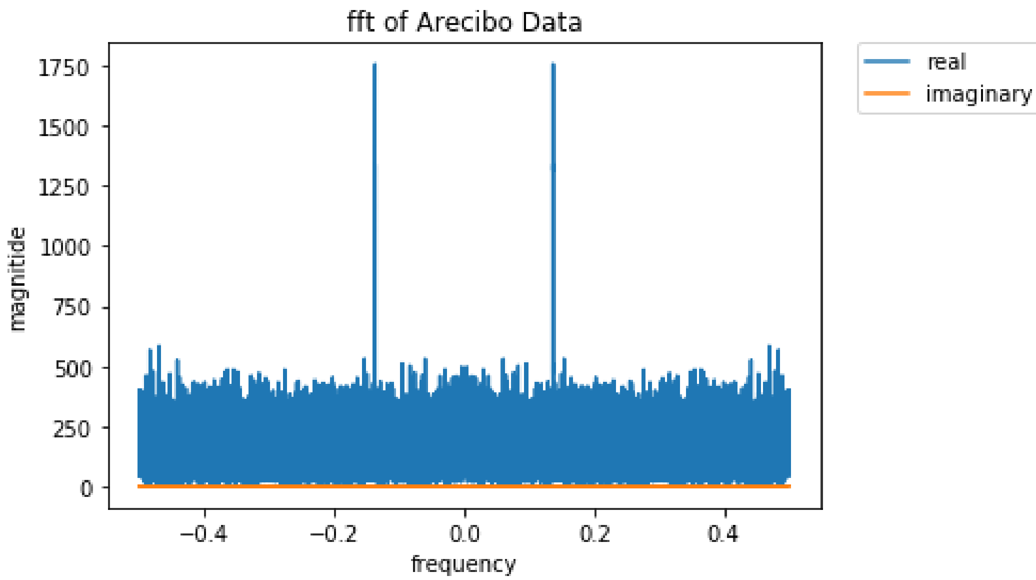


Figure 7: When we use the fft function on the dataset reveals two peaks. the peak with the highest magnitude is located at 136.993408203125Hz

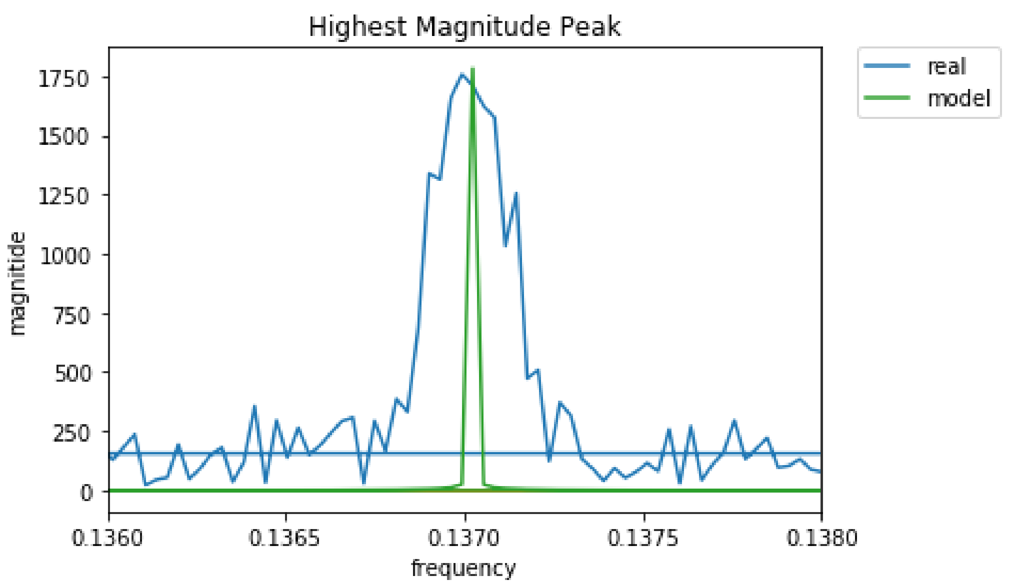


Figure 8: Focusing in on this peak we see that it is rounded. We model this peak as $e^{\frac{-(t-.137)^2}{(.092)^2}}$.

Part III

Now we will explore the Lomb-Scargle algorithm.

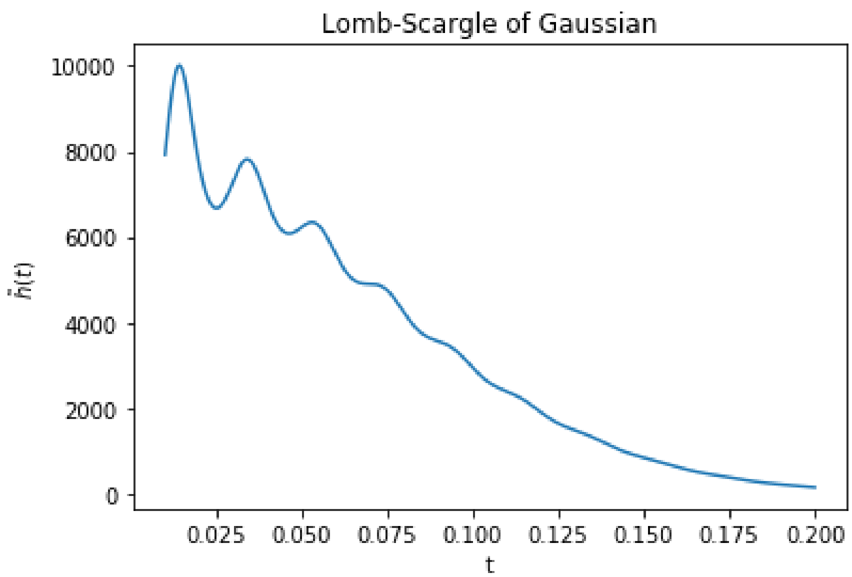


Figure 9: The Scipy Lomb-Scargle routine run on the Gaussian $h^2(t)$

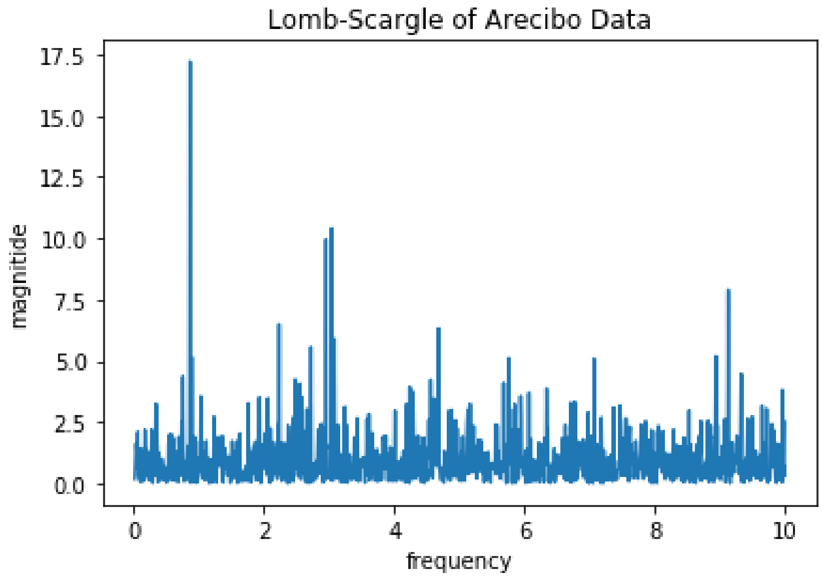


Figure 10: The Scipy Lomb-Scargle routine run on the Arecibo data

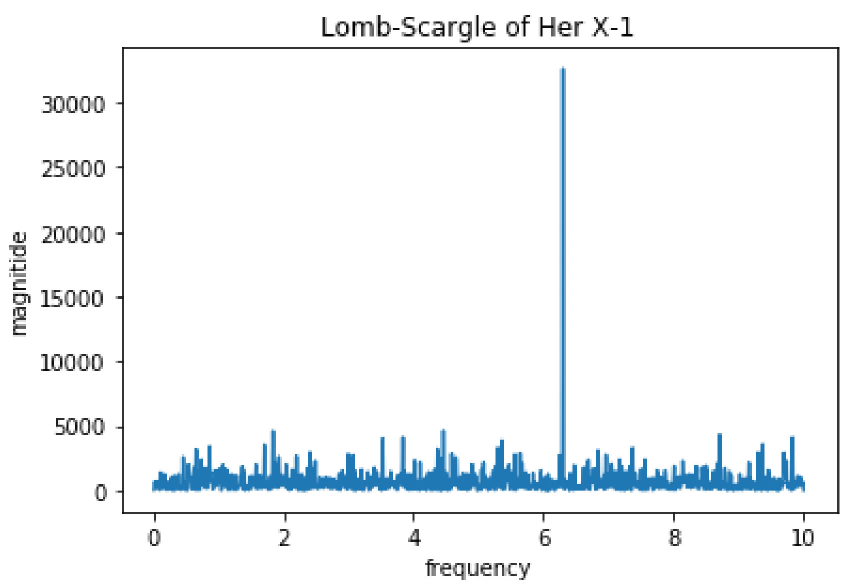


Figure 11: The Scipy Lomb-Scargle routine run on Her X-1 data collected from the Catalina Real Time Survey.