Partition Theory

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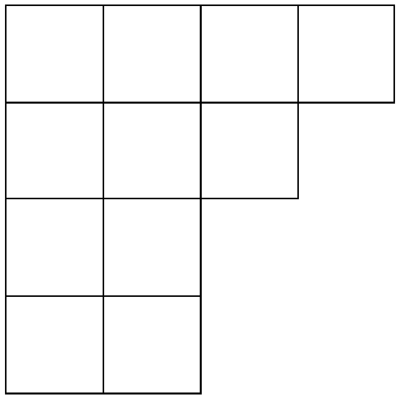
What are Partitions?

- A partition of positive *n* is a set of nonzero positive numbers which sum to *n.* Partitions are not distinct by order.
- For example, the partitions of 4 are:

4
$3+1$
$2+2$
$2+1+1$
$1+1+1+1$

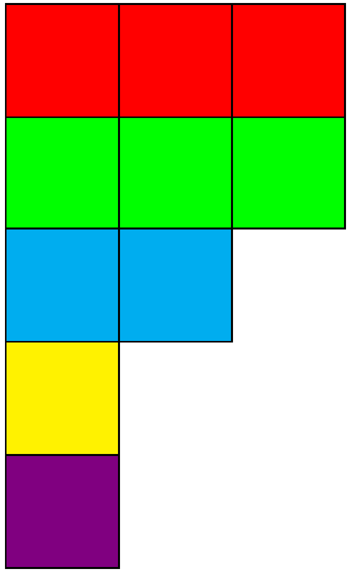
- We denote *p(n)* to be the number of partitions of n. So, *p(*4*)*=5. We assign *p(*0*)*=1, the *empty partition* $\varnothing$.
- If a bijection (an invertible function) $f : A \to B$ exists, then $|A| = |B|$
- We visualize partitions with Young Diagrams. We arrange parts in weakly decreasing order. Then, every part *p* has its own row with *p* squares. We can do transformations on these to obtain partition identities.



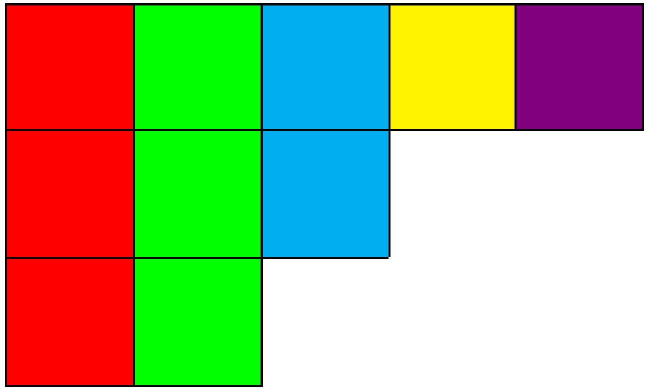
Partition: $4+3+2+2$ of 11

Conjugating Partitions

- We can *conjugate* a partition to get a new partition. To conjugate a partition, swap the rows and columns of its Young diagram. Conjugation is an *involution* – it is its own inverse, so it is a bijection.



Partition: $3 + 3 + 2 + 1 + 1$ of 10



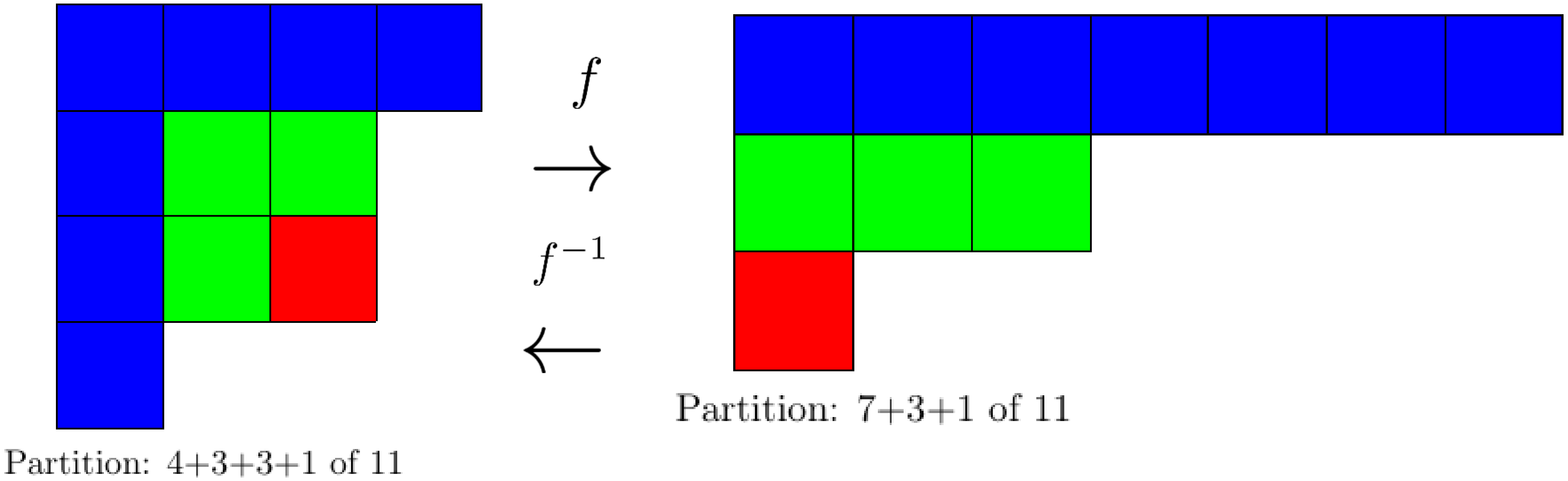
Conjugate Partition: $5 + 3 + 2$ of 10

- Observe that the first part of the conjugate partition is the number of parts in the original partition, and the other parts are weakly decreasing. This gives a partition identity since conjugation is bijective:

$$p(n \mid m \text{ parts}) = p(n \mid \text{greatest part is } m)$$

Partition Identities

- A partition is *self-conjugate* if it is its own conjugate. There is a bijection between self-conjugate partitions and partitions with distinct odd parts:



- Here, the bijection f is given by dividing the partition into hook shapes which get their own row.
- Notice every hook has an odd number of parts that are distinct: two equal parts imply the Young diagram is drawn incorrectly.
- An inverse is given by rearranging these odd parts into the hook shapes and nesting the hooks to recreate the original partition, so this is a bijection.

Generating Function of *p(n)*

- Generating functions are used to keep track of sequences in combinatorics. For sequence a_n the generating function is: $$\sum_{n=0}^{\infty} a_n x^n$$
- The generating function for *p(n)* is: $\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$
- Observe that any partition can be uniquely represented by the number of 1s, 2s, 3s, and so on. So, a partition like $5+4+2+2+1$ has one 5, one 4, zero 3s, two 2s, and one 1.
- We want an expression which will count every possible partition exactly once. We will devote one factor to the occurrences of each natural number:

$$(1+x+x^2+x^3+\dots)(1+x^2+x^4+x^6+\dots)\dots(1+x^k+x^{2k}+x^{3k}+\dots)\dots$$

- In this product, every partition will occur as a power of x. Every partition is finite, so we must select finitely many terms which are not 1 (so the exponent is finite). Every factor in this product is a geometric series.
- The partition $5+4+2+2+1$ is $(x^1)(x^{2(2)})(x^{0(3)})(x^4)(x^5)(1)\dots(1)\dots$

  $\sum_{n\geq 0} x^{nk} = \frac{1}{1-x^k}$ gives the above formula.

The Rogers-Ramanujan Identities

- We define parts as *d-distinct* if every parts differs by at least *d*.
- First Rogers-Ramanujan Identity:

$$p(n \mid \text{parts} \equiv 1 \text{ or } 4 \pmod 5) = p(n \mid \text{2-distinct parts})$$

- Second Rogers-Ramanujan Identity:

$$p(n \mid \text{parts} \equiv 2 \text{ or } 3 \pmod 5) = p(n \mid \text{2-distinct parts} \geq 2)$$

- From the generating functions for each kind of partition, we can derive analytic forms for these identities:

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})} = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)(1-q^2)\ldots(1-q^m)},\ |q| < 1 \qquad \text{(First)}$$

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})} = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2+m}}{(1-q)(1-q^2)\ldots(1-q^m)},\ |q| < 1 \qquad \text{(Second)}$$

- The generating function for *p(n)* converges for $|q| < 1$.

Proving the Rogers-Ramanujan Identities

- The first bijective proof was found by Garsia and Milne and is 51 pages long.
- One proof requires using the q-binomial numbers. The q-binomial numbers $\binom{n}{k}_q$ are generating functions for partitions which fit into a $k \times (n-k)$ rectangle.

$$s_n(q) = \sum_{j=0}^{n} q^{j^2} \binom{n}{j}_q \quad t_n(q) = \sum_{j=0}^{n} q^{j^2+j} \binom{n}{j}_q \quad \tau_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(5j-3)}{2}} \binom{2n+1}{n+2j}_q$$

$$\sigma_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(5j+1)}{2}} \binom{2n}{n+2j}_q \qquad \sigma_n^*(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(5j+1)}{2}} \binom{2n}{n+1+2j}_q$$

- By induction and recursion relations we show that $s_n(q) = \sigma_n(q) = \sigma_n^*(q)$ and $t_n(q) = \tau_n(q)$. Then we take limits of these functions as $n \to \infty$ to obtain both Rogers-Ramanujan identities.
- There is a generalization of the Rogers-Ramanujan identities to certain odd moduli.

Conclusions

- Partition theory is a remarkable and intriguing subject. It has implications throughout combinatorics, and it has been used to solve problems in physics and statistical mechanics.
- This presentation has only scratched the surface of partition theory. I encourage you to read further into the subject, especially on identities of the Rogers-Ramanujan kind.

Further Readings and References

Integer Partitions, G.E. Andrews and K. Eriksson

The Theory of Partitions, G.E. Andrews

A Rogers-Ramanujan Bijection, A.M. Garsia and S.C. Milne

Partitions and Durfee Dissection, G.E. Andrews

An Analytic Generalization of the Rogers-Ramanujan Identities for Odd Moduli, G.E. Andrews

Acknowledgements

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- Thanks to the UMN Math Department for being awesome!

And remember,

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{sinh\left(\frac{2\pi\sqrt{x-\frac{1}{24}}}{3k} \right)}{\sqrt{x-\frac{1}{24}}} \right]_{x=n}$$

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi inh}{k}}$$

$\omega_{h,k}$ a particular 24kth root of unity

The Hardy-Ramanujan-Rademacher expansion of *p(n)*, a convergent series.
Proof through analytic number theory.