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Exercise Sheet 6 @ MfCS 1

Problem 1

a)

Pick a monotone increasing sequence $a_n \in X$ without loss of generality (we can take the elements from a set in any reasonable order, because sets are unordered). Clearly, $a_n \leq a_{n+1} \leq \sup X$. The problem thus reduces to proving $\lim_{n\to\infty} a_n = \sup X$. For every $\varepsilon > 0$ there exists m such that $a_m > \sup X - \varepsilon$. Otherwise, $\sup X - \varepsilon$ would be the upper bound of X, which is a contradiction. This would also violate our initial premise. Because a_n is increasing, for every $n > m$, $|c - a_n| \leq |c - a_m| < \varepsilon$. QED.

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The same proof can be formulated for minimising sequences by picking a monotone decreasing sequence and a set X which is bounded below. The proof is conducted analogically and omitted for brevity.

b)

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Consider $r \in \mathbb{R}$ and $n \in \mathbb{N}$. As such, there is a rational number $q_n \in \mathbb{Q}$ such that $r - \frac{1}{n} < q_n < r + \frac{1}{n}$, because $\mathbb{Q}$ is dense in $\mathbb{R}$ per the remark on page 66 of the lecture notes. By simple arithmetic manipulation we obtain $-\frac{1}{n} < q_n - r < \frac{1}{n} \Leftrightarrow |q_n - r| < \frac{1}{n}$, thus, q_n converges to r.

Problem 2

Let's take a closer look at this clever device:

$$\lim_{n\to\infty} \lim_{k\to\infty} (\cos n!\pi x)^{2k}$$

Observe that if x is a rational number, so $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$, then $n!$ as $n \to \infty$ will certainly contain a factor of q, meaning that $n!x$ is integer. If x is irrational, this does not happen, and $n!x$ is never an integer. Furthermore, it is obvious that $\cos k\pi \in \mathbb{Z} \Leftrightarrow k \in \mathbb{Z}$. Knowing that $\cos k\pi \in [-1, 1]$ and if $k \in \mathbb{Z}$ then $\cos k\pi \neq 0$ (because k would have to be equal to $n - 0.5$ for some $n \in \mathbb{Z}$, so it would not be integer), it follows that if $k \in \mathbb{Z}$ then $\cos k\pi \in \{-1, 1\}$ and if $k \notin \mathbb{Z}$ then $\cos k\pi \in (-1, 1)$.

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This has an interesting consequence. $(\cos n!\pi x)^2 = 1$ if x is integer and $0 \leq (\cos n!\pi x)^2 < 1$ otherwise. It is clear that $\lim_{k\to\infty} a^k = 1$ if $a = 1$ and $\lim_{k\to\infty} a^k = 0$ if $0 \leq a < 1$. Thus, the double limit converges to 1 if x is rational and to 0 if x is irrational.

Problem 3

a)

$$\lim_{n\to\infty}\left(\frac{n^3}{\sqrt{n^2+a}}-\frac{n^3}{\sqrt{n^2+b}}\right)=\lim_{n\to\infty}\left(\frac{n^3\left(\sqrt{n^2+b}-\sqrt{n^2+a}\right)}{\sqrt{n^2+a}\sqrt{n^2+b}}\right)$$
$$=\lim_{n\to\infty}\left(\frac{\sqrt{n^8+bn^6}-\sqrt{n^8+an^6}}{\sqrt{n^2+a}\sqrt{n^2+b}}\right)$$
$$=\lim_{n\to\infty}\left(\frac{\sqrt{n^8+bn^6}-\sqrt{n^8+an^6}}{\sqrt{n^2+a}\sqrt{n^2+b}}*\frac{\sqrt{n^8+bn^6}+\sqrt{n^8+an^6}}{\sqrt{n^8+bn^6}+\sqrt{n^8+an^6}}\right)$$
$$=\lim_{n\to\infty}\left(\frac{n^8+bn^6-n^8-an^6}{\sqrt{n^2+a}\sqrt{n^2+b}}*\frac{1}{\sqrt{n^8+bn^6}+\sqrt{n^8+an^6}}\right)$$
$$=\lim_{n\to\infty}\left(\frac{bn^6-an^6}{\sqrt{n^2+a}\sqrt{n^2+b}}*\frac{1}{n^3\sqrt{n^2+b}+n^3\sqrt{n^2+a}}\right)$$
$$=\lim_{n\to\infty}\left(\frac{bn^6-an^6}{\sqrt{n^2+a}\sqrt{n^2+b}}*\frac{1}{n^3\sqrt{n^2+b}+n^3\sqrt{n^2+a}}\right)$$
$$=\lim_{n\to\infty}\left(\frac{n^6(b-a)}{n^3\sqrt{n^2+a}\sqrt{n^2+b}(\sqrt{n^2+b}+\sqrt{n^2+a})}\right)$$
$$=(b-a)*\lim_{n\to\infty}\left(\frac{n^3}{(n^2+b)\sqrt{n^2+a}+(n^2+a)\sqrt{n^2+b}}\right)$$

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$$=(b-a)*\lim_{n\to\infty}\left(\frac{n^3}{n^2(1+\frac{b}{n^2})\sqrt{n^2+a}+n^2(1+\frac{a}{n^2})\sqrt{n^2+b}}\right)$$
$$=(b-a)*\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^2+a}+\sqrt{n^2+b}}\right)$$
$$=(b-a)*\lim_{n\to\infty}\left(\frac{n}{n\sqrt{1+an^{-2}}+n\sqrt{1+bn^{-2}}}\right)$$
$$=(b-a)*\lim_{n\to\infty}\left(\frac{n}{2n}\right)$$
$$=\frac{(b-a)}{2}$$

b)

$$\lim_{n\to\infty}\frac{n!}{n^n}=\lim_{n\to\infty}\left(\frac{n}{n}*\frac{n-1}{n}*\frac{n-2}{n}*\ \dots\ *\frac{1}{n}\right)$$

Generally speaking, for $a\in\mathbb{N}$:

$$\lim_{n\to\infty}\frac{n-a}{n}=1$$

Thus:

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$$\lim_{n\to\infty}\left(1*1*1*\ \dots\ *\frac{1}{n}\right)=\lim_{n\to\infty}\frac{1}{n}=0$$

c)

We exploit the fact that for continuous function f, $\lim_{n\to n_0} f(g(n)) = f\left(\lim_{n\to n_0} g(n)\right)$:

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$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln\left(n^{\frac{1}{n}}\right)} = e^{\lim_{n\to\infty}\ln\left(n^{\frac{1}{n}}\right)} = e^{\lim_{n\to\infty}\frac{\ln n}{n}} = e^0 = 1$$

✓

d)

The approach is the same as previously.

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$$\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln\left(a^{\frac{1}{n}}\right)} = e^{\lim_{n\to\infty}\ln\left(a^{\frac{1}{n}}\right)} = e^{\lim_{n\to\infty}\frac{\ln a}{n}} = e^0 = 1$$

✓

e)

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{a^n+b^n} &= \lim_{n\to\infty} a\sqrt[n]{1+\frac{b^n}{a^n}}\\ &= a\lim_{n\to\infty}\sqrt[n]{1+\left(\frac{b}{a}\right)^n}\end{aligned}$$

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$a > b$ clearly implies that $0 < \frac{b}{a} < 1$ so $\lim_{n\to\infty}\left(\frac{b}{a}\right)^n = 0$.

$$a\lim_{n\to\infty}\sqrt[n]{1+\left(\frac{b}{a}\right)^n} = a\lim_{n\to\infty}\sqrt[n]{1} = a$$

✓

Problem 4

i)

We prove that $x_{n+1} \in [2,4]$. Substitute the recursive formula:

$1 + \dfrac{6}{x_n} \in [2,4]$ would imply that $\dfrac{6}{x_n} \in [1,3]$.

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Notice that if $x_n \in [2,4]$, this is always true: The value of $\dfrac{6}{x_n}$ dimnishes when x_n grows, and grows when x_n dimnishes, thus checking the endpoints is enough to support our claim: $\dfrac{6}{2} = 3$ and $\dfrac{6}{4} = 1.5$.

✓

ii)

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$$\begin{aligned}x_{n+2} &= 1+\frac{6}{x_{n+1}} = 1+\frac{6}{1+\frac{6}{x_n}}\\ &= 1+\frac{6}{\frac{6+x_n}{x_n}} = 1+\frac{6x_n}{6+x_n}\\ &= \frac{7x_n+6}{x_n+6} = \frac{7(x_n+6)-36}{x_n+6} = 7-\frac{36}{x_n+6}\end{aligned}$$

✓

iii & iv)

Let's prepare x_{n+1} and a few following terms:

$x_{n+1} = 1 + \frac{6}{x_n}$.

$x_{n+2} = 1 + \frac{6x_n}{6+x_n}$.

$x_{n+3} = 1 + \frac{6}{1+\frac{6x_n}{6+x_n}} = 1 + \frac{6(x_n+6)}{7x_n+6}$.

iii: $x_{n+2} \geq x_n \Leftrightarrow x_{n+3} \leq x_{n+1}$.

Notice that $x_{n+2} \geq x_n$ happens only if $x_n \in [2, 3]$ (or more generally $[-2, 3]$, but a big part of this interval is outside of our domain).

To conduct a proof by contradiction, let's look at the second inequality now:

$$\frac{6(x_n+6)}{7x_n+6} > \frac{6}{x_n}$$

$$\frac{6(x_n+6)}{7x_n+6} * \frac{x_n}{6} > 1$$

$$\frac{x_n^2 - x_n - 6}{7x_n+6} > 0$$

The bottom term is clearly always greater than zero. Let's get rid of it then by multiplying sides:

$$x_n^2 - x_n - 6 > 0$$

Clearly the quadratic function is negative at both of the endpoints of our domain - 2 and 3, thus per the properties of the quadratic function it's also negative on the entire interval between them. This leads to a contradiction, meaning that $x_{n+2} \geq x_n \Leftrightarrow x_{n+3} \leq x_{n+1}$.

iv: The outline of the proof is exactly the same.

$x_{n+2} \leq x_n$ happens if $x_n \in [3, 4]$ (analogically to our previous result). In our proof by contradiction we arrive at a very similar final equation:

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$$x_n^2 - x_n - 6 < 0$$

Clearly the quadratic function is positive at both of the endpoints of our domain - 3 and 4, thus per the properties of the quadratic function it's also positive on the entire interval between them. This leads to a contradiction, meaning that $x_{n+2} \leq x_n \Leftrightarrow x_{n+3} \geq x_{n+1}$