Exercise sheet 7 @ MfCS1

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Problem 1

Consider the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$.

a)

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\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} \\
&= \left(1+\frac{1}{n+1}\right) * \frac{\left(1+\frac{1}{n+1}\right)^n}{\left(1+\frac{1}{n}\right)^n} \\
&= \left(1+\frac{1}{n+1}\right) * \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^n \\
&= \left(\frac{n+2}{n+1}\right) * \left(\frac{\frac{n+2}{n+1}}{\frac{n+1}{n}}\right)^n \\
&= \left(\frac{n+2}{n+1}\right) * \left(\frac{n+2}{n+1} * \frac{n}{n+1}\right)^n \\
&= \left(\frac{n+2}{n+1}\right) * \left(\frac{n(n+2)}{(n+1)^2}\right)^n \\
&= \frac{n+2}{n+1} * \frac{n^n(n+2)^n}{(n+1)^{2n}} \\
&= \frac{n^n(n+2)^{n+1}}{(n+1)^{2n+1}} \\
&= \frac{n^{n+1}(n+2)^{n+1}}{(n+1)^{2n+2}} * \frac{n+1}{n} \\
&= \left(\frac{(n+1)^2-1}{(n+1)^2}\right)^{n+1} * \frac{n+1}{n} \\
&= \left(1-\frac{1}{(n+1)^2}\right)^{n+1} * \frac{n+1}{n}
\end{aligned}
$$

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b)

Set $x = -\dfrac{1}{(n+1)^2}, k = n+1.$

Using Bernoulli's inequality:

$$(1+x)^k \geq 1 + kx$$

$$\left(1-\frac{1}{(n+1)^2}\right)^{n+1} \geq 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Thusly:

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$$\left(1 - \frac{1}{(n+1)^2}\right)^{n+1} * \frac{n+1}{n} \geq \frac{n}{n+1} * \frac{n+1}{n} = 1$$

$$\frac{a_{n+1}}{a_n} \geq 1$$

c)

In the binomial expansion, set $x = \frac{1}{n}$:

$$a_n = \left(1 + \frac{1}{n}\right)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!n^j}$$

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Use the hint to deduce that the expression $\frac{n!}{(n-j)!n^j} \leq 1$. Because of this, $\sum_{j=0}^{n} \frac{n!}{j!(n-j)!n^j} \leq \sum_{j=0}^{n} \frac{1}{j!}$. Thus, $a_n \leq \sum_{j=0}^{n} \frac{1}{j!}$.

d)

Compare the series obtained in point C,

$$a_n \leq \sum_{j=0}^{n} \frac{1}{j!} = 1 + \frac{1}{1} + \frac{1}{1*2} + \frac{1}{1*2*3} + \ldots$$

and

$$b_n = 1 + \sum_{j=0}^{n} \frac{1}{2^n} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2*2} + \frac{1}{2*2*2}$$

It is immediately obvious that $a_n \leq b_n$. Notice how the sum in b_n is the sum of geometric series with $g_1 = 1$ and $q = \frac{1}{2}$. As such, we obtain a closed form formula for b_n:

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$$b_n = 1 + g_1 \frac{1-q^n}{1-q} = 1 + \frac{1-q^n}{1-q} = 1 + \frac{1-2^{-n}}{\frac{1}{2}} = 1 + 2 - 2^{-n+1} = 3 - 2^{-n+1}$$

Thus, we determine that:

$$a_n \leq \sum_{j=0}^{n} \frac{1}{j!} \leq 3 - 2^{-n+1} \leq 3$$

e)

We want to prove that

$$2 < \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n < 3$$

Use the result from point C:

$$2 < \lim_{n\to\infty} \left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!n^j}\right) < 3$$

Notice how $\dfrac{n!}{(n-j)!n^j} = \dfrac{n}{n} * \dfrac{n-1}{n} * \dfrac{n-2}{n} * \ldots * \dfrac{n-j+1}{n}$, so taking the limit as $n \to \infty$, $\dfrac{n!}{(n-j)!n^j} \to 1$. We conclude:

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$$2 < \sum_{j=0}^{\infty} \frac{1}{j!} < 3$$

Notice that $2 < \sum_{j=0}^{\infty} \dfrac{1}{j!} = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{2*3} + \ldots < 1 + \sum_{j=0}^{\infty} \dfrac{1}{2^n} = 3$ (per the result obtianed in point D).

Problem 2

Assuming we denote the original sequences as a_n, assume a family of sequences for some i and k:

$$b_n = \begin{cases} a_n & \text{if } n < i \\ k & \text{otherwise} \end{cases}$$

a)

Notice how we can take i elements from this sequence verbatim and after the i-th element, ignore every element other than -1 and 1. For each $i \in \mathbb{N}$ and $k \in \{-1, 1\}$, b_n are the convergent subsequences of our original sequence. Additionally, the constant sequences $c_n = -1$ and $d_n = 1$ are also convergent subsequences of the original sequence. By transitivity, any subsequence of b also works.

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b)

Apply the same logic as before, but pick some $p \in \mathbb{N}$ to use as k. There are also infinitely many constant sequences $a_n = p$ that are convergent subsequences of the original sequence.

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c)

Notice that this sequence is bounded between zero and one: $0 < a_n < 1$ for any given n. Additionally, notice that this sequence contains every rational number in this range. Thus, we find subsequences converging to $\alpha \in [0, 1]$ in this set.

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Problem 3

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}$$

Using partial fraction decomposition we arrive at:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} &= \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} \right) \\
&= \sum_{k=1}^{n} \left(\frac{1}{2} * \frac{1}{k} - \frac{1}{2} * \frac{1}{k+1} - \frac{1}{2} * \frac{1}{k+1} + \frac{1}{2} * \frac{1}{k+2} \right) \\
&= \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} - \frac{1}{k+1} + \frac{1}{k+2} \right)
\end{aligned}$$

Observe that we are dealing here with two telescoping series:

$$\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k+2}-\frac{1}{k+1}\right)$$

That is, as k grows in the sums, the terms tend to cancel out! Take for example the first sum:

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\dots$$

It's easy to observe that factors in the middle of the telescoping series cancel out, so we can sum the series immediately now:

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k+2}-\frac{1}{k+1}\right)&=\frac{1}{2}\left(1-\frac{1}{n+1}\right)+\frac{1}{2}\left(\frac{1}{n+2}-\frac{1}{2}\right)\\&=\frac{1}{2}-\frac{1}{2n+2}+\frac{1}{2n+4}-\frac{1}{4}\\&=\frac{1}{4}-\frac{1}{2n+2}+\frac{1}{2n+4}\\&=\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}\end{aligned}$$

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~~Problem 4~~ ?

Apply the root test:

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{n(n+1)(n+2)}}=\lim_{n\to\infty}\frac{1}{\sqrt[n]{n(n+1)(n+2)}}=0$$

Knowing that the series is convergent, we proceed:

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)(k+2)}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\lim_{n\to\infty}\left(\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}\right)=\frac{1}{4}$$

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