$\frac{49}{50}$

# Exercise Sheet 4 @ MfCS 1

val## Problem 1

a) Prove using mathematical induction:

$$\sum_{i=1}^{n} \log\left(1+\frac{1}{i}\right) = \log(1+n)$$

*Base case:* Consider the $n=1$ case - the sum has only element so we drop it immediately and substitute $i=1$:
$\log\left(1+\frac{1}{1}\right) = \log(1+1)$
which is true because $\log 2 = \log 2$.

*Induction step:* Show that for every $k>0$, if $f(k)$ holds then $f(k+1)$ also holds.
Assume that

$$\sum_{i=1}^{n} \log\left(1+\frac{1}{i}\right) = \log(1+n)$$

holds for some $n$. It follows that:

5

$$\log\left(1+\frac{1}{n+1}\right) + \sum_{i=1}^{n} \log\left(1+\frac{1}{i}\right) = \log(1+n) + \log\left(1+\frac{1}{n+1}\right)$$

Collapse the logarithm term on the left into the sum. Rewrite the RHS using logarithm sum rule.

$$\sum_{i=1}^{n+1} \log\left(1+\frac{1}{i}\right) = \log(1+n) + \log\left(1+\frac{1}{n+1}\right) = \log\left((1+n) * \left(1+\frac{1}{n+1}\right)\right)$$

Multiply out the contents of the logarithm on the right.

$$\sum_{i=1}^{n+1} \log\left(1+\frac{1}{i}\right) = \log(2+n)$$

Meaning that $f(n+1)$ also holds true, which completes the inductive proof. QED.

b) Prove using mathematical induction:

$$\prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{2^n n!}$$

*Base case:*

$$(2*1-1) = \frac{(2*1)!}{2^1 1!}$$

$$1 = \frac{2}{2^1 1!} = \frac{2}{2} = 1$$

LHS=RHS, hence the base case is true.
*Inductive step:* Show that for every $k > 0$, if $f(k)$ holds then $f(k+1)$ also holds.
From

$$\prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{2^n n!}$$

it follows that

$$(2n+1) * \prod_{i=1}^{n}(2i-1) = (2n+1) * \frac{(2n)!}{2^n n!}$$

Collapse into the product:

$$\prod_{i=1}^{n+1}(2i-1) = (2n+1) * \frac{(2n)!}{2^n n!}$$

Multiply the term on the right by a special $1$:

$$\prod_{i=1}^{n+1}(2i-1) = \frac{(2n+1)(2n+2)}{(2n+2)} * \frac{(2n)!}{2^n n!}$$

Notice that we can collapse the product in the numerator into the factorial:

$$\prod_{i=1}^{n+1}(2i-1) = \frac{1}{(2n+2)} * \frac{(2n+2)!}{2^n n!}$$

5

Focus on the denominator:

$$\prod_{i=1}^{n+1}(2i-1) = \frac{(2n+2)!}{2(n+1)2^n n!}$$

Distribute the $2$ into $2^n$ to get $2^{n+1}$ and $n+1$ into $n!$ to get $(n+1)!$.

$$\prod_{i=1}^{n+1}(2i-1) = \frac{(2n+2)!}{2^{n+1}(n+1)!}$$

c) Prove using mathematical induction:

$$n^2 \leq 2^n \leq n!$$

for $n \in \mathbb{N}$ and $n \geq 4$.
*Base case* ($n = 4$):

$$4^2 \leq 2^4 \leq 4!$$

$$16 \leq 16 \leq 24$$

*Inductive case:* Show that for every $k > 4$, if $f(k)$ holds then $f(k+1)$ also holds.
It follows that

5

$$n^2 \leq 2^n \leq n!$$

$$n^2 \leq 2^n \wedge 2^n \leq n!$$

Let's consider $n^2 \leq 2^n$ first. Notice how $2^{n+1} = 2 * 2^n$, and assuming $f(k)$ holds, $2 * 2^n > 2 * n^2 = n^2 + n^2 > n^2 + 2n + 1 = (n+1)^2$, thus $2^{n+1} \geq (n+1)^2$ holds.
Consider $2^n \leq n!$. Notice how $(n+1)! = (n+1)n!$, assuming $f(k)$ holds, $(n+1)n! \geq n * 2^n > 2 * 2^n = 2^{n+1}$, thus $(n+1)! \geq 2^{n+1}$ holds.
QED.
d) Prove using mathematical induction:

$$6|(2^n + 3^n - 5^n)$$

Consider

$$6x = 2^n + 3^n - 5^n$$

Rephrase the problem as proving that $x \in \mathbb{Z}$.
*Base case:* Consider $n = 1$:

$$2 + 3 - 5 = 0$$

$0 = 6x$ when $x = 0$, so $x \in \mathbb{Z}$.

*Inductive case*: Show that for every $k > 0$, if $f(k)$ holds then $f(k+1)$ also holds.
Prove the statement for $f(k+1)$, considering that $2^n = 6x - 3^n + 5^n$:

$$2^{n+1} + 3^{n+1} - 5^{n+1} = 2 * 2^n + 3 * 3^n - 5 * 5^n = 2 * (6x - 3^n + 5^n) + 3 * 3^n - 5 * 5^n$$

Multiply out and rearrange terms, then group:

$$\begin{aligned} 12x - 2 * 3^n + 2 * 5^n + 3 * 3^n - 5 * 5^n &= 12x + 3 * 3^n - 2 * 3^n + 2 * 5^n - 5 * 5^n \\ &= 12x + (3-2)3^n + (-3)5^n \\ &= 12x + 3^n - 3 * 5^n \\ &= 12x + 3 * 3^{n-1} - 3 * 5^n \\ &= 12x + 3 * (3^{n-1} - 5^n) \\ &= 3 * 4x + 3 * (3^{n-1} - 5^n) \\ &= 3 * (4x + 3^{n-1} - 5^n) \end{aligned}$$

If $3^{n-1} - 5^n$ is divisible by two, then we can take the 2 factor out of $4x$ and $3^{n-1} - 5^n$, yielding us $6 * (u \in \mathbb{Z})$, meaning that $2^{n+1} + 3^{n+1} - 5^{n+1}$ is divisible by 6, which would complete the inductive proof. Let's prove that for $n \in \mathbb{N}$, $3^{n-1} - 5^n = 2x$ where $x \in \mathbb{Z}$ using mathematical induction now.
*Base case:* Consider $n = 1$:

$$3^0 - 5^1 = 1 - 5 = -4$$

$-4 = 2x$ when $x = -2$, so $x \in \mathbb{Z}$.
*Inductive case:* Show that for every $k > 0$, if $f(k)$ holds then $f(k+1)$ also holds. Also consider $5^n = -2x + 3^{n-1}$.

5

$$\begin{aligned}3^n - 5^{n+1} &= 3^n - 5 * 5^n \\ &= 3^n - 5 * (-2x + 3^{n-1}) \\ &= 3^n + 10x - \frac{5}{3}3^n \\ &= 3^n(1 - \frac{5}{3}) + 10x \\ &= 2 * 3^{n-1} + 10x \\ &= 2(3^{n-1} + 5x)\end{aligned}$$

Because $3^{n-1} + 5x$ is integer, $3^n - 5^{n+1}$ is divisible by two, which proves the first hypothesis.

## Problem 2

a)

Assume without loss of generality that $m = 1$, because it's the smallest possible value of $m$ and it only guarantees $\mathcal{P}(1)$ to be true.

4

From it, the base inductive case trivially follows.

Strong inductive case: If $k > 1$ then $\mathcal{P}(1)..\mathcal{P}(k-1)$ implies $\mathcal{P}(k)$.

$k$ is always greater than $m$. $\mathcal{P}(k)$ is true only when $\mathcal{P}(j)$ is true for all $j < k$, meaning that $P(k)$ is true only if $\mathcal{P}(1)..\mathcal{P}(k)$ is true, which is the inductive hypothesis.

QED.

b)

2x1 rectangle: 2x1 block; 1 way

2x2 rectangle: 2x2 block, two 2x1 blocks, two 1x2 blocks; 3 ways

So $T_1 = 1, T_2 = 3$. The recursive formula for $T$ is $T_n = T_{n-1} + 2 * T_{n-2}$.

The reasoning is rather simple.

If the last tile of $2 * n$ board is a vertical bar, then the amount of possible tilings is exactly the same as $T_{n-1}$, because we can just ignore the last domino.

If the last two dominoes are a horizontal bar, then the amount is exctly $T_{n-2}$. This also happens when the last domino is a 2x2 rectangle, hence the formula $T_n = T_{n-1} + 2 * T_{n-2}$.

Suppose we had another 2x2 rectangle with a dot in the middle to distinguish it from the other 2x2 rectangle. Then the amount of the possible tilings would be $T_n = T_{n-1} + 3 * T_{n-2}$. We could also assume a 3xN tile, etc... - you get the drill.

c)

Prove the validity of the formula:

$$T_n = \frac{1}{3}\left[2^{n+1} + (-1)^n\right]$$

*Base case*:

$$T_1 = \frac{1}{3}\left[2^2 - 1\right] = \frac{1}{3} * 3 = 1$$

Which is consistent with our previous finding :).

*Strong inductive case*: Assume, that $T_n = \frac{1}{3}\left[2^{n+1} + (-1)^n\right]$ holds for $n$ from $1$ to $k$. Prove that the same formula for $T_{k+1}$ also holds.

Use the recursive formula for $T_{n+1}$, that is:

8

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\begin{aligned}
T_{n+1} = T_n + 2 * T_{n-1} &= \frac{1}{3}\left[2^{n+1} + (-1)^n\right] + \frac{2}{3}\left[2^n + (-1)^{n-1}\right] \\
&= \frac{1}{3} * 2^{n+1} + \frac{1}{3} * (-1)^n + \frac{2}{3} * 2^n + \frac{2}{3} * (-1)^{n-1} \\
&= \frac{1}{3} * 2^{n+1} + \frac{1}{3} * 2^{n+1} + \frac{1}{3} * (-1)^n + \frac{2}{3} * (-1)^{n-1} \\
&= \frac{1}{3} * 2^{n+2} + \frac{1}{3} * (-1)^n - \frac{2}{3} * (-1)^n \\
&= \frac{1}{3} * 2^{n+2} - \frac{1}{3} * (-1)^n \\
&= \frac{1}{3}\left[2^{n+2} + (-1)^{n+1}\right]
\end{aligned}
$$

Which completes the proof.

## Problem 3

a)

There are exactly 6 different results of computing $n \bmod 6$ (0, 1, 2, 3, 4, 5).
By the pigeonhole principle, in a collection of 7 integers, we're placing 7 objects into 6 pigeonholes (results of remainder by 6), meaning that one pigeonhole will have an extra element, so there must be at least two numbers with the same remainder when divided by 6. Call this remainder $k$. The numbers must be in form $6n + k$ and $6m + k$ for integer $m, n, k$. Their difference is $6(n - m)$, thus it's divisible by $6$.

b)

The area of a square with side $n$ has the area of $n^2$. If, without loss of generality, we distribute $n^2 + 1$ points on the entire square at points between $(a - 1, b - 1)$ and $(a, b)$ such that $a, b \in \{1, 2, .., n\}$, meaning that we put each point in a 1x1 box, by the pigeonhole principle at least one point will end up sharing a box with some other point, meaning that their distance will be under $\sqrt{2}$.

12

c)

There are precisely 50 pairs of numbers from 1 to 100 that sum to $101$: $100 + 1, 99 + 2$, etc...
Because we're provided 51 numbers from this interval, we must eventually pick two elements that together form one of the pairs above by the pigeonhole principle, since we have 50 pigeonholes and 51 pigeons.

## Problem 4

Suppose that there are $n$ prime numbers. Thus, the set of prime numbers would be $\mathbb{P} = \{p_1, p_2, \ldots, p_n\}$.
The problem statement lets us assume that a number is either prime or a product of prime numbers.
Consider a number:

$$k = 1 + \prod_{p \in \mathbb{P}} p$$

5

$k$ is either:

- a prime, which is not present in $\mathbb{P}$.
- a composite number itself. Then, it must be divisible by some prime number, which can't be present in $\mathbb{P}$

Proving that there's at least one prime number missing from $\mathbb{P}$, thus the set of prime numbers is infinite.