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Exercise sheet 3 @ MfCS 1

Problem 1

Define subtraction as $a - b = a + (-b)$ and division $a/b = a * b^{-1}, b \neq 0$ in a field $(K, +, *)$.

Lemma 1

$(a * b)^{-1} = a^{-1} * b^{-1}$.
Proof:
Take the inverse element definition, $a * a^{-1} = e$ and substitute $a * b$:
$(a * b) * (a * b)^{-1} = e$.
Use the associativity rule and multiply both sides by a^{-1}. Notice how $e * a^{-1} = a^{-1}$ on the right and we use associativity and commutativity to cancel out a and a^{-1} on the LHS:
$b * (a * b)^{-1} = a^{-1}$.
Apply the same rule for b:
$(a * b)^{-1} = a^{-1} * b^{-1}$.

✓

Lemma 2

$a/b = (a * c)/(b * c)$ for some c being an element of K.
Proof:

$$a/b = (a * c)/(b * c)$$

Substitute the definitions and apply the law of associativity of multiplication on the righthand side:

$$a * b^{-1} = a * (c * (b * c)^{-1})$$

Use Lemma 1:

$$a * b^{-1} = a * (c * (b^{-1} * c^{-1}))$$

Use associativity and inverse element definition:

$$a * b^{-1} = a * (b^{-1} * (c * c^{-1}))$$

$$a * b^{-1} = a * (b^{-1} * e)$$

Use the existence of multiplicative identity law stating that $a * e = a$.

$$a * b^{-1} = a * b^{-1}$$

✓ LHS=RHS, QED.

Lemma 3

Prove that $a^{-1^{-1}} = a$.
Proof:
Multiply sides of the definition of inverse element by $a^{-1^{-1}}$.

$(a * a^{-1}) * a^{-1^{-1}} = e * a^{-1^{-1}}$.
Use the associative property:
$a * (a^{-1} * a^{-1^{-1}}) = e * a^{-1^{-1}}$.
Use the definition of identity element:
$a * e = e * a^{-1^{-1}}$.
Use the definition of identity element on both sides:
$a = a^{-1^{-1}}$.
QED.

Proof 1

Suppose that a, b, c, d are elements of K and $b, d \neq 0$.
Prove that $a/b - c/d = (a * d - b * c)/(b * d)$.
Proof:
Use lemma 2 on the LHS:

$$(a * d)/(b * d) - (b * c)/(b * d) = (a * d - b * c)/(b * d)$$

Use the definition of subtraction and division.

$$(a * d) * (b * d)^{-1} + (-(b * c) * (b * d)^{-1}) = (a * d + (-b * c)) * (b * d)^{-1}$$

Use the distributive law:

$$((a * d) + (-b * c)) * (b * d)^{-1} = (a * d + (-b * c)) * (b * d)^{-1}$$

QED.

Proof 2

Prove that $(a/b)/(d/c) = (a * c)/(b * d)$.
Proof:
Use the definition of division.

$$(a * b^{-1}) * (d * c^{-1})^{-1} = (a * c) * (b * d)^{-1}$$

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Use lemma 1 and lemma 3:

$$(a * b^{-1}) * d^{-1} * c = (a * c) * (b^{-1} * d^{-1})$$

Reorder the terms (associative property):

$$(a * c) * (b^{-1} * d^{-1}) = (a * c) * (b^{-1} * d^{-1})$$

Problem 2

Part 1

Prove that $x * e = x$ for all $x \in X$, given that (Ax1) the element $e \in X$ satisfies $e * x = x$ and the operation $*$ is associative.
Proof:

Suppose $x * e \neq x$ is true.
Multiply sides by x.
$(x * e) * x \neq x * x$.
Use associativity.
$x * (e * x) \neq x * x$.
Use Ax1:
$x * x \neq x * x$
Contradiction. Thus, $x * e = x$ must hold.

Part 2

Prove that $x * x^{-1} = e$, given that (Ax2) for each $x \in X$ there exists element x^{-1} with $x^{-1} * x = e$ and the operation $*$ is assocative.
Proof:
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Suppose $x * x^{-1} \neq e$ is true.
Multiply sides by x:
$(x * x^{-1}) * x \neq e * x$.
Use associativity:
$x * (x^{-1} * x) \neq e * x$.
Use Ax2:
$x * e \neq e * x$.
Use Ax1:
$x * e \neq x$.
Use part 1 result:
$x \neq x$.
Contradiction. Thus, $x * x^{-1} = e$ must hold.

Problem 3

A) $x^2 - 1$ is in the denominator, thus $x \neq -1 \wedge x \neq 1$.

$$\frac{4x-5}{x^2-1} < 5$$

If $x^2 - 1 < 0$, that is, $x^2 < 1$, so $x > -1 \wedge x < 1$:

$$4x - 5 > 5(x^2 - 1) \Leftrightarrow 4x - 5 > 5x^2 - 5 \Leftrightarrow 4x > 5x^2$$

If $x = 0$, then the inequality is false. If $x > 0$ then $4 > 5x$ is true when $x < \frac{4}{5}$. Conversely, if $x < 0$ then $4 < 5x$ is true when $x > \frac{4}{5}$, but this never holds, because x can not be smaller than one and greater than 0.8 simultaineously. Thus, this branch tells us that the inequality holds when $x \in (0, \frac{4}{5})$.
If $x^2 - 1 > 0$, so $x^2 > 1$, meaning that $x > 1 \vee x < -1$:

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$$4x - 5 < 5(x^2 - 1) \Leftrightarrow 4x - 5 < 5x^2 - 5 \Leftrightarrow 4x < 5x^2$$

If $x > 0$, then $4 < 5x$ which is true when $x > \frac{4}{5}$. If $x < 0$, then $4 > 5x$, which holds when $x < \frac{4}{5}$, meaning that the inequality in this branch holds when $x \in (-\infty, 0) \cup (\frac{4}{5}, \infty)$. To account for the domain we obtianed by solving $x^2 - 1 > 0$, we know that the equation holds when $x \in (-\infty, -1) \cup (1, \infty)$.

Thus, the entire equation holds when $x \in (-\infty, -1) \cup (0, \frac{4}{5}) \cup (1, \infty)$.

B) The denominators are $5x - 1$ and $2x + 1$, meaning that $x \neq \frac{1}{5}$ and $x \neq -\frac{1}{2}$.

$$\frac{5}{5x-1} < \frac{2}{2x+1}$$

$$\frac{5}{5x-1} - \frac{2}{2x+1} < 0$$

$$\frac{5*(2x+1)}{(2x+1)*(5x-1)} - \frac{2*(5x-1)}{(2x+1)*(5x-1)} < 0$$

$$\frac{10x+5}{(2x+1)*(5x-1)} - \frac{10x-2}{(2x+1)*(5x-1)} < 0$$

$$\frac{3}{(2x+1)*(5x-1)} < 0$$

$$\frac{3}{(x-\frac{1}{5})*(x+\frac{1}{2})} < 0$$

The denominator of this equation is a parabola with zeroes in $-\frac{1}{2}$ and $\frac{1}{5}$. The a coefficient of the parabola is positive (suppose we multiplied this out to a $ax^2 + bx + c$ form, a would be equal to one), thus $(x-\frac{1}{5})*(x+\frac{1}{2}) > 0 \Leftrightarrow x \in (-\infty, -\frac{1}{2}) \cup (\frac{1}{5}, \infty)$, $(x-\frac{1}{5})*(x+\frac{1}{2}) < 0 \Leftrightarrow x \in (-\frac{1}{2}, \frac{1}{5})$, because the positive x^2 term will eventually dominate the x term and the vertex of the parabola is in its global minimum, meaning that the function is negative between the roots, zero in the roots, and positive everywhere else. This result is important and I will use it multiple times in the following problems.

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If we multiply sides by the denominator, we get either $3 < 0$, which never holds, or $3 > 0$ which always holds. To accomplish the latter, the denominator must be negative to flip the sign, thus $x \in (-\frac{1}{2}, \frac{1}{5})$.

C) The denominators are $2x + 3$ and $x + 1$, so $x \neq -1$ and $x \neq -\frac{3}{2}$.

$$\frac{3x+2}{2x+3} < \frac{x}{x+1}$$

$$\frac{3x+2}{2x+3} - \frac{x}{x+1} < 0$$

$$\frac{(3x+2)*(x+1)}{(2x+3)*(x+1)} - \frac{(2x+3)*x}{(2x+3)*(x+1)} < 0$$

$$\frac{(3x+2)*(x+1) - (2x+3)*x}{(2x+3)*(x+1)} < 0$$

$$\frac{3x^2+5x+2-2x^2-3x}{(2x+3)*(x+1)} < 0$$

$$\frac{x^2+2x+2}{(2x+3)*(x+1)} < 0$$

The quadratic function in the numerator is always greater than zero (since the discriminant is less than zero and the a coefficient is positive). Thus we can divide the sides by it.

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$$\frac{1}{(2x+3)*(x+1)} < 0$$

The quadratic function in the denominator has roots -1 and -1.5. We proceed analogically to the last step in point B, because this quadratic function has the same properties as the one I've discussed previously. This approach tells us that $x \in (-1.5, -1)$.

D)

$$\left|\frac{(x-1)(2x-3)}{x(x-5)}\right| > 1$$

$$\frac{(x-1)(2x-3)}{x(x-5)} > 1 \vee \frac{(x-1)(2x-3)}{x(x-5)} < -1$$

$$\frac{(x-1)(2x-3)-x(x-5)}{x(x-5)} > 0 \vee \frac{(x-1)(2x-3)+x(x-5)}{x(x-5)} < 0$$

$$\frac{(2x^2-5x+3)-(x^2-5x)}{x(x-5)} > 0 \vee \frac{(2x^2-5x+3)+(x^2-5x)}{x(x-5)} < 0$$

$$\frac{x^2+3}{x(x-5)} > 0 \vee \frac{3x^2-10x+3}{x(x-5)} < 0$$

x^2+3 has no roots and it's always positive so we can divide sides by it.

$$\frac{1}{x(x-5)} > 0 \vee \frac{3x^2-10x+3}{x(x-5)} < 0$$

Factor the quadratic function on the right.
I can do it in my memory, and my approximate chain of reasoning follows: clearly it has a root in 3 ($3*9-30+3 = 30-30 = 0$), so the result is in the form $(ax-b)(x-3)$, but a must be equal to 3 (to yield $3x^2$ in the end), and b must be equal to -1 (because $-1*-3 = 3$ as in the original polynomial).

$$\frac{1}{x(x-5)} > 0 \vee \frac{(3x-1)(x-3)}{x(x-5)} < 0$$

The quadratic function on the left must be greater than zero for the inequality to hold. $x(x-5)$ obviously has zeroes in 0 and 5, and because the coefficient next to x^2 in x^2-5x is greater than zero, the quadratic function is greater than zero unless $x \in [0,5]$. Speaking of the right side, we will assume that $x \in (0,5)$ for efficiency: if x is outside of this interval, then it has already been covered in the previous case. This means that the quadratic function in the denominator is negative, so multiplying sides by it changes the type of inequality:

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$$x < 0 \vee x > 5 \vee (3x-1)(x-3) > 0$$

Clearly the polynomial on the right has roots in $\frac{1}{3}$ and 3. The a coefficient of this quadratic function is positive, thus the function is greater than zero outside of the $[\frac{1}{3}, 3]$ interval. From this, we can deduce that the initial inequality is true when $x \in (\frac{1}{3}, 0) \cup (0, 3)$ (we need to exclude 0 from the results as it would yield in division by zero in the initial inequality).

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E)

$$\log(\frac{2-x}{12+4x}) > 0$$

Since e^x is monotonically increasing, because $\frac{d}{dx}e^x = e^x$ and $\forall_{x>0} e^x > 0$, we can e^x both sides without loss of generality:

$$\frac{2-x}{12+4x} > 1$$

$$\frac{2-x}{12+4x} - \frac{12+4x}{12+4x} > 0$$

$$\frac{2-x-12-4x}{12+4x} > 0$$

$$\frac{-5x-10}{4x+12} > 0$$

$4x+12$ has a root in $x=-3$. If $x < -3$ then:

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$$-5x - 10 < 0$$

If $x > -3$ then:

$$-5x - 10 > 0$$

$y = -5x - 10$ has a root in -2. Thus, if $x < -2$ then $y > 0$, if $x > -2$ then $y < 0$.
Thus, the equation holds if $x < -3$ and $x > -2$ (first case), or $x > -3$ and $x < -2$ (second case).
The first case always resolves to $x \in \emptyset$ (no overlap) and the second case resolves to $x \in (-3, -2)$.
F)

$$e^x > 3^{x^2}$$

Since $\log x$ is monotonically increasing, because $\frac{d}{dx}\log x = \frac{1}{x}$ and $\forall_{x>0} \frac{1}{x} > 0$, we can $\log x$ both sides:

$$x > \log 3^{x^2}$$

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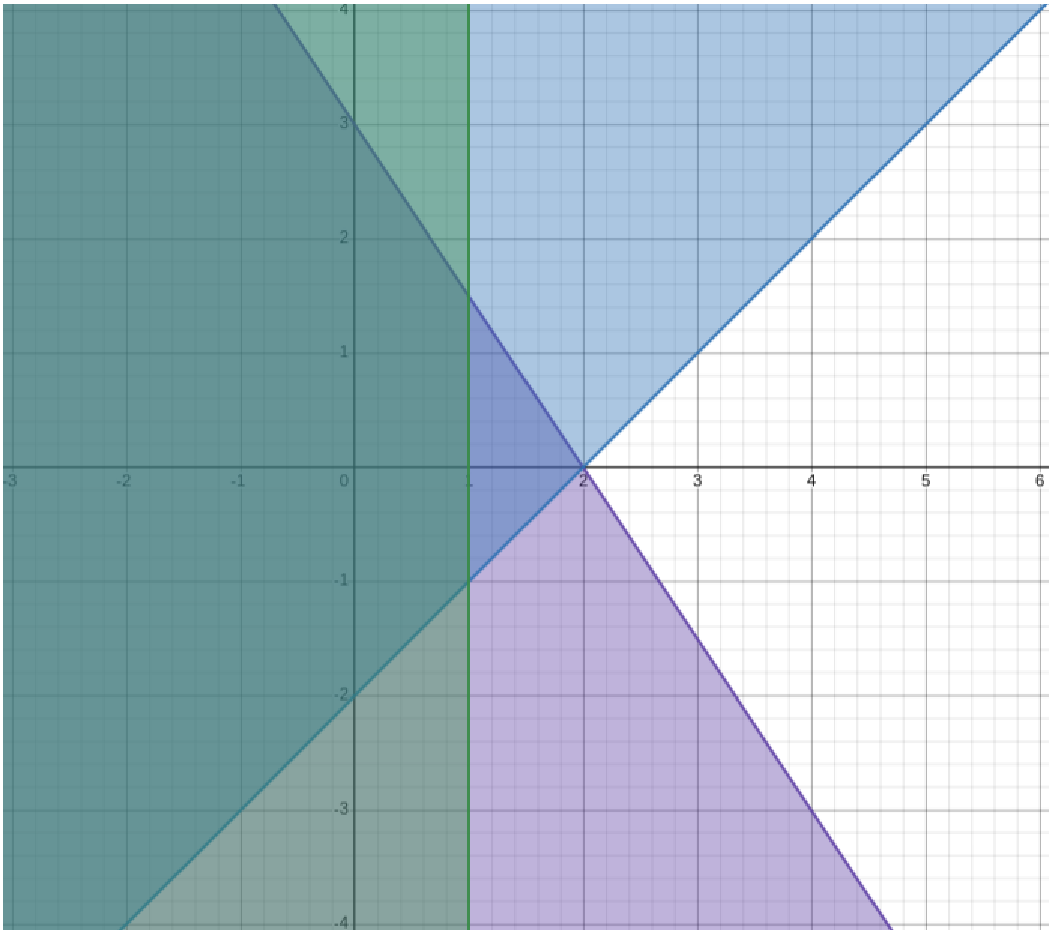
$$x > x^2 \log 3$$

If $x = 0$ then the inequality is false. If $x > 0$ then $1 > x \log 3 \Leftrightarrow \frac{1}{\log 3} > x$, if $x < 0$ then $1 < x \log 3 \Leftrightarrow \frac{1}{\log 3} < x$.
Thus, $x \in (0, \frac{1}{\log 3})$ can be dediced from the first case and the second case is contradictionary (because $0 < \frac{1}{\log 3}$).

Problem 4

A) First two of these are linear inequalities in disguise, so we can plot a linear function and then cover the area above/under the graph.
$3x + 2y \leq 6 \Leftrightarrow 2y \leq 6 - 3x \Leftrightarrow y \leq 3 - 1.5x.$
$x - y \leq 2 \Leftrightarrow -y \leq 2 - x \Leftrightarrow y \geq x - 2.$
The last one marks all x lesser or equal to one. We can plot all of these sets now (purple, blue and green are the functions respectively).



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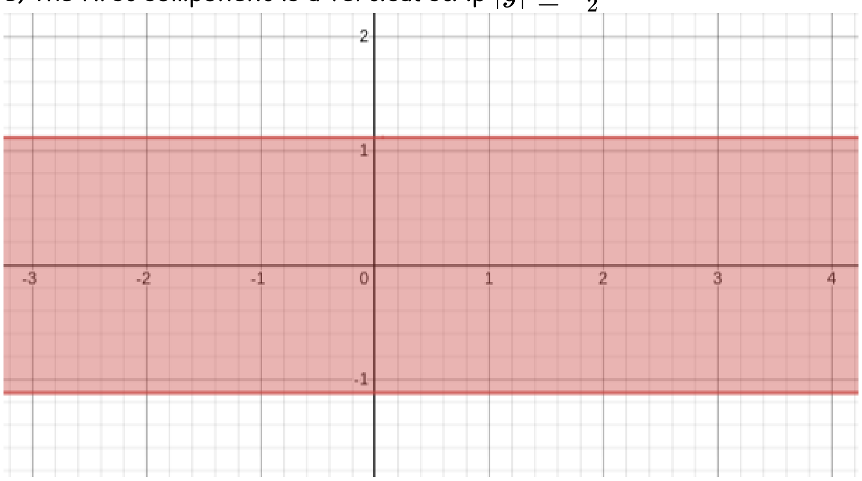
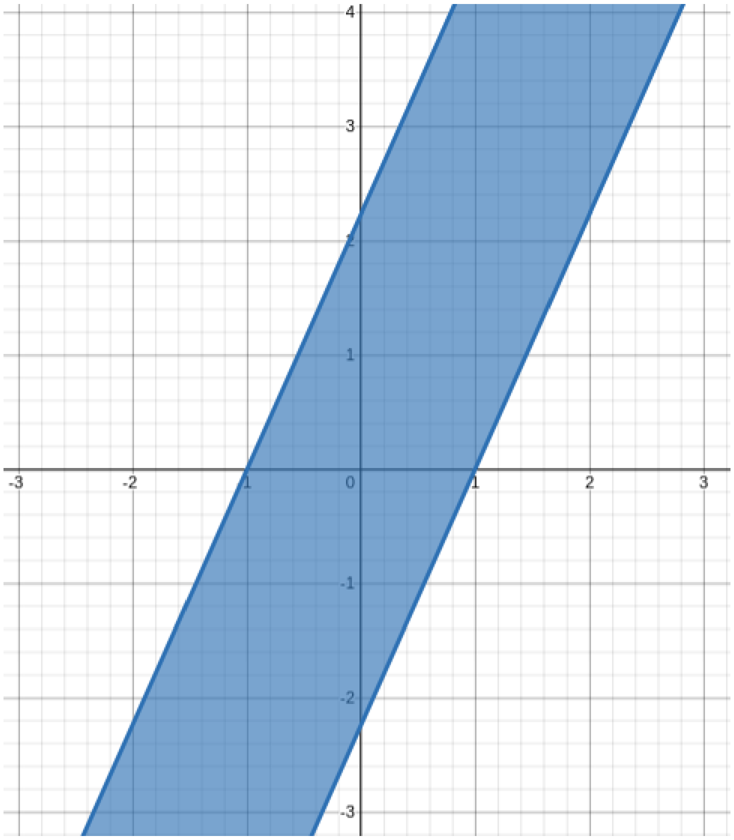
B) The first component is a vertical strip $|y| \leq \frac{\sqrt{5}}{2}$:

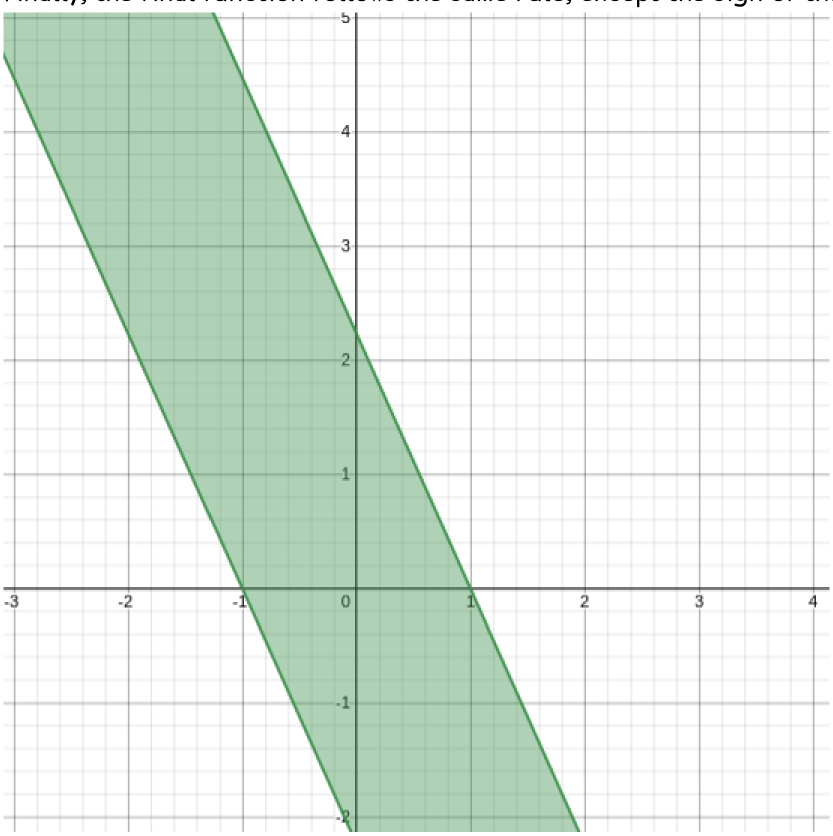


The reasoning is that for $|y| < c$ to hold, $-c < y < c$ must hold.

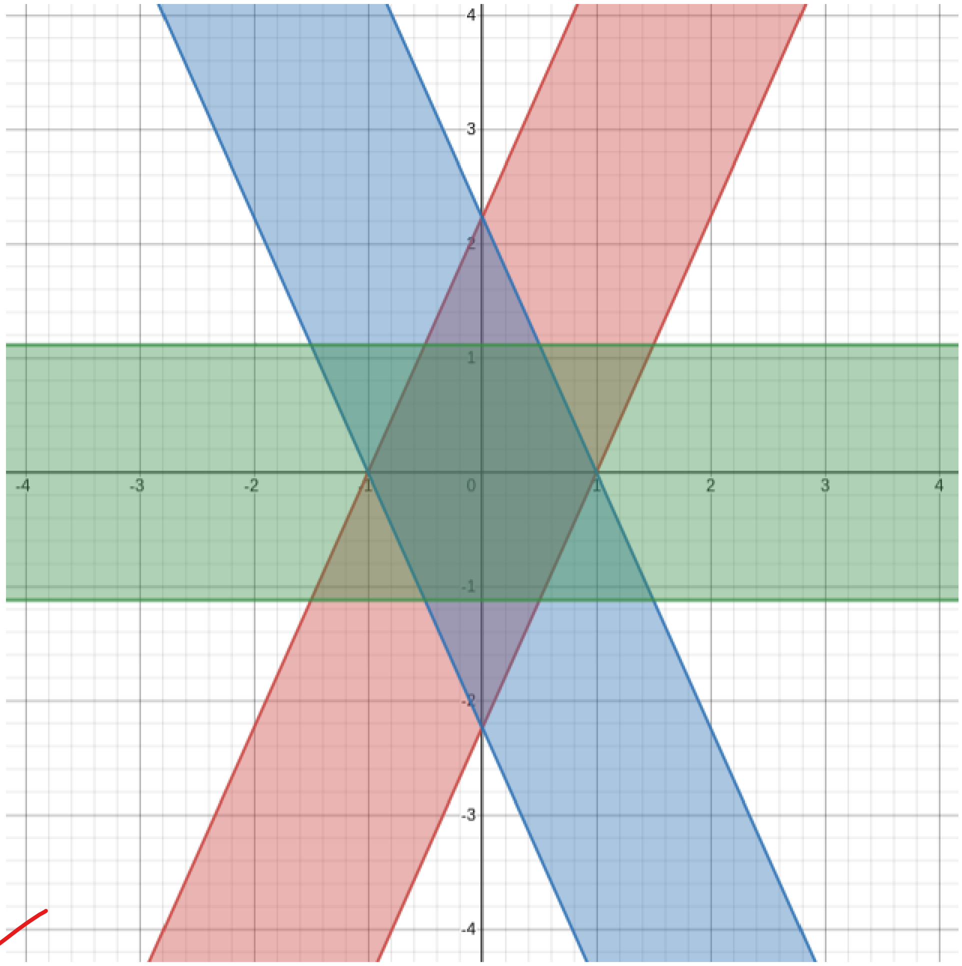
$\left|y - \sqrt{5}x\right| \leq \sqrt{5}$ is a skewed strip, because $-\sqrt{5} \leq y - \sqrt{5}x \leq \sqrt{5}$, so in essence we mark the area between two linear functions: $y - \sqrt{5}x = \pm\sqrt{5} \Leftrightarrow y = \sqrt{5}x \pm \sqrt{5}$.



Finally, the final function follows the same rule, except the sign of the x term is swapped, so the function we get is



An union of all these sets follows:



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