Solved Problems

4.1 Derive the transformation that rotates an object point $\theta°$ about the origin. Write the matrix representation for this rotation.

SOLUTION

Refer to Fig. 4-13. Definition of the trigonometric functions sin and cos yields

$$x' = r\cos(\theta + \phi) \qquad y' = r\sin(\theta + \phi)$$

and

$$x = r\cos\phi \qquad y = r\sin\phi$$

Using trigonometric identities, we obtain

$$r\cos(\theta + \phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi) = x\cos\theta - y\sin\theta$$

and

$$r\sin(\theta + \phi) = r(\sin\theta\cos\phi + \cos\theta\sin\phi) = x\sin\theta + y\cos\theta$$

or

$$x' = x\cos\theta - y\sin\theta \qquad y' = x\sin\theta + y\cos\theta$$

Writing $P' = \begin{pmatrix} x' \\ y' \end{pmatrix}$, $P = \begin{pmatrix} x \\ y \end{pmatrix}$, and

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

we can now write $P' = R_\theta \cdot P$.

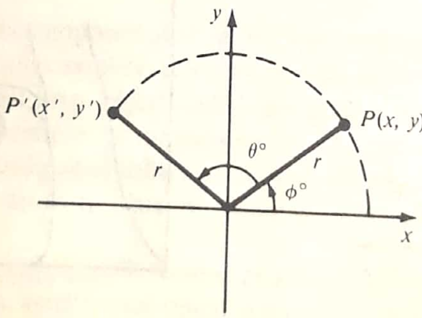

Fig. 4-13

4.2 (*a*) Find the matrix that represents rotation of an object by 30° about the origin.
(*b*) What are the new coordinates of the point $P(2, -4)$ after the rotation?

SOLUTION

(*a*) From Prob. 4.1:

$$R_{30°} = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(*b*) So the new coordinates can be found by multiplying:

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} \sqrt{3}+2 \\ 1-2\sqrt{3} \end{pmatrix}$$

4.3 Describe the transformation that rotates an object point, $Q(x, y)$, θ degrees about a fixed center of rotation $P(h, k)$ (Fig. 4-14).

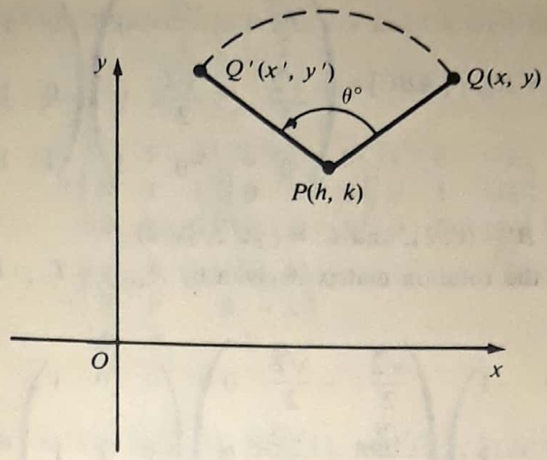

Fig. 4-14

SOLUTION

We determine the transformation $R_{\theta,P}$ in three steps: (1) translate so that the center of rotation P is at the origin, (2) perform a rotation of θ degrees about the origin, and (3) translate the origin back to P. Using $\mathbf{v} = h\mathbf{I} - k\mathbf{J}$ as the translation vector, we build $R_{\theta,P}$ by composition of transformations:

$$R_{\theta,O'} = T_{-\mathbf{v}} \cdot R_\theta \cdot T_{\mathbf{v}}$$

4.4 Write the general form of the matrix for rotation about a point $P(h, k)$.

SOLUTION

Following Prob. 4.3, we write $R_{\theta,P} = T_{-\mathbf{v}} \cdot R_\theta \cdot T_{\mathbf{v}}$, where $\mathbf{v} = -h\mathbf{I} - k\mathbf{J}$. Using the 3×3 homogeneous coordinate form for the rotation and translation matrices, we have

$$R_{\theta,P} = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta) & -\sin(\theta) & [-h\cos(\theta) + k\sin(\theta) + h] \\ \sin(\theta) & \cos(\theta) & [-h\sin(\theta) - k\cos(\theta) + k] \\ 0 & 0 & 1 \end{pmatrix}$$

4.5 Perform a 45° rotation of triangle $A(0, 0)$, $B(1, 1)$, $C(5, 2)$ (a) about the origin and (b) about $P(-1, -1)$.

SOLUTION

We represent the triangle by a matrix formed from the homogeneous coordinates of the vertices:

$$\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

(a) The matrix of rotation is

$$R_{45^\circ} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the coordinates $A'B'C'$ of the rotated triangle ABC can be found as

$$[A'B'C'] = R_{45°} \cdot [ABC] = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{array}{c} \begin{matrix} A' & B' & C' \end{matrix} \\ \begin{pmatrix} 0 & 0 & \frac{3\sqrt{2}}{2} \\ 0 & 2 & \frac{7\sqrt{2}}{2} \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

Thus $A' = (0, 0)$, $B' = (0, 2)$, and $C' = (\frac{3}{2}\sqrt{2}, \frac{7}{2}\sqrt{2})$.

(*b*) From Prob. 4.4, the rotation matrix is given by $R_{45°,P} = T_{-\mathbf{v}} \cdot R_{45°} \cdot T_{\mathbf{v}}$, where $\mathbf{v} = \mathbf{I} + \mathbf{J}$. So

$$R_{45°,P} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & (\sqrt{2}-1) \\ 0 & 0 & 1 \end{pmatrix}$$

Now

$$[A'B'C'] = R_{45°,P} \cdot [ABC]$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & (\sqrt{2}-1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & (3\sqrt{2}-1) \\ (\sqrt{2}-1) & (2\sqrt{2}-1) & \left(\frac{9}{2}\sqrt{2}-1\right) \\ 1 & 1 & 1 \end{pmatrix}$$

So $A' = (-1, \sqrt{2}-1)$, $B' = (-1, 2\sqrt{2}-1)$, and $C' = (3\sqrt{2}-1, \frac{9}{2}\sqrt{2}-1)$.

4.6 Find the transformation that scales (with respect to the origin) by (*a*) a units in the X direction, (*b*) b units in the Y direction, and (*c*) simultaneously a units in the X direction and b units in the Y direction.

SOLUTION

(*a*) The scaling transformation applied to a point $P(x, y)$ produces the point (ax, y). We can write this in matrix form as $S_{a,1} \cdot P$, or

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ y \end{pmatrix}$$

(*b*) As in part (*a*), the required transformation can be written in matrix form as $S_{1,b} \cdot P$. So

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ by \end{pmatrix}$$

(*c*) Scaling in both directions is described by the transformation $x' = ax$ and $y' = by$. Writing this in matrix form as $S_{a,b} \cdot P$, we have

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

4.7 Write the general form of a scaling matrix with respect to a fixed point $P(h, k)$.

SOLUTION

Following the same general procedure as in Probs. 4.3 and 4.4, we write the required transformation with $\mathbf{v} = -h\mathbf{I} - k\mathbf{J}$ as

$$S_{a,b,P} = T_{-\mathbf{v}} \cdot S_{a,b} \cdot T_{\mathbf{v}}$$

$$= \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & l \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & -ah + h \\ 0 & b & -bk + k \\ 0 & 0 & 1 \end{pmatrix}$$

4.8 Magnify the triangle with vertices $A(0, 0)$, $B(1, 1)$, and $C(5, 2)$ to twice its size while keeping $C(5, 2)$ fixed.

SOLUTION

From Prob. 4.7, we can write the required transformation with $\mathbf{v} = -5\mathbf{I} - 2\mathbf{J}$ as

$$S_{2,2,C} = T_{-\mathbf{v}} \cdot S_{2,2} \cdot T_{\mathbf{v}}$$

$$= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

3x3 3x3 3x1

Representing a point P with coordinates (x, y) by the column vector $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, we have

$$S_{2,2,C} \cdot A = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

$$S_{2,2,C} \cdot B = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$$S_{2,2,C} \cdot C = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

So $A' = (-5, -2)$, $B' = (-3, 0)$, and $C' = (5, 2)$. Note that since the triangle ABC is completely determined by its vertices, we could have saved much writing by representing the vertices using a 3×3 matrix

$$[ABC] = \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

and applying $S_{2,2,C}$ to this. So

$$S_{2,2,C} \cdot [ABC] = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -3 & 5 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} = [A'B'C']$$

4.9 Describe the transformation M_L which reflects an object about a line L.

SOLUTION

Let line L in Fig. 4-15 have a y intercept $(0, b)$ and an angle of inclination θ degrees (with respect to the x axis). We reduce the description to known transformations:

1. Translate $(0, b)$ to the origin.
2. Rotate by $-\theta$ degrees so that line L aligns with the x axis.
3. Mirror-reflect about the x axis.
4. Rotate back by θ degrees.
5. Translate the origin back to the point $(0, b)$.

In transformation notation, we have

$$M_L = T_{-\mathbf{v}} \cdot R_\theta \cdot M_x \cdot R_{-\theta} \cdot T_{\mathbf{v}}$$

where $\mathbf{v} = -b\mathbf{J}$.

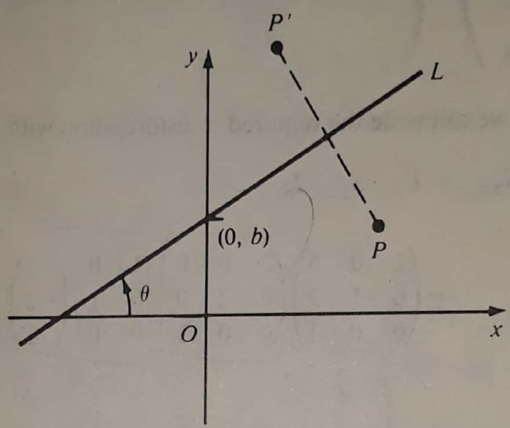

Fig. 4-15

4.10 Find the form of the matrix for reflection about a line L with slope m and y intercept $(0, b)$.

SOLUTION

Following Prob. 4.9 and applying the fact that the angle of inclination of a line is related to its slope m by the equation $\tan(\theta) = m$, we have with $\mathbf{v} = -b\mathbf{J}$,

$$M_L = T_{-\mathbf{v}} \cdot R_\theta \cdot M_x \cdot R_{-\theta} \cdot T_{\mathbf{v}},$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

Now if $\tan(\theta) = m$, standard trigonometry yields $\sin(\theta) = m/\sqrt{m^2+1}$ and $\cos(\theta) = 1/\sqrt{m^2+1}$. Substituting these values for $\sin(\theta)$ and $\cos(\theta)$ after matrix multiplication, we have

$$M_L = \begin{pmatrix} \dfrac{1-m^2}{m^2+1} & \dfrac{2m}{m^2+1} & \dfrac{-2bm}{m^2+1} \\ \dfrac{2m}{m^2+1} & \dfrac{m^2-1}{m^2+1} & \dfrac{2b}{m^2+1} \\ 0 & 0 & 1 \end{pmatrix}$$

4.11 Reflect the diamond-shaped polygon whose vertices are $A(-1,0)$, $B(0,-2)$, $C(1,0)$, and $D(0,2)$ about (*a*) the horizontal line $y=2$, (*b*) the vertical line $x=2$, and (*c*) the line $y=x+2$.

SOLUTION

We represent the vertices of the polygon by the homogeneous coordinate matrix

$$V=\begin{pmatrix}-1 & 0 & 1 & 0\\ 0 & -2 & 0 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}$$

From Prob. 4.9, the reflection matrix can be written as

$$M_L = T_{-\mathbf{v}}\cdot R_\theta \cdot M_x \cdot R_{-\theta}\cdot T_{\mathbf{v}}$$

(*a*) The line $y=2$ has y intercept $(0,2)$ and makes an angle of 0 degrees with the x axis. So with $\theta=0$ and $\mathbf{v}=-2\mathbf{J}$, the transformation matrix is

$$M_L=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 4\\ 0 & 0 & 1\end{pmatrix}$$

This same matrix could have been obtained directly by using the results of Prob. 4.10 with slope $m=0$ and y intercept $b=0$. To reflect the polygon, we set

$$M_L\cdot V=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 4\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0 & 1 & 0\\ 0 & -2 & 0 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}=\begin{array}{c}\begin{matrix}A' & B' & C' & D'\end{matrix}\\ \begin{pmatrix}-1 & 0 & 1 & 0\\ 4 & 6 & 4 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}\end{array}$$

Converting from homogeneous coordinates, $A'=(-1,4)$, $B'=(0,6)$, $C'=(1,4)$, and $D'=(0,2)$.

(*b*) The vertical line $x=2$ has no y intercept and an infinite slope! We can use M_y, reflection about the y axis, to write the desired reflection by (1) translating the given line two units over to the y axis, (2) reflect about the y axis, and (3) translate back two units. So with $\mathbf{v}=-2\mathbf{I}$,

$$M_L = T_{-\mathbf{v}}\cdot M_y\cdot T_{\mathbf{v}},$$

$$=\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}-1 & 0 & 4\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Finally

$$M_L\cdot V=\begin{pmatrix}-1 & 0 & 4\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0 & 1 & 0\\ 0 & -2 & 0 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}=\begin{pmatrix}5 & 4 & 3 & 4\\ 0 & -2 & 0 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}$$

or $A'=(5,0)$, $B'=(4,-2)$, $C'=(3,0)$, and $D'=(4,2)$.

(*c*) The line $y=x+2$ has slope 1 and a y intercept $(0,2)$. From Prob. 4.10, with $m=1$ and $b=2$, we find

$$M_L=\begin{pmatrix}0 & 1 & -2\\ 1 & 0 & 2\\ 0 & 0 & 1\end{pmatrix}$$

The required coordinates A', B', C', and D' can now be found.

$$M_L\cdot V=\begin{pmatrix}0 & 1 & -2\\ 1 & 0 & 2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0 & 1 & 0\\ 0 & -2 & 0 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}=\begin{pmatrix}-2 & -4 & -2 & 0\\ 1 & 2 & 3 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}$$

So $A'=(-2,1)$, $B'=(-4,2)$, $C'=(-2,3)$, and $D'=(0,2)$.

4.12 The matrix $\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$ defines a transformation called a *shearing*. The special case when $b = 0$ is called *shearing in the x direction*. When $a = 0$, we have *shearing in the y direction*. Illustrate the effect of these shearing transformations on the square $A(0,0)$, $B(1,0)$, $C(1,1)$, and $D(0,1)$ when $a = 2$ and $b = 3$.

SOLUTION

Figure 4-16(*a*) shows the original square, Fig. 4-16(*b*) shows shearing in the *x* direction, Fig. 4-16(*c*) shows shearing in the *y* direction, and Fig. 4-16(*d*) shows shearing in both directions.

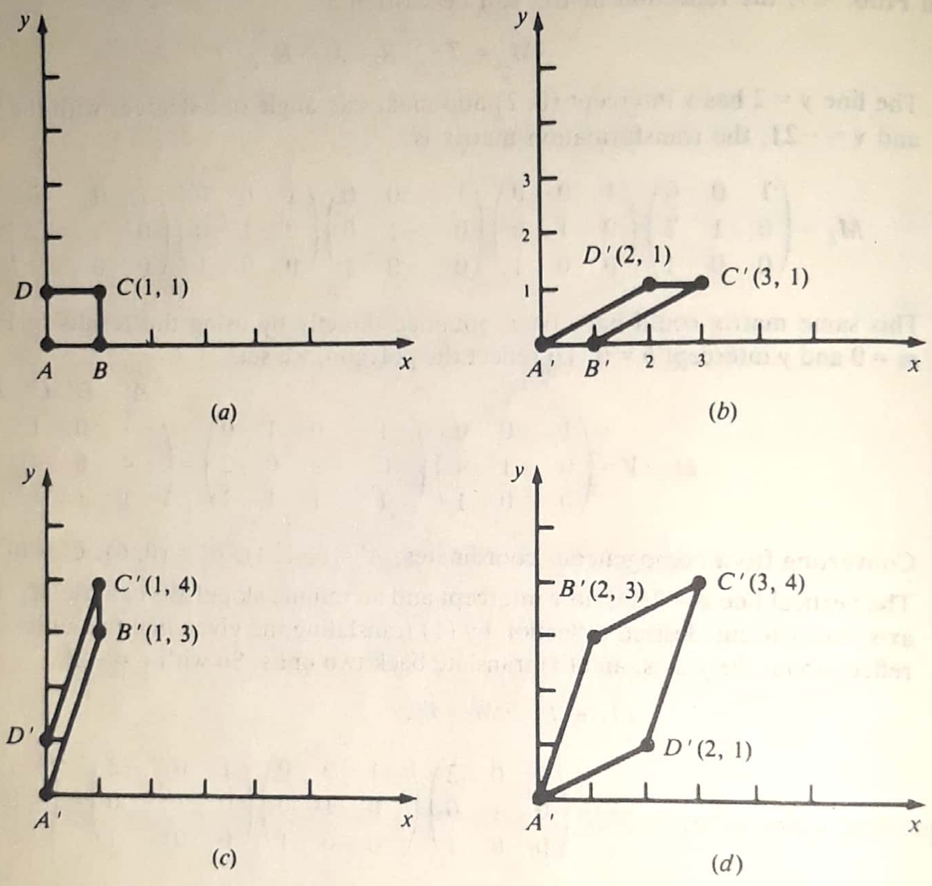

Fig. 4-16

4.13 An observer standing at the origin sees a point $P(1, 1)$. If the point is translated one unit in the direction $\mathbf{v} = \mathbf{I}$, its new coordinate position is $P'(2, 1)$. Suppose instead that the observer stepped back one unit along the x axis. What would be the apparent coordinates of P with respect to the observer?

SOLUTION

The problem can be set up as a transformation of coordinate systems. If we translate the origin O in the direction $\mathbf{v} = -\mathbf{I}$ (to a new position at O') the coordinates of P in this system can be found by the translation $\bar{T}_{\mathbf{v}}$:

$$\bar{T}_{\mathbf{v}} \cdot P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

So the new coordinates are $(2, 1)'$. This has the following interpretation: a displacement of one unit in a given direction can be achieved by either moving the object forward or stepping back from it.

4.14 An object is defined with respect to a coordinate system whose units are measured in feet. If an observer's coordinate system uses inches as the basic unit, what is the coordinate transformation used to describe object coordinates in the observer's coordinate system?

SOLUTION

Since there are 12 in to a foot, the required transformation can be described by a coordinate scaling transformation with $s = \frac{1}{12}$ or

$$\bar{S}_{1/12} = \begin{pmatrix} \dfrac{1}{1/12} & 0 \\ 0 & \dfrac{1}{1/12} \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$$

and so

$$\bar{S}_{1/12} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12x \\ 12y \end{pmatrix}$$

4.15 Find the equation of the circle $(x')^2 + (y')^2 = 1$ in terms of xy coordinates, assuming that the $x'y'$ coordinate system results from a scaling of a units in the x direction and b units in the y direction.

SOLUTION

From the equations for a coordinate scaling transformation (see pp. 83 and 85), we find

$$x' = \frac{1}{a} \cdot x \qquad y' = \frac{1}{a} \cdot y$$

Substituting, we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 .$$

Notice that as a result of scaling, the equation of the circle is transformed to the equation of an ellipse in the xy coordinate system.

4.16 Find the equation of the line $y' = mx' + b$ in xy coordinates if the $x'y'$ coordinate system results from a $90°$ rotation of the xy coordinate system.

SOLUTION

The rotation coordinate transformation equations can be written as

$$x' = x \cos(90°) + y \sin(90°) = y \qquad y' = -x \sin(90°) + y \cos(90°) = -x$$

Substituting, we find $-x = my + b$. Solving for y, we have $y = (-1/m)x - b/m$.

4.17 Find the instance transformation which places a half-size copy of the square $A(0, 0), B(1, 0), C(1, 1), D(0, 1)$ [Fig. 4-17(a)] into a master picture coordinate system so that the center of the square is at $(-1, -1)$ [Fig. 4-17(b)].

SOLUTION

The center of the square $ABCD$ is at $P(\frac{1}{2}, \frac{1}{2})$. We shall first apply a scaling transformation while keeping P fixed (see Prob. 4.7). Then we shall apply a translation that moves the center P to $P'(-1, -1)$. Taking $t_x = (-1) - (\frac{1}{2}) = -\frac{3}{2}$ and similarly $t_y = -\frac{3}{2}$ (so $\mathbf{v} = -\frac{3}{2}\mathbf{I} + -\frac{3}{2}\mathbf{J}$), we obtain

$$N_{\text{picture, square}} = T_{\mathbf{v}} \cdot S_{1/2,1/2,P} = \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{5}{4} \\ 0 & \frac{1}{2} & -\frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

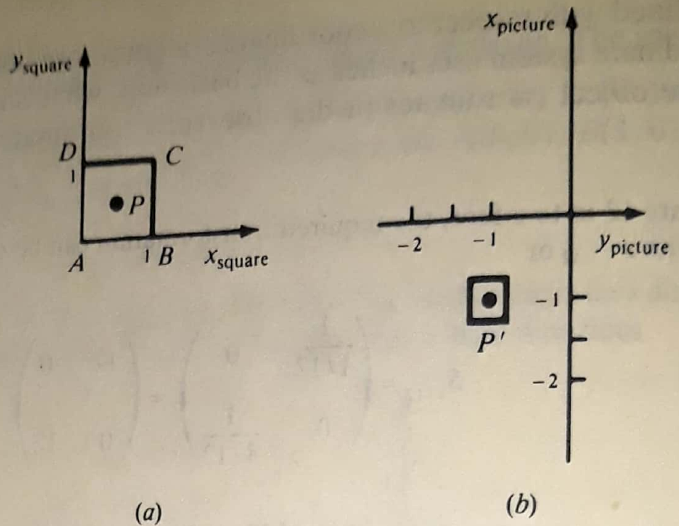

Fig. 4-17

4.18 Write the current transformation that creates the design in Fig. 4-19 from the symbols in Fig. 4-18.

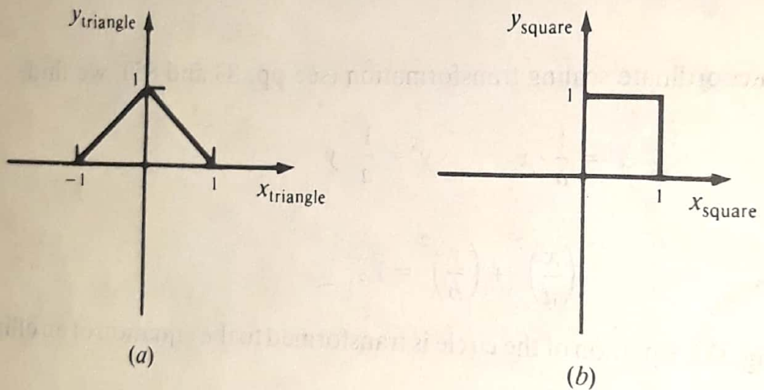

Fig. 4-18

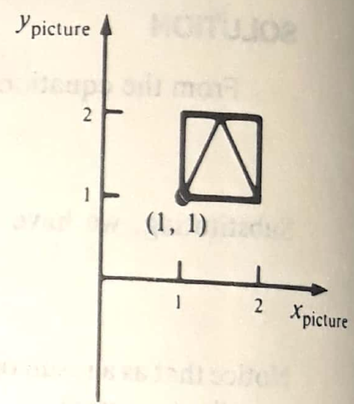

Fig. 4-19

SOLUTION

First we create an instance of the triangle [Fig. 4-18(*a*)] in the square [Fig. 4-18(*b*)]. Since the base of the triangle must be halved while keeping the height fixed at one unit, the appropriate instance transformation is the scaling $N_{\text{square, triangle}} = S_{1/2,1}$.

The instance transformation needed to place the square at the desired position in the picture coordinate system (Fig. 4-19) is a translation in the direction $\mathbf{v} = \mathbf{I} + \mathbf{J}$.

$$N_{\text{picture, square}} = T_{\mathbf{v}}$$

Then the current transformation for placing the triangle into the picture is

$$C_{\text{picture, triangle}} = N_{\text{picture, square}} \cdot N_{\text{square, triangle}}$$

and the current transformation to place the square into the picture is

$$C_{\text{picture, square}} = N_{\text{picture, square}}$$