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Lecture 21
Recursion

FIT 1008
Introduction to Computer Science

MONASH University
Information Technology

Objectives for this lecture

- To re-visit the concept of **recursive algorithm**
- To understand how to **implement** recursive algorithms.
- To be able to reason about **recursive algorithms**

Recursive algorithms

- Solve a Large problem by solving subproblems
  - ➡ of the same kind as the original.
  - ➡ simpler to solve

Each **subproblem** is solved with the **same algorithm** …

… **until** subproblems are so "<u>simple</u>" that they can be solved without further reductions (**base case**)

Candidate problems for recursion.

1. Must be possible to **decompose** them into simpler **similar problems**

2. **At some point**, the problems must become so simple that can be solved with **no further decomposition**

3. Once all subproblems are solved, the solution to the original problem can be computed by combining these solutions

General recursive structure

Simpler / Smaller

```
def solve(problem):
    if problem is simple:
        Solve problem directly
    else:
        Decompose problem into subproblems p1, p2,…
        solve(p1)
        solve(p2)
        solve(p3)…
        Combine the subsolutions to solve problem
```

Base cases

Recursive calls

That of a **function** that **calls itself** (directly or via others).

Factorial: a recursive approach?

Examples:

```
0! = 1
1! = 1
2! = 1*2
3! = 1*2*3
4! = 1*2*3*4
…
(n-1)! = 1*2*3*4*…*(n-1)
    n! = 1*2*3*4*…*(n-1)*n
```

1. Must be possible to **decompose** them into simpler **similar problems**
2. **At some point**, the problems must become so simple that can be solved with **no further decomposition**
3. Once all subproblems are solved, the solution to the original problem can be computed by combining these solutions

```
n! = (n-1)!*n
```

Subproblems: smaller, <u>converge</u> to base case.

Factorial: a recursive approach

```
def factorial(n):
    if n == 0:  # base case
        return 1
    else:
        return n*factorial(n-1) # recursive call
```

Convergence

Combination

Key idea for complexity of recursion:

- <u>How many recursive calls do we make in each version?</u>
- <u>How much work do we do per call?</u>

Terminology

Arity:

- Unary: a single recursive call (all previous code)
- Binary: two recursive calls (recursive sorts, later…)
- n-ary: n recursive calls

Direct vs Indirect recursion:

- Direct: recursive calls are calls to the same function (all previous examples)
- Indirect (or mutual): recursion through two or more methods (e.g., method p calls method q which in turn calls p again)

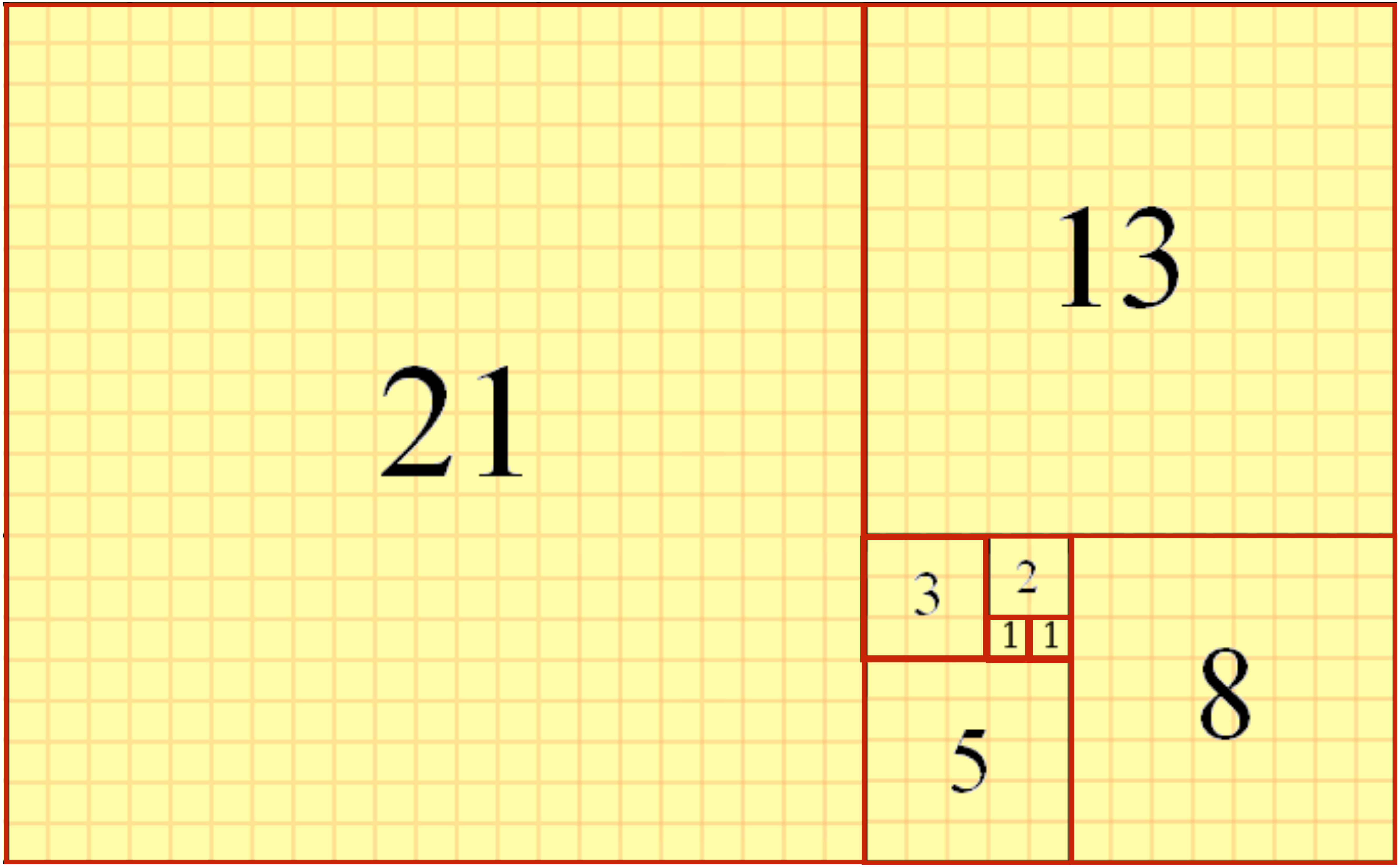
Tail-recursion:

- Where the result of the recursive call is the result of the function. Nothing is done in the “way back”. Closest to iteration - can be transformed *without “storing”*. Useful for compiler optimisation.

Tail recursive version of factorial

```python
def factorial(n):
    return factorial_aux(n,1)


def factorial_aux(n, result):
    if n == 0:
        return result
    else:
        return factorial_aux(n-1, result*n)
```


21
13
3
2
1
1
8
5

n = 0	**0**
n = 1	**1**
n = 2	**1**
n = 3	**2**
n = 4	**3**
n = 5	**5**

Fib(5)

Fibonacci

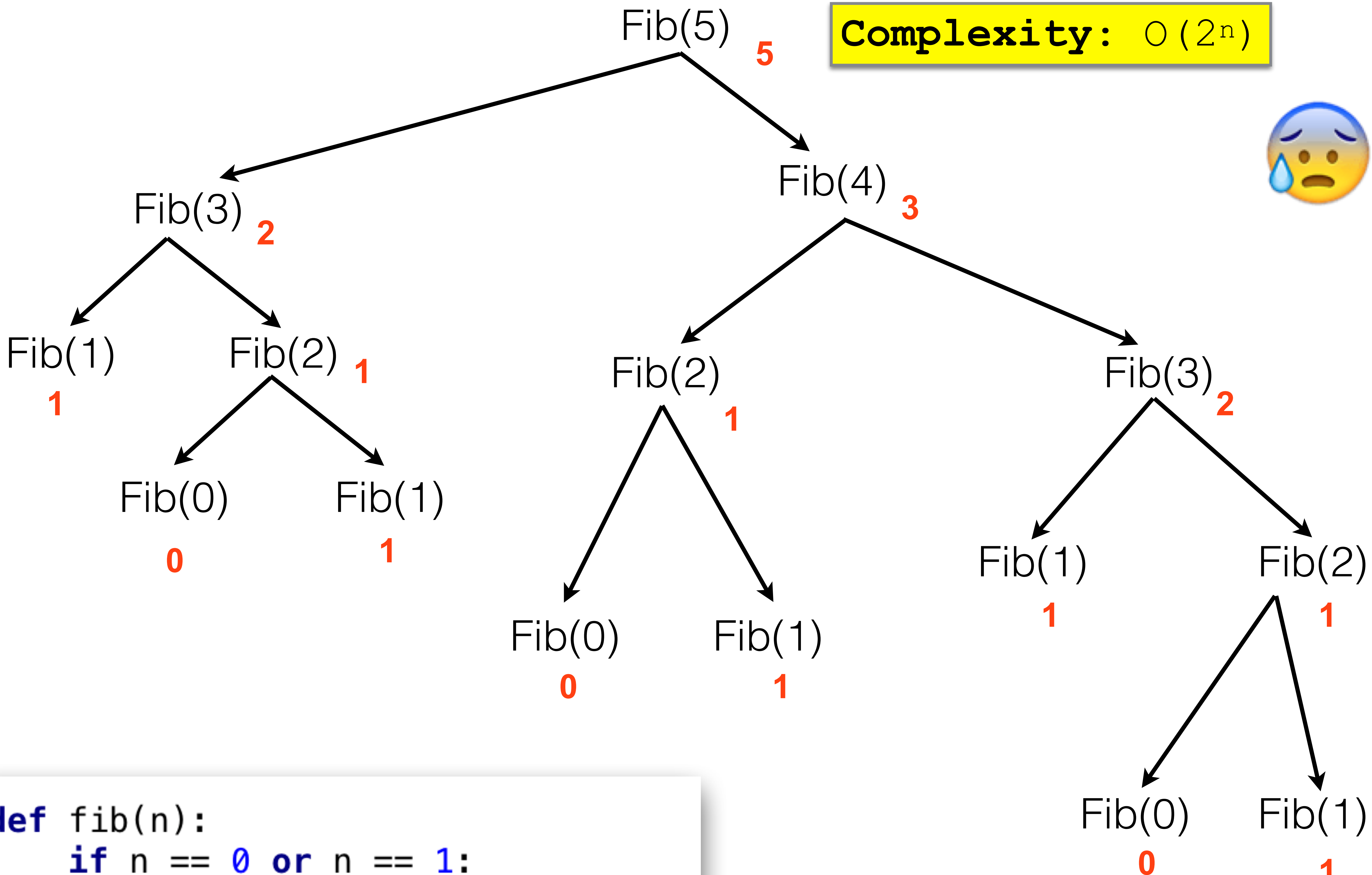
n = 0	0
n = 1	1
n = 2	1
n = 3	2
n = 4	3
n = 5	5

```python
def fib(n):
    if n == 0 or n == 1:
        return n
    else:
        return fib(n-2) + fib(n-1)
```

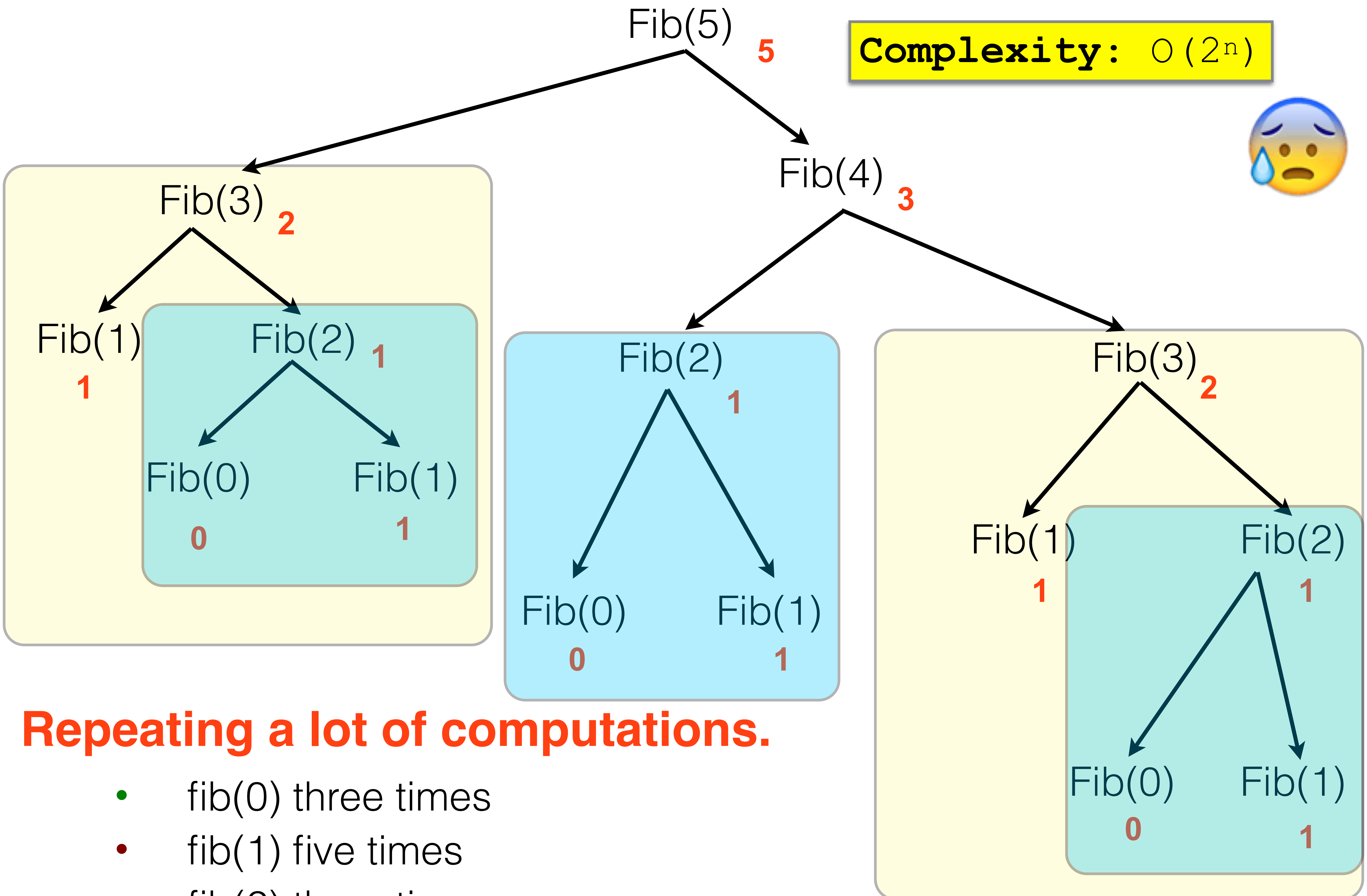
First.

Second.

Complexity: $O(2^n)$




```python
def fib(n):
    if n == 0 or n == 1:
        return n
    else:
        return fib(n-2) + fib(n-1)
```



Repeating a lot of computations.

- fib(0) three times
- fib(1) five times
- fib(2) three times
- fib(3) two times

```
                 before_last   last
fib(7)
fib_aux(7, 0, 1)
fib_aux(6, 1, 1)
fib_aux(5, 1, 2)
fib_aux(4, 2, 3)
fib_aux(3, 3, 5)
fib_aux(2, 5, 8)
fib_aux(1, 8, 13)
fib_aux(0, 13, 21)
```

Observations

- n decreases each call
- before_last = last
- last = before_last +last
- base case n =0
- return before_last

0	1	2	3	4	5	6	7	8
0	1	1	2	3	5	8	13	21

```python
def fib(n):
    return fib_aux(n, 0, 1)


def fib_aux(n, before_last, last):
    if n == 0:
        return before_last
    else:
        return fib_aux(n-1, last, before_last+last)
```

```python
def fib(n):
    return fib_aux(n, 0, 1)


def fib_aux(n, before_last, last):
    if n == 0:
        return before_last
    else:
        return fib_aux(n-1, last, before_last+last)
```

fib(5)

fib_aux(5, 0, 1)

fib_aux(4, 1, 0+1)

fib_aux(3, 1, 1+1)

fib_aux(2, 2, 2+1)

fib_aux(1, 3, 3+2)

fib_aux(0, 5, 5+3)

return 5

Complexity: O(n)

What happens with Recursion at the Low Level?

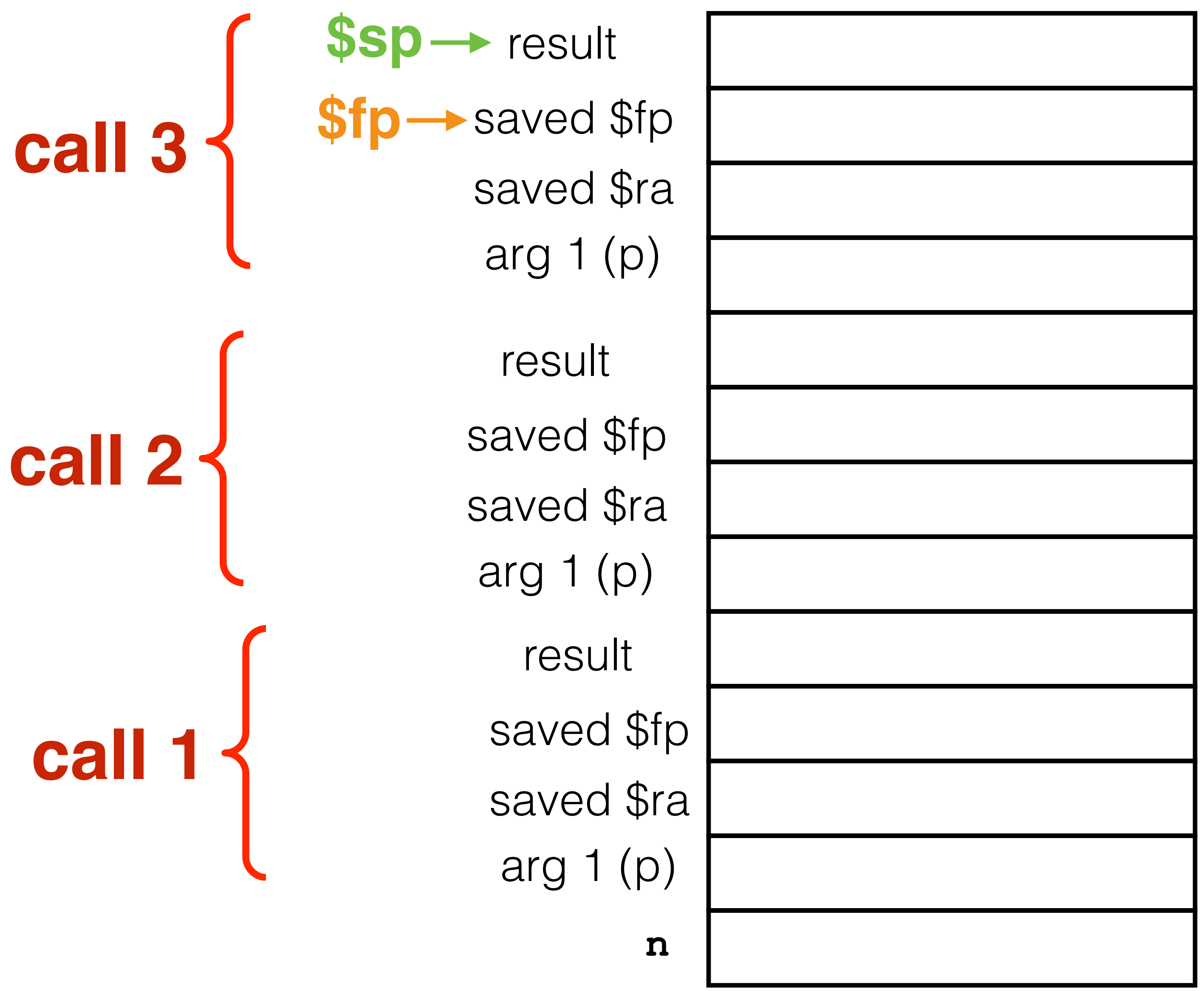
MIPS **Recursion**:

a function calling a function

(simply happen to be the same functions)

↓

We do not need to learn anything new.


call 3
call 2
call 1
$sp
$fp
result
saved $fp
saved $ra
arg 1 (p)
result
saved $fp
saved $ra
arg 1 (p)
result
saved $fp
saved $ra
arg 1 (p)
n

Tail-recursion:

- Where the result of the recursive call is the result of the function. Nothing is done in the “way back”. Closest to iteration - can be transformed *without “storing”*. **Useful for compiler optimisation.**

MIPS & Tail recursion

Fibonacci: clarity vs efficiency

- First recursive version is:
  - Clear
  - **Descriptive**
  - Inefficient

- Tail recursion with accumulators:
  - Not so clear at first
  - **More Efficient**
  - It can be easily transformed into an iterative version

Straightforward recursion is natural, but not necessarily the most efficient.

Summary

- Recursive algorithms and their implementation
- Complexity of recursive algorithms
- Reasoning about recursion