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FIT3155 S1/2020: Advanced Algorithms and Data Structures

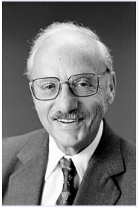
Week 10: Linear Programming

Faculty of Information Technology, Monash University

Linear Programming introduction

- Arose out of efforts dating back to World War-II when Mathematicians were asked to solve logistics problems for military operations.
- Linear Programming is an combinatorial optimization method to operate specific systems **optimally** under resource **constraints**.
- It was introduced by **Dantzig** in 1947 in a paper entitled "**Programming in Linear Structure**"
- A method of solving such problems (**simplex method**) was proposed in 1949.

Dantzig



"George Dantzig will go down in history as one of the founders and chief contributors to the field of mathematical programming, and as the creator of the simplex algorithm for linear programming, perhaps the most important algorithm developed in the 20th century." – Richard Karp.

Consider this problem

- A small manufacturer makes two products A and B
- Each of these products require 2 resources R_1 and R_2

- Each unit of A requires **1 unit** of R_1 and **3 units** of R_2
- Each unit of B requires **1 unit** of R_1 and **2 units** of R_2

- But the manufacturer has only **5 units** of R_1 and **12 units** of R_2

- On every unit of A sold the manufacturer makes $\$6$ **profit**
- On every unit of B sold the manufacturer makes $\$5$ **profit**

Problem statement

How many units of the product A and B should the manufacturer produce to **maximize** profit?

Formalizing the problem

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$x = \#$units of product A produced

$y = \#$units of product B produced

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$$z = 6x + 5y$$

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$$x + y \leq 5$$

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$$3x + 2y \leq 12$$

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Formalism

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y = #units of product B produced

$$z = 6x + 5y$$

$$x + y \leq 5$$

$$3x + 2y \leq 12$$

$$x, y \geq 0$$

Formalizing the problem

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x = #units of product A produced
y = #units of product B produced

$$maximize(profit)\ z = 6x + 5y$$

$$x + y \leq 5$$

$$3x + 2y \leq 12$$

$$x, y \geq 0$$

Now let's introduce some technical jargon

Decision Variables

The variable x and y are called **decision variables**

Objective function

The profit function ($z = 6x + 5y$) that is being optimized is called the **objective function.**

Constraints

The restrictions

$$x + y \leq 5$$

$$3x + 2y \leq 12$$

$$x, y \geq 0$$

are called (linear) **constraints**. These restrict the values the decision variables can take.

We now have a linear programming formulation

x = number of units of product A produced
y = number of units of product B produced

maximize (profit) $z = 6x + 5y$

$$x + y \leq 5$$
$$3x + 2y \leq 12$$
$$x, y \geq 0$$

All standard Linear Programming (LP) problems have...

A **`linear`** objective function

A set of **`linear`** constraints

A non-negativity constraint on all decision variables

Linear Programming problem in its standard form

Maximize

$$z = c_1 * x_1 + c_2 * x_2 + \cdots + c_n * x_n$$

Subject to the constraints:

$$a_{11} * x_1 + a_{12} * x_2 + \cdots + a_{1n} * x_n \leq b_1$$
$$a_{21} * x_1 + a_{22} * x_2 + \cdots + a_{2n} * x_n \leq b_2$$
$$\vdots$$
$$a_{m1} * x_1 + a_{m2} * x_2 + \cdots + a_{mn} * x_n \leq b_m$$

and

$$x_1, x_2, \cdots x_n \geq 0$$

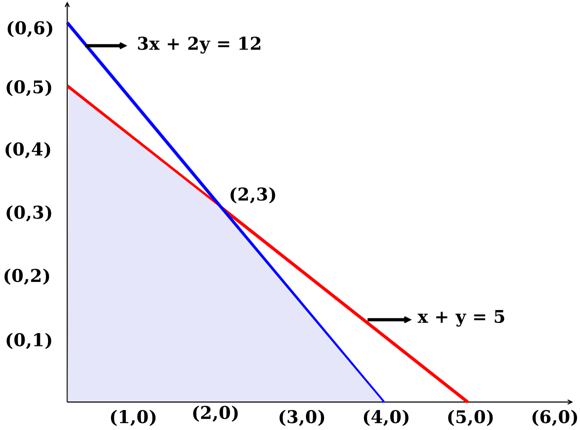
Gaining insights from a graphical solution – although it is practically useless!

A graphical solution is applicable in practice to solutions involving 2 (or even 3 decision variables).

Let's use the example linear programming problem we have considered so far to gain some insights about solutions of **general** linear programs.

Graphical Solution

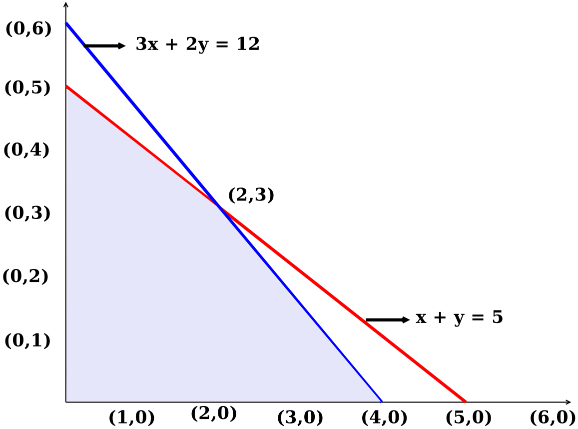
$$\text{maximize profit:} \quad z = 6x + 5y$$
$$x + y \leq 5$$
$$3x + 2y \leq 12$$
$$x, y \geq 0$$



Graphical Solution Insights

Feasible region

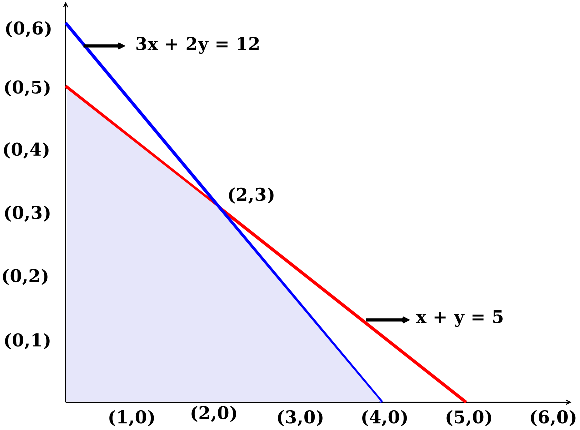
...is the region where the given linear program has **feasible solutions** **satisfying** all its constraints



Graphical Solution Insights

Insight 1

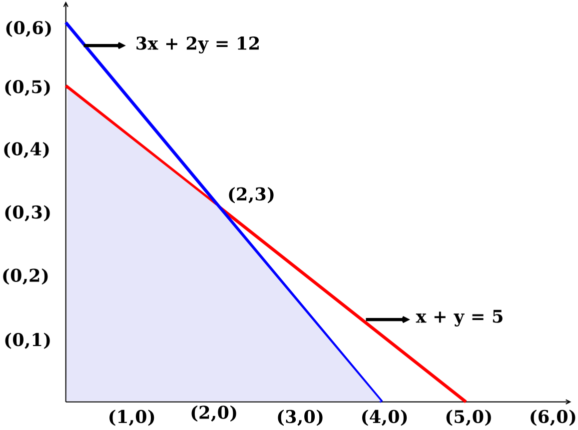
Feasible region is a convex polyhedron.



Graphical Solution Insights

Insight 2

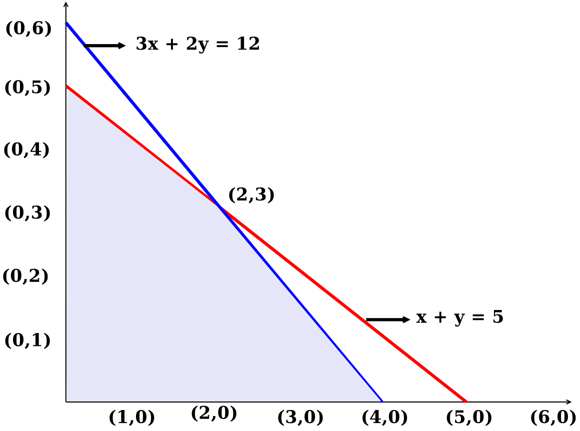
For **any point** in the **feasible region**, **there is always a point on the boundary** of the feasible region that will **dominate** it (i.e., will give a better evaluation of the **objective function**)



Graphical Solution Insights

Insight 3

Further, **among all points on the boundary** of the feasible regions, there is **at least** `one corner point` of the feasible region that will always **`dominate`**, given the objective function.



Now lets approach this Graphical solution with some Algebra

- We will approach a solution algebraically – this is quite simple!
- An important step here is to convert the **`inequality`** constraints into **`equality`** constraints.
- This is achieved by adding **`slack variables`** to convert an inequality to an equality.

Algebraic (graphical) solution – changing inequality constraints to equality constraints

First constraint: $x + y \leq 5$ becomes

$$x + y + s = 5$$

where $s \geq 0$

Similarly the second constraint $3x + 2y \leq 12$ becomes

$$3x + 2y + t = 12$$

where $t \geq 0$

Objective function can be written as:

$$\text{Maximize } z = 6x + 5y$$

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Objective function can be written as:

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Algebraic (graphical) solution – Formulation of the problem

Therefore, solving this problem:

Maximize

$$z = 6x + 5y$$

Subject to the constraints:

$$x + y \leq 5$$

$$3x + 2y \leq 12$$

and $x, y \geq 0$

...is same as solving this problem:

Maximize

$$z = 6x + 5y$$

Subject to the constraints:

$$x + y + s = 5$$

$$3x + 2y + t = 12$$

and $x, y, s, t \geq 0$

Algebraic (graphical) solution – Can a solution to be found in this setup?

Maximize

$$z = 6x + 5y$$

Subject to the constraints:

$$x + y + s = 5$$

$$3x + 2y + t = 12$$

and $x, y, s, t \geq 0$

There are 4 variables and 2 equations! Can a solution be found?

Algebraic (graphical) solution – Can a solution to be found in this setup?

Maximize

$$z = 6x + 5y$$

Subject to the constraints:

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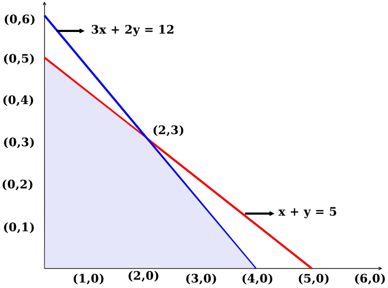
There are 4 variables and 2 equations! Can a solution be found?

Solutions can be found for any 2 variables, if the remaining 2 variables are 'fixed'!

Algebraic (graphical) solution – possible ways of fixing variables

Fix x, y	and solve for s, t
Fix x, s	and solve for y, t
Fix x, t	and solve for y, s
Fix y, s	and solve for x, t
Fix y, t	and solve for x, s
Fix s, t	and solve for x, y



Algebraic (graphical) solution

Maximize

$$z = 6x + 5y$$

Subject to the constraints:

$$x + y + s = 5$$

$$3x + 2y + t = 12$$

and $x, y, s, t \geq 0$

By fixing the variables in the left hand column to zeros, we get:

Non-Basic Variables	**Basic Variables**
Fixing $x = 0, y = 0$	We get $s = 5, t = 12$
Fixing $x = 0, s = 0$	We get $y = 5, t = 2$
Fixing $x = 0, t = 0$	We get $y = 6, s = -1$
Fixing $y = 0, s = 0$	We get $x = 5, t = -3$
Fixing $y = 0, t = 0$	We get $x = 4, s = 1$
Fixing $s = 0, t = 0$	We get $x = 2, y = 3$

Counter-intuitive terminology alert!!!

In the context of linear programming, we get to hear the following terms that might appear counter-intuitive:

Non-basic variables

- **`Non-basic`** variables are the variable that are **`fixed`** to zero
- They are so called, because they are OUTSIDE the basis of the linear programming problem – you can ignore this part.

Basic variables

- The remaining **`free`** variables are called **`basic variables`**.
- Basic variables are those that are WITHIN the basis of the linear programming problem – you can ignore this part.

Algebraic (graphical) Solution – exhaustive search

Objective

$$z = 6x + 5y$$

Now let's evaluate the objective function corresponding to all possible ways of choosing/fixing variables to become non-basic variables:

Non-Basic Variables	**Basic Variables**	value of objective
Fix $x = 0, y = 0$	$s = 5, t = 12$	

Algebraic (graphical) Solution – exhaustive search

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Non-Basic Variables	Basic Variables	value of objective
Fix $x = 0, y = 0$	$s = 5, t = 12$	$z = 0$
Fix $x = 0, s = 0$	$y = 5, t = 2$	

Algebraic (graphical) Solution – exhaustive search

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Now let's evaluate the objective function corresponding to all possible ways of choosing/fixing variables to become non-basic variables:

Non-Basic Variables	**Basic Variables**	value of objective
Fix $x = 0, y = 0$	$s = 5, t = 12$	$z = 0$
Fix $x = 0, s = 0$	$y = 5, t = 2$	$z = 25$
Fix $x = 0, t = 0$	$y = 6, s = -1$	

Algebraic (graphical) Solution – exhaustive search

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Non-Basic Variables	**Basic Variables**	value of objective
Fix $x = 0, y = 0$	$s = 5, t = 12$	$z = 0$
Fix $x = 0, s = 0$	$y = 5, t = 2$	$z = 25$
Fix $x = 0, t = 0$	$y = 6, s = -1$	Infeasible
Fix $y = 0, s = 0$	$x = 5, t = -3$	

Algebraic (graphical) Solution – exhaustive search

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Fix $y = 0, s = 0$	$x = 5, t = -3$	Infeasible
Fix $y = 0, t = 0$	$x = 4, s = 1$	

Algebraic (graphical) Solution – exhaustive search

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Fix $x = 0, s = 0$	$y = 5, t = 2$	$z = 25$
Fix $x = 0, t = 0$	$y = 6, s = -1$	Infeasible
Fix $y = 0, s = 0$	$x = 5, t = -3$	Infeasible
Fix $y = 0, t = 0$	$x = 4, s = 1$	$z = 24$
Fix $s = 0, t = 0$	$x = 2, y = 3$	

Algebraic (graphical) Solution – exhaustive search

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Fix $y = 0, s = 0$	$x = 5, t = -3$	Infeasible
Fix $y = 0, t = 0$	$x = 4, s = 1$	$z = 24$
Fix $s = 0, t = 0$	$x = 2, y = 3$	$z = 27$

Algebraic (graphical) solution – exhaustive search

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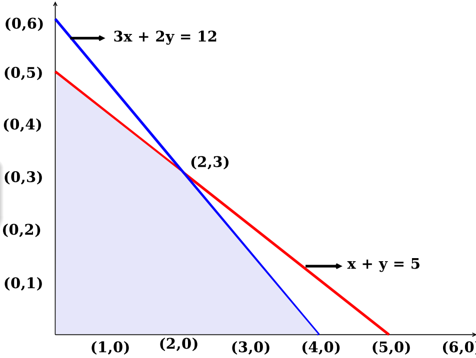
Did you notice that the values x and y above correspond to the corner points!



Now generalizing this Algebraic (graphical) method

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- Assign N out of $N + m$ (decision+slack) variables to 0, and solve for the remaining m variables.

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- There are ${}^{N+m}C_N$ such possibilities, sigh!

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- Substitute the values of that N **decision variables** take into the objective function.

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- There are ${}^{N+m}C_N$ such possibilities, sigh!
- Substitute the values of that N **decision variables** take into the objective function.
- Once all possibilities are explored, choose the **optimum** over all the solutions.

Disadvantages of Algebraic (graphical) method

- Grows combinatorially for large N
- Evaluates infeasible solutions.
- Does not tell you if the optima has been reached. The search is exhaustive (brute-force).
- Does not give progressively better solutions (in terms of the objective function) – the search is fairly random.

Simplex Method to solving linear programming problems

Simplex Method is a way to solve this problem. Advantages of simplex are:

- Polynomial time (in practice, but not in worst case)
- Does NOT evaluate infeasible solutions.
- Explores **`progressively better solutions`**.
- Tells you when the optima has been reached and stops.

Simplex Method – explored in two forms

We will now see two forms of the simplex method using the example we have been considering:

- Algebraic form of simplex: **this will give you the insights of what it really is doing**!
- Tabular (or tableau) form of simplex: **this will allow you to crank-turn rather easily** and implement a program in the lab!

Simplex method – Algebraic form

Maximize

$$z = 6x + 5y$$

Subject to the constraints:

$$x + y + s = 5$$

$$3x + 2y + t = 12$$

and $x, y, s, t \geq 0$

Simplex method – Algebraic form

Maximize

$$z = 6x + 5y$$

Subject to the constraints:

$$x + y + s = 5$$

$$3x + 2y + t = 12$$

and $x, y, s, t \geq 0$

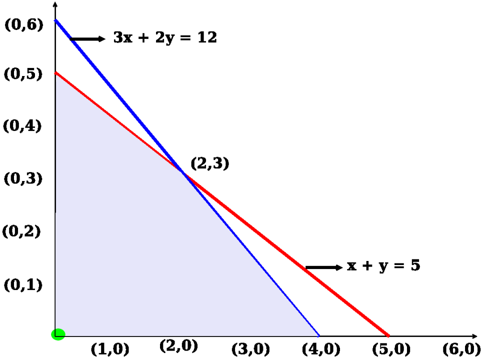
Start

Start by choosing x and y as **`non-basic variables`** (i.e., variables that are set to zero.)

This gives the solutions to s and t as $s = 5$ and $t = 12$ respectively.

Simplex method – Algebraic form

Visualize what this solution of $(x, y) = (0, 0)$ implies:



Simplex method – Algebraic form

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First iteration

`Rewrite the constraints in terms of` s **`and`** t (now basic variables)

Simplex method – Algebraic form

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First iteration

Rewrite the constraints in terms of s and t (now basic variables)

$$s = 5 - x - y$$

Simplex method – Algebraic form

Maximize

$$z = 6x + 5y$$

Subject to the constraints:

$$x + y + s = 5$$

$$3x + 2y + t = 12$$

and $x, y, s, t \geq 0$

First iteration

`Rewrite the constraints in terms of` s **`and`** t (now basic variables)

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

Simplex method – Algebraic form

Maximize

$$z = 6x + 5y$$

Subject to the constraints:

$$x + y + s = 5$$

$$3x + 2y + t = 12$$

and $x, y, s, t \geq 0$

First iteration

`Rewrite the constraints in terms of` s **`and`** t (now basic variables)

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

Objective is simply: $z = 6x + 5y$

Simplex method – Algebraic form

Problem as we transformed it in the previous slide

Maximize

$$z = 6x + 5y$$

Subject to the constraints (notice we have rewritten the constraints as in the previous slide):

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

and $x, y, s, t \geq 0$

- Recall, currently $x = 0$ and $y = 0$. Therefore the objective function, substituting these values, is $z = 0$.

---

* To increase the objective function, if the coefficient of the variable in the objective function is positive, you increase the variable. Else, if its coefficient is negative you decrease the variable.

Simplex method – Algebraic form

Problem as we transformed it in the previous slide

Maximize

$$z = 6x + 5y$$

Subject to the constraints (notice we have rewritten the constraints as in the previous slide):

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

and $x, y, s, t \geq 0$

- Recall, currently $x = 0$ and $y = 0$. Therefore the objective function, substituting these values, is $z = 0$. Can we improve z?

* To increase the objective function, if the coefficient of the variable in the objective function is positive, you increase the variable. Else, if its coefficient is negative you decrease the variable.

Simplex method – Algebraic form

Problem as we transformed it in the previous slide

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- Yes. We can either **increase*** the value of x or **increase*** the value of y.

* To increase the objective function, if the coefficient of the variable in the objective function is positive, you increase the variable. Else, if its coefficient is negative you decrease the variable.

Simplex method – Algebraic form

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- Recall, currently $x = 0$ and $y = 0$. Therefore the objective function, substituting these values, is $z = 0$. Can we improve z?
- Yes. We can either **`increase`*** the value of x or **`increase`*** the value of y.
- In simplex, you improve only **`one variable`** at a time.

---

*To increase the objective function, if the coefficient of the variable in the objective function is positive, you increase the variable. Else, if its coefficient is negative you decrease the variable.

Simplex method – Algebraic form

Problem as it stands!

Maximize

$$z = 6x + 5y$$

Subject to the constraints

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

and $x, y, s, t \geq 0$

- Choose to increase x **(since it has a larger coefficient of the two choices, hence gives higher increase of the objective)**.

Simplex method – Algebraic form

Problem as it stands!

Maximize

$$z = 6x + 5y$$

Subject to the constraints

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

and $x, y, s, t \geq 0$

- Choose to increase x **`(since it has a larger coefficient of the two choices, hence gives higher increase of the objective)`**.
- However, having chosen to increase x, we can either increase it:

Simplex method – Algebraic form

Problem as it stands!

Maximize

$$z = 6x + 5y$$

Subject to the constraints

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

and $x, y, s, t \geq 0$

- Choose to increase x **(since it has a larger coefficient of the two choices, hence gives higher increase of the objective)**.
- However, having chosen to increase x, we can either increase it:
  - from $x = 0$ to $x = 5$ according to the **first** constraint.

Simplex method – Algebraic form

Problem as it stands!

Maximize

$$z = 6x + 5y$$

Subject to the constraints

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

and $x, y, s, t \geq 0$

- Choose to increase x **(since it has a larger coefficient of the two choices, hence gives higher increase of the objective)**.
- However, having chosen to increase x, we can either increase it:
  - from $x = 0$ to $x = 5$ according to the **first** constraint. Anything more will violate the **non-negativity constraint** for the slack variable s. Or

Simplex method – Algebraic form

Problem as it stands!

Maximize

$$z = 6x + 5y$$

Subject to the constraints

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

and $x, y, s, t \geq 0$

- Choose to increase x **(since it has a larger coefficient of the two choices, hence gives higher increase of the objective)**.
- However, having chosen to increase x, we can either increase it:
  - from $x = 0$ to $x = 5$ according to the **first** constraint. Anything more will violate the **non-negativity constraint** for the slack variable s. Or
  - from $x = 0$ to $x = 4$ according to the **second** constraint.

Simplex method – Algebraic form

Problem as it stands!

Maximize

$$z = 6x + 5y$$

Subject to the constraints

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

and $x, y, s, t \geq 0$

- Choose to increase x **(since it has a larger coefficient of the two choices, hence gives higher increase of the objective)**.
- However, having chosen to increase x, we can either increase it:
  - from $x = 0$ to $x = 5$ according to the **first** constraint. Anything more will violate the **non-negativity constraint** for the slack variable s. Or
  - from $x = 0$ to $x = 4$ according to the **second** constraint.Anything more will violate the **non-negativity constraint** for t.

Simplex method – Algebraic form

Problem as it stands!

Maximize

$$z = 6x + 5y$$

Subject to the constraints

$$s = 5 - x - y$$

$$t = 12 - 3x - 2y$$

and $x, y, s, t \geq 0$

- Choose to increase x **(since it has a larger coefficient of the two choices, hence gives higher increase of the objective)**.
- However, having chosen to increase x, we can either increase it:
  - from $x = 0$ to $x = 5$ according to the **first** constraint. Anything more will violate the **non-negativity constraint** for the slack variable s. Or
  - from $x = 0$ to $x = 4$ according to the **second** constraint.Anything more will violate the **non-negativity constraint** for t.
- Choosing the **minimum** of $x = 5$ and $x = 4$ will ensure that both $s, t \geq 0$.

Simplex method – Algebraic form

- Since we have chosen the **minimum increase** of x from $x = 0$ to $x = 4$, x enters the basis, while t exits the basis ($x = 4, \implies t = 0$).

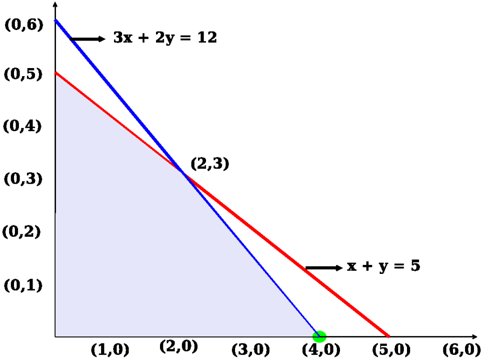
Simplex method – Algebraic form

- Since we have chosen the **minimum increase** of x from $x = 0$ to $x = 4$, x enters the basis, while t exits the basis ($x = 4, \implies t = 0$).
- The new set of **non-basic variables** are therefore y **and** t, and **basic variables** are x **and** s. That is, $x = 4, s = 1, y = 0, t = 0$.

Simplex method – Algebraic form

- Since we have chosen the **minimum increase** of x from $x = 0$ to $x = 4$, x enters the basis, while t exits the basis ($x = 4, \implies t = 0$).
- The new set of **non-basic variables** are therefore y **and** t, and **basic variables** are x **and** s. That is, $x = 4, s = 1, y = 0, t = 0$.

Visualize what this solution of $(x, y) = (4, 0)$ implies



Simplex method – Algebraic form

- Since we have chosen the **minimum increase** of x from $x = 0$ to $x = 4$, x enters the basis, while t exits the basis ($x = 4, \implies t = 0$).
- The new set of **non-basic variables** are therefore y **and** t, and **basic variables** are x **and** s. That is, $x = 4, s = 1, y = 0, t = 0$.

Rewrite the constraints in terms of the new basic variables

Rewrite the second constraint ($t = 12 - 3x - 2y$), now with x as the **basic variable**:

$$x = 4 - \frac{2}{3}y - \frac{1}{3}t$$

The first constraint in the previous iteration was $s = 5 - x - y$. Rewriting this by substituting x from above, becomes:

$$s = 1 - \frac{1}{3}y + \frac{1}{3}t$$

Simplex method – Algebraic form

Problem as it is now transformed – second iteration

$$x = 4 - \frac{2}{3}y - \frac{1}{3}t$$

$$s = 1 - \frac{1}{3}y + \frac{1}{3}t$$

Objective was previously to maximize:

$$z = 6x + 5y$$

now becomes (in terms of the new **`non-basic`** variables y and t)

$$z = 24 + y - 2t$$

Simplex method – Algebraic form

Problem as it is now transformed – second iteration

$$x = 4 - \frac{2}{3}y - \frac{1}{3}t$$

$$s = 1 - \frac{1}{3}y + \frac{1}{3}t$$

Objective was previously to maximize:

$$z = 6x + 5y$$

now becomes (in terms of the new **non-basic** variables y and t)

$$z = 24 + y - 2t$$

Notice how the objective function is growing

Previously, $z = 0$ (with x and y as the non-basic variables). Now, $z = 24$ when $y = 0$ and $t = 0$ are the non-basic variables

Simplex method – Algebraic form

Problem as it is now transformed – second iteration

$$x = 4 - \frac{2}{3}y - \frac{1}{3}t$$

$$s = 1 - \frac{1}{3}y + \frac{1}{3}t$$

Objective was previously to maximize:

$$z = 6x + 5y$$

now becomes (in terms of the new **non-basic** variables y and t)

$$z = 24 + y - 2t$$

Notice how the objective function is growing

Previously, $z = 0$ (with x and y as the non-basic variables). Now, $z = 24$ when $y = 0$ and $t = 0$ are the non-basic variables

Can z be increased further?

Simplex method – Algebraic form

(Note: What we are doing here is a **repetition** of the steps/logic undertaken in the previous iteration)

- To increase z further, we can either **increase** y (since coefficient in the objective is **positive**) or **decrease** t (since the coefficient in the objective is **negative**.
- Decreasing t is infeasible because t is already zero and any further decrease will violate non-negativity constraint. Therefore, our only choice here is to **increase** y
- However, y can only be increased from $y = 0$ to $y = 3$ (based on the second equation) for the solution to be feasible.

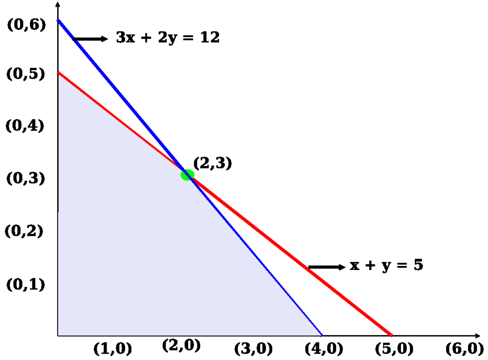
Simplex method – Algebraic form

Since we have chosen to **increase y from 0 to 3**, the new **basic variables** are x, y whose current values are $x = 2$ and $y = 3$ and the non-basic variables are $s = 0, t = 0$.

Simplex method – Algebraic form

Since we have chosen to **increase y from 0 to 3**, the new **basic variables** are x, y whose current values are $x = 2$ and $y = 3$ and the non-basic variables are $s = 0, t = 0$.

Visualize what this solution of $(x, y) = (2, 3)$ implies:



Simplex method – Algebraic form

Since we have chosen to **increase y from 0 to 3**, the new **basic variables** are x, y whose current values are $x = 2$ and $y = 3$ and the non-basic variables are $s = 0, t = 0$.

Rewrite the constraints in terms of the new basic variables

we rewrite the second equation ($s = 1 - \frac{1}{3}y + \frac{1}{3}t$) with y as the **basic variable**:

$$y = 3 - 3s + t$$

But, the first equation in the iteration is $x = 4 - \frac{2}{3}y - \frac{1}{3}t$. Rewriting this by substituting y from above, we get:

$$x = 2 + 2s - t$$

Simplex method – Algebraic form

Problem as it is now transformed

$$x = 2 + 2s - t$$

$$y = 3 - 3s + t$$

Objective was to maximize:

$$z = 24 + y - 2t$$

now becomes (in terms of new **`non-basic`** variables)

$$z = 27 - 3s - t$$

Simplex method – Algebraic form

Problem as it is now transformed

$$x = 2 + 2s - t$$

$$y = 3 - 3s + t$$

Objective was to maximize:

$$z = 24 + y - 2t$$

now becomes (in terms of new **non-basic** variables)

$$z = 27 - 3s - t$$

Observe that $z = 27$ when $s = 0$ and $t = 0$ as the new **non-basic variables**

Simplex method – Algebraic form

Problem as it is now transformed

$$x = 2 + 2s - t$$

$$y = 3 - 3s + t$$

Objective was to maximize:

$$z = 24 + y - 2t$$

now becomes (in terms of new **non-basic** variables)

$$z = 27 - 3s - t$$

Observe that $z = 27$ when $s = 0$ and $t = 0$ as the new **non-basic variables**
Can z be increased further?

Simplex method – Algebraic form

Problem as it is now transformed

$$x = 2 + 2s - t$$

$$y = 3 - 3s + t$$

Objective was to maximize:

$$z = 24 + y - 2t$$

now becomes (in terms of new **non-basic** variables)

$$z = 27 - 3s - t$$

Observe that $z = 27$ when $s = 0$ and $t = 0$ as the new **non-basic variables**
Can z be increased further?

The only way to increase z further is to either **decrease** s and/or **decrease** t.
Both both these choices are infeasible Therefore STOP!

Simplex method – Algebraic form

When simplex method stopped, the values of **basic variables** were $x = 2$ and $y = 3$ (we get these by substituting $s = 0$ and $t = 0$ (**non-basic variables**) into the constraints in the previous slide.

These are the **optimum values** of the decision variables x and y for this example problem.

Visualize what this solution of $(x, y) = (2, 3)$ implies:

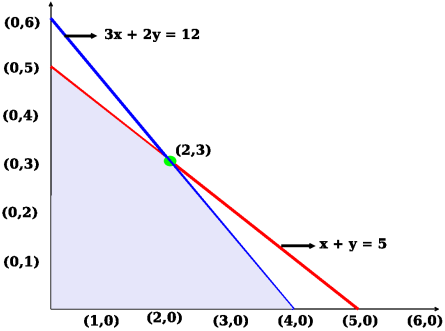


Tableau form of Simplex does **exactly** the same thing but in a convenient tabular way (that can be converted into a computer program)

Now let's see the same simplex method now in **`tableau form`** (This is now a crank-turning exercise).

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

	x	y	s	t	RHS	θ

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

	6	5	0	0		
	x	y	s	t	RHS	θ

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

	6	5	0	0		
	x	y	s	t	RHS	θ
s						
t						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s						
0	t						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	
0	t						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	
0	t	3	2	0	1	12	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	
0	t	3	2	0	1	12	
	$c_j - z_j$						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	
0	t	3	2	0	1	12	
	$c_j - z_j$	6					

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	
0	t	3	2	0	1	12	
	$c_j - z_j$	6	5				

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	
0	t	3	2	0	1	12	
	$c_j - z_j$	6	5	0			

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	
0	t	3	2	0	1	12	
	$c_j - z_j$	6	5	0	0		

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	
0	t	3	2	0	1	12	
	$c_j - z_j$	6	5	0	0	0	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	
	$c_j - z_j$	6	5	0	0	0	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
	s						
	x						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s						
6	x						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s						
6	x	1	2/3	0	1/3	4	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	
6	x	1	2/3	0	1/3	4	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	
6	x	1	2/3	0	1/3	4	
	$c_j - z_j$						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	
6	x	1	2/3	0	1/3	4	
	$c_j - z_j$	0	1	0	-2	24	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	
6	x	1	2/3	0	1/3	4	
	$c_j - z_j$	0	1	0	-2	24	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	3
6	x	1	2/3	0	1/3	4	6
	$c_j - z_j$	0	1	0	-2	24	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	3
6	x	1	2/3	0	1/3	4	6
	$c_j - z_j$	0	1	0	-2	24	
	y						
	x						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	3
6	x	1	2/3	0	1/3	4	6
	$c_j - z_j$	0	1	0	-2	24	
	y	0	1	3	-1	3	
	x						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		x (6)	y (5)	s (0)	t (0)	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	3
6	x	1	2/3	0	1/3	4	6
	$c_j - z_j$	0	1	0	-2	24	
	y	0	1	3	-1	3	
	x	1	0	-2	1	2	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	3
6	x	1	2/3	0	1/3	4	6
	$c_j - z_j$	0	1	0	-2	24	
5	y	0	1	3	-1	3	
6	x	1	0	-2	1	2	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	3
6	x	1	2/3	0	1/3	4	6
	$c_j - z_j$	0	1	0	-2	24	
5	y	0	1	3	-1	3	
6	x	1	0	-2	1	2	
	$c_j - z_j$						

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	3
6	x	1	2/3	0	1/3	4	6
	$c_j - z_j$	0	1	0	-2	24	
5	y	0	1	3	-1	3	
6	x	1	0	-2	1	2	
	$c_j - z_j$	0	0	-3	-1	27	

Simplex – Tableau Form

Listen to the lecture recording while going through this tableau method

		6	5	0	0		
		x	y	s	t	RHS	θ
0	s	1	1	1	0	5	5
0	t	3	2	0	1	12	4
	$c_j - z_j$	6	5	0	0	0	
0	s	0	1/3	1	-1/3	1	3
6	x	1	2/3	0	1/3	4	6
	$c_j - z_j$	0	1	0	-2	24	
5	y	0	1	3	-1	3	
6	x	1	0	-2	1	2	
	$c_j - z_j$	0	0	-3	-1	27	

STOP!!!

-=o0o=-

END

-=o0o=-