Laurent series

Consider a function f (z) that is not analytic at the points z_0, z_1, z_2, $z_3, \ldots, z_k$. Then we expand the function f (z) at any point say z_0 in a series which contains both positive and negative powers of $z - z_\infty$.
This series is called as the Laurent series of the f (z).

Consider power series of the form

$$\sum_{n=0}^{\infty} C_n (z - z_0)^n \quad -------\to (1)$$

$$\sum_{n=0}^{\infty} \frac{C_n}{(z - z_0)^n} \quad -------\to (2)$$

Let radius of convergence of the series (1) is 'r_1' and this series converges to some analytic function $f_1(z)$

$$\therefore \; f_1(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n, \quad |z - z_0| < r_1$$

Let $\xi = \dfrac{1}{z - z_0}$ then (2) $\Rightarrow \sum_{n=0}^{\infty} C_n \xi^n$ which is a power series in ξ and converges to some analytic function $\phi(\xi)$ within its circle of convergence. Let its radius of convergence $\dfrac{1}{r_2}$.

$$\Rightarrow . \; f_2(z) = \sum_{n=0}^{\infty} \frac{C_n}{(z - z_0)^n}, \quad |z - z_0| > r_2$$

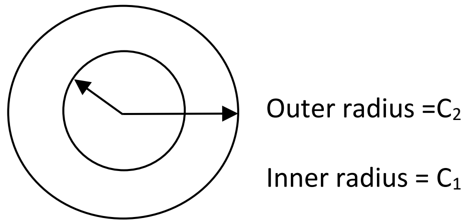
Thus the region of convergence of the series (2) is the region exterior to the circle $|z - z_0| > r_2$. Let $r_1 < r_2$. Then the intersection of two regions of convergence is $r_1 > |z - z_0| > r_2$. Therefore the series of positive & negative powers of $(z - z_0)$ converges in $r_1 > |z - z_0| > r_2$ to analytic function $f_1(z) + f_2(z)$

i.e. $f(z) = f_1(z) + f_2(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n, \quad r_1 > |z - z_0| > r_2$.

This series is called as Laurent series.

Laurent Series :-

Let C_1 and C_2 denote the concentric circles $|z - z_0| = r_1$ and $|z - z_0| = r_2$ respectively with $r_1 < r_2$. Let f (z) be analytic in a region containing the circular annulus $r_1 < |z - z_0| < r_2$. Then f (z) be represented as convergent series of positive and negative powers of $z - z_0$ given by



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$$

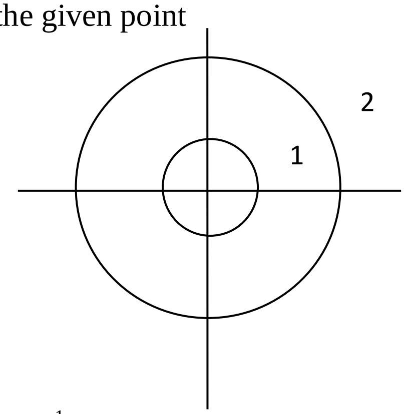
Where $a_n = \dfrac{1}{2\pi i} \oint_{c_2} \dfrac{f(z)dz}{(z - z_0)^{n+1}}$, $b_n = \dfrac{1}{2\pi i} \oint_{c_1} \dfrac{f(z)dz}{(z - z_0)^{n-1}}$

Note:

1) The Coefficients of the positive powers of $z - z_0$. In the Laurent series cannot be replaced by the derivative expressions $\dfrac{f^{(n)}(z_0)}{n!}$, although they are identical in form as in Taylor Series, since f (z) is not analytic throughout the region inside c_2 & the Cauchy integral formula for derivatives cannot be used.
2) In the Laurent series let $r_1 \to 0$. Then f (z) is analytic in $|z - z_0| < r_2$ except at z_0.
   ∴ The region of convergence is $0 < |z - z_0| < r_2$. If f (z) is analytic at z_0 also then the Laurent Series is the same as Taylor Series.
3) In Laurent Series, let $r_2 \to \infty$, r then the region of convergence is $|z - z_0| < r_1$.
4) Laurent Series expansion of f (z) in the annulus region $r_1 > |z - z_0| > r_2$ is unique.

Example 1: Obtain the Laurent series expansion of the following series about the given point

$$f(z) = \frac{1}{z^2 - 3z + 2}$$



Solution: $f(z) = \dfrac{1}{(z-1)(z-2)} \Rightarrow$ Singular points are 1 & 2

$\therefore$ Possible regions are $|z| < 1$, $1 < |z| < 2$ & $|z| > 2$.

f (z) is analytic within the region $|z| < 1$, $1 < |z| < 2$ & $|z| > 2$

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2(1-\frac{z}{2})} + \frac{1}{1-z} = -\frac{1}{2}(1-\frac{z}{2})^{-1} + (1-z)^{-1}$$

$$= \frac{-1}{2}\left[1 + \frac{z}{2} + \frac{z}{2}^2 + \frac{z^3}{2} + \frac{z^4}{2} + \dots\dots\dots\right] + \left[1 + z + z^2 + z^3 + z^4 + \dots\dots\right] = \left[\frac{1}{2} + \frac{3z}{4} + \frac{7z^2}{8} + \frac{15z^4}{16} + \dots\dots\dots\right]$$

$\therefore$ f (z) has Taylor expansion in region $|z| < 1$