Final assignment-1: Modelling ultrasonic backscatter strength

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[ ]:
```python
import numpy as np
from numpy.linalg import inv
import matplotlib.pyplot as plt
from IPython.display import display, Markdown, Latex
import pandas as pd
from scipy import stats
```

Analytical model for the ultrasound backscatter:

$\mu = c_1 + c_2(100 - g - s)d50 + c_3 g + c_4 s$

Parameters are $x = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}^T$. With 25 measurements of the Lambert parameter, $y = \begin{bmatrix} \mu_{01} & \mu_{02} & \cdots & \mu_{25} \end{bmatrix}^T$. Model matrix is:

$$A = \begin{bmatrix} 1 & (100 - g_{01} - s_{01})d50_{01} & g_{01} & s_{01} \\ 1 & (100 - g_{02} - s_{02})d50_{02} & g_{02} & s_{02} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (100 - g_{25} - s_{25})d50_{25} & g_{25} & s_{25} \end{bmatrix}$$

[ ]:
```python
#Import the raw data

data = np.loadtxt("final_assgn_data.txt")
data_pd = pd.DataFrame(data)
data_pd.columns = ['d50', 'g', 's', 'mu']
display(Markdown("Background scatter data" + f"{data_pd.
 ↪to_latex(float_format='%.4g')}"))
```

Background scatter data

	d50	g	s	mu
0	33	0	0.5	-22
1	221	0	0.5	-16
2	100	0	0.5	-16
3	28	0	0.5	-22
4	501	56.6	0.8	-8
5	404	0	0.2	-19
6	281	0	0.2	-18
7	233	0	0.1	-19
8	33	0	0.01	-23
9	30	0	0.01	-22
10	238	0	0.01	-21
11	337	0.04	0.02	-22
12	405	14.3	0.5	-12
13	399	7.75	0.18	-12
14	41	0	0.01	-23
15	318	0	0.01	-21
16	432	0.4	0.3	-20
17	506	70.2	0.7	-8
18	437	49.95	0.71	-7
19	519	32	2	-10
20	398	30	5.1	-5
21	357	48	0.4	-9
22	389	49.5	2.1	-8
23	367	0.07	0.17	-20
24	399	39.5	1.5	-11

[ ]:
```
y = data[:,3]

A1 = np.ones(len(y))
A2 = (100 - data[:,1] - data[:,2])*data[:,0]
A3 = data[:,1]
A4 = data[:,2]
A = np.column_stack((A1, A2, A3, A4))
```

1 Question-a

$\hat{x} = (A^T A)^{-1} A^T y$

[ ]:
```
x_hat = inv(A.T @ A) @ A.T @ y

display(Markdown("$\hat{x}=$" + f"{pd.DataFrame(x_hat).
↪to_latex(index=False,header=False,float_format='%.4G')}"))
```

$\hat{x} =$

-22.14
9.537E-05
0.1925
1.729

$c_1 = -22.14, c_2 = 9.537 \times 10^{-5}, c_3 = 0.1925, c_4 = 1.729$

2 Question-b

$\hat{y} = A\hat{x}$

$\hat{e} = y - \hat{y}$

[ ]:
```
y_hat = A @ x_hat
e_hat = y - y_hat

display(Markdown("$\hat{e}=$" + f"{pd.DataFrame(e_hat).
 ↪to_latex(index=False,header=False,float_format='%.4G')}"))
```

$\hat{e} =$

-1.042
3.174
4.322
-0.9947
-0.178
-1.056
1.115
0.7424
-1.197
-0.1681
-1.152
-3.119
3.227
4.829
-1.273
-1.915
-2.552
-1.992
2.237
-0.7485
0.08121
1.448
-0.8186
-1.664
-1.306

$\sigma_y^2 = \frac{\hat{e}^T\hat{e}}{m-n}$

$Q_y = \sigma_y^2 I_m$

[ ]:
```
m = len(y)
n = 4
dof = m - n

display(Markdown(f"Degrees of freedom $m - n = {dof}$"))
```

Degrees of freedom $m - n = 21$

[ ]:
```
var = e_hat.T @ e_hat / dof

display(Markdown(f"$\sigma^2={var:.4G}$"))
```

$\sigma^2 = 5.171$

[ ]:
```
Qy = var * np.eye(m)
```

$Q_y = 5.171 I_{25}$

3 Question-c

For unweighted least-squares solution:

$Q_{\hat{x}} = (A^T A)^{-1} A^T Q_y A (A^T A)^{-1}$

Substituting $Q_y = \sigma^2 I$ above:

$Q_{\hat{x}} = (A^T A)^{-1} A^T Q_y A (A^T A)^{-1}$

$Q_{\hat{x}} = \sigma^2 (A^T A)^{-1}$

[ ]:
```
Q_x_hat = inv(A.T @ A)*var

display(Markdown("$Q_{\hat{x}}=$" + f"{pd.DataFrame(Q_x_hat).
↪to_latex(index=False,header=False,float_format='%.4G')}"))
```

$Q_{\hat{x}} =$

1.015	-3.064E-05	-0.005956	-0.04235
-3.064E-05	1.385E-09	5.25E-08	-1.663E-06
-0.005956	5.25E-08	0.0005052	-0.004795
-0.04235	-1.663E-06	-0.004795	0.2289

[ ]:
```
sigma_x_hat = np.sqrt(np.diag(Q_x_hat))

display(Markdown("$\sigma_{\hat{x}}=$" + f"{pd.DataFrame(sigma_x_hat).
↪to_latex(index=False,header=False,float_format='%.4G')}"))
```

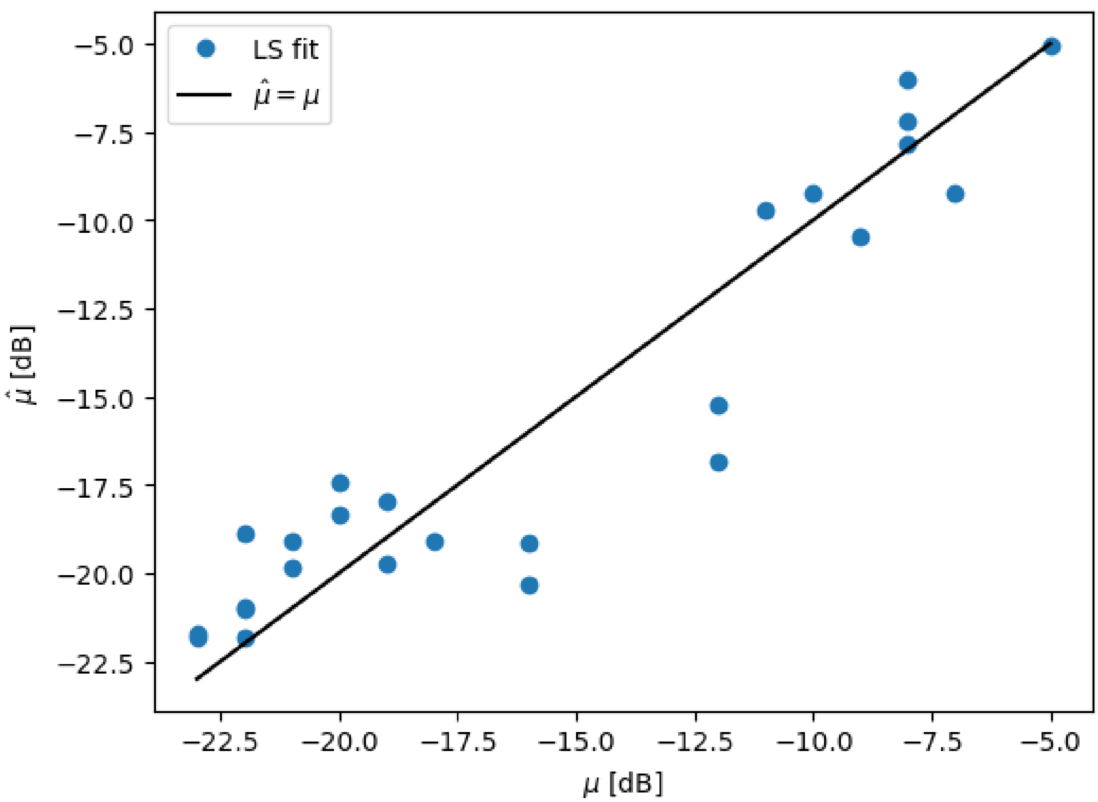
$\sigma_{\hat{x}} =$

1.008
3.721E-05
0.02248
0.4784

4 Question-d

[ ]:
```
fig_d, ax_d = plt.subplots()
ax_d.plot(y, y_hat, 'o', label="LS fit")
ax_d.plot(np.sort(y), np.sort(y), 'k', label=r"$\hat{\mu}=\mu$")
ax_d.set_xlabel(r'$\mu$ [dB]')
ax_d.set_ylabel(r'$\hat{\mu}$ [dB]')
ax_d.legend()
```

[ ]: <matplotlib.legend.Legend at 0x7efe8d19f100>



5 Question-e

For unweighted least-squares solution:

$Q_{\hat{e}} = P^{\perp} Q_y (P^{\perp})^T$

$P = A(A^T A)^{-1} A^T$

$P^{\perp} = I_m - P$

[ ]:
```
P = A @ inv(A.T @ A) @ A.T
P_orth = np.eye(m) - P

Q_e_hat = P_orth @ Qy @ P_orth.T

sigma_e_hat = np.sqrt(np.diag(Q_e_hat))
norm_e_hat = e_hat / sigma_e_hat

display(Markdown("$w =$" + f"{pd.DataFrame(norm_e_hat).
 ↪to_latex(index=False,header=False,float_format='%.4G')}"))
```

$w =$

-0.5007
1.44
2.009
-0.4794
-0.0871
-0.5028
0.5082
0.3373
-0.5742
-0.0808
-0.524
-1.446
1.481
2.248
-0.6079
-0.8826
-1.232
-1.073
1.067
-0.3538
0.08037
0.6952
-0.3916
-0.7781
-0.6016

6 Question-f

Significance level $\alpha = 0.1$

[ ]:
```
from scipy.stats import t

alpha = 0.1

#For 2-tailed test
q = 1 - alpha/2

#Quantile function aka inverse cdf for t-distribution
k_alpha = t.ppf(q, dof)

display(Markdown(f"$k_{{0.95}} = {k_alpha:.4G}$"))
```

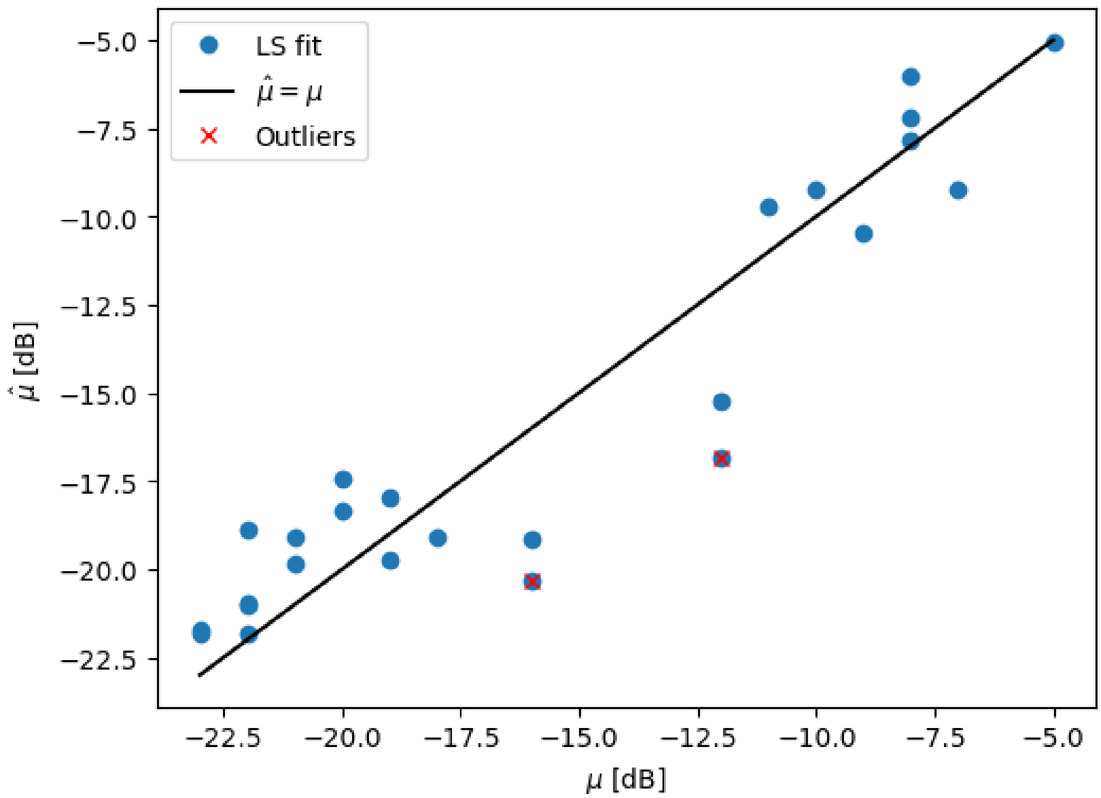
$k_{0.95} = 1.721$

[ ]:
```
data_pd_outliers = data_pd.iloc[np.argwhere(np.abs(norm_e_hat) > k_alpha).
    ↪flatten()]

display(Markdown("Outliers" + f"{data_pd_outliers.to_latex(float_format='%.
    ↪4g')}"))
```

Outliers

	d50	g	s	mu
2	100	0	0.5	-16
13	399	7.75	0.18	-12

[ ]:
```
y_hat_outliers = y_hat[np.argwhere(np.abs(norm_e_hat) > k_alpha)]
y_outliers = y[np.argwhere(np.abs(norm_e_hat) > k_alpha)]

ax_d.plot(y_outliers, y_hat_outliers, 'rx', label="Outliers")
ax_d.legend()
display(fig_d)
```


LS fit
$\hat{\mu} = \mu$
Outliers
$\hat{\mu}$ [dB]
μ [dB]
−5.0
−7.5
−10.0
−12.5
−15.0
−17.5
−20.0
−22.5
−22.5
−20.0
−17.5
−15.0
−12.5
−10.0
−7.5
−5.0