Assignment 4: Gauss Markov Theorem

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[ ]:
```
import numpy as np
from numpy.linalg import inv
from matplotlib import pyplot as plt
```

1 Question 1

$E(\underline{y}) = Ax$

$D(\underline{y}) = Q_y = \sigma^2 I_3$

$x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$

$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$

[ ]:
```
A = np.array([[1,0],[-1,1],[0,-1]])
inv_AT_A = inv(A.T @ A)
sqrt_inv_AT_A = np.sqrt(inv_AT_A)
```

$(A^T A)^-1 = \begin{bmatrix} 0.67 & 0.33 \\ 0.33 & 0.67 \end{bmatrix}$

$Q_{\hat{x}} = \sigma^2 (A^T A)^{-1}$

$Q_{\hat{x}} = \begin{bmatrix} 0.67\sigma^2 & 0.33\sigma^2 \\ 0.33\sigma^2 & 0.67\sigma^2 \end{bmatrix}$

$\sigma_{x_1} = 0.82\sigma$

2 Question 2

$\underline{y} = \begin{bmatrix} \underline{y}_1 & \underline{y}_2 & \underline{y}_3 & \underline{y}_4 \end{bmatrix}^T$

$E(\underline{y}) = \begin{bmatrix} x & x & x & x \end{bmatrix}^T$

$Q_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

2.1 2-a

$E(\underline{y}) = Ax$

$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$

2.2 2-b

$\underline{\hat{x}} = (A^T Q_y^{-1} A)^{-1} A^T Q_y^{-1} \underline{y}$

[ ]:
```
A = np.array([[1],[1],[1],[1]])
Qy = np.array([[1,0,0,0],[0,2,0,0],[0,0,3,0],[0,0,0,4]])

W = inv(Qy)
inv_AT_W_A = inv(A.T @ W @ A)
AT_W = A.T @ W
xhat_matrix = inv_AT_W_A @ AT_W
```

$\underline{\hat{x}} = \begin{bmatrix} 0.48 & 0.24 & 0.16 & 0.12 \end{bmatrix} \begin{bmatrix} \underline{y}_1 & \underline{y}_2 & \underline{y}_3 & \underline{y}_4 \end{bmatrix}^T$

$\underline{\hat{x}} = 0.48\underline{y}_1 + 0.24\underline{y}_2 + 0.16\underline{y}_3 + 0.12\underline{y}_4$

2.3 2-c

$Q_{\hat{x}} = (A^T Q_y^{-1} A)^{-1}$

[ ]:
```
Qxhat = inv_AT_W_A
```

$Q_{\hat{x}} = 0.48$

3 Question 3

$\underline{y} = \begin{bmatrix} 319 & 325 & 321 \end{bmatrix}^T$

$Q_y = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$

[ ]:
```
y = np.array([[319],[325],[321]])
A = np.array([[1],[1],[1]])
Qy = np.array([[9,0,0],[0,9,0],[0,0,1]])

W = inv(Qy)
Qxhat = inv(A.T @ W @ A)
xhat = Qxhat @ A.T @ W @ y
std_xhat = np.sqrt(Qxhat)
```

$\underline{\hat{x}} = (A^T Q_y^{-1} A)^{-1} A^T Q_y^{-1} \underline{y}$

$\hat{\underline{x}} = 321.18$ cm

$\sigma_x = 0.905$ cm

4 Question 4

$\underline{y} = \begin{bmatrix} \underline{y}_1 & \underline{y}_2 & \underline{y}_3 \end{bmatrix}^T$

$x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

$E(\underline{y}) = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Q_y = \sigma^2 I_3$

4.1 4-a

$\hat{\underline{x}} = (A^T Q_y^{-1} A)^{-1} A^T Q_y^{-1} \underline{y}$

$\hat{\underline{x}} = (A^T A)^{-1} A^T \underline{y}$

[ ]:
```
A = np.array([[1,0],[0,1],[1,1]])
xhat_matrix = inv(A.T @ A) @ A.T
```

$\hat{\underline{x}} = \begin{bmatrix} 0.67 & -0.33 & 0.33 \\ -0.33 & 0.67 & 0.33 \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \end{bmatrix}$

$\hat{\underline{x}}_1 = 0.67\underline{y}_1 - 0.33\underline{y}_2 + 0.33\underline{y}_3$

$\hat{\underline{x}}_2 = -0.33\underline{y}_1 + 0.67\underline{y}_2 + 0.33\underline{y}_3$

4.2 4-b

$Q_{\hat{x}} = \sigma^2 (A^T A)^{-1}$

[ ]:
```
inv_AT_A = inv(A.T @ A)
```

$Q_{\hat{x}} = \sigma^2 \begin{bmatrix} 0.67 & -0.33 \\ -0.33 & 0.67 \end{bmatrix}$

Let $\hat{\underline{x}}_3 = \hat{\underline{x}}_1 + \hat{\underline{x}}_2$

$\hat{\underline{x}}_3 = B \begin{bmatrix} \hat{\underline{x}}_1 \\ \hat{\underline{x}}_2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \end{bmatrix}$

For linear case, the variance propagation law is $Q_{\hat{x}_3} = BQ_{\hat{x}}B^T$

[ ]:
```
Qxhat_matrix = inv_AT_A
B = np.array([[1,1]])

Qx3hat_matrix = B @ Qxhat_matrix @ B.T
```

$Q_{\hat{x}_3} = 0.67\sigma^2$

5 Question 5

$\underline{y} = \begin{bmatrix}\underline{y}_1 & \underline{y}_2\end{bmatrix}^T$

$E(\underline{y}) = \begin{bmatrix}x & x\end{bmatrix}^T$

$A = \begin{bmatrix}1 & 1\end{bmatrix}^T$

$Q_y = \sigma^2 \begin{bmatrix}1 & \rho \\ \rho & 1\end{bmatrix}; \rho = 0, \pm 1$

5.1 5-a

$Q_{\hat{x}} = (A^T Q_y^{-1} A)^{-1}$

$Q_y^{-1} = \sigma^{-2}(1-\rho^2)^{-1} \begin{bmatrix}1 & -\rho \\ -\rho & 1\end{bmatrix}$

$Q_{\hat{x}} = \left(\begin{bmatrix}1 & 1\end{bmatrix} \sigma^{-2}(1-\rho^2)^{-1} \begin{bmatrix}1 & -\rho \\ -\rho & 1\end{bmatrix} \begin{bmatrix}1 \\ 1\end{bmatrix}\right)^{-1}$

$Q_{\hat{x}} = \sigma^2(1-\rho^2) \left(\begin{bmatrix}1 & 1\end{bmatrix} \begin{bmatrix}1-\rho \\ 1-\rho\end{bmatrix}\right)^{-1}$

$$Q_{\hat{x}} = \frac{\sigma^2(1+\rho)}{2}$$

5.2 5-b

$$Q_{\hat{x}}(\rho = 0) = \frac{\sigma^2}{2}$$

For $\rho = 0$, the measurements are independent. $Q_{\hat{x}}$ is equivalent to the variance of the mean of the independent measurements.

$$Q_{\hat{x}}(\rho = 1) = \sigma^2$$
$$Q_{\hat{x}}(\rho = -1) = 0$$

For $\rho = \pm 1$, The measurments are perfectly correlated. If the correlation is perfectly positive ($\rho = 1$) then the variance of BLUE of x achieves highest possible value. This is because, for perfect positive correlation, the uncertainities get added up in the BLUE, and the variance of the mean in this case will be equl to the variance of the individual measurement.

If the correlation is perfectly negative ($\rho = -1$), then the variance of BLUE of x acheives lowest possible value. This is because, for perfect negative correlation, the uncertainities get cancelled in the BLUE, causing the variance of the mean to become zero.