Random Numbers

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I. Uniform Random Variables

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution Download the following C file.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gen_uniform.
   c
```

And run it using

```
cc gen_uniform.c
./a.out > uni.dat
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat.
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_uni.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/uni.dat
```

And run it using

```
python3 cdf_uni.py
```

It will generate the plot in Figure (1).

1.3 Find a theoretical expression for the CDF of U.
Solution The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

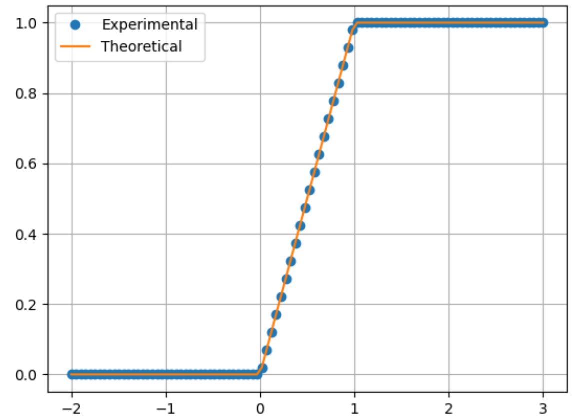

Fig. 1: CDF of U

And so we can find the CDF,

$$F_U(x) = \int_{-\infty}^{x} p_U(x)dx \tag{2}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{3}$$

1.4 Write a C program to find the mean and variance of U.
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/mean_var.c
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/uni.dat
```

And run it using

```
cc mean_var.c
./a.out < uni.dat
```

It will give the output

```
Mean : 0.499631
Variance: 0.083320
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4}$$

Solution We use the fact that

$$dF_U(x) = p_U(x)dx \tag{5}$$

So from eq (1), we have

$$E\left[U^k\right] = \int_{-\infty}^{+\infty} x^k p_U(x)dx \tag{6}$$

$$= \int_0^1 x^k dx \tag{7}$$

$$= \left(\frac{x^{k+1}}{k+1}\right)\Bigg|_0^1 \tag{8}$$

$$= \frac{1}{k+1} \tag{9}$$

Now using eq (9), we can find the mean of U

$$\mu = E[U] = \frac{1}{2} \tag{10}$$

and the variance

$$\text{var}[U] = E\left[U^2\right] - E[U]^2 \tag{11}$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{12}$$

$$= \frac{1}{12} = 0.83333.. \tag{13}$$

We see that the theoretical mean and variance match with the experimental values.

II. Central Limit Theorem

Let X be a random variable defined as

$$X = \sum_{i=1}^{12} U_i - 6 \tag{14}$$

2.1 Generate 10^6 samples of X using a C program, and save into a file called gau.dat
Solution Download the following C file.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gen_gaussian
   .c
```

And run it using

```
cc gen_gaussian.c
./a.out > gau.dat
```

2.2 Load the gau.dat file into python and plot the empirical CDF of X using the samples in gau.dat. What properties does the CDF have?
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_gau.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gau.dat
```

And run it using

```
python3 cdf_gau.py
```

It will generate the plot in Figure (2).

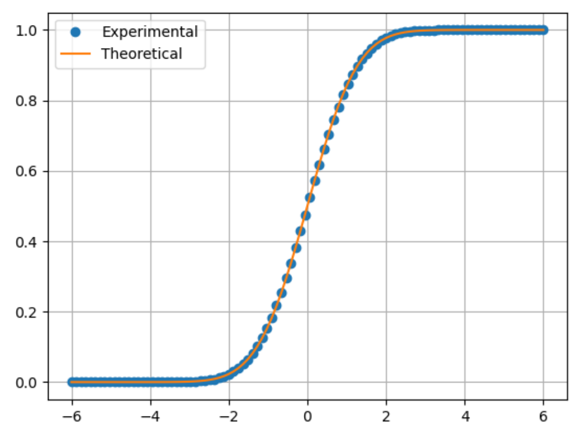

Fig. 2: CDF of X

Properties of the CDF:

- $\lim_{x \to -\infty} F_X(x) = 0$
- $\lim_{x \to \infty} F_X(x) = 1$
- $F_X(0) = \frac{1}{2}$
- $F_X(x) + F_X(-x) = 1$

2.3 Load the gau.dat file into python and plot the empirical PDF of X using the samples in gau.dat. What properties does the PDF have?
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/pdf_gau.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gau.dat
```

And run it using

```
python3 pdf_gau.py
```

It will generate the plot in Figure (3).

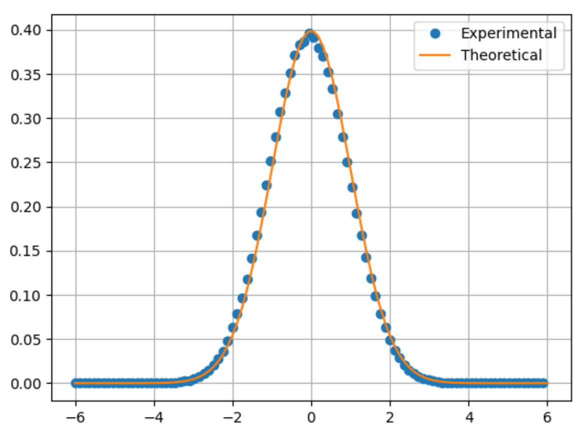

Fig. 3: PDF of X

Properties of the PDF:

- $F_X(x) = F_X(-x)$ i.e., symmetric about 0
- PDF is bell shaped
- Peak of PDF is also the mean

2.4 Write a C program to find the mean and variance of X.
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/mean_var.c
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gau.dat
```

And run it using

```
cc mean_var.c
./a.out < gau.dat
```

It will give the output

```
Mean : 0.000635
Variance: 0.999490
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{15}$$

repeat the above exercise theoretically.
Solution The mean is given by

$$E\left[X\right] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{16}$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{17}$$

Since this is the integral of an odd function over an odd interval, and the function goes to zero as x diverges,

$$E\left[U\right] = 0 \tag{18}$$

To calculate variance of X

$$\text{var}(X) = E\left[X - E[X]\right]^2 \tag{19}$$

$$= E\left[X^2\right] \tag{20}$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{21}$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{22}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot x e^{-\frac{x^2}{2}} dx \tag{23}$$

Integrating by parts, we get

$$\text{var}(X) = \frac{1}{\sqrt{2\pi}} \left(-x \exp\left(\frac{-x^2}{2}\right) + \int \exp\left(\frac{-x^2}{2}\right) dx \right) \Bigg|_{-\infty}^{\infty} \tag{24}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx \tag{25}$$

(26)

Substituting the Gaussian integral,

$$\text{var}(X) = 1 \tag{27}$$

III. From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{28}$$

and plot its CDF

Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_v.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/uni.dat
```

And run it using

```
python3 cdf_v.py
```

It will generate the plot in Figure (4).

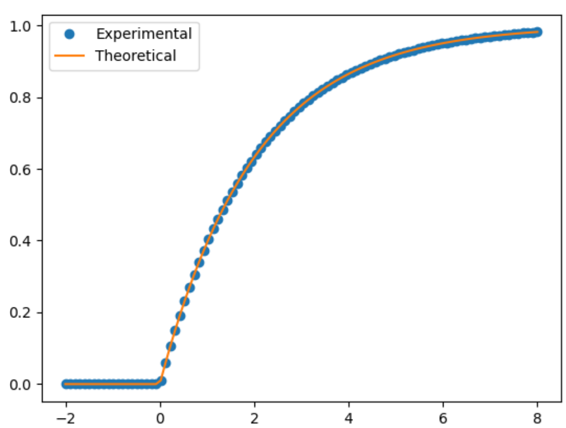

Fig. 4: CDF of V

3.2 Find a theoretical expression for $F_V(x)$

Solution

$$F_V(x) = \Pr(x \le V) \tag{29}$$

$$= \Pr(x \le -2\log(1-U)) \tag{30}$$

$$= \Pr\left(\log(1-U) \le \frac{-x}{2}\right) \tag{31}$$

$$= \Pr\left(1-U \le \exp\left(\frac{-x}{2}\right)\right) \tag{32}$$

$$= \Pr\left(1-\exp\left(\frac{-x}{2}\right) \le U\right) \tag{33}$$

$$= F_U\left(1-\exp\left(\frac{-x}{2}\right)\right) \tag{34}$$

We know $F_U(x)$ from eq (3), so we have

$$F_V(x) = F_U\left(1-\exp\left(\frac{-x}{2}\right)\right) \tag{35}$$

$$= \begin{cases} 0 & 1-e^{\frac{-x}{2}} < 0 \\ 1-e^{\frac{-x}{2}} & 0 < 1-e^{\frac{-x}{2}} < 1 \\ 1 & 1-e^{\frac{-x}{2}} > 1 \end{cases} \tag{36}$$

Simplifying, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1-\exp\left(\frac{-x}{2}\right) & x \ge 0 \end{cases} \tag{37}$$

IV. Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{38}$$

Solution Download the following C file.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/
   gen_triangular.c
```

And run it using

```
cc gen_triangular.c
./a.out > tri.dat
```

4.2 Find the CDF of T

Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_tri_num.
   py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/tri.dat
```

And run it using

```
python3 cdf_tri_num.py
```

It will generate the plot in Figure (5).

4.3 Find the PDF of T

Solution Download the following files

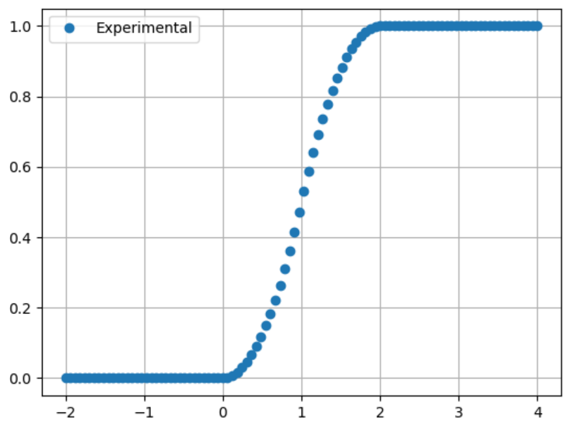

Fig. 5: Experimental CDF of V

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/pdf_tri_num.
   py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/tri.dat
```

And run it using

```
python3 pdf_tri_num.py
```

It will generate the plot in Figure (6).

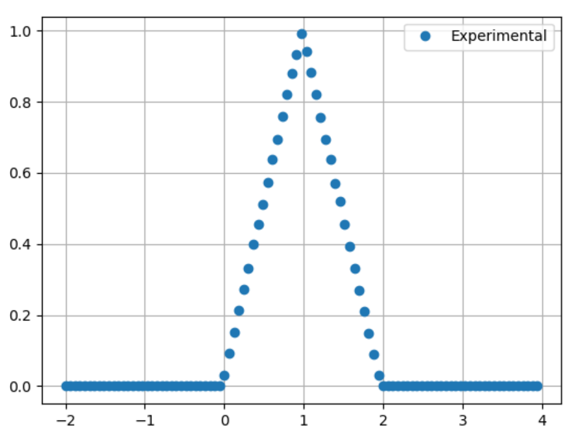

Fig. 6: Experimental PDF of X

4.4 Find the theoretical expressions for the CDF and PDF of T

Solution

$$T = U_1 + U_2 \tag{39}$$

So we have

$$p_T(t) = (p_{U_1} * p_{U_2})(t) \tag{40}$$

Since U_1 and U_2 are i.i.d., $p_{U_1}(t) = p_{U_2}(t) = p_U(t)$, which is given by eq (1).

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau) p_U(t-\tau) d\tau \tag{41}$$

When $\tau < 0$ or $\tau > 1$, the integral evaluates to 0.

$$p_T(t) = \int_0^1 p_U(\tau) p_U(t-\tau) d\tau \tag{42}$$

$$= \int_0^1 p_U(t-\tau) d\tau \tag{43}$$

When $t < 0$ or $t > 2$, the integral evaluates to 0. When $0 < t < 1$,

$$p_T(t) = \int_0^1 p_U(t-\tau) d\tau \tag{44}$$

$$= \int_0^t d\tau + \int_t^1 0 \cdot d\tau \tag{45}$$

$$= t \tag{46}$$

When $1 < t < 2$,

$$p_T(t) = \int_0^1 p_U(t-\tau) d\tau \tag{47}$$

$$= \int_0^{t-1} 0 \cdot d\tau + \int_{t-1}^1 d\tau \tag{48}$$

$$= 2 - t \tag{49}$$

Therefore the PDF is

$$p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < 1 \\ 2-t & 1 < t < 2 \\ 0 & t > 2 \end{cases} \tag{50}$$

The CDF of T is given by

$$F_T(t) = \int_0^t p_T(t) dt \tag{51}$$

Simplifying, we get

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 < t < 1 \\ -\frac{t^2}{2} + 2t - 1 & 1 < t < 2 \\ 1 & t > 2 \end{cases} \tag{52}$$

4.5 Verify your results through a plot
Solution Download the following files

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/pdf_tri.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_tri.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/tri.dat
```

And run them using

```
python3 pdf_tri.py
python3 cdf_tri.py
```

They will generate the plots in Figures (7) and (8).

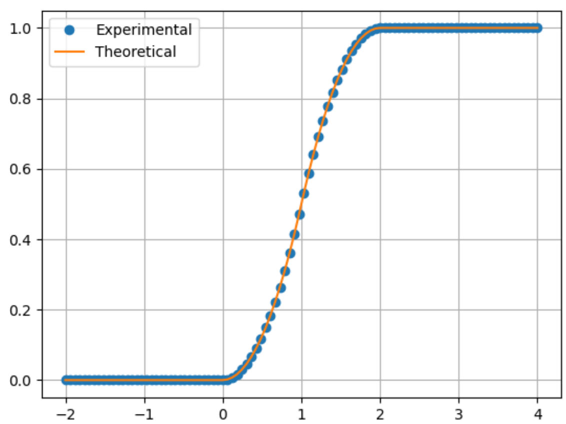

Fig. 7: CDF of X

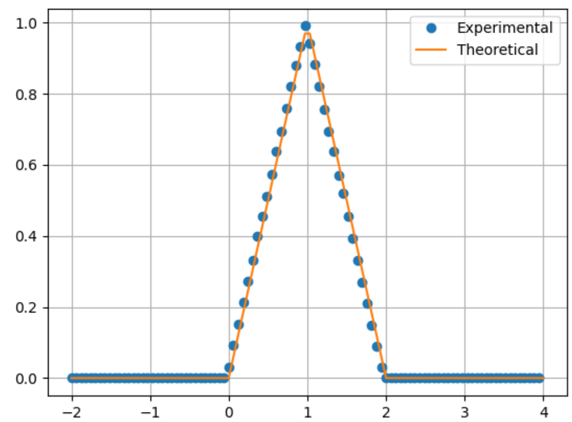

Fig. 8: PDF of X