Random Numbers

Kartheek Tammana

I. Uniform Random Variables

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution Download the following C file.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gen_uniform.
   c
```

And run it using

```
cc gen_uniform.c
./a.out > uni.dat
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat.
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_uni.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/uni.dat
```

And run it using

```
python3 cdf_uni.py
```

It will generate the plot in Figure (1).

1.3 Find a theoretical expression for the CDF of U.
Solution The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

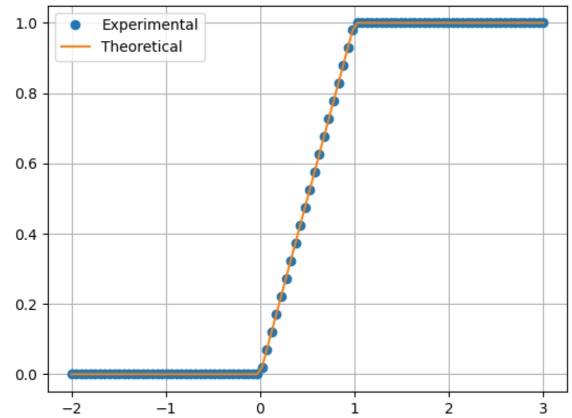

Fig. 1: CDF of U

And so we can find the CDF,

$$F_U(x) = \int_{-\infty}^{x} p_U(x) dx \tag{2}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{3}$$

1.4 Write a C program to find the mean and variance of U.
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/mean_var.c
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/uni.dat
```

And run it using

```
cc mean_var.c
./a.out < uni.dat
```

It will give the output

```
Mean : 0.499631
Variance: 0.083320
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4}$$

Solution We use the fact that

$$dF_U(x) = p_U(x)dx \tag{5}$$

So from eq (1), we have

$$E\left[U^k\right] = \int_{-\infty}^{+\infty} x^k p_U(x)dx \tag{6}$$

$$= \int_0^1 x^k dx \tag{7}$$

$$= \left(\frac{x^{k+1}}{k+1}\right)\Big|_0^1 \tag{8}$$

$$= \frac{1}{k+1} \tag{9}$$

Now using eq (9), we can find the mean of U

$$\mu = E\left[U\right] = \frac{1}{2} \tag{10}$$

and the variance

$$\text{var}\left[U\right] = E\left[U^2\right] - E\left[U\right]^2 \tag{11}$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{12}$$

$$= \frac{1}{12} = 0.83333.. \tag{13}$$

We see that the theoretical mean and variance match with the experimental values.

II. Central Limit Theorem

Let X be a random variable defined as

$$X = \sum_{i=1}^{12} U_i - 6 \tag{14}$$

2.1 Generate 10^6 samples of X using a C program, and save into a file called gau.dat
Solution Download the following C file.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gen_gaussian
   .c
```

And run it using

```
cc gen_gaussian.c
./a.out > gau.dat
```

2.2 Load the gau.dat file into python and plot the empirical CDF of X using the samples in gau.dat. What properties does the CDF have?
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_gau.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gau.dat
```

And run it using

```
python3 cdf_gau.py
```

It will generate the plot in Figure (2).

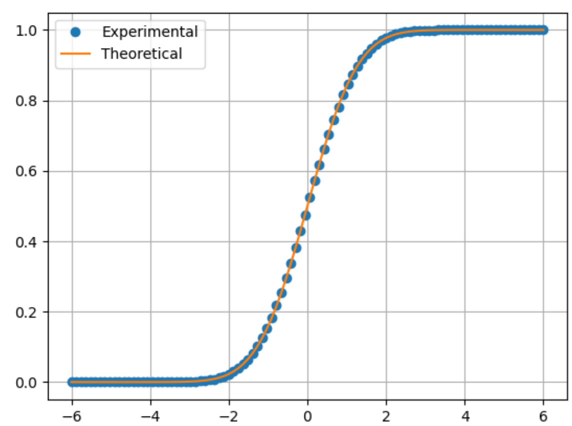

Fig. 2: CDF of X

Properties of the CDF:

- $\lim_{x\to-\infty} F_X(x) = 0$
- $\lim_{x\to\infty} F_X(x) = 1$
- $F_X(0) = \frac{1}{2}$
- $F_X(x) + F_X(-x) = 1$

2.3 Load the gau.dat file into python and plot the empirical PDF of X using the samples in gau.dat. What properties does the PDF have?
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/pdf_gau.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gau.dat
```

And run it using

```
python3 pdf_gau.py
```

It will generate the plot in Figure (3).

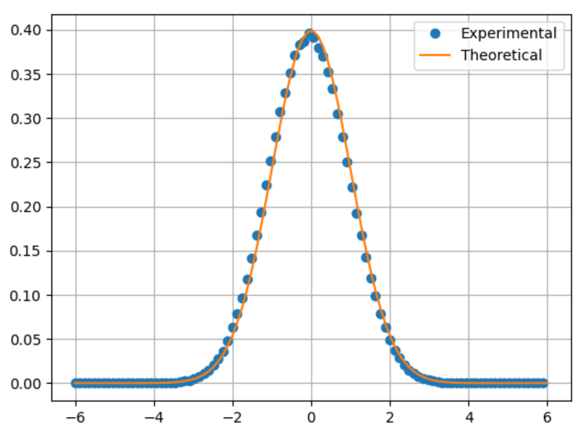

Fig. 3: PDF of X

Properties of the PDF:
- $F_X(x) = F_X(-x)$ i.e., symmetric about 0
- PDF is bell shaped
- Peak of PDF is also the mean

2.4 Write a C program to find the mean and variance of X.
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/mean_var.c
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gau.dat
```

And run it using

```
cc mean_var.c
./a.out < gau.dat
```

It will give the output

```
Mean : 0.000635
Variance: 0.999490
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{15}$$

repeat the above exercise theoretically.
Solution The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{16}$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{17}$$

Since this is the integral of an odd function over an odd interval, and the function goes to zero as x diverges,

$$E[U] = 0 \tag{18}$$

To calculate variance of X

$$\text{var}(X) = E\left[X - E[X]\right]^2 \tag{19}$$

$$= E\left[X^2\right] \tag{20}$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{21}$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{22}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot x e^{-\frac{x^2}{2}} dx \tag{23}$$

Integrating by parts, we get

$$\text{var}(X) = \frac{1}{\sqrt{2\pi}} \left(-x \exp\left(\frac{-x^2}{2}\right) + \int \exp\left(\frac{-x^2}{2}\right) dx \right)\Bigg|_{-\infty}^{\infty} \tag{24}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx \tag{25}$$

$$\tag{26}$$

Substituting the Gaussian integral,

$$\text{var}(X) = 1 \tag{27}$$

III. From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{28}$$

and plot its CDF
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_v.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/uni.dat
```

And run it using

```
python3 cdf_v.py
```

It will generate the plot in Figure (4).

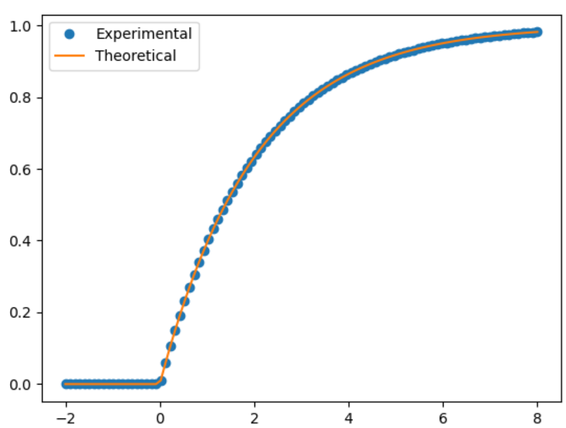

Fig. 4: CDF of V

3.2 Find a theoretical expression for $F_V(x)$
Solution

$$F_V(x) = \Pr(x \le V) \tag{29}$$
$$= \Pr(x \le -2\log(1-U)) \tag{30}$$
$$= \Pr\left(\log(1-U) \le \frac{-x}{2}\right) \tag{31}$$
$$= \Pr\left(1-U \le \exp\left(\frac{-x}{2}\right)\right) \tag{32}$$
$$= \Pr\left(1-\exp\left(\frac{-x}{2}\right) \le U\right) \tag{33}$$
$$= F_U\left(1-\exp\left(\frac{-x}{2}\right)\right) \tag{34}$$

We know $F_U(x)$ from eq (3), so we have

$$F_V(x) = F_U\left(1-\exp\left(\frac{-x}{2}\right)\right) \tag{35}$$
$$= \begin{cases} 0 & 1-e^{\frac{-x}{2}} < 0 \\ 1-e^{\frac{-x}{2}} & 0 < 1-e^{\frac{-x}{2}} < 1 \\ 1 & 1-e^{\frac{-x}{2}} > 1 \end{cases} \tag{36}$$

Simplifying, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1-\exp\left(\frac{-x}{2}\right) & x \ge 0 \end{cases} \tag{37}$$

IV. Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{38}$$

Solution Download the following C file.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/
   gen_triangular.c
```

And run it using

```
cc gen_triangular.c
./a.out > tri.dat
```

4.2 Find the CDF of T
Solution Download the following files.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_tri_num.
   py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/tri.dat
```

And run it using

```
python3 cdf_tri_num.py
```

It will generate the plot in Figure (5).

4.3 Find the PDF of T
Solution Download the following files

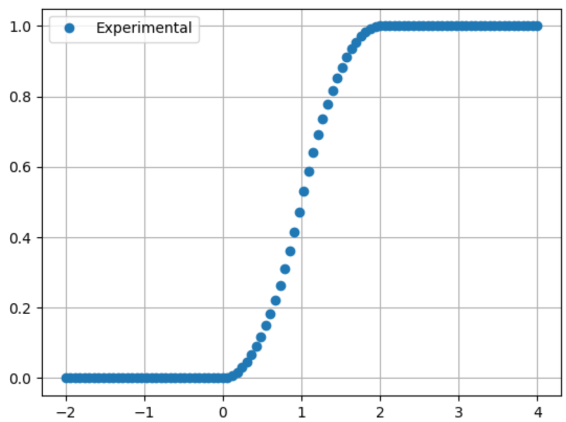

Fig. 5: Experimental CDF of V

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/pdf_tri_num.
   py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/tri.dat
```

And run it using

```
python3 pdf_tri_num.py
```

It will generate the plot in Figure (6).

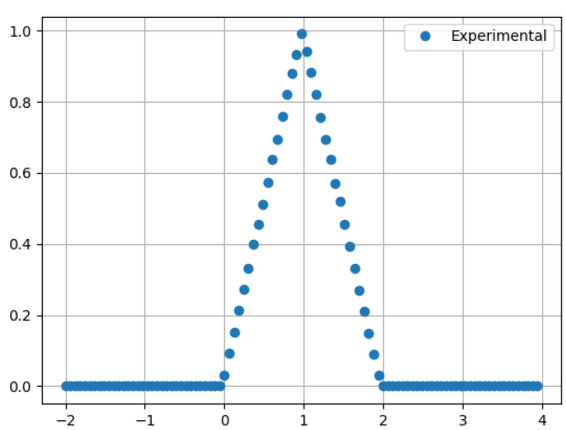

Fig. 6: Experimental PDF of X

4.4 Find the theoretical expressions for the CDF and PDF of T
Solution

$$T = U_1 + U_2 \tag{39}$$

So we have

$$p_T(t) = (p_{U_1} * p_{U_2})(t) \tag{40}$$

Since U_1 and U_2 are i.i.d., $p_{U_1}(t) = p_{U_2}(t) = p_U(t)$, which is given by eq (1).

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau)p_U(t-\tau)d\tau \tag{41}$$

When $\tau < 0$ or $\tau > 1$, the integral evaluates to 0.

$$p_T(t) = \int_0^1 p_U(\tau)p_U(t-\tau)d\tau \tag{42}$$

$$= \int_0^1 p_U(t-\tau)d\tau \tag{43}$$

When $t < 0$ or $t > 2$, the integral evaluates to 0. When $0 < t < 1$,

$$p_T(t) = \int_0^1 p_U(t-\tau)d\tau \tag{44}$$

$$= \int_0^t d\tau + \int_t^1 0 \cdot d\tau \tag{45}$$

$$= t \tag{46}$$

When $1 < t < 2$,

$$p_T(t) = \int_0^1 p_U(t-\tau)d\tau \tag{47}$$

$$= \int_0^{t-1} 0 \cdot d\tau + \int_{t-1}^1 d\tau \tag{48}$$

$$= 2 - t \tag{49}$$

Therefore the PDF is

$$p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < 1 \\ 2-t & 1 < t < 2 \\ 0 & t > 2 \end{cases} \tag{50}$$

The CDF of T is given by

$$F_T(t) = \int_0^t p_T(t)dt \tag{51}$$

Simplifying, we get

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 < t < 1 \\ -\frac{t^2}{2} + 2t - 1 & 1 < t < 2 \\ 1 & t > 2 \end{cases} \tag{52}$$

4.5 Verify your results through a plot
Solution Download the following files

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/pdf_tri.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/cdf_tri.py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/tri.dat
```

And run them using

```
python3 pdf_tri.py
python3 cdf_tri.py
```

They will generate the plots in Figures (7) and (8).

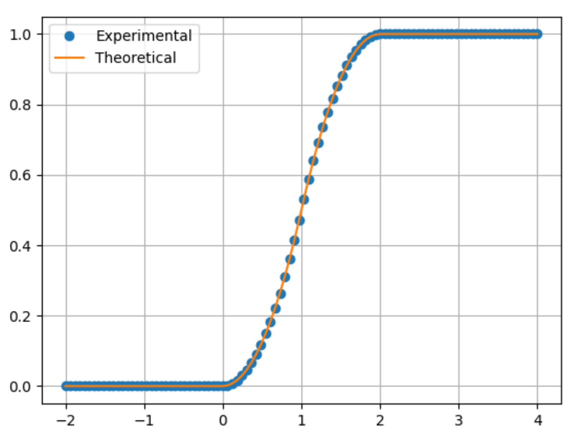

Fig. 7: CDF of X

V. Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution Download the following C file

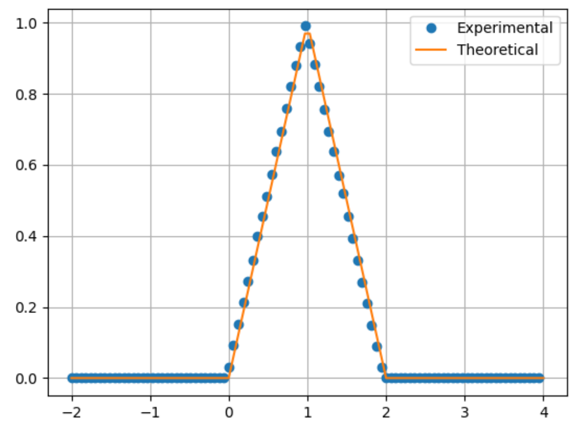

Fig. 8: PDF of X

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/
   gen_bernoulli.c
```

And run it using

```
cc gen_bernoulli.c
./a.out > ber.dat
```

5.2 Generate

$$Y = AX + N \tag{53}$$

where A = 5 dB, and $N \sim \mathcal{N}(0, 1)$.
Solution Download the following files

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gen_y.py
```

And run it using

```
python gen_y.py
```

It will create the file `maxlike.dat` containing Y.

5.3 Plot Y using a scatter plot.
Solution Download the following files

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/scatter_y.py
```

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/maxlike.dat
```

And run it using

```
python scatter_y.py
```

It will generate the plot in Figure (9).

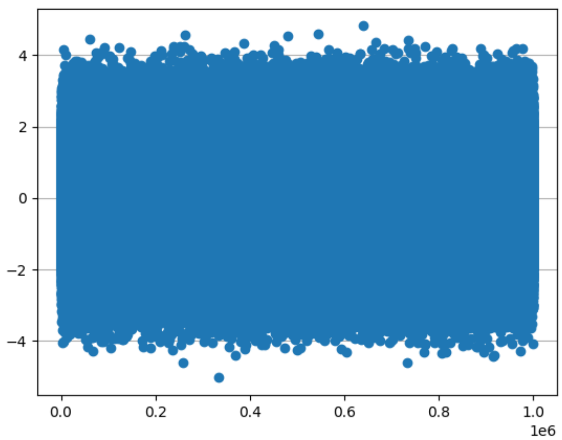

Fig. 9: Scatter plot of Y

5.4 Guess how to estimate X from Y.
Solution We can estimate X using

$$\hat{X} = \begin{cases} 1 & Y \geq 0 \\ -1 & Y < 0 \end{cases} \tag{54}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{55}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = 0\right) \tag{56}$$

Solution We have

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{57}$$
$$= \Pr\left(AX + N < 0|X = 1\right) \tag{58}$$
$$= \Pr\left(A + N < 0\right) \tag{59}$$
$$= \Pr\left(N < -A\right) \tag{60}$$
$$= F_N(-A) \tag{61}$$
$$= Q(A) \tag{62}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{63}$$
$$= \Pr\left(AX + N \geq 0|X = -1\right) \tag{64}$$
$$= \Pr\left(-A + N \geq 0\right) \tag{65}$$
$$= \Pr\left(N \geq A\right) \tag{66}$$
$$= Q(A) \tag{67}$$

So we have our result

$$P_{e|0} = P_{e|1} = Q(A) \tag{68}$$

5.6 Find P_e assuming X has equiprobable symbols
Solution We need to find

$$P_e = P_{e|0} \cdot \Pr\left(X = -1\right) + P_{e|1} \cdot \Pr\left(X = 1\right) \tag{69}$$
$$= Q(A) \cdot \frac{1}{2} + Q(A) \cdot \frac{1}{2} \tag{70}$$
$$P_e = Q(A) \tag{71}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB
Solution Download the following files

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/error_prob.
   py
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/ber.dat
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gau.dat
```

And run it using

```
python scatter_y.py
```

It will generate the plot in Figure (10).

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution We can define our guess as

$$\hat{X} = \begin{cases} 1 & Y \geq \delta \\ -1 & Y < \delta \end{cases} \tag{72}$$

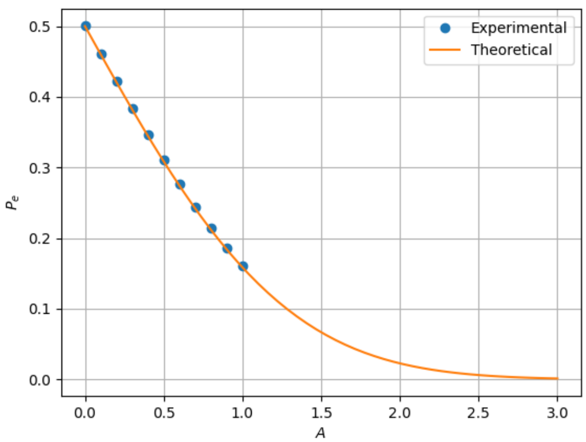

Fig. 10: Scatter plot of Y

Now,

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{73}$$
$$= \Pr\left(AX + N < \delta|X = 1\right) \tag{74}$$
$$= \Pr\left(A + N < \delta\right) \tag{75}$$
$$= \Pr\left(N < -A + \delta\right) \tag{76}$$
$$= F_N(-(A - \delta)) \tag{77}$$
$$= Q(A - \delta) \tag{78}$$

And similarly,

$$P_{e|1} = Q(A + \delta) \tag{79}$$

So we have,

$$P_e = P_{e|0} \cdot \Pr\left(X = -1\right) + P_{e|1} \cdot \Pr\left(X = 1\right) \tag{80}$$
$$= Q(A - \delta) \cdot \frac{1}{2} + Q(A + \delta) \cdot \frac{1}{2} \tag{81}$$

To find the minimal P_e, we differentiate Eq (81) wrt δ and equate it to 0

$$0 = \frac{d}{d\delta}\left(\frac{1}{2}\left(Q(A - \delta) + Q(A + \delta)\right)\right) \tag{82}$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(A-\delta)^2}{2}} - e^{-\frac{(A+\delta)^2}{2}}\right) \tag{83}$$

So we have

$$(A - \delta)^2 = (A + \delta)^2 \tag{84}$$
$$\implies \delta = 0 \tag{85}$$

So P_e is minimal when the threshold is $\delta = 0$.

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{86}$$

Solution We have

$$P_e = P_{e|0}p + P_{e|1}(1 - p) \tag{87}$$
$$= Q(A - \delta)p + Q(A + \delta)(1 - p) \tag{88}$$

Differentiating wrt δ to get minimum,

$$0 = \frac{d}{d\delta}(Q(A - \delta)p + Q(A + \delta)(1 - p)) \tag{89}$$
$$= pe^{-\frac{(A-\delta)^2}{2}} - (1 - p)e^{-\frac{(A+\delta)^2}{2}} \tag{90}$$

Taking ln,

$$\ln p - \frac{(A - \delta)^2}{2} = \ln(1 - p) - \frac{(A + \delta)^2}{2} \tag{91}$$
$$2A\delta = \ln\frac{1 - p}{p} \tag{92}$$
$$\delta = \frac{1}{2A}\ln\frac{1 - p}{p} \tag{93}$$

And so we have our result

5.10 Repeat the above exercise using the MAP criterion.
Solution Assume $\Pr\left(X = -1\right) = p$.
According to the MAP criterion, when

$$p_{X|Y}(-1|y) > p_{X|Y}(1|y) \tag{94}$$

We should guess -1, else we guess 1. Now,

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} \tag{95}$$

Now,

$$p_{Y|X}(y|-1) = p_{(-A+N)}(y) \tag{96}$$

Since A is a constant,

$$p_{Y|X}(y|-1) = p_N(y + A) \tag{97}$$
$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{(y+A)^2}{2}} \tag{98}$$

Substituting into (95),

$$p_{X|Y}(-1|y) = \frac{p_{Y|X}(y|-1)p_X(-1)}{p_Y(y)} \tag{99}$$
$$p_{X|Y}(-1|y) = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+A)^2}{2}}p}{p_Y(y)} \tag{100}$$

Similarly,

$$p_{X|Y}(1|y) = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}}(1-p)}{p_Y(y)} \tag{101}$$

Finally, substituting Eqs (100) and (101) into Eq (94), we should guess that $X = -1$ when

$$\frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} p}{p_Y(y)} > \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}}(1-p)}{p_Y(y)} \tag{102}$$

$$\frac{p}{1-p} > e^{2Ay} \tag{103}$$

$$y < \frac{1}{2A} \ln \frac{p}{1-p} \tag{104}$$

And so we have our guess,

$$\hat{X} = \begin{cases} -1 & y < \delta \\ 1 & \text{otherwise} \end{cases} \tag{105}$$

where $\delta = \frac{1}{2A} \ln \frac{p}{1-p}$.
Consider the special case $p = \frac{1}{2}$

$$\delta = \frac{1}{2A} \ln 1 \tag{106}$$

$$\delta = 0 \tag{107}$$

So our guess is,

$$\hat{X} = \begin{cases} -1 & y < 0 \\ 1 & y > 0 \end{cases} \tag{108}$$