I. UNIFORM RANDOM VARIABLES

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat

Solution Download the following C file.

```
wget https://raw.
   githubusercontent.com/kst164
   /AI1110-Assignments/main/
   rand_nums/codes/gen_uniform.
   c
```

And run it using

```
cc gen_uniform.c
./a.out > uni.dat
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat.

Solution Download the following files.

```
wget https://raw.
    githubusercontent.com/kst164
    /AI1110-Assignments/main/
    rand_nums/codes/cdf_uni.py
wget https://raw.
    githubusercontent.com/kst164
    /AI1110-Assignments/main/
    rand_nums/codes/uni.dat
```

And run it using

It will generate the plot in Figure (1).

1.3 Find a theoretical expression for the CDF of U.

Solution The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & 0 \le x \le 1\\ 0 & \text{otherwise} \end{cases} \tag{1}$$

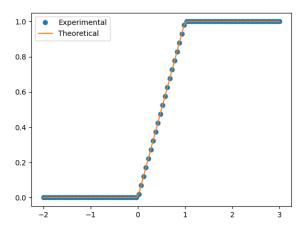
And so we can find the CDF.

$$F_U(x) = \int_{-\infty}^x p_U(x)dx \tag{2}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$
 (3)

1.4 Write a C program to find the mean and variance of U.

Solution Download the following files.



1

Fig. 1: CDF of U

wget https://raw.
 githubusercontent.com/kst164
 /AI1110-Assignments/main/
 rand_nums/codes/mean_var.c
wget https://raw.
 githubusercontent.com/kst164
 /AI1110-Assignments/main/
 rand_nums/codes/uni.dat

And run it using

```
cc mean_var.c
./a.out < uni.dat</pre>
```

It will give the output

Mean : 0.499631 Variance: 0.083320

1.5 Verify your result theoretically given that

$$E\left[U^{k}\right] = \int_{-\infty}^{\infty} x^{k} dF_{U}(x) \tag{4}$$

Solution We use the fact that

$$dF_U(x) = p_U(x)dx \tag{5}$$

So from eq (1), we have

$$E\left[U^{k}\right] = \int_{-\infty}^{+\infty} x^{k} p_{U}(x) dx \tag{6}$$

$$= \int_0^1 x^k dx \tag{7}$$

$$= \left(\frac{x^{k+1}}{k+1}\right)\Big|_0^1 \tag{8}$$

$$=\frac{1}{k+1}\tag{9}$$

Now using eq (9), we can find the mean of U

$$\mu = E[U] = \frac{1}{2} \tag{10}$$

and the variance

$$var[U] = E[U^2] - E[U]^2$$
 (11)

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{12}$$

$$= \frac{1}{12} = 0.83333.. \tag{13}$$

We see that the theoretical mean and variance match with the experimental values.

II. CENTRAL LIMIT THEOREM

Let X be a random variable defined as

$$X = \sum_{i=1}^{12} U_i - 6 \tag{14}$$

2.1 Generate 10⁶ samples of X using a C program, and save into a file called gau.dat **Solution** Download the following C file.

And run it using

2.2 Load the gau.dat file into python and plot the empirical CDF of X using the samples in gau.dat. What properties does the CDF have? **Solution** Download the following files.

```
wget https://raw.
  githubusercontent.com/kst164
  /AI1110-Assignments/main/
  rand nums/codes/cdf gau.py
wget https://raw.
  githubusercontent.com/kst164
  /AI1110-Assignments/main/
  rand nums/codes/gau.dat
```

And run it using

It will generate the plot in Figure (2).

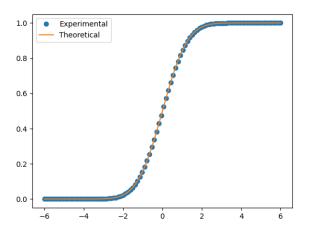


Fig. 2: CDF of X

Properties of the CDF:

- $F_X(x) = \frac{1}{2\pi} \int_{-\infty}^x \exp\left(-\frac{u^2}{2}\right) du$

- $\lim_{x \to -\infty} F_X(x) = 0$ $\lim_{x \to \infty} F_X(x) = 1$ $F_X(0) = \frac{1}{2}$ $F_X(x) + F_X(-x) = 1$
- 2.3 Load the gau.dat file into python and plot the empirical PDF of X using the samples in gau.dat. What properties does the PDF have? **Solution** Download the following files.

```
wget https://raw.
  githubusercontent.com/kst164
  /AI1110-Assignments/main/
  rand_nums/codes/pdf_gau.py
wget https://raw.
  githubusercontent.com/kst164
   /AI1110-Assignments/main/
  rand_nums/codes/gau.dat
```

And run it using

python3 pdf_gau.py

It will generate the plot in Figure (3).

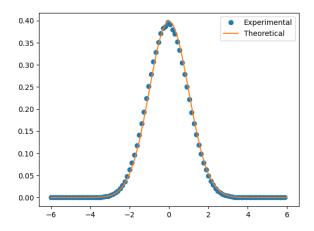


Fig. 3: PDF of X

Properties of the PDF:

- $F_X(x) = F_X(-x)$ i.e., symmetric about 0
- PDF is bell shaped
- Peak of PDF is also the mean
- 2.4 Write a C program to find the mean and variance of X.

Solution Download the following files.

wget https://raw.
 githubusercontent.com/kst164
 /AI1110-Assignments/main/
 rand_nums/codes/mean_var.c
wget https://raw.
 githubusercontent.com/kst164
 /AI1110-Assignments/main/
 rand_nums/codes/gau.dat

And run it using

cc mean_var.c ./a.out < gau.dat

It will give the output

Mean: 0.000635 Variance: 0.999490

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{15}$$

repeat the above exercise theoretically.

Solution The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{16}$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad (17)$$

Since this is the integral of an odd function over an odd interval, and the function goes to zero as x diverges,

$$E[U] = 0 (18)$$

To calculate variance of X

$$\operatorname{var}(X) = E\left[X - E[X]\right]^{2} \tag{19}$$

$$=E\left[X^{2}\right] \tag{20}$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{21}$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad (22)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) dx \tag{23}$$

Integrating by parts, we get

$$\operatorname{var}(X) = \frac{1}{\sqrt{2\pi}} \left(-x \exp\left(\frac{-x^2}{2}\right) + \int \exp\left(\frac{-x^2}{2}\right) dx \right) \tag{24}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx \qquad (25)$$
(26)

Substituting the Gaussian integral,

$$var(X) = 1 \tag{27}$$

III. FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{28}$$

and plot its CDF

Solution Download the following files.

And run it using

It will generate the plot in Figure (4).

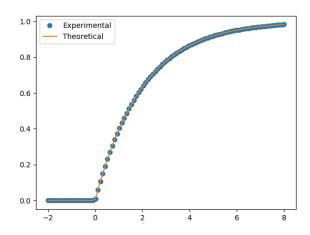


Fig. 4

3.2 Find a theoretical expression for $F_V(x)$ **Solution**

$$F_{V}(x) = \Pr(x \le V)$$

$$= \Pr(x \le -2\log(1 - U))$$

$$= \Pr\left(\log(1 - U) \le \frac{-x}{2}\right)$$

$$= \Pr\left(1 - U \le \exp\left(\frac{-x}{2}\right)\right)$$

$$= \Pr\left(1 - \exp\left(\frac{-x}{2}\right)\right)$$

$$= F_{U}\left(1 - \exp\left(\frac{-x}{2}\right)\right)$$

$$= (34)$$

We know $F_U(x)$ from eq (3), so we have

$$F_{V}(x) = F_{U}\left(1 - \exp\left(\frac{-x}{2}\right)\right)$$

$$= \begin{cases} 0 & 1 - \exp\left(\frac{-x}{2}\right) < 0\\ 1 - \exp\left(\frac{-x}{2}\right) & 0 < 1 - \exp\left(\frac{-x}{2}\right) < 1\\ 1 & 1 - \exp\left(\frac{-x}{2}\right) > 1 \end{cases}$$

$$(36)$$

Simplifying, we get

$$F_V(x) = \begin{cases} 0 & x < 0\\ 1 - \exp\left(\frac{-x}{2}\right) & x \ge 0 \end{cases}$$
 (37)