

# Random Numbers

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## I. UNIFORM RANDOM VARIABLES

Let  $U$  be a uniform random variable between 0 and 1.

- 1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat

**Solution** Download the following C file.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gen_uniform.c
```

And run it using

```
cc gen_uniform.c
./a.out > uni.dat
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat.

**Solution** Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_uni.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/uni.dat
```

And run it using

```
python3 cdf_uni.py
```

It will generate the plot in Figure (1).

- 1.3 Find a theoretical expression for the CDF of  $U$ .

**Solution** The PDF of  $U$  is given by

$$p_U(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

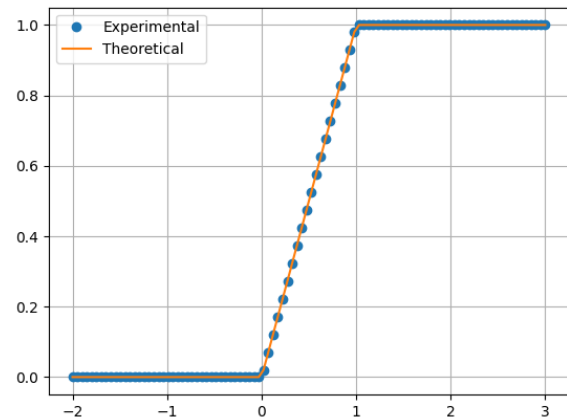


Fig. 1: CDF of  $U$

And so we can find the CDF,

$$F_U(x) = \int_{-\infty}^x p_U(x) dx \quad (2)$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (3)$$

- 1.4 Write a C program to find the mean and variance of  $U$ .

**Solution** Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/mean_var.c
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/uni.dat
```

And run it using

```
cc mean_var.c
./a.out < uni.dat
```

It will give the output

Mean : 0.499631  
Variance: 0.083320

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (4)$$

**Solution** We use the fact that

$$dF_U(x) = p_U(x)dx \quad (5)$$

So from eq (1), we have

$$E[U^k] = \int_{-\infty}^{+\infty} x^k p_U(x) dx \quad (6)$$

$$= \int_0^1 x^k dx \quad (7)$$

$$= \left( \frac{x^{k+1}}{k+1} \right) \Big|_0^1 \quad (8)$$

$$= \frac{1}{k+1} \quad (9)$$

Now using eq (9), we can find the mean of  $U$

$$\mu = E[U] = \frac{1}{2} \quad (10)$$

and the variance

$$\text{var}[U] = E[U^2] - E[U]^2 \quad (11)$$

$$= \frac{1}{3} - \left( \frac{1}{2} \right)^2 \quad (12)$$

$$= \frac{1}{12} = 0.083333.. \quad (13)$$

We see that the theoretical mean and variance match with the experimental values.

## II. CENTRAL LIMIT THEOREM

Let  $X$  be a random variable defined as

$$X = \sum_{i=1}^{12} U_i - 6 \quad (14)$$

2.1 Generate  $10^6$  samples of  $X$  using a C program, and save into a file called gau.dat

**Solution** Download the following C file.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gen_gaussian.c
```

And run it using

```
cc gen_gaussian.c
./a.out > gau.dat
```

2.2 Load the gau.dat file into python and plot the empirical CDF of  $X$  using the samples in gau.dat. What properties does the CDF have?

**Solution** Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_gau.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gau.dat
```

And run it using

```
python3 cdf_gau.py
```

It will generate the plot in Figure (2).

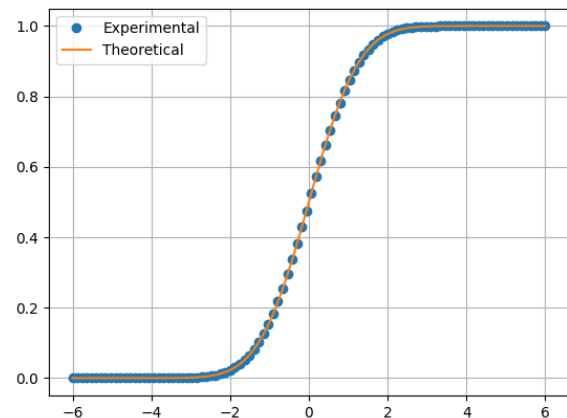


Fig. 2: CDF of  $X$

Properties of the CDF:

- $\lim_{x \rightarrow -\infty} F_X(x) = 0$
- $\lim_{x \rightarrow \infty} F_X(x) = 1$
- $F_X(0) = \frac{1}{2}$
- $F_X(x) + F_X(-x) = 1$

2.3 Load the gau.dat file into python and plot the empirical PDF of  $X$  using the samples in gau.dat. What properties does the PDF have?

**Solution** Download the following files.

```
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/pdf_gau.py
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/gau.dat
```

And run it using

```
python3 pdf_gau.py
```

It will generate the plot in Figure (3).

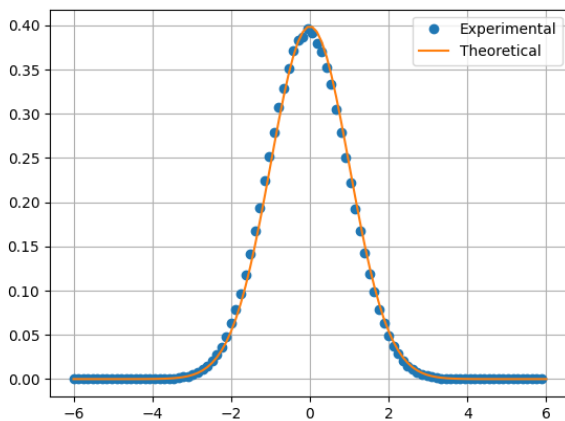


Fig. 3: PDF of  $X$

Properties of the PDF:

- $F_X(x) = F_X(-x)$  i.e., symmetric about 0
- PDF is bell shaped
- Peak of PDF is also the mean

2.4 Write a C program to find the mean and variance of  $X$ .

**Solution** Download the following files.

```
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/mean_var.c
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/gau.dat
```

And run it using

```
cc mean_var.c
./a.out < gau.dat
```

It will give the output

```
Mean : 0.000635
Variance: 0.999490
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (15)$$

repeat the above exercise theoretically.

**Solution** The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (16)$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (17)$$

Since this is the integral of an odd function over an odd interval, and the function goes to zero as  $x$  diverges,

$$E[U] = 0 \quad (18)$$

To calculate variance of  $X$

$$\text{var}(X) = E[X - E[X]]^2 \quad (19)$$

$$= E[X^2] \quad (20)$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (21)$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (22)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot x e^{-\frac{x^2}{2}} dx \quad (23)$$

Integrating by parts, we get

$$\text{var}(X) = \frac{1}{\sqrt{2\pi}} \left( -x \exp\left(\frac{-x^2}{2}\right) \right. \quad (24)$$

$$\left. + \int \exp\left(\frac{-x^2}{2}\right) dx \right) \Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx \quad (25)$$

$$(26)$$

Substituting the Gaussian integral,

$$\text{var}(X) = 1 \quad (27)$$

### III. FROM UNIFORM TO OTHER

#### 3.1 Generate samples of

$$V = -2\ln(1 - U) \quad (28)$$

and plot its CDF

**Solution** Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_v.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/uni.dat
```

And run it using

```
python3 cdf_v.py
```

It will generate the plot in Figure (4).

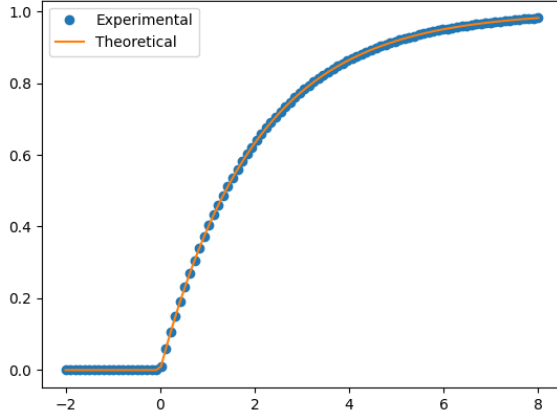


Fig. 4: CDF of  $V$

#### 3.2 Find a theoretical expression for $F_V(x)$

**Solution**

$$F_V(x) = \Pr(x \leq V) \quad (29)$$

$$= \Pr(x \leq -2\log(1 - U)) \quad (30)$$

$$= \Pr\left(\log(1 - U) \leq \frac{-x}{2}\right) \quad (31)$$

$$= \Pr\left(1 - U \leq \exp\left(\frac{-x}{2}\right)\right) \quad (32)$$

$$= \Pr\left(1 - \exp\left(\frac{-x}{2}\right) \leq U\right) \quad (33)$$

$$= F_U\left(1 - \exp\left(\frac{-x}{2}\right)\right) \quad (34)$$

We know  $F_U(x)$  from eq (3), so we have

$$F_V(x) = F_U\left(1 - \exp\left(\frac{-x}{2}\right)\right) \quad (35)$$

$$= \begin{cases} 0 & 1 - e^{\frac{-x}{2}} < 0 \\ 1 - e^{\frac{-x}{2}} & 0 < 1 - e^{\frac{-x}{2}} < 1 \\ 1 & 1 - e^{\frac{-x}{2}} > 1 \end{cases} \quad (36)$$

Simplifying, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(\frac{-x}{2}\right) & x \geq 0 \end{cases} \quad (37)$$

### IV. TRIANGULAR DISTRIBUTION

#### 4.1 Generate

$$T = U_1 + U_2 \quad (38)$$

**Solution** Download the following C file.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gen_triangular.c
```

And run it using

```
cc gen_triangular.c
./a.out > tri.dat
```

#### 4.2 Find the CDF of $T$

**Solution** Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_tri_num.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/tri.dat
```

And run it using

```
python3 cdf_tri_num.py
```

It will generate the plot in Figure (5).

#### 4.3 Find the PDF of $T$

**Solution** Download the following files

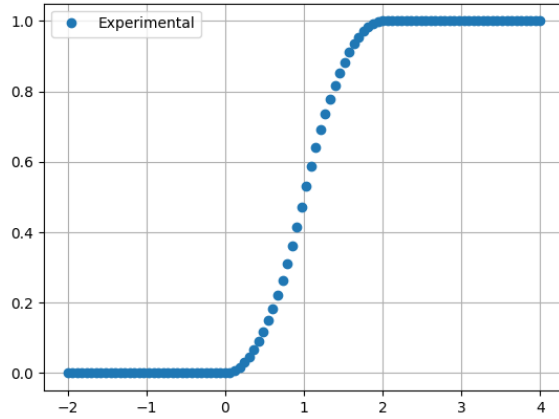


Fig. 5: Experimental CDF of V

```
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/pdf_tri_num.
py
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/tri.dat
```

And run it using

```
python3 pdf_tri_num.py
```

It will generate the plot in Figure (6).

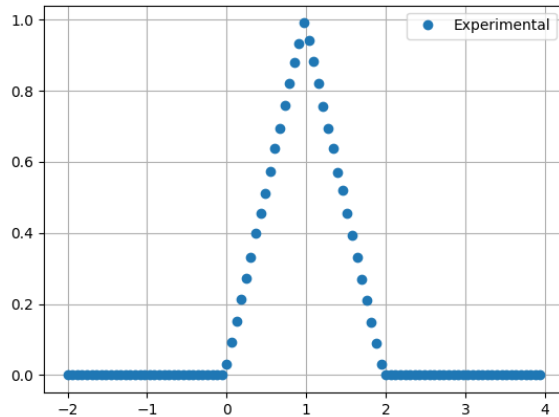


Fig. 6: Experimental PDF of X

4.4 Find the theoretical expressions for the CDF and PDF of  $T$

**Solution**

$$T = U_1 + U_2 \quad (39)$$

So we have

$$p_T(t) = (p_{U_1} * p_{U_2})(t) \quad (40)$$

Since  $U_1$  and  $U_2$  are i.i.d.,  $p_{U_1}(t) = p_{U_2}(t) = p_U(t)$ , which is given by eq (1).

$$p_T(t) = \int_{-\infty}^{\infty} p_U(\tau) p_U(t - \tau) d\tau \quad (41)$$

When  $\tau < 0$  or  $\tau > 1$ , the integral evaluates to 0.

$$p_T(t) = \int_0^1 p_U(\tau) p_U(t - \tau) d\tau \quad (42)$$

$$= \int_0^1 p_U(t - \tau) d\tau \quad (43)$$

When  $t < 0$  or  $t > 2$ , the integral evaluates to 0. When  $0 < t < 1$ ,

$$p_T(t) = \int_0^1 p_U(t - \tau) d\tau \quad (44)$$

$$= \int_0^t d\tau + \int_t^1 0 \cdot d\tau \quad (45)$$

$$= t \quad (46)$$

When  $1 < t < 2$ ,

$$p_T(t) = \int_0^1 p_U(t - \tau) d\tau \quad (47)$$

$$= \int_0^{t-1} 0 \cdot d\tau + \int_{t-1}^1 d\tau \quad (48)$$

$$= 2 - t \quad (49)$$

Therefore the PDF is

$$p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < 1 \\ 2 - t & 1 < t < 2 \\ 0 & t > 2 \end{cases} \quad (50)$$

The CDF of  $T$  is given by

$$F_T(t) = \int_0^t p_T(t) dt \quad (51)$$

Simplifying, we get

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 < t < 1 \\ -\frac{t^2}{2} + 2t - 1 & 1 < t < 2 \\ 1 & t > 2 \end{cases} \quad (52)$$

#### 4.5 Verify your results through a plot

**Solution** Download the following files

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/pdf_tri.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_tri.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/tri.dat
```

And run them using

```
python3 pdf_tri.py
python3 cdf_tri.py
```

They will generate the plots in Figures (7) and (8).

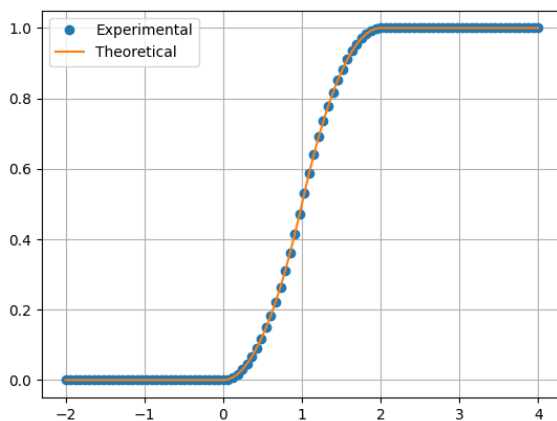


Fig. 7: CDF of  $X$

## V. MAXIMUM LIKELIHOOD

### 5.1 Generate equiprobable $X \in \{1, -1\}$ .

**Solution** Download the following C file

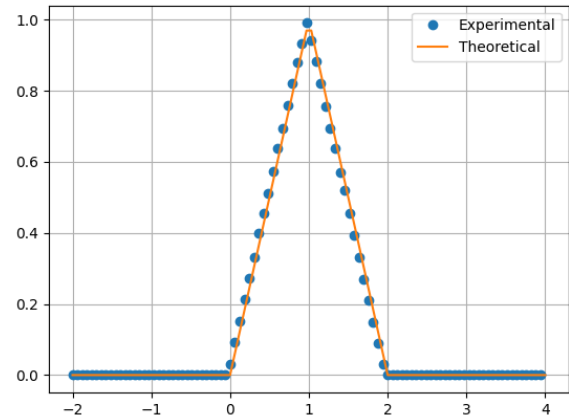


Fig. 8: PDF of  $X$

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gen_bernoulli.c
```

And run it using

```
cc gen_bernoulli.c
./a.out > ber.dat
```

### 5.2 Generate

$$Y = AX + N \quad (53)$$

where  $A = 5$  dB, and  $N \sim \mathcal{N}(0, 1)$ .

**Solution** Download the following files

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gen_y.py
```

And run it using

```
python gen_y.py
```

It will create the file `maxlike.dat` containing  $Y$ .

### 5.3 Plot $Y$ using a scatter plot.

**Solution** Download the following files

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/scatter_y.py
```

```
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/maxlike.dat
```

And run it using

```
python scatter_y.py
```

It will generate the plot in Figure (9).

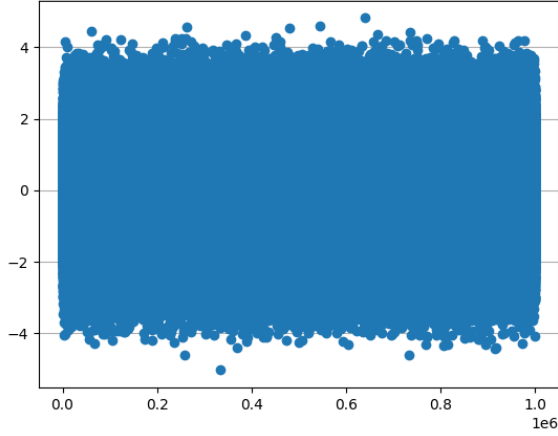


Fig. 9: Scatter plot of  $Y$

5.4 Guess how to estimate  $X$  from  $Y$ .

**Solution** We can estimate  $X$  using

$$\hat{X} = \begin{cases} 1 & Y \geq 0 \\ -1 & Y < 0 \end{cases} \quad (54)$$

5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (55)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = 0) \quad (56)$$

**Solution** We have

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (57)$$

$$= \Pr(AX + N < 0|X = 1) \quad (58)$$

$$= \Pr(A + N < 0) \quad (59)$$

$$= \Pr(N < -A) \quad (60)$$

$$= F_N(-A) \quad (61)$$

$$= Q(A) \quad (62)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (63)$$

$$= \Pr(AX + N \geq 0|X = -1) \quad (64)$$

$$= \Pr(-A + N \geq 0) \quad (65)$$

$$= \Pr(N \geq A) \quad (66)$$

$$= Q(A) \quad (67)$$

So we have our result

$$P_{e|0} = P_{e|1} = Q(A) \quad (68)$$

5.6 Find  $P_e$  assuming  $X$  has equiprobable symbols

**Solution** We need to find

$$P_e = P_{e|0} \cdot \Pr(X = -1) + P_{e|1} \cdot \Pr(X = 1) \quad (69)$$

$$= Q(A) \cdot \frac{1}{2} + Q(A) \cdot \frac{1}{2} \quad (70)$$

$$P_e = Q(A) \quad (71)$$

5.7 Verify by plotting the theoretical  $P_e$  with respect to  $A$  from 0 to 10 dB

**Solution** Download the following files

```
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/error_ml.py
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/ber.dat
wget https://raw.githubusercontent.com/kst164/
AI1110-Assignments/main/
rand_nums/codes/gau.dat
```

And run it using

```
python error_ml.py
```

It will generate the plot in Figure (10).

5.8 Now, consider a threshold  $\delta$  while estimating  $X$  from  $Y$ . Find the value of  $\delta$  that maximizes the theoretical  $P_e$ .

**Solution** We can define our guess as

$$\hat{X} = \begin{cases} 1 & Y \geq \delta \\ -1 & Y < \delta \end{cases} \quad (72)$$

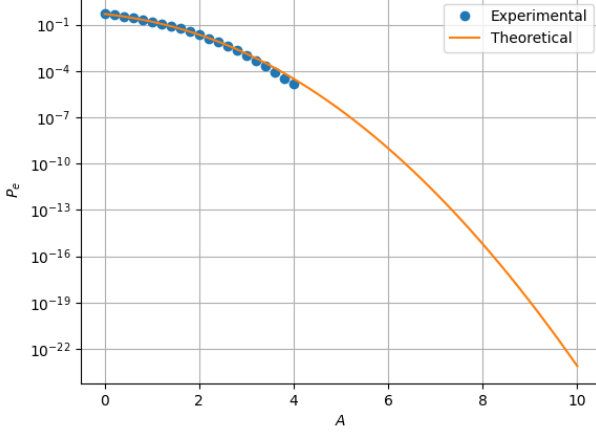


Fig. 10: semilog plot of  $P_e$

Now,

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (73)$$

$$= \Pr(A\hat{X} + N < \delta | X = 1) \quad (74)$$

$$= \Pr(A + N < \delta) \quad (75)$$

$$= \Pr(N < -A + \delta) \quad (76)$$

$$= F_N(-(A - \delta)) \quad (77)$$

$$= Q(A - \delta) \quad (78)$$

And similarly,

$$P_{e|1} = Q(A + \delta) \quad (79)$$

So we have,

$$P_e = P_{e|0} \cdot \Pr(X = -1) + P_{e|1} \cdot \Pr(X = 1) \quad (80)$$

$$= Q(A - \delta) \cdot \frac{1}{2} + Q(A + \delta) \cdot \frac{1}{2} \quad (81)$$

To find the minimal  $P_e$ , we differentiate Eq (81) wrt  $\delta$  and equate it to 0

$$0 = \frac{d}{d\delta} \left( \frac{1}{2} (Q(A - \delta) + Q(A + \delta)) \right) \quad (82)$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \left( e^{-\frac{(A-\delta)^2}{2}} - e^{-\frac{(A+\delta)^2}{2}} \right) \quad (83)$$

So we have

$$(A - \delta)^2 = (A + \delta)^2 \quad (84)$$

$$\implies \delta = 0 \quad (85)$$

So  $P_e$  is minimal when the threshold is  $\delta = 0$ .

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (86)$$

**Solution** We have

$$P_e = P_{e|0}p + P_{e|1}(1 - p) \quad (87)$$

$$= Q(A - \delta)p + Q(A + \delta)(1 - p) \quad (88)$$

Differentiating wrt  $\delta$  to get minimum,

$$0 = \frac{d}{d\delta} (Q(A - \delta)p + Q(A + \delta)(1 - p)) \quad (89)$$

$$= pe^{-\frac{(A-\delta)^2}{2}} - (1 - p)e^{-\frac{(A+\delta)^2}{2}} \quad (90)$$

Taking ln,

$$\ln p - \frac{(A - \delta)^2}{2} = \ln(1 - p) - \frac{(A + \delta)^2}{2} \quad (91)$$

$$2A\delta = \ln \frac{1 - p}{p} \quad (92)$$

$$\delta = \frac{1}{2A} \ln \frac{1 - p}{p} \quad (93)$$

And so we have our result

5.10 Repeat the above exercise using the MAP criterion.

**Solution** Assume  $\Pr(X = -1) = p$ .

According to the MAP criterion, when

$$p_{X|Y}(-1|y) > p_{X|Y}(1|y) \quad (94)$$

We should guess -1, else we guess 1. Now,

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} \quad (95)$$

Now,

$$p_{Y|X}(y|-1) = p_{(-A+N)}(y) \quad (96)$$

Since  $A$  is a constant,

$$p_{Y|X}(y|-1) = p_N(y + A) \quad (97)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} \quad (98)$$

Substituting into (95),

$$p_{X|Y}(-1|y) = \frac{p_{Y|X}(y|-1)p_X(-1)}{p_Y(y)} \quad (99)$$

$$p_{X|Y}(-1|y) = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} p}{p_Y(y)} \quad (100)$$



Similarly,

$$p_{X|Y}(1|y) = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}} (1-p)}{p_Y(y)} \quad (101)$$

Finally, substituting Eqs (100) and (101) into Eq (94), we should guess that  $X = -1$  when

$$\frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} p}{p_Y(y)} > \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}} (1-p)}{p_Y(y)} \quad (102)$$

$$\frac{p}{1-p} > e^{2Ay} \quad (103)$$

$$y < \frac{1}{2A} \ln \frac{p}{1-p} \quad (104)$$

And so we have our guess,

$$\hat{X} = \begin{cases} -1 & y < \delta \\ 1 & \text{otherwise} \end{cases} \quad (105)$$

where  $\delta = \frac{1}{2A} \ln \frac{p}{1-p}$ .

Consider the special case  $p = \frac{1}{2}$

$$\delta = \frac{1}{2A} \ln 1 \quad (106)$$

$$\delta = 0 \quad (107)$$

So our guess is,

$$\hat{X} = \begin{cases} -1 & y < 0 \\ 1 & y > 0 \end{cases} \quad (108)$$

## VI. GAUSSIAN TO OTHER

6.1 Let  $X_1 \sim \mathcal{N}(0, 1)$  and  $X_2 \sim \mathcal{N}(0, 1)$ . Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (109)$$

**Solution** Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_chi.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/pdf_chi.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/chi.dat
```

And run them using

```
python3 cdf_chi.py
python3 pdf_chi.py
```

They will generate the plots in Figures (11) and (12).

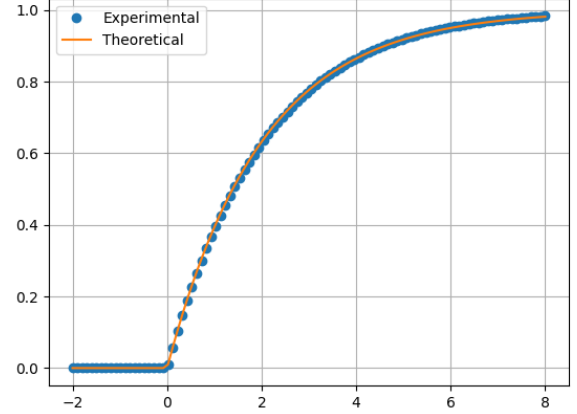


Fig. 11: CDF of  $V$

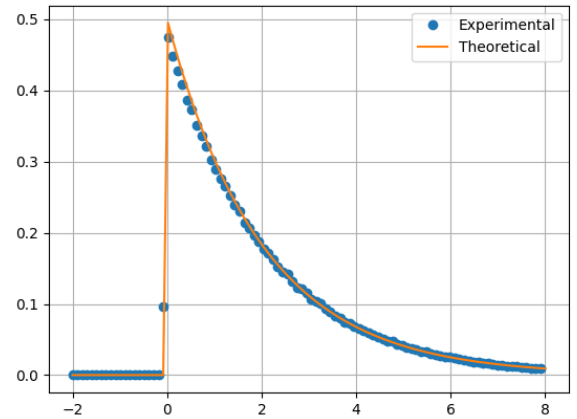


Fig. 12: PDF of  $V$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (110)$$

Find  $\alpha$ .

**Solution** We define the random variables

$$X_1 = R \cos \Theta \quad (111)$$

$$X_2 = R \sin \Theta \quad (112)$$

The Jacobian matrix is given by

$$J = \begin{bmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Theta} \end{bmatrix} \quad (113)$$

$$J = \begin{bmatrix} \cos \Theta & -R \sin \Theta \\ \sin \Theta & R \cos \Theta \end{bmatrix} \quad (114)$$

$$\Rightarrow |J| = R \quad (115)$$

Also, since  $X_1$  and  $X_2$  are iid,

$$p_{X_1, X_2}(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2) \quad (116)$$

$$= \frac{1}{2\pi} e^{-\frac{x_1^2 + x_2^2}{2}} \quad (117)$$

$$= \frac{1}{2\pi} e^{-\frac{r^2}{2}} \quad (118)$$

Now since

$$p_{R, \Theta}(r, \theta) = |J|p_{X_1, X_2}(x_1, x_2) \quad (119)$$

From Eqs (115) and (118), we have

$$p_{R, \Theta}(r, \theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \quad (120)$$

$$\Rightarrow p_R(r) = \int_0^{2\pi} p_{R, \Theta}(r, \theta) d\theta \quad (121)$$

$$= r e^{-\frac{r^2}{2}} \quad (122)$$

We then have

$$F_R(r) = \int_0^r p_R(r) \quad (123)$$

$$= \int_0^r r e^{-\frac{r^2}{2}} dr \quad (124)$$

$$= \left( -e^{-\frac{r^2}{2}} \right)_0^r \quad (125)$$

$$= 1 - e^{-\frac{r^2}{2}} \quad (126)$$

Finally,

$$F_V(x) = \Pr(V \leq x) \quad (127)$$

$$= \Pr(X_1^2 + X_2^2 \leq x) \quad (128)$$

$$= \Pr(R^2 \leq x) = \Pr(R \leq \sqrt{x}) \quad (129)$$

$$= F_R(\sqrt{x}) \quad (130)$$

$$= 1 - e^{-\frac{x}{2}} \quad (131)$$

$$= 1 - e^{-\alpha x} \quad (132)$$

And so we have our result

$$\alpha = \frac{1}{2} \quad (133)$$

### 6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (134)$$

**Solution** Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_ray.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/pdf_ray.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/chi.dat
```

And run them using

```
python3 cdf_ray.py
python3 pdf_ray.py
```

They will generate the plots in Figures (13) and (14).

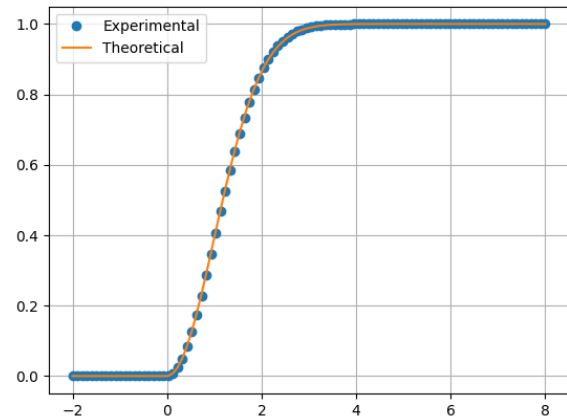


Fig. 13: CDF of  $V$

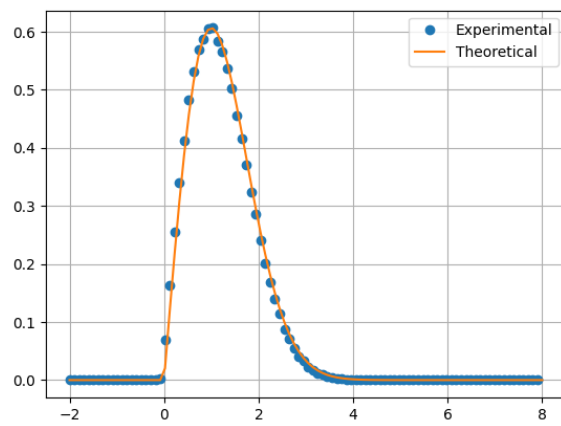


Fig. 14: PDF of  $V$