Random Numbers

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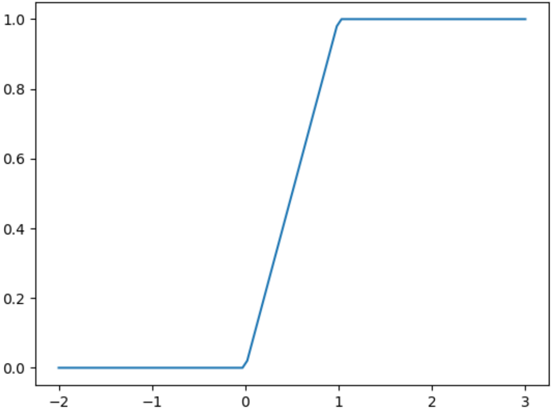
Outline

1.1 - Generating U

The random number generation is done in `./codes/gen_uniform.c`.

1.2 - Graph of CDF of U



1.3 - CDF of U

The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases} \tag{1}$$

And so we can find the CDF,

$$F_U(x) = \int_{-\infty}^{x} p_U(x)dx \tag{2}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{3}$$

1.4 - Mean and variance of U

The mean and variance are calculated in `./codes/mean_var.c`, if `./codes/uni.dat` is piped into `stdin`.

1.5 - Verification

We need to find

$$E\left[U^k\right] = \int_{-\infty}^{+\infty} x^k dF_U(x) \tag{4}$$

But

$$dF_U(x) = p_U(x)dx \tag{5}$$

So from eq (1), we have

$$E\left[U^k\right] = \int_{-\infty}^{+\infty} x^k p_U(x)dx \tag{6}$$

$$= \int_0^1 x^k dx \tag{7}$$

$$= \left(\frac{x^{k+1}}{k+1}\right)\Big|_0^1 \tag{8}$$

$$= \frac{1}{k+1} \tag{9}$$

1.5 - Verification (contd.)

Now using eq (9), we can find the mean of U

$$\mu = E\left[U\right] = \frac{1}{2} \tag{10}$$

and the variance

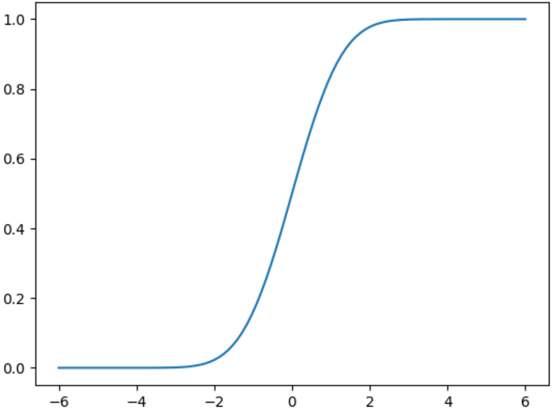
$$\begin{aligned} \text{var}\left[U\right] &= E\left[U^2\right] - E\left[U\right]^2 \\ &= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \\ &= \frac{1}{12} = 0.83333.. \end{aligned} \tag{11–13}$$

These values match with the experimental values of 0.500169 and 0.83395, respectively.

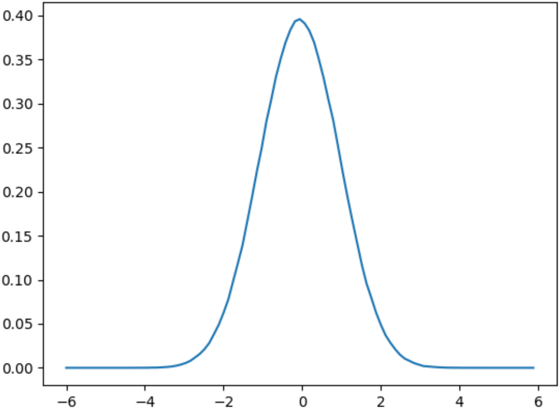
2.1 - Generating X

The random number generation is done in `./codes/gen_gaussian.c`.

2.2 - Graph of CDF of X



2.3 - Graph of PDF of X



2.4 - Mean and variance of X

The mean and variance are calculated in `./codes/mean_var.c`, if `./codes/gau.dat` is piped into `stdin`.

2.5 - Mean and variance of a Gaussian distribution

We have

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{14}$$

The mean is given by

$$E\left[X\right] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{15}$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{16}$$

Since this is the integral of an odd function over an odd interval, and the function goes to zero as x diverges,

$$E\left[U\right] = 0 \tag{17}$$

2.5 (contd)

To calculate variance of X

$$
\begin{aligned}
\text{var}(X) &= E\left[X - E[X]\right]^2 && (18)\\
&= E\left[X^2\right] && (19)\\
&= \int_{-\infty}^{\infty} x^2 p_X(x) dx && (20)\\
&= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx && (21)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) dx && (22)
\end{aligned}
$$

2.5 (contd)

Integrating by parts, we get

$$\text{var}(X) = \frac{1}{\sqrt{2\pi}}\left(-x\exp\left(\frac{-x^2}{2}\right) + \int \exp\left(\frac{-x^2}{2}\right) dx\right)\Bigg|_{-\infty}^{\infty} \tag{23}$$

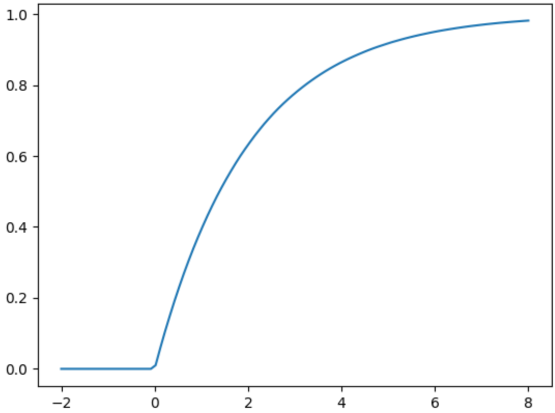
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx \tag{24}$$

$$\tag{25}$$

Substituting the Gaussian integral,

$$\text{var}(X) = 1 \tag{26}$$

3.1 - Graph of CDF of V



3.2 - CDF of V

$$F_V(x) = \Pr(x \leq V) \tag{27}$$
$$= \Pr(x \leq -2\log(1-U)) \tag{28}$$
$$= \Pr\left(\log(1-U) \leq \frac{-x}{2}\right) \tag{29}$$
$$= \Pr\left(1-U \leq \exp\left(\frac{-x}{2}\right)\right) \tag{30}$$
$$= \Pr\left(1-\exp\left(\frac{-x}{2}\right) \leq U\right) \tag{31}$$
$$= F_U\left(1-\exp\left(\frac{-x}{2}\right)\right) \tag{32}$$

3.2 - (contd)

We know $F_U(x)$ from eq (3), so we have

$$F_V(x) = F_U\left(1 - \exp\left(\frac{-x}{2}\right)\right) \tag{33}$$

$$= \begin{cases} 0 & 1 - \exp\left(\frac{-x}{2}\right) < 0 \\ 1 - \exp\left(\frac{-x}{2}\right) & 0 < 1 - \exp\left(\frac{-x}{2}\right) < 1 \\ 1 & 1 - \exp\left(\frac{-x}{2}\right) > 1 \end{cases} \tag{34}$$

Simplifying, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(\frac{-x}{2}\right) & x \geq 0 \end{cases} \tag{35}$$