Random Numbers

Kartheek Tammana

June 25, 2022





Outline

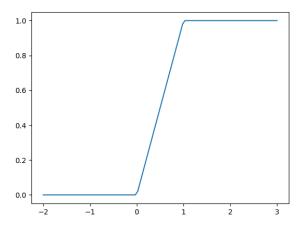
- Question 1
 - Question 1.1
 - Question 1.2
 - Question 1.3
 - Question 1.4
 - Question 1.5
- Question 2
 - Question 2.1
 - Question 2.2
 - Question 2.3
 - Question 2.4
 - Question 2.5
- Question 2.:
- Question 3
 - Question 3.1
 - Question 3.2

1.1 - Generating U

The random number generation is done in ./codes/gen_uniform.c.



1.2 - Graph of CDF of U





1.3 - CDF of U

The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{1}$$

And so we can find the CDF,

$$F_U(x) = \int_{-\infty}^{x} p_U(x) dx \tag{2}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$
 (3)



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1.4 - Mean and variance of U

The mean and variance are calculated in ./codes/mean_var.c, if ./codes/uni.dat is piped into stdin.



1.5 - Verification

We need to find

$$E\left[U^{k}\right] = \int_{-\infty}^{+\infty} x^{k} dF_{U}(x) \tag{4}$$

But

$$dF_U(x) = p_U(x)dx (5)$$

So from eq (1), we have

$$E\left[U^{k}\right] = \int_{-\infty}^{+\infty} x^{k} \rho_{U}(x) dx \tag{6}$$

$$= \int_0^1 x^k dx \tag{7}$$

$$= \left(\frac{x^{k+1}}{k+1}\right)\Big|_0^1 \tag{8}$$

$$=\frac{1}{k+1}\tag{9}$$

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1.5 - Verification (contd.)

Now using eq (9), we can find the mean of U

$$\mu = E[U] = \frac{1}{2} \tag{10}$$

and the variance

$$var[U] = E[U^2] - E[U]^2$$
 (11)

$$=\frac{1}{3}-\left(\frac{1}{2}\right)^2\tag{12}$$

$$=\frac{1}{12}=0.83333.. (13)$$

These values match with the experimental values of 0.500169 and 0.83395, respectively.

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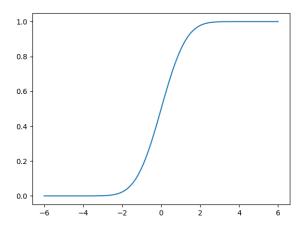
2.1 - Generating X

The random number generation is done in ./codes/gen_gaussian.c.



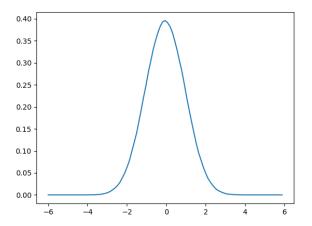
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2.2 - Graph of CDF of X





2.3 - Graph of PDF of X





2.4 - Mean and variance of X

The mean and variance are calculated in ./codes/mean_var.c, if ./codes/gau.dat is piped into stdin.



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2.5 - Mean and variance of a Gaussian distribution

We have

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{14}$$

The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{15}$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{16}$$

Since this is the integral of an odd function over an odd interval, and the function goes to zero as x diverges,

$$E\left[U\right] = 0\tag{17}$$

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2.5 (contd)

To calculate variance of X

$$var(X) = E[X - E[X]]^{2}$$
 (18)

$$= E\left[X^2\right] \tag{19}$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{20}$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{21}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) dx \tag{22}$$



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2.5 (contd)

Integrating by parts, we get

$$\operatorname{var}(X) = \frac{1}{\sqrt{2\pi}} \left(-x \exp\left(\frac{-x^2}{2}\right) + \int \exp\left(\frac{-x^2}{2}\right) dx \right) \Big|_{-\infty}^{\infty}$$
 (23)

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx \tag{24}$$

(25)

Substituting the Gaussian integral,

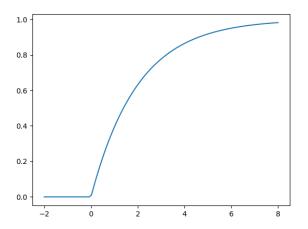
$$var(X) = 1 \tag{26}$$

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3.1 - Graph of CDF of V





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3.2 - CDF of V

$$F_V(x) = \Pr\left(x \le V\right) \tag{27}$$

$$= \Pr\left(x \le -2\log(1-U)\right) \tag{28}$$

$$= \Pr\left(\log(1-U) \le \frac{-x}{2}\right) \tag{29}$$

$$= \Pr\left(1 - U \le \exp\left(\frac{-x}{2}\right)\right) \tag{30}$$

$$= \Pr\left(1 - \exp\left(\frac{-x}{2}\right) \le U\right) \tag{31}$$

$$=F_{U}\left(1-\exp\left(\frac{-x}{2}\right)\right)\tag{32}$$



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3.2 - (contd)

We know $F_U(x)$ from eq (3), so we have

$$F_{V}(x) = F_{U}\left(1 - \exp\left(\frac{-x}{2}\right)\right)$$

$$= \begin{cases} 0 & 1 - \exp\left(\frac{-x}{2}\right) < 0\\ 1 - \exp\left(\frac{-x}{2}\right) & 0 < 1 - \exp\left(\frac{-x}{2}\right) < 1\\ 1 & 1 - \exp\left(\frac{-x}{2}\right) > 1 \end{cases}$$
(33)

Simplifying, we get

$$F_V(x) = \begin{cases} 0 & x < 0\\ 1 - \exp\left(\frac{-x}{2}\right) & x \ge 0 \end{cases}$$
 (35)

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