

I. UNIFORM RANDOM VARIABLES

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat

Solution Download the following C file.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gen_uniform.c
```

And run it using

```
cc gen_uniform.c
./a.out > uni.dat
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat.

Solution Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_uni.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/uni.dat
```

And run it using

```
python3 cdf_uni.py
```

It will generate the plot in Figure (1).

- 1.3 Find a theoretical expression for the CDF of U .

Solution The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

And so we can find the CDF,

$$F_U(x) = \int_{-\infty}^x p_U(x) dx \quad (2)$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (3)$$

- 1.4 Write a C program to find the mean and variance of U .

Solution Download the following files.

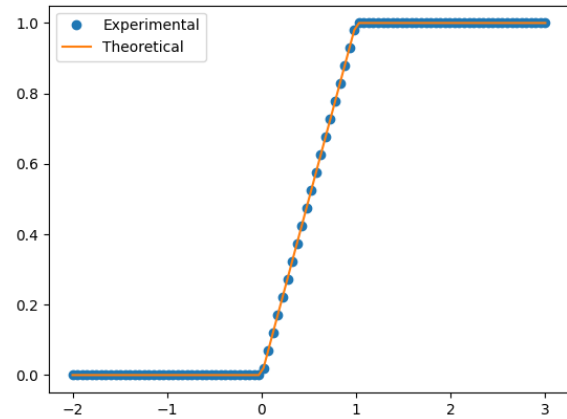


Fig. 1: CDF of U

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/mean_var.c
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/uni.dat
```

And run it using

```
cc mean_var.c
./a.out < uni.dat
```

It will give the output

```
Mean : 0.499631
Variance: 0.083320
```

- 1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (4)$$

Solution We use the fact that

$$dF_U(x) = p_U(x) dx \quad (5)$$

So from eq (1), we have

$$E[U^k] = \int_{-\infty}^{+\infty} x^k p_U(x) dx \quad (6)$$

$$= \int_0^1 x^k dx \quad (7)$$

$$= \left(\frac{x^{k+1}}{k+1} \right) \Big|_0^1 \quad (8)$$

$$= \frac{1}{k+1} \quad (9)$$

Now using eq (9), we can find the mean of U

$$\mu = E[U] = \frac{1}{2} \quad (10)$$

and the variance

$$\text{var}[U] = E[U^2] - E[U]^2 \quad (11)$$

$$= \frac{1}{3} - \left(\frac{1}{2} \right)^2 \quad (12)$$

$$= \frac{1}{12} = 0.83333.. \quad (13)$$

We see that the theoretical mean and variance match with the experimental values.

II. CENTRAL LIMIT THEOREM

Let X be a random variable defined as

$$X = \sum_{i=1}^{12} U_i - 6 \quad (14)$$

- 2.1 Generate 10^6 samples of X using a C program, and save into a file called gau.dat

Solution Download the following C file.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gen_gaussian.c
```

And run it using

```
cc gen_gaussian.c
./a.out > gau.dat
```

- 2.2 Load the gau.dat file into python and plot the empirical CDF of X using the samples in gau.dat. What properties does the CDF have?

Solution Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_gau.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gau.dat
```

And run it using

```
python3 cdf_gau.py
```

It will generate the plot in Figure (2).

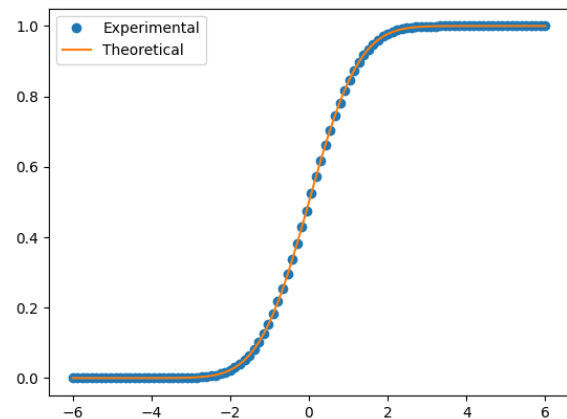


Fig. 2: CDF of X

Properties of the CDF:

- $F_X(x) = \frac{1}{2\pi} \int_{-\infty}^x \exp\left(-\frac{u^2}{2}\right) du$
- $\lim_{x \rightarrow -\infty} F_X(x) = 0$
- $\lim_{x \rightarrow \infty} F_X(x) = 1$
- $F_X(0) = \frac{1}{2}$
- $F_X(x) + F_X(-x) = 1$

- 2.3 Load the gau.dat file into python and plot the empirical PDF of X using the samples in gau.dat. What properties does the PDF have?

Solution Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/pdf_gau.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gau.dat
```

And run it using

```
python3 pdf_gau.py
```

It will generate the plot in Figure (3).

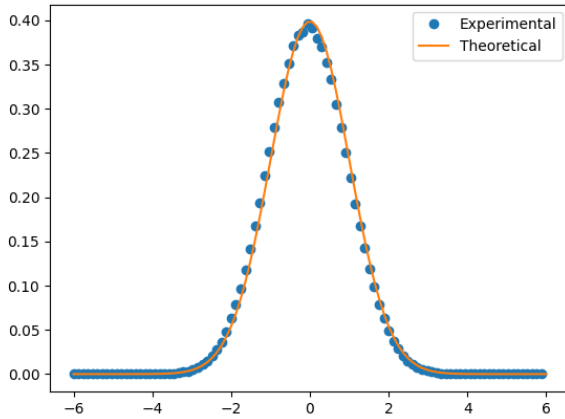


Fig. 3: PDF of X

Properties of the PDF:

- $F_X(x) = F_X(-x)$ i.e., symmetric about 0
- PDF is bell shaped
- Peak of PDF is also the mean

2.4 Write a C program to find the mean and variance of X .

Solution Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/mean_var.c
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/gau.dat
```

And run it using

```
cc mean_var.c
./a.out < gau.dat
```

It will give the output

```
Mean : 0.000635
Variance: 0.999490
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (15)$$

repeat the above exercise theoretically.

Solution The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (16)$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (17)$$

Since this is the integral of an odd function over an odd interval, and the function goes to zero as x diverges,

$$E[U] = 0 \quad (18)$$

To calculate variance of X

$$\text{var}(X) = E[X - E[X]]^2 \quad (19)$$

$$= E[X^2] \quad (20)$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (21)$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (22)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) dx \quad (23)$$

Integrating by parts, we get

$$\text{var}(X) = \frac{1}{\sqrt{2\pi}} \left(-x \exp\left(-\frac{x^2}{2}\right) + \int \exp\left(-\frac{x^2}{2}\right) dx \right) \quad (24)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad (25)$$

$$(26)$$

Substituting the Gaussian integral,

$$\text{var}(X) = 1 \quad (27)$$

III. FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2\ln(1 - U) \quad (28)$$

and plot its CDF

Solution Download the following files.

```
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/cdf_v.py
wget https://raw.githubusercontent.com/kst164/AI1110-Assignments/main/rand_nums/codes/uni.dat
```

And run it using

```
python3 cdf_v.py
```

It will generate the plot in Figure (4).

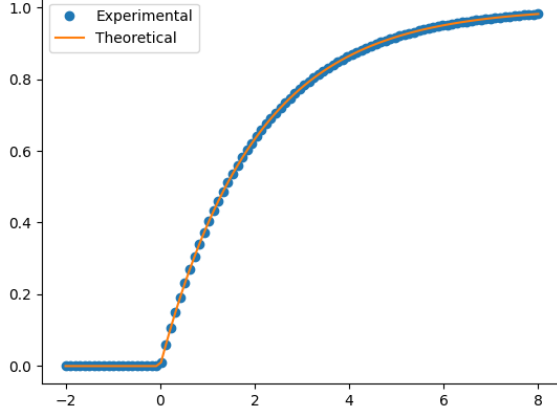


Fig. 4

3.2 Find a theoretical expression for $F_V(x)$

Solution

$$F_V(x) = \Pr(x \leq V) \quad (29)$$

$$= \Pr(x \leq -2 \log(1 - U)) \quad (30)$$

$$= \Pr\left(\log(1 - U) \leq \frac{-x}{2}\right) \quad (31)$$

$$= \Pr\left(1 - U \leq \exp\left(\frac{-x}{2}\right)\right) \quad (32)$$

$$= \Pr\left(1 - \exp\left(\frac{-x}{2}\right) \leq U\right) \quad (33)$$

$$= F_U\left(1 - \exp\left(\frac{-x}{2}\right)\right) \quad (34)$$

We know $F_U(x)$ from eq (3), so we have

$$F_V(x) = F_U\left(1 - \exp\left(\frac{-x}{2}\right)\right) \quad (35)$$

$$= \begin{cases} 0 & 1 - \exp\left(\frac{-x}{2}\right) < 0 \\ 1 - \exp\left(\frac{-x}{2}\right) & 0 < 1 - \exp\left(\frac{-x}{2}\right) < 1 \\ 1 & 1 - \exp\left(\frac{-x}{2}\right) > 1 \end{cases} \quad (36)$$

Simplifying, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(\frac{-x}{2}\right) & x \geq 0 \end{cases} \quad (37)$$